Flavors of Rigidity
Flavor II - Global Rigidity
(following the generic religion)

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October 2014

Stress

An important idea in rigidity is the notion of a stress, which is a scalar $\omega_{ij} = \omega_{ji}$ assigned to each member $\{i, j\}$ of the graph G. (For non-members $\omega_{ij} = 0$.) Denote the whole stress as a row vector by $\omega = (\dots, \omega_{ij}, \dots)$. We say that ω is in *equilibrium* with respect to a configuration $\mathbf{p}$ if for each vertex j,

$$\sum_i \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0.$$

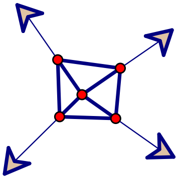
With respect to the rigidity matrix R, this is equivalent to $\omega R(\mathbf{p}) = 0$. In other words ω is in the cokernel of R when it is in equilibrium. The matrix $R(\mathbf{p})$ is of the form below:

$$R(\mathbf{p}) = \begin{pmatrix} 0 & \dots & \mathbf{p}_i - \mathbf{p}_j & 0 & \dots & 0 & \mathbf{p}_j - \mathbf{p}_i & & \dots \\ 0 & \dots & \mathbf{p}_k - \mathbf{p}_j & 0 & \dots & 0 & 0 & & \mathbf{p}_j - \mathbf{p}_k \end{pmatrix}$$

Rigidity Matrix Calculations

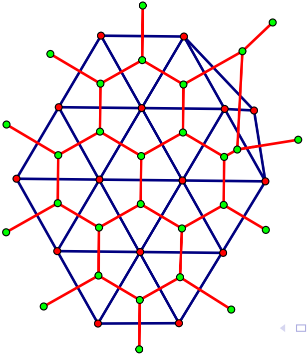
The rows of the rigidity matrix correspond to the members of G, and the columns to the vertices of G.

The equilibrium condition for a stress is shown graphically. (This is the basis of a method called 'graphical statics' that was promoted by J. Clerk Maxwell in that 1800's and used extensively to calculate forces in various structures.)



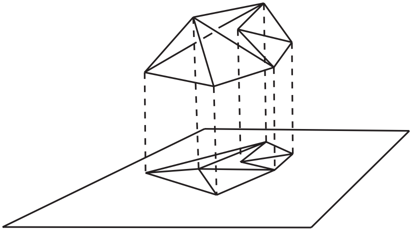
Reciprocal

This shows the reciprocal diagram, where the faces of the planar graph G correspond to the vertices of the reciprocal, edges of the reciprocal correspond to and are perpendicular to the edges of G, and faces of reciprocal correspond to the edges of the reciprocal. The ratio of the oriented length of an edge in the reciprocal to the corresponding oriented length in G is the equilibrium stress in G.



Maxwell-Cremona

One very nice way to create stresses in $\mathbb{R}^3$ (or the plane) is by the following correspondence. Suppose that $(G, \mathbf{p})$ are vertices and edges of a simply-connected 2-manifold in $\mathbb{R}^3$, whose faces are regarded a rigid plates, and $\mathbf{p}'$ is an infinitesimal flex such that restricted to each of the facial plates, is trivial. Then there is a bijection between such flexes and equilibrium stresses ω such that $\omega_{ij} > 0$ when the dihedral angle at edge $\{i, j\}$ is increasing and $\omega_{ij} < 0$ when the angle at edge $\{i, j\}$ is decreasing.



Schläfli's and Alexander's Theorems

Theorem (Schläfli-Ralph Alexander)

Any infinitesimal flex of a closed polyhedral surface in $\mathbb{R}^3$ leaves the total mean curvature invariant.

Theorem (I. Sabitov 1995)

Any continuous flex of a closed polyhedral surface in $\mathbb{R}^3$ leaves the volume it bounds constant.

Stresses in tensegrities

Stresses and (infinitesimal) motions are dual concepts, and stresses can be thought of as "blocking" motions. For example, an infinitesimal motion $\mathbf{p}'$ of a tensegrity $(G, \mathbf{p})$ must satisfy:

$$(\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \leq 0, \;\; (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) = 0, \;\; (\mathbf{p}_i - \mathbf{p}_j)\cdot(\mathbf{p}'_i - \mathbf{p}'_j) \geq 0,$$

for a cable, bar, strut respectively. A stress ω for $\mathbf{p}$ is *proper* if

$$\omega_{ij} \geq 0, \;\; \omega_{ij} \leq 0,$$

for a cable and strut respectively. (No condition for a bar.)

Proposition (Roth-Whiteley)

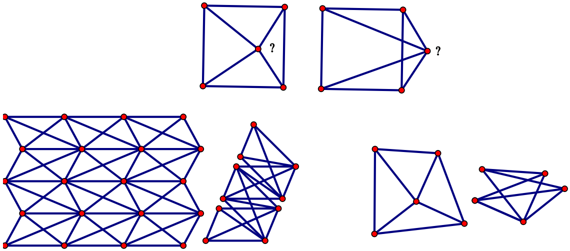
A tensegrity framework $(G, \mathbf{p})$ is infinitesimally rigid if and only if there is a proper equilibrium stress ω such that every non-trivial infinitesimal flex $\mathbf{p}'$ is blocked by ω in the sense that

$$\omega R(\mathbf{p})\mathbf{p}' = \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j) < 0.$$

Global Rigidity

A (tensegrity) framework $(G, \mathbf{p})$ is *globally rigid* in $\mathbb{R}^d$ if when $(G, \mathbf{q})$ satisfies the constraints of $(G, \mathbf{p})$, where $\mathbf{q}$ is a configuration in $\mathbb{R}^d$, then $\mathbf{p}$ is congruent to $\mathbf{q}$. That is cables don't get longer, struts don't get shorter, and bars stay the same length.

Problem: Where in the plane should a vertex, connected to each vertex of a square, be placed to get a globally rigid bar farmework?



The Miura fold here can be used to fold the framework flat in 3-space.

Detection

Question (Saxe 1979)

How do you tell when a given framework (or tensegrity) is globally rigid?

This is known to be hard. For example, for a simple cone on a cycle, the framework is globally rigid in the plane if and only if there is an assignment of pluses and minuses to the central angles such that the signed angles add to a multiple of 2π. This is equivalent to the knapsack or subset-sum problem that is NP complete.

The Generic Option

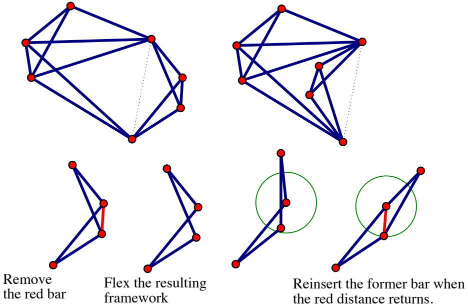
A bar framework $(G, \mathbf{p})$ is a *redundantly rigid* if it is locally rigid and it remains so after the removal of any bar. A graph G is $(d+1)$-connected, if it is connected and it remains so after the removal of no more than d vertices.

Theorem (Hendrickson 1992)

If $(G, \mathbf{p})$ is a globally rigid bar framework in $\mathbb{R}^d$, not a simplex, where $\mathbf{p}$ is generic, then it is redundantly rigid and vertex $(d+1)$-connected.

Clearly $(d+1)$-connectivity is needed since otherwise the framework can be reflected about a hyperplane through the separating vertices. If $(G, \mathbf{p})$ is not redundantly rigid, then remove a bar and watch it flex. When the distance between the endpoints returns to the original distance, it will be in a non-congruent configuration.

Characterizing Global Rigidity

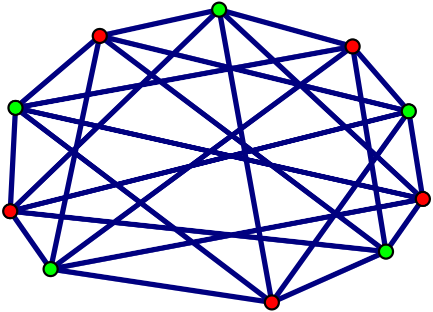


Conjecture (Hendrickson 1986)

For a generic configuration $\mathbf{p}$ *in* $\mathbb{R}^d$*, the graph G being redundantly rigid and* $(d+1)$*-connected is implies that* $(G, \mathbf{p})$ *is globally rigid in* $\mathbb{R}^d$*.*

The Bad News

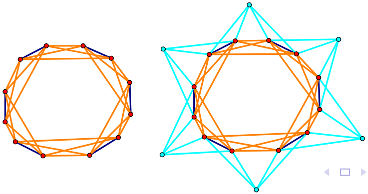
Hendrickson's Conjecture is false in $\mathbb{R}^d$ for $d \geq 3$.
For a while, the only known example in $\mathbb{R}^3$ was the complete bipartite graph $K(5,5)$.



Another Counterexample

This example is due, very recently, to Tibor Jordan, coming from his study of body and hinge structures. The framework on the left consists of 6 tetrahedra joined cyclicly along opposite edges (bars). It has 12 vertices, and each vertex is adjacent to 5 other vertices, so it has the same vertex-edge count as the regular icosahedron and thus $m = 3n - 6$, namely $m = 5 \cdot 12/2 = 30$, and $n = 12$. It turns out this framework is generically isostatic, and by Hendrickson's Theorem is not globally rigid in $\mathbb{R}^3$. But when one takes the cone over each of the tetrahedra as in the framework on the right, it becomes redundantly rigid and vertex 4 connected, and another counterexample to Hendrickson's conjecture in $\mathbb{R}^3$.

Problem: Find a geometric configuration where the framework on the left is infinitesimally rigid.



The Good News

Theorem (Jackson-Jordan 2005)

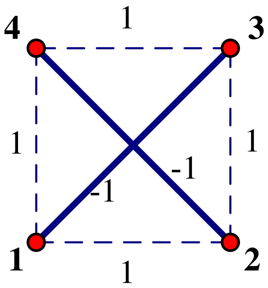
Hendrickson's Conjecture is true in $\mathbb{R}^2$.

The pebble game algorithms for the Hendrickson conditions work well and so they work in reasonable polynomial-time.

But there is still a problem. Can you tell when the configuration is at one of the exceptional configurations? For local rigidity, the generic condition can be replaced by the condition that rank of the rigidity matrix be maximal. For global rigidity, it is not that easy. A key concept is the notion of a stress matrix and stress-energy.

The Stress Matrix

Suppose that $\omega = (\dots, \omega_{ij}, \dots)$ is an equilibrium stress for a framework $(G, \mathbf{p})$ with n nodes. The corresponding stress matrix Ω is an n-by-n symmetric matrix, with $\{i, j\}$ entry $-\omega_{ij}$, and diagonal entries such that its row and column sums are 0. For example for the square framework with diagonals, you get the following stress matrix:



$$\Omega = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

Matrix Formulation

For any configuration $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ in $\mathbb{R}^d$, define the following $(d+1)$-by-n *configuration matrix*

$$\hat{P} = \begin{pmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \ldots & \mathbf{p}_n \\ 1 & 1 & \ldots & 1 \end{pmatrix},$$

where each $\mathbf{p}_i$ is an n-by-1 column vector/matrix. Then the equilibrium equation for the stress ω becomes

$$\hat{P}\Omega = 0.$$

The rows of $\hat{P}$ are in the (co-)kernel of Ω, and the rank of $\hat{P}$ is the dimension of the affine span of the configuration p. So if the rank of Ω is $n - d - 1$, and $\mathbf{p}$ has a d-dimensional affine span in $\mathbb{R}^d$, the rows of $\hat{P}$ form a basis for the cokernel of Ω.

Universal Configurations

For a given equilibrium stress ω for a configuration $\mathbf{p}$, with corresponding stress matrix Ω, we say that $\mathbf{p}$ is *universal with respect to* $\mathbf{p}$ if any other configuration $\mathbf{q}$, that is in equilibrium with respect to ω, is an affine image of $\mathbf{p}$. The following is basic linear algebra.

Proposition

A configuration $\mathbf{p}$ *is universal with respect to an equilibrium stress* ω *if and only if the rows of the configuration matrix* $\hat{P}$ *are a basis for the cokernel of the stress matrix* Ω.

An affine map of the configuration $\mathbf{p}$ in $\mathbb{R}^d$ to a configuration $\mathbf{q}$ can be regarded as a linear map of $\mathbb{R}^{d+1}$ that takes $\hat{P}$ to $\hat{Q}$. Note also that also that affine maps preserve equilibrium stresses.

Generic Global Rigidity

Recall that an equilibrium stress ω, as a row vector, can be regarded as an element of the cokernel of the rigidity matrix, namely $\omega R(\mathbf{p}) = 0$. Suppose that the configuration $\mathbf{p}$ is generic in $\mathbb{R}^d$. Then the rigidity map $f_G = f : \mathbb{R}^{dn} \to \mathbb{R}^m$ of a neighborhood of $\mathbf{p}$ in $\mathbb{R}^{dn}$ to squared member-length space $\mathbb{R}^m$ is non-singular, modulo congruences. If there is another configuration $\mathbf{q}$ in $\mathbb{R}^d$ with the same member lengths, then $f(\mathbf{p}) = f(\mathbf{q})$. Let $U_{\mathbf{p}}$ and $U_{\mathbf{q}}$ be neighborhoods of $\mathbf{p}$ and $\mathbf{q}$ respectively in $\mathbb{R}^{dn}$. If $f(U_{\mathbf{p}}) \cap f(U_{\mathbf{q}})$ is not top dimensional in $\mathbb{R}^m$, then $f(\mathbf{p})$ lives in a lower-dimensional subspace of $\mathbb{R}^m$, contradicting the generic nature of the configuration $\mathbf{p}$. (Technically this uses a kind of Tarsky-Seidenberg elimination theory.)

$$\begin{array}{ccc} U_{\mathbf{p}} & \stackrel{f}{\to} & \mathbb{R}^m \\ \downarrow & \nearrow & \\ U_{\mathbf{q}} & & \end{array}$$

The Transfer Map

The upshot of the previous argument is that there is a non-singular diffeomorphism $g : U_{\mathbf{p}} \to U_{\mathbf{q}}$, and by taking differentials of the previous diagram $f = f \circ g : U_{\mathbf{p}} \to \mathbb{R}^m$, we get $df = df \circ dg$. Thus the cokernels of f at $\mathbf{p}$ and at $\mathbf{q}$ are the same, and so if the rank of the stress matrix is maximal at $n - d - 1$, then the configuration $\mathbf{q}$ has to be an affine image of $\mathbf{p}$.

Notice how the "forbidden configurations," those that are to be avoided here, are hard to define explicitly.

Next: We determine when affine maps can preserve the member lengths.

The Affine equivalences

Any affine map $\alpha : \mathbb{R}^d \to \mathbb{R}^d$ is such that $\alpha(\mathbf{p}_i) = A\mathbf{p}_i + \mathbf{b}$, where $\mathbf{b}$ is a constant vector in $\mathbb{R}^d$. The translation given by $\mathbf{b}$ is a congruence, so we will concentrate on the d-by-d matrix A. The member length for the $\{i, j\}$ member squared is

$$\begin{aligned}(A\mathbf{p}_i - A\mathbf{p}_j) \cdot (A\mathbf{p}_i - A\mathbf{p}_j) &= A(\mathbf{p}_i - \mathbf{p}_j) \cdot A(\mathbf{p}_i - \mathbf{p}_j) \\ &= [A(\mathbf{p}_i - \mathbf{p}_j)]^T A(\mathbf{p}_i - \mathbf{p}_j) \\ &= (\mathbf{p}_i - \mathbf{p}_j)^T A^T A(\mathbf{p}_i - \mathbf{p}_j)\end{aligned}$$

So when there is an affine motion, not a congruence, that preserves member lengths, there is a symmetric matrix Q where $(\mathbf{p}_i - \mathbf{p}_j)^T Q(\mathbf{p}_i - \mathbf{p}_j) = 0$, where $Q = A^T A - I$, since $Q = 0$ exactly when A an orthogonal matrix.

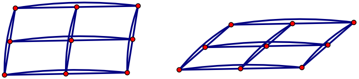
Conic at Infinity

For a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ with affine span all of $\mathbb{R}^d$, we say that the member directions lie on a *conic at infinity* if there is a non-zero d-by-d symmetric matrix Q such that $(\mathbf{p}_i - \mathbf{p}_j)^T Q(\mathbf{p}_i - \mathbf{p}_j) = 0$ for all $\{i, j\}$ members of G.

Theorem

For a bar framework $(G, \mathbf{p})$ in $\mathbb{R}^d$, it has a non-trivial affine flex, if and only if it has is a non-congruent affine image if and only if the member directions do not lie on a conic at infinity.

Problem: Show the "if" direction.



In the plane, conics at infinity have 2 points. The affine flex occurs since there are just two member directions. (The curved bars are just to indicate overlapping bars.)

The Stress Condition for Global Rigidity

Theorem

If p is a generic configuration in $\mathbb{R}^d$ with n vertices, and the framework $(G, \mathbf{p})$ has an equilibrium stress with stress matrix Ω of rank $n - d - 1$, then $(G, \mathbf{p})$ is globally rigid in $\mathbb{R}^d$.

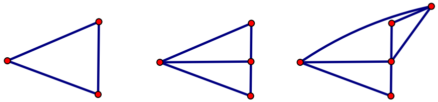
The argument above implies that any other framework $(G, \mathbf{q})$ with corresponding bar lengths the same must have the same set of equilibrium stresses. It is easy to show that the member directions do not lie on a conic at infinity due to the generic condition. Thus $\mathbf{q}$ must be congruent to $\mathbf{p}$.

Application: Edge Splitting

For a bar framework in $\mathbb{R}^d$ choose a point $\mathbf{r}$ along the line connecting the endpoints of a bar $\{\mathbf{p}_i, \mathbf{p}_j\}$, not at the vertices. Remove the bar, and connect $\mathbf{r}$ to $\mathbf{p}_i, \mathbf{p}_j$ and $d-1$ other points in the framework such that all $d+1$ of these points, not counting $\mathbf{r}$, are affine independent (i.e. no d of them lie in a hyperplane). This is called an *edge splitting* along the edge $\{i, j\}$.

Theorem (Henneberg 1911)

If the framework $(H, \mathbf{q})$ is obtained from the infinitesimally rigid framework $(G, \mathbf{p})$ by edge splitting, then $(H, \mathbf{q})$ is infinitesimally rigid.



Edge Splitting and Global Rigidity

Edge splitting preserves infinitesimal rigidity; passing to the generic case it preserves generic rigidity as well. A consequence of Laman's Theorem is that any generic framework in the plane can be obtained from a triangle by sequentially attaching a vertex to two other vertices and edge splitting, collectively called *Henneberg* moves.

Proposition

If ω is an equilibrium stress for a bar framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ with a stress matrix Ω of rank $n - d - 1$, then the rank of Ω increases by one after an edge-splitting.

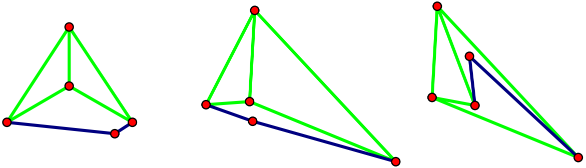
In the plane, I conjectured the following result:

Theorem (Jackson-Jordan 2005)

Any vertex 3-connected generically redundantly graph G can be obtained from K_4, the complete graph on 4 vertices, by a sequence of edge splittings and edge insertions, thus verifying Hendrickson's conjecture in the plane.

Non-Generic Example

A priori, it is not clear that global rigidity is a generic property. For example, the following framework $(G, \mathbf{p})$ has a sub framework $(H, \mathbf{q})$, in green, that is relatively globally rigid in the sense that any other equivalent configuration of $(G, \mathbf{p})$ restricted to the vertices of $(H, \mathbf{q})$ is a congruence. Nevertheless, there are other generic configurations of $(G, \mathbf{p})$ where this is not true.



On the other hand, the stress matrix rank property is a generic property.

Stress Matrix Necessity

Theorem (Gortler-Healy-Thurston 2010)

If a bar framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ is globally rigid at a generic configuration $\mathbf{p}$, then either $(G, \mathbf{p})$ is a complete graph (with $d+1$ or fewer vertices), or there exists a stress ω with stress matrix Ω of rank $n-d-1$, where n is the number of vertices of G.

So global rigidity is a generic property after all. Furthermore, there is an algorithm to compute whether a given graph G with n vertices is generically globally rigid in $\mathbb{R}^d$:

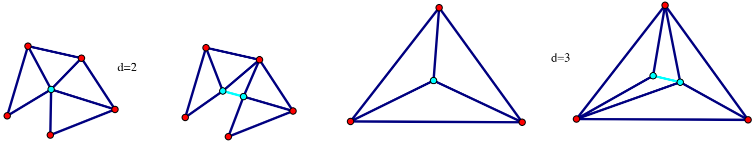
- Choose a "random" configuration $\mathbf{p}$ in $\mathbb{R}^d$, and compute the rank of its rigidity matrix $R(\mathbf{p})$. If the rank is $nd - d(d+1)$, continue. If not, G is probably not even locally rigid.
- Compute a "random" equilibrium stress ω, and compute the rank of the stress matrix Ω. If it is $n-d-1$, you know with certainty that G is globally rigid at generic configurations, but maybe not at the configuration $\mathbf{p}$. If it is not $n-d-1$, then G is probably not generically globally rigid.

Vertex Splitting

At any vertex $\mathbf{p}_0$ of a framework $(G, \mathbf{p})$ in $\mathbb{R}^d$, divide the vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$, for $k \geq d$, adjacent to $\mathbf{p}_0$ into two subsets A and B with exactly $d-1$ vertices in common. Remove $\mathbf{p}_0$, replace it with two other vertices, $\mathbf{q}_1, \mathbf{q}_2$ and join $\mathbf{q}_1$ to A, and $\mathbf{q}_2$ to B, as well as $\mathbf{q}_1$ and $\mathbf{q}_2$ to each other. Call the resulting framework $(H, \mathbf{q})$, the *vertex splitting* of $(G, \mathbf{p})$.

Theorem (Whiteley 1990)

Vertex splitting preserves infinitesimal rigidity with proper placement of $\mathbf{q}_1, \mathbf{q}_2$.



2-Manifolds

For an integer $p \geq 2$, an abstract simplicial complex X is called a p-cycle complex if it is the support complex of a non-trivial p-cycle. It is called a minimal p-cycle complex if it is the support complex of a nontrivial minimal p-cycle.

Theorem (Fogelsanger 1988)

The 1-skeleton of a minimal $(d-1)$-cycle complex, $d \geq 3$, is generically rigid in $\mathbb{R}^d$.

The proof is a very insightful application of vertex splitting. Note that minimal cycle complexes in dimension 3 include triangulated 2-manifolds. Those with genus greater than 1 were not known to be generically rigid in $\mathbb{R}$ until this result.

Conjectures

Conjecture

If G is generically globally rigid in $\mathbb{R}^d$ and H is obtained by a vertex splitting such that each subset (A and B above) has at least $d+1$ vertices, then H is generically globally rigid as well.

Question

Is a vertex 3-connected triangulated surface of genus $g \geq 2$ generically globally rigid in $\mathbb{R}^3$?

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