

# Flavors of Rigidity

## Flavor I – Local Rigidity

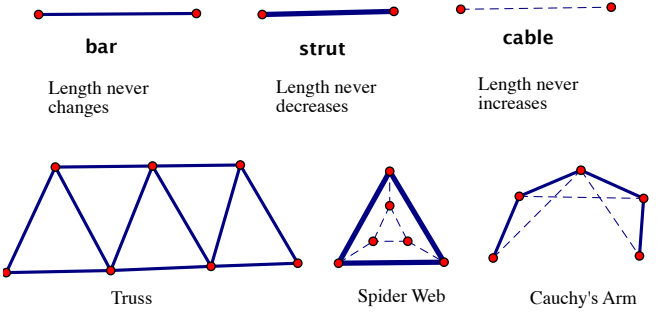
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October 2014

# The model: Finite frameworks



The basic members of a framework with some examples.

# Notation

The configuration of points is denoted as

$$\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n),$$

where  $\mathbf{p}_i$  is in  $\mathbb{R}^d$ . The underlying graph  $G$  is simple (no multiple edges or loops) but each edge or *member* is 'colored' as a bar, strut or cable.

All together the *framework* (or *tensegrity*) is denoted as  $(G, \mathbf{p})$ . Think of the members as imposing constraints of the configuration, and the "flavor" of rigidity is determined by the result of those constraints.

# The fundamental questions

**Rigidity** Given a (tensegrity) framework, what are the other configurations (locally, in the given Euclidean space, or in any higher-dimensional Euclidean space).

**Existence** Given the (tensegrity) constraints on a graph  $G$  what are the configurations (if any) that satisfy those constraints.

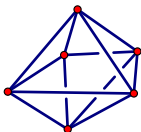
# Flavor 1: Local Rigidity

We say  $(G, \mathbf{p})$  is (*locally*) *rigid* if any of the following equivalent statements hold:

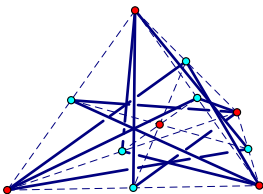
- Any continuous motion  $\mathbf{p}(t), 0 \leq t \leq 1$  of the configuration satisfying the member constraints is a congruence.
- Any analytic motion  $\mathbf{p}(t), 0 \leq t \leq 1$  of the configuration satisfying the member constraints is a congruence.
- For every  $\epsilon > 0$  such that when  $|\mathbf{q}_i - \mathbf{q}_j| = |\mathbf{p}_i - \mathbf{p}_j|$  for  $\{i, j\}$  a bar of  $G$ , and  $|\mathbf{p} - \mathbf{q}| < \epsilon$ , then  $\mathbf{p}$  is congruent to  $\mathbf{q}$ . (The equality constraint is replaced by the appropriate inequality constraint for struts and cables.)

A configuraton  $\mathbf{p}$  is *congruent* to a configuraton  $\mathbf{q}$  if  $|\mathbf{p}_i - \mathbf{p}_j| = |\mathbf{q}_i - \mathbf{q}_j|$  for all  $i, j$ .

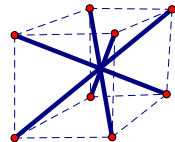
# Examples that are Locally Rigid



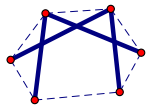
Convex  
Triangulated  
Polytope



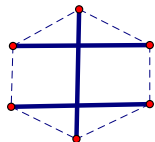
Complete Bipartite  
Graph  $K(6,5)$



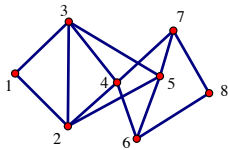
Cube with Long  
Diagonals



A Cauchy Polygon



A Desargues'  
Framework



A Trilaterated Framework  
(Nodes attached in order.)

# Infinitesimal Flexes

We say that  $\mathbf{p}' = (\mathbf{p}'_1, \dots, \mathbf{p}'_n)$  is an *infinitesimal flex* of  $(G, \mathbf{p})$  if  $R(\mathbf{p})(\mathbf{p}') = 0$ , which means that for all bars  $\{i, j\}$  of  $G$ , a bar framework,

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) = 0.$$

For cables and struts, this constraint is replaced by

$$(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) \begin{cases} \leq 0 & : \{i, j\} \text{ is a cable} \\ \geq 0 & : \{i, j\} \text{ is a strut.} \end{cases}$$

We say that an infinitesimal flex  $\mathbf{p}'$  of  $(G, \mathbf{p})$  is *trivial* in  $\mathbb{R}^d$  if it is the time 0 derivative of a differentiable path of congruences of  $\mathbb{R}^d$ . This turns out to be equivalent to saying that  $\mathbf{p}'_i = A\mathbf{p}_i + \mathbf{b}$ , where  $A = -A^T$  is a skew symmetric  $d$ -by- $d$  matrix, and  $\mathbf{b}$  is a constant infinitesimal translation.  $()^T$  is transpose.

# Infinitesimal Rigidity

We say that  $(G, \mathbf{p})$  is *infinitesimally rigid* if all its infinitesimal flexes are trivial. The following is fundamental.

## Theorem (Rank Condition)

*A bar framework  $(G, \mathbf{p})$  is infinitesimally rigid in  $\mathbb{R}^d$  for  $n \geq d$  if and only if the rank of  $R(\mathbf{p})$  is  $nd - d(d + 1)/2$ .*

## Theorem (Infinitesimal implies Local)

*If a bar framework  $(G, \mathbf{p})$  is infinitesimally rigid in  $\mathbb{R}^d$ , then it is locally rigid in  $\mathbb{R}^d$ .*



# Infinitesimal Rigidity

One consequence of this definition is that certain counts on the members and nodes are necessary (but not always sufficient) for infinitesimal rigidity.

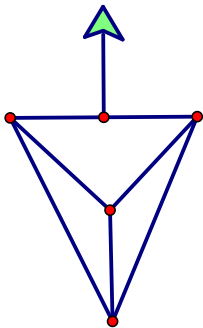
## Theorem (Infinitesimal Counts)

*If a bar framework  $(G, \mathbf{p})$  is infinitesimally rigid in  $\mathbb{R}^d$  with  $n \geq d$  vertices and  $m$  bars, then*

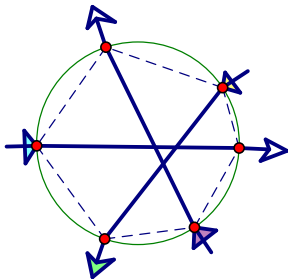
$$m \geq dn - d(d + 1)/2$$

For the plane, these conditions are  $m \geq 2n - 3$ , and for 3-space  $m \geq 3n - 6$ .

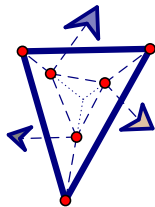
# Infinitesimal Flexes



Clothes line

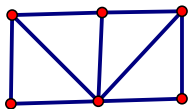


Bipartite push-me pull-you



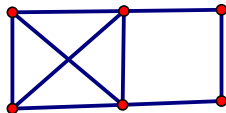
Desargues'  
Framework

# Examples and Non-examples of Infinitesimally Rigid Frameworks



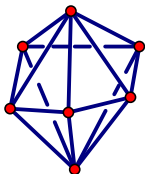
In the plane

Infinitesimally rigid



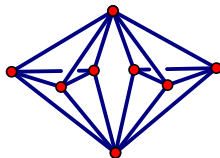
Not infinitesimally rigid

$$m=9=2n-3=2 \times 6-3$$



In 3-space

Any strictly convex  
triangulated polytope



The double banana

$$m=9=3n-6$$

# Rigidity Map

The *rigidity map* is

$$f_G : \mathbb{R}^{nd} \rightarrow \mathbb{R}^m$$

given by  $f_G(\mathbf{p}) = (\dots, |\mathbf{p}_i - \mathbf{p}_j|^2, \dots)$ , where  $\{i, j\}$  is a member (edge) of the graph  $G$  with  $n$  nodes and  $m$  members, and the configuration  $\mathbf{p}$ , regarded as a single column vector of  $n$  blocks of size  $d$  is in  $\mathbb{R}^{nd}$ . One half times the differential of  $f_G$  is

$$\frac{1}{2}df_G = R(\mathbf{p}),$$

which is defined to be the *rigidity matrix* for the graph  $G$ . So

$$R(\mathbf{p})(\mathbf{p}') = (\dots, (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j), \dots).$$

Note that the infinitesimal flexes  $\mathbf{p}'$  are simply the vectors in the kernel of the matrix.

# Proof that Infinitesimal implies Local Rigidity

Call  $\mathcal{T}$  the equivalence relation on  $\mathbb{R}^{nd}$  that identifies two configurations if they are congruent. Then the rigidity map  $f_G$  defines the map

$$\hat{f}_G : \mathbb{R}^{nd} / \mathcal{T} \rightarrow \mathbb{R}^m.$$

Then when  $(G, \mathbf{p})$  is a bar framework, the inverse function theorem applies when the rank of  $\hat{f}_G$  is  $m$ . If the affine span of the points  $\mathbf{p} = (\mathbf{p}_1, \dots, \mathbf{p}_n)$  are full dimensional in  $\mathbb{R}^d$ , then  $\hat{f}_G$  will be injective in a neighborhood of  $\mathbf{p}$ . The dimension of the space of congruences of  $\mathbb{R}^d$  is  $d(d-1)/2$  (the dimension of the orthogonal rotation group) plus  $d$  (the dimension of the space of translations) which is  $d(d+1)/2$  total. So, in particular, if  $(G, \mathbf{p})$  is infinitesimally rigid as a bar framework and  $n \geq d$ , then  $m \geq nd - d(d+1)/2$ .

# Generic Hypothesis

One temptation (in life) is to assume that nothing is special. Nothing can be measured exactly, anyway, so why not just assume that any given configuration has no symmetries, and that each point is independent of all the others. :( Rigorously, we say that a configuration  $\mathbf{p}$  in  $\mathbb{R}^d$  is *generic* if there is no non-zero polynomial relation over the integers among the coordinates of  $\mathbf{p}$ .

## Proposition

*The generic configurations in  $\mathbb{R}^d$  are of full measure.*

For example, if a configuration  $\mathbf{p}$  is generic in  $\mathbb{R}^d$  for  $d = 2$ , no 3 of its points are collinear, no 6 of its points lie on a conic, no 3 of its points form an isosceles triangle, etc.

# Laman's Theorem

We say that a bar framework  $(G, \mathbf{p})$  with  $n$  nodes and  $m$  bars in  $\mathbb{R}^2$  is a *Laman graph* if  $m = 2n - 3$  and any subgraph  $G'$  of  $G$  with  $n'$  nodes and  $m'$  bars is such that  $m' \leq 2n' - 3$ .

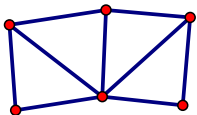
## Theorem (Laman 1970)

*If  $G$  is a graph with  $m = 2n - 3$  and  $\mathbf{p}$  is generic in  $\mathbb{R}^2$ , then  $(G, \mathbf{p})$  is infinitesimally rigid in  $\mathbb{R}^2$  if and only if  $G$  is a Laman graph if and only if  $(G, \mathbf{p})$  is locally rigid in  $\mathbb{R}^2$ .*

A consequence of this result is that there is an algorithm (in its most popular form it is called the *pebble game*) which decides whether  $G$  is a Laman graph and thus is generically (locally) rigid in  $cn^2$  steps, where  $c$  is a constant.

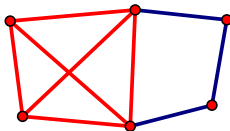
# Laman Examples

Think generic



Infinitesimally rigid

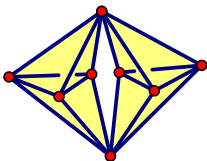
In the plane



$$m=9=2n-3=2 \times 6-3$$

Not infinitesimally rigid with the over constrained subgraph guaranteed by Laman's Theorem.

In 3-space



$$m=3n-6$$

The double banana satisfies the Laman-type condition, but it is generically flexible



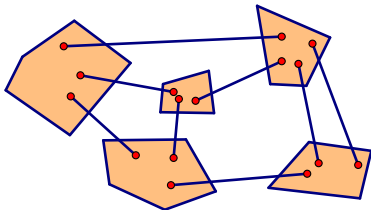
# Big Question

## Question

*Is there a polynomial time combinatorial algorithm to determine generic local rigidity for bar frameworks in  $\mathbb{R}^3$  or  $\mathbb{R}^d$  for  $d \geq 3$  for that matter?*

There are classes of frameworks, body-bar frameworks for example, where there is a combinatorial polynomial-time algorithm, to determine generic local rigidity. (See Tay-Whiteley)

Body-Bar Frameworks



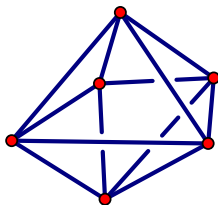
All the bodies are rigid objects, but all the vertices are generically attached to the bodies at distinct points.

# Convexity

There are classes of frameworks, where geometric conditions, such as convexity, implies infinitesimal rigidity.

## Theorem (Max Dehn 1916)

*A convex polyhedron, with all its faces triangles, is infinitesimally rigid in  $\mathbb{R}^3$ .*



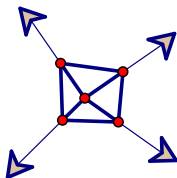
Convex Triangulated Polytope



# Rigidity Matrix Calculations

The rows of the rigidity matrix correspond to the members of  $G$ , and the columns to the vertices of  $G$ .

The equilibrium condition for a stress is shown graphically. (This is the basis of a method called 'graphical statics' that was promoted by J. Clerk Maxwell in that 1800's and used extensively to calculate forces in various structures.)



# Problems

- 1 Show that if the number of bars of a bar framework  $(G, \mathbf{p})$  in  $\mathbb{R}^d$  is  $m = nd - d(d + 1)/2$  and  $n \geq d$ , then it is infinitesimally rigid in  $\mathbb{R}^d$  if and only if the only equilibrium stress is the 0 stress. Such a bar framework is called *isostatic*.
- 2 Suppose that  $P$  is a convex polytope in  $\mathbb{R}^3$  with  $n$  vertices and  $m$  edges, where all the  $f$  faces are triangles. Show that  $m = 3n - 6$ . Thus  $P$  is isostatic if and only if it has only the 0 equilibrium stress.
- 3 Show that if there is a non-zero equilibrium stress at, at least some, of the edges (bars) adjacent to a vertex  $\mathbf{p}_i$  of a convex polytope with only triangular faces, then there are at least four changes in sign as one proceeds cyclicly around  $\mathbf{p}_i$ .

# Problems

Let  $H$  be any embedded connected graph, with no loops or multiple edges, on the surface of a sphere. Let  $f_i$  be the number of faces with  $i$  sides in the sphere,  $n$  the number of vertices of  $H$  and  $m$  the number of edges in  $H$ .

- 4 Use the Euler characteristic to show that

$$n - m + (f_3 + f_4 + f_5 + f_6 + \dots) = 2.$$

- 5 Show that if each edge is labeled with a plus or minus, then the number of sign changes in faces with  $i$  sides is at most  $i$  if  $i$  is even, and it is at most  $i - 1$  if  $i$  is odd.
- 6 Put these problems together to prove Dehn's Theorem.

# References



Max Dehn, Über die Starrheit konvexer Polyeder  
*Math. Ann.*, 77, 466–473, 1916.



Tiong-Seng Tay, and Walter Whiteley, Recent advances in the generic rigidity of structures,  
*Structural Topology*, 9, 31–88, 1984.