Flavors of Rigidity
Flavor III - Universal Rigidity and Tensegrities

Discrete Networks
University of Pittsburgh

Bob Connelly

Cornell University

October 2014

Stress-Energy Form

Recall that a tensegrity framework $(G, \mathbf{p})$ is *universally rigid* if when $(G, \mathbf{q})$ satisfies the constraints of $(G, \mathbf{p})$, where $\mathbf{q}$ is a configuration in any $\mathbb{R}^D$, then $\mathbf{p}$ is congruent to $\mathbf{q}$. That is, cables don't get longer, struts don't get shorter, and bars stay the same length.

If $\omega = (\ldots, \omega_{ij}, \ldots)$ is a stress assigned to the edges of a graph G, the following quadratic form is defined for all configurations $\mathbf{p} = (\mathbf{p_1}, \ldots, \mathbf{p_n})$ for all $\mathbf{p}_i$ in any Euclidean space $\mathbb{R}^d \subset \mathbb{R}^{d+1} \subset \mathbb{R}^{d+2} \ldots,$

$$E_\omega(\mathbf{p}) = \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)^2.$$

Stress-Energy Basics

With respect to the standard basis for $\mathbb{R}^d$, the matrix of E_ω is $\Omega \otimes I^d$. But with a permutation of the indices we see that

$$\begin{pmatrix} \Omega & 0 & \dots & 0 \\ 0 & \Omega & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & \Omega \end{pmatrix} = I^d \otimes \Omega.$$

So E_ω is positive semi-definite (PSD) if and only if Ω is PSD.

The Fundamental Criterion

For a tensegrity graph G, we say that a stress ω is *proper* if $\omega_{ij} \geq 0$ for $\{i, j\}$ a cable, and $\omega_{ij} \leq 0$ for $\{i, j\}$ a strut.

Theorem (Stress-Energy)

Let $(G, \mathbf{p})$ be configuration with n vertices in Euclidean space with d-dimensional affine span, a proper equilibrium stress ω and corresponding stress matrix Ω such that

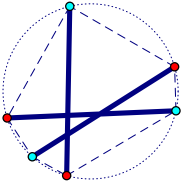
1. *rank $\Omega = n - d - 1$,*
2. *Ω is PSD, and*
3. *the member directions with non-zero stress and bar directions do not lie on a conic at infinity in the affine span of $\mathbf{p}$.*

Then $(G, \mathbf{p})$ is universally rigid.

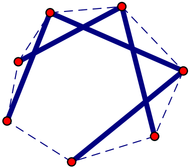
Notice that there is no hedging about the configuration being generic. I call a tensegrity that satisfies the conditions above *super stable.*

Tensegrity Examples

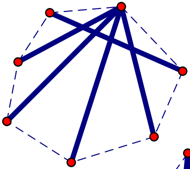
K(3,3) on a conic



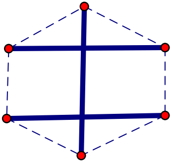
Cauchy Polygon

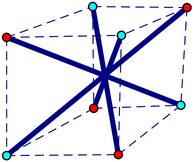


Grunbaum Polygon

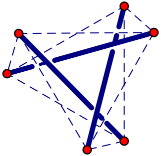


A Desargues' Framework

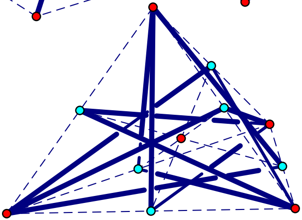




Cube with Long Diagonals, K(4,4)

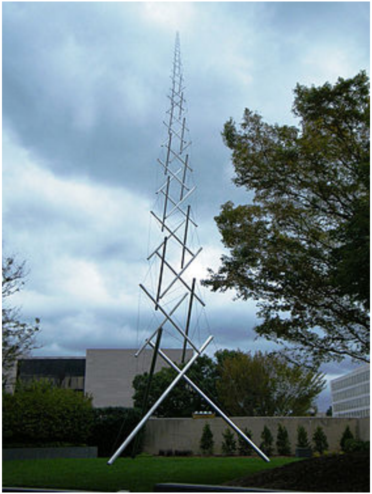


Kenneth Snelson's Original Tensegrity



Complete Bipartite Graph K(6,5)

Snelson Tower



The artist Kenneth Snelson has several tensegrity sculptures worldwide. This 60 ft. tensegrity Needle Tower is at the Hirshhorn Museum in Washington, D.C.

Principle of Least Energy

The proof of the Fundamental Theorem uses the principle of least work (my favorite). Suppose that $(G, \mathbf{q})$ is a framework, in any dimension, that has cables no longer, struts no shorter, and bars the same length as $(G, \mathbf{p})$, which has a proper PSD equilibrium stress ω of rank $n - d - 1$. Then

$$E_\omega(\mathbf{q}) = \sum_{i<j} \omega_{ij}(\mathbf{q}_i - \mathbf{q}_j)^2 \leq \sum_{i<j} \omega_{ij}(\mathbf{p}_i - \mathbf{p}_j)^2 = 0.$$

Since Ω and thus E_ω is PSD, $\mathbf{q}$ is in the kernel of E_ω and $(G, \mathbf{q})$ is in equilibrium with respect to ω. Since $\mathbf{p}$ is universal with respect to ω, $\mathbf{q}$ is an affine image of $\mathbf{p}$, and since the stressed directions do not lie on a conic at infinity, $\mathbf{q}$ is congruent to $\mathbf{p}$. $\square$

Planar Examples

The planar planar tensegrities above are examples of the following:

Theorem

If $(G, \mathbf{p})$ is obtained from a convex planar polygon with cables on the external edges, struts along some of the internal diagonals, and a proper non-zero equilibrium stress, then it is super stable.

Any tensegrity in this class must have a kernel of rank at most 3. So the signature of any linear combination of stress matrices must remain constant, and any PSD example implies they all are PSD of maximal rank.

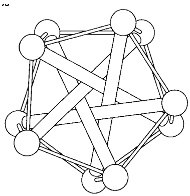
Problem: For a convex polygon find one example, with any arrangement of strut diagonals, that is super stable.

3D Examples

A *prysmic tensigrid* is obtained from two regular polygons in parallel planes, both centered on the z-axis. Cables join adjacent vertices of the polygons, and each vertex is connected by a cable to a vertex in the other polygon and by a strut to one other vertex in the other polygon, maintaining symmetric (dihedral) symmetry.

Theorem (RC and Maria Terrell 1995)

For a prysmic tensigrid, when one polygon is rotated so that there is a non-zero proper equilibrium stress, then the tensegrity is super stable.

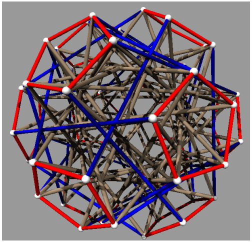


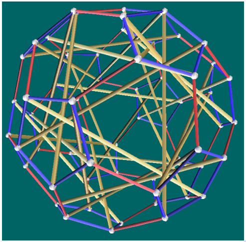
$P_5(3,1)$

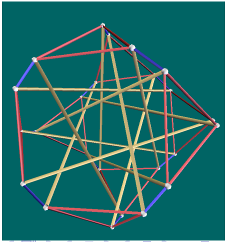
Symmetric Examples-Form Finding

Many of the examples used by artists are highly symmetric. One example of that is when the point group of symmetries acts transitively on the vertices and there are exactly two transitivity classes of cables and one for struts. In this case, the stress matrix is simply an element of the group algebra for the representations of the abstract group of symmetries. Each irreducible representation corresponds to vectors in the kernel of the stress matrix. When it is 3-dimensional it provides a configuration that is super stable. The ratio between the cable stresses is a parameter that the user can choose.

http://www.math.cornell.edu/~tens/

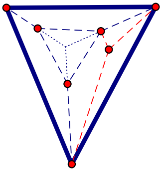




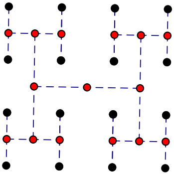


Spiderwebs

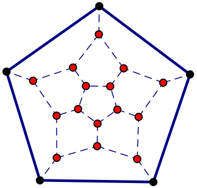
There are many other examples of universally rigid tensegrity frameworks where there is no single stress that rigidifies the structure. It is easy to construct examples with *spiderwebs*, which are tensegrity frameworks where some nodes are pinned or rigidified, while all the members are cables, as in the examples below. Any positive equilibrium stress will rigidify the tensegrity.



The red cables are unstressed but they are attached to a universally rigid subframework.



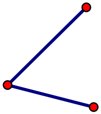
The black nodes are pinned. The process of successively attaching more nodes is iterated.

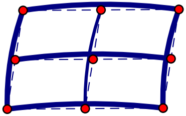


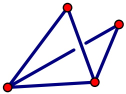
The projection of any convex polytope into one of its faces creates an equilibrium spiderweb.

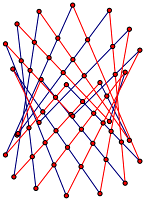
Dimensional Rigidity

A (tensegrity) framework $(G, \mathbf{p})$ is called (by Alfakih) *dimensionally rigid* if its affine span is d dimensional and any equivalent framework $(G, \mathbf{q})$ (with bars the same length, cables no longer, struts no shorter) has an affine span of dimension at most d. Note that a framework may be dimensionally rigid, and yet not be even locally rigid, as in the examples below. (*Dimensionally maximal* would be a better name.)









A ruled hyperboloid

Dimensional Rigidity and Universal Rigidity

Clearly dimensional rigidity is weaker than universal rigidity, but they are very closely related.

Theorem (Alfakih 2007)

If $(G, \mathbf{p})$ has n vertices, with affine span d dimensional, and is dimensionally rigid, then $d = (n-1)$ or it has a (proper) PSD equilibrium stress. Furthermore, any equivalent framework is an affine image of $\mathbf{p}$.

If $(G, \mathbf{q})$ is equivalent to $(G, \mathbf{p})$ place $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^d \times 0$ and $0 \times \mathbb{R}^d$ respectively. Then $\mathbf{p}_i(t) = ((\cos t)\mathbf{p}_i, (\sin t)\mathbf{q}_i)$ for each vertex i of G, for $0 \leq t \leq \pi/2$, is a *monotone flex* of $\mathbf{p}$ to $\mathbf{q}$. That is, the distance $|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$ changes monotonically in t. (E.g. it stays constant for bars.) The configuration $\mathbf{p}_i(\pi/4)$ lies in a d-dimensional affine space, by the dimensional rigidity, so the graph of the map from $\mathbf{p}$ to $\mathbf{q}$ lies in a d-dimensional affine subspace of $\mathbb{R}^d \times \mathbb{R}^d$, and the correspondence extends to an affine map.

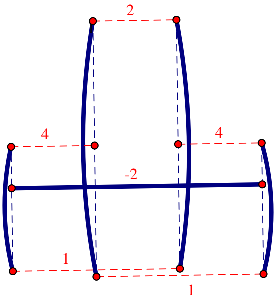
Iterated Universal Rigidity

Corollary

A framework $(G, \mathbf{p})$ is universally rigid if and only if it is dimensionally rigid and the member directions with a non-zero stress and bars do not lie on a conic at infinity.

The question remains as to what about frameworks $(G, \mathbf{p})$ that are universally rigid, but whose stress matrices have too low a rank. It turns out that in that case there is a sequence of stresses with stress matrices $\Omega_1, \Omega_2, \ldots, \Omega_k$, where each Ω_i is defined on those configurations in the kernel of the previous Ω_j for $j < i$. If the final space of configurations is d-dimensional, then $(G, \mathbf{p})$ is d-dimensionally rigid. This is essentially an application of facial reduction in convexity theory. See Connelly-Gortler.

Example of Iterated Universal Rigidity



This shows an example of a framework that is dimensionally rigid in the plane, with a 3-step iteration. The second-level stresses are shown, which turn out to be lever arm balancing forces. The first and second level stresses are in the vertical members. A computation shows that the sequence of stress matrices, when restricted to the kernel of the previous stresses, are each PSD, but after the first level these higher-level stresses are not PSD on the whole space of configurations.

Projective Invariance

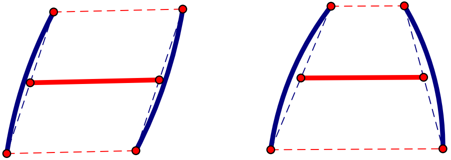
A projective map on real projective space can be regarded as a map as follows: For each point $\mathbf{p}_i \in \mathbb{R}^d$ define $\alpha(\mathbf{p}_i) = A\hat{p}_i/z_i \in \mathbb{R}^d \times 1$, where $z_i \neq 0$ is the last coordinate of $A\hat{p}_i$, and A is a $(d+1)$-by-$(d+1)$ matrix. If $\mathbf{p} = (\mathbf{p_1}, \dots, \mathbf{p_n})$ is a configuration in $\mathbb{R}^d$, and Ω is an equilibrium stress matrix for the stress ω for $\mathbf{p}$, then

$$A\hat{P}D^{-1}D\Omega D = 0$$

shows that $D\Omega D$ is another equilibrium stress matrix for $\alpha(\mathbf{p})$, where D is the diagonal matrix with entries z_i. The new stress matrix $D\Omega D$ has the same signature for the configuration as Ω, where each stress ω_{ij} is replaced with $z_i\omega_{ij}z_j$. Note that the sign of ω_{ij} changes if the (projective) image of the member crosses the line/space at infinity. This shows that infinitesimal rigidity is a projective invariant, and using this we can show that dimensional rigidity is projectively invariant. However ...

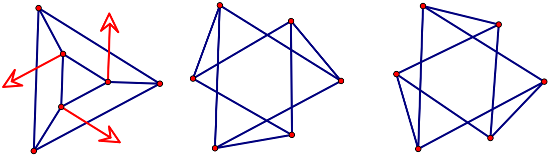
Universal Rigidity is not Projectively Invariant

The ladder on the left is not universally rigid, since it has an affine flex. Whereas the orchard ladder, a projective image of the straight ladder, on the right is universally rigid since there are 3 distinct stressed directions.



Averaging-deaveraging Method

A method of relating infinitesimal rigidity to global rigidity going back many years is the following: Suppose that $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are two equivalent frameworks, i.e. corresponding bars have the same lengths. Then $\mathbf{p}'_i = \mathbf{p}_i - \mathbf{q}_i, i = 1, \ldots, n$ defines an infinitesimal flex on $(G, \bar{\mathbf{p}})$, where $\bar{\mathbf{p}}_i = (\mathbf{p}_i + \mathbf{q}_i)/2$ is the average of $\mathbf{p}$ and $\mathbf{q}$. If $\mathbf{p}$ and $\mathbf{q}$ are not congruent, then $\mathbf{p}'$ is not trivial, and if they are congruent and have d-dimensional affine span in $\mathbb{R}^d$, then $\mathbf{p}'$ is trivial. Conversely, if $\mathbf{p}'$ is an infinitesimal flex of $(G, \mathbf{p})$, then $(G, \mathbf{p} + \mathbf{p}')$ is equivalent to $(G, \mathbf{p} - \mathbf{p}')$.



Problem: Find an example of congruent frameworks $(G, \mathbf{p} + \mathbf{p}'), (G, \mathbf{p} - \mathbf{p}')$ even though $\mathbf{p}'$ is a non-trivial infinitesimal flex of $(G, \mathbf{p})$.

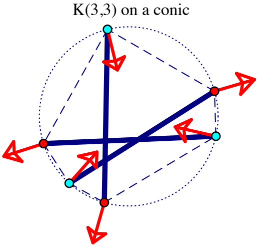
Bipartite Frameworks

A very extensive analysis of the equilibrium stresses of a complete bipartite graph in any dimension was given in Bolker-Roth. For example:

Theorem (Bolker-Roth 1980)

A complete bipartite graph $K_{m,n}(\mathbf{p},\mathbf{q})$ in $\mathbb{R}^d$ for which the affine span of $\mathbf{p}$ and $\mathbf{q}$ are both d-dimensional and $m = d+1$, $n = d(d+1)/2$, is infinitesimally rigid unless its $m+n = (d+1)(d+2)/2$ vertices lie on a quadric surface.

Walter Whiteley showed how the infinitesimal flexes can be found from the quadric surface.



Universally Rigid Bipartite Frameworks

A natural question is to find a characterization of when a bipartite framework is universally rigid. The following is partial information.

Theorem (Connelly-Gortler)

If $(K(m,n),(\mathbf{p},\mathbf{q}))$ is a bipartite framework in $\mathbb{R}^d$, with $m+n \geq d+2$, such that the partition vertices $(\mathbf{p},\mathbf{q})$ are strictly separated by a quadric, then it is not universally rigid.

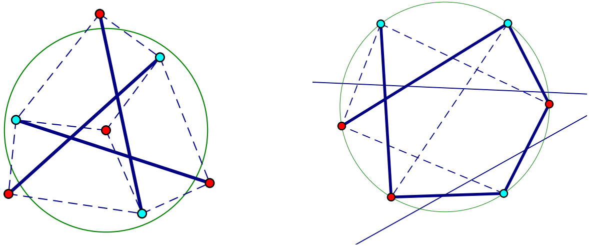
Conversely, if $(K(m,n),(\mathbf{p},\mathbf{q}))$ cannot have its vertices $\mathbf{p}$ and $\mathbf{q}$ separated by a quadric, then there is an equilibrium stress with stress matrix with positive diagonal entries. In $\mathbb{R}^d$ the existence of a PSD maximal rank stress matrix reduces to the case when $m+n=(d+1)(d+2)/2$.

Conjecture

For all $d \geq 1$, $(K(m,n),(\mathbf{p},\mathbf{q}))$ in $\mathbb{R}^d$ is universally rigid if and only if $\mathbf{p}$ and $\mathbf{q}$ cannot be strictly separated by a quadric.

Universally Rigid Bipartite Frameworks

The conjecture is true for $d = 1$, by T. Jordan and V-H Nguyen, and for $d = 2$. For $d = 3$, calculations strongly suggest that it is true for $K(6,5)$, the critical case.



The tensegrity on the left is centrally symmetric and so any strictly separating conic has to be a circle centered at the red center point, and it is easy to see that there is no such circle. For the tensegrity on the right the red and blue points are separated by the degenerate conic consisting of two lines.

References

- Abdo Alfakih, On dimensional rigidity of bar-and-joint frameworks, *Discrete Appl. Math., The Journal of Combinatorial Algorithms, Informatics and Computational Sciences*, 155, no. 10, 1-59, 2007.
- E. D. Bolker and B. Roth, When is a Bipartite Graph a Rigid Framework?, *Pacific Journal of Mathemstics*, Vol. 90, No. 1, 27–44, 1980.
- Robert Connelly and Steven Gortler, Iterative Universal Rigidity, *arXiv:1401.7029*, 1-41, 2014.
- Robert Connelly and Maria Terrell, Globally Rigid Symmetric Tensegrities, *Structural Topology*, No. 21, 59–78, (1995).
- Tibor Jordán and Viet-Hng Nguyen, On universally rigid frameworks on the line., *Technical Report TR-2012-10, Egerv?ary Research Group on Combinatorial Optimization*, Budapest Hungary, pages 1-13, July 14, 2012., No. 21, 59–78, (1995).
- Walter Whiteley, Infinitesimal motions of a bipartite framework., *Pacific Journal of Mathemstics*, Vol. 110, No. 1, 233-255, 1984.