Flavors of Rigidity
Flavor IV - Reconstructing

Discrete Networks
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Main Problem

Given a graph G and positive scalars d_{ij} assigned to each edge $\{i, j\}$ in G, construct a configuration $\mathbf{p}$ in $\mathbb{R}^d$ so that $|\mathbf{p}_i - \mathbf{p}_j| = d_{ij}$. When is it possible, and how can you construct it if it is possible?

Easy first case is a triangle: The condition is that the triangle inequalities must hold.

$$d_{12} \leq d_{23} + d_{31}, \quad d_{23} \leq d_{31} + d_{12}, \quad d_{31} \leq d_{12} + d_{23}.$$

Equivalently the Cayley-Menger determinant is

$$\det \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & d_{12}^2 & d_{31}^2 \\ 1 & d_{12}^2 & 0 & d_{23}^2 \\ 1 & d_{31}^2 & d_{23}^2 & 0 \end{pmatrix} = -16A^2 \leq 0,$$

which is a form of Heron's formula for the area A of the triangle.

The simplex

The next easy case is for the case of the complete graph. There are Cayley-Menger constraints, but the Gram matrix is more direct, although it is less democratic in that one vertex, say $\mathbf{p}_0 = 0$ for convenience. Then let P be the configuration matrix (without inserting 1's in an extra coordinate) for the rest of the vertices. So the realizability condition is that

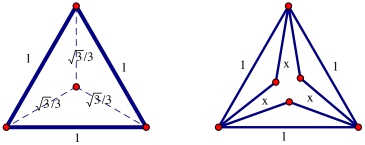
$$P^T P = (\mathbf{p}_i \cdot \mathbf{p}_j)$$

is PSD of rank d if $\mathbf{p}$ has affine span in $\mathbb{R}^d$, where $\mathbf{p}_i \cdot \mathbf{p}_j = (d_{0i}^2 + d_{0j}^2 - d_{ij}^2)/2$. Since this matrix just depends on the distances d_{ij}, when it is PSD, one can factor it as $P^T P$ obtaining a configuration $\mathbf{p}$.

The Cayley-Menger determinants, which alternate in sign, are just a few row and column operations from the Gram matrix condition above.

Tetrangle Inequality

The triangle inequality is not enough to insure the existence of a configuration with given edge lengths. For example, one way to show that a given set of proposed edge lengths are not feasible in any Euclidean space is to compare them to an appropriate tensegrity.



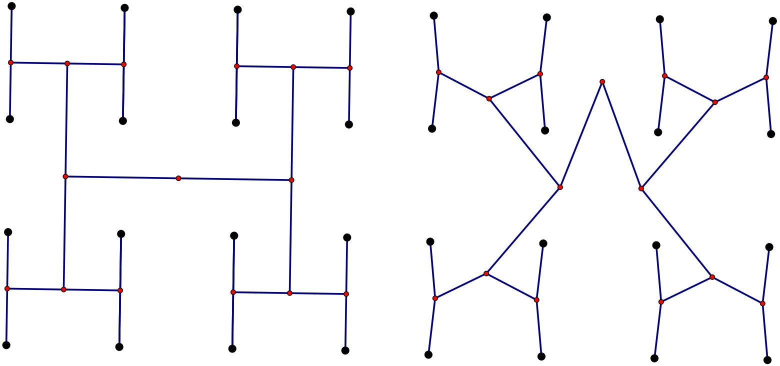
Whenever $1/2 \leq x < \sqrt{3}/3$, the metric described on the right cannot be realized in any Euclidean space even though the triangle inequality holds for all triangles. The tensegrity on the left, since it is universally rigid, is a certificate that the distances indicated on the right are not realizable.

Semi-Definite Programming

Determining when a general graph can be realized in $\mathbb{R}^d$, even for $d = 1$ can be difficult. For example, when G is a cycle of n vertices with given edge lengths, to determine when it can be realized in $\mathbb{R}^1$ is equivalent to the backpack problem. However, if you do not mind rising to the occasion and simply realizing in some possibly much higher dimension, there are methods that can work with reasonably sized graphs. Another approach to graph realization is to take the given member lengths and apply an algorithm that uses semi-definite programming (SDP) to find a configuration with a maximal dimensional affine span for the given edge lengths, starting with $(G, \mathbf{p})$. If it returns the configuration $\mathbf{p}$ again, you can conclude that $(G, \mathbf{p})$ is universally rigid. The problem is that this process only converges to a dimensionally rigid example, and the measure of success is how close the calculated lengths are to the given lengths, which can be problematic as the following example shows. The question of whether there is an "algorithm" to "compute" a given metric is partly tied up with the question of how the configuration itself is defined.

Is the problem itself well-defined?

Sloppy Realizations



The black vertices are pinned, while the members on the right have been increased by less than 1%.

Bipartite Realizations

Suppose that the graph one wants to reconstruct is a complete bipartite graph. If the partitions can be strictly separated by a quadric, then SDP methods will provide a realization, but only in higher dimensions. On the other hand if the partitions cannot be separated, then, if our conjecture is correct, the SDP algorithm will provide a realization in the desired dimension.

But how do you know if the configuration can be separated by a quadric if you don't know the configuration?

ABBIE-Think Globally, Act Locally

One method, due to Bruce Hendrickson, is to split the graph into two globally rigid pieces and recursively realize each piece in $\mathbb{R}^d$. Then combine the two pieces into the larger desired realization of the whole graph. This was "in honor of Abbie Hoffman for his admonition to "think globally, act locally," although it is doubtful he had nonlinear optimization in mind!"

This has two difficulties: First, it assumes that the configuration is generic, and second it needs that not only is the whole graph globally rigid, the graph has to have enough edges to be assured of the existence of the globally rigid subgraphs.

The motivation for this "molecule problem" was to reconstruct moderately large proteins where Nuclear Magnetic Resonance (NMR) data gave distance information about certain pairs of atoms, and these graphs had at least some parts that were not even locally rigid.

Traditional Reconstuction Methods

One method is to fill in estimates of upper and lower bounds on all the pairs of distances of the graph, and then improve them as much as possible using the triangle inequality, tetrangle inequality, etc. (Bound smoothing.) Then when no further improvements are possible, project onto $\mathbb{R}^3$ using least square methods and then do some energy minimization. This is explained in Crippen and Havel's book.

3-Realizable Graphs

There are two steps in the realizability problem. Finding a realization in some possibly higher dimensional Euclidean space, and pushing that realization down to a lower-dimension, such as dimension 3 or 2 still preserving its edge lengths. So we can assume the first step, that the graph is realized in some $\mathbb{R}^D$, for some large dimension D. If it always happens that the realization can then be realized in dimension d, we say the graph is *d-realizable*.

If a finite graph H is obtained from a finite graph G by a sequence of edge contractions and deletions, we say that H is a *minor* of G. A property of graphs is called *minor monotone* if when G has the property every minor H of G also has the property. A *forbidden minor* for a given graph property is (class of graphs) G that do not have the property.

Theorem (Robertson-Seymour)

For any graph property that is minor monotone, there is a finite list of forbidden minors, and there is a polynomial-time algorithm for determining when a given graph has the property.

3-Realizable Graphs

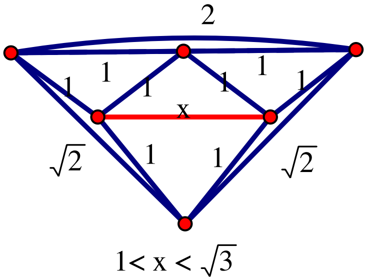
It is easy to check that d-realizability is minor monotone, so the quest is to determine a list of minimal forbidden minors.
The Robertson-Seymour Theorem is great for reassurance, but it does not tell you how to find the minimal minors. It is like a cheering section, but does not enter the playing itself.

- If $d = 1$, there is one minimal forbidden minor, the triangle. So a graph is 1 realizable if and only if it is a forest, a finite union of trees.
- If $d = 2$, there is still only one forbidden minor, $K(4)$, the complete graph on 4 vertices. These are sometimes called series parallel graphs.
- If $d = 3$, then the only forbidden minors are $K(5)$ and the one-skeleton of the octahedron. See [Connelly-Belk, and Belk].
- $d \geq 4$ it is not known what the forbidden minors are, although Laurent-Varvitsoitis do have a result for the cone over a graph in dimension 4.

Problem: Find a minimal forbidden minor for 4-realizability other than $K(6)$.

Forbidden Octahedron

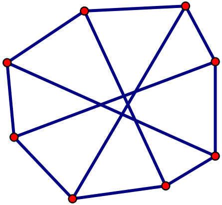
This shows an example of an octahedron with the indicated edge lengths that is only realizable in dimension 4.

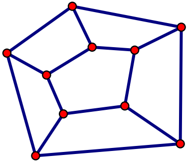


The 4 outside vertices in this picture are forced to be planar, and the next two can be placed in $\mathbb{R}^4$ when the length of the red edge is between 1 and $\sqrt{3}$. Notice that this framework is universally rigid. The 10 member directions do not lie on a quadric in $\mathbb{RP}^3$.

Some 3-Realizable Graphs

The following two graphs, V_8 and $C_5 \times C_2$, are 3-realizable. They represent critical graphs that have to be shown how they are 3-realizable by hand. See [Belk].





Back to Local Rigidity

There are several locally rigid frameworks and tensegrities that are not infinitesimally rigid, not globally rigid, and not universally rigid. The basic concept is *prestress stability*. This means roughly that the framework rests at local minimum of some sort of energy function that is the sum of energy functions of pairs of distances and the minimum exists because it passes a second derivative test. To be more specific let f_{ij} be a real-valued function assigned to each member $\{i, j\}$ of the graph G, such that it is monotone increasing for cables of a tensegrity graph, monotone decreasing for the struts, and at a minimum for a bar and all these functions are assumed to be concave up. Then for any configuration $\mathbf{q}$ in $\mathbb{R}^d$ sufficiently close to a configuration $\mathbf{p}$ in $\mathbb{R}^d$, define

$$E(\mathbf{q}) = \sum_{i<j} f_{ij}((\mathbf{q}_i - \mathbf{q}_j)^2).$$

Prestress stability

It is clear (by the principle of least work again) that if the configuration $\mathbf{p}$ is a strict local minimum point for E, modulo rigid congruences, then $(G, \mathbf{p})$ is locally rigid. Define a stress for $(G, \mathbf{p})$ by $\omega_{ij} = f'_{ij}((\mathbf{p}_i - \mathbf{p}_j)^2)$. Furthermore define *stiffness coefficients* for E by $c_{ij} = f''_{ij}((\mathbf{p}_i - \mathbf{p}_j)^2)$. Then the Hessian of this potential function for the configuration $\mathbf{p}$ is

$$H(\mathbf{p}') = \sum_{i<j} 2\omega_{ij}(\mathbf{p}'_i - \mathbf{p}'_j)^2 + \sum_{i<j} 4c_{ij}(\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}'_i - \mathbf{p}'_j).$$

The matrix of this quadratic form is

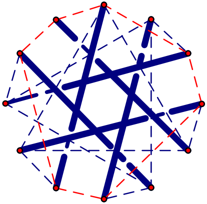
$$H = 2\Omega \otimes I^d + 4R(\mathbf{p})^T CR(\mathbf{p}),$$

where C is the m-by-m diagonal matrix of stiffness coefficients, for the m members.

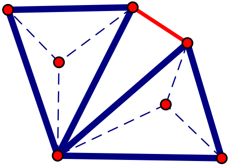
Prestress stability

So if one wants to show local rigidity for a given framework $(G, \mathbf{p})$, one might as well choose positive stiffness coefficients and one needs only to look at the stress energy function restricted to the kernel of the rigidity matrix $R(\mathbf{p})$. If one is given the potential functions f_{ij}, then the stiffness coefficients are determined by the material properties, for example, but the stress coefficients are determined by how a prestress is applied. So $(G, \mathbf{p})$ is *prestress stable* if there is an equilibrium stress ω such that H is PSD with only the trivial infinitesimal flexes in its kernel (for C the identity). Most practical locally rigid structures are at least prestress stable.

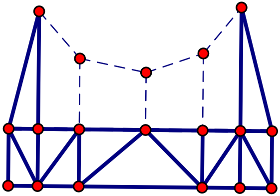
Prestress examples



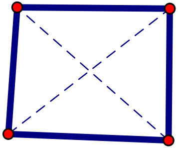
Two Snelsons concatenated.
The red polygon is planar.



The two triangles are stressed, but the red bar is not. This is infinitesimally rigid as a tensegrity, but the red is a bar not a cable or strut.



A suspension bridge



This is infinitesimally rigid as a tensegrity, but the stress matrix works against stability in the plane. The stiffness matrix dominates.

Second-order Rigidity

A secondary flavor of local rigidity is *second-order rigidity*. This is defined as follows:
A second-order flex $(\mathbf{p}', \mathbf{p}'')$ of a framework $(G, \mathbf{p})$ is when $\mathbf{p}'$ is a an infinitesimal flex of $(G, \mathbf{p})$ and for each bar $\{i, j\}$ of G

$$(\mathbf{p}'_i - \mathbf{p}'_j)^2 - (\mathbf{p}_i - \mathbf{p}_j) \cdot (\mathbf{p}''_i - \mathbf{p}''_j) = 0.$$

We say that $(G, \mathbf{p})$ is second-order rigid in $\mathbb{R}^d$ if every non-trivial infinitesimal flex $\mathbf{p}'$ of $(G, \mathbf{p})$ does not extend to a second-order flex of $(G, \mathbf{p})$.

Theorem (RC)

If $(G, \mathbf{p})$ is second-order rigid in $\mathbb{R}^d$, then it is locally rigid in $\mathbb{R}^d$.

Second-order Demon

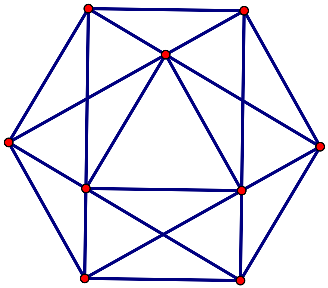
An equivalent dual version of second-order rigidity and prestress stability relates to equilibrium stresses. We say a stress ω *blocks an infinitesimal motion* $\mathbf{p}'$ of a bar framework $(G, \mathbf{p})$ in $\mathbb{R}^d$ if

$$\mathbf{p}'^T \Omega \mathbf{p}' > 0$$

for all non-trivial infinitesimal flexes $\mathbf{p}'$ of $(G, \mathbf{p})$, where Ω is the stress matrix corresponding to ω. Prestress stability is equivalent to having one stress block all non-trivial infinitesimal flexes $\mathbf{p}'$ of $(G, \mathbf{p})$. Second-order rigidity is equivalent to having the property that each infinitesimal flex $\mathbf{p}'$ of $(G, \mathbf{p})$ has its own personal stress ω that blocks it. It is as if there is a demon in the framework that adjusts the stress to block any potential infinitesimal flex.

A Non-prestressable Second-order Rigid Example

The following example in the plane is second-order rigid $\mathbb{R}^3$.



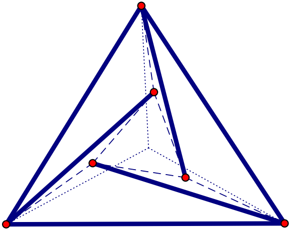
The triangle with tabs

But it flexes to another folded configuration in $\mathbb{R}^2$ through $\mathbb{R}^4$.

Applications of Second-order rigidity

Theorem

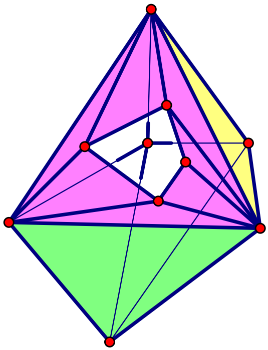
Any triangulation of a convex polytope in $\mathbb{R}^3$ *is second-order rigid, and indeed prestress stable.*



This is an example of a triangulation of a triangle, that is universally rigid, indeed super stable, and could be part of a triangulation of a face of a convex polytope. If the internal vertices are on the indicated sides of the dotted lines the internal stresses cannot all be positive.

Holey Polytopes

It is even possible to put convex holes in the faces of the polytope, triangulate the rest, and the resulting surface with boundary is prestress stable if the holes are placed so that there is another convex polytope that projects onto the given polytope with a flat face projection onto each hole.



References

- Maria Belk, Realizability of graphs in three dimensions,
  Discrete Comput Geom, no. 2, 139-162 (2007).

- Maria Belk and Robert Connelly, Realizability of Graphs,
  Discrete Comput Geom, 37:125137 (2007).

- Robert Connelly, The rigidity of certain cabled frameworks and the second-order rigidity of arbitrarily triangulated convex surfaces,
  Adv. in Math., 37 , no. 3, 272299 (1980).

- Robert Connelly and Walter Whiteley, Second-order rigidity and prestress stability for tensegrity frameworks,
  SIAM J. Discrete Math., 9, no. 3, 365-376, (1996).

- G. M. Crippen and T. F. Havel, Distance geometry and molecular conformation.,
  Chemometrics Series, 15., Research Studies Press, Ltd., Chichester; John Wiley & Sons, Inc., New York.,,x+541 pp, 1988.

References

- Bruce Hendrickson, The molecule problem: exploiting structure in global optimization,
  SIAM J. Optim., no. 4, 835-857, 1995.
- M. Laurent and A. Varvitsiotis, Positive semidefinite matrix completion, universal rigidity and the strong Arnold property,
  Linear Algebra Appl. , 452, 292-317, 2014.
- Neil Robertson and P. D. Seymour, Graph minors. XX. Wagner's conjecture,
  J. Combin. Theory Ser. B , 92, no. 2, 325-357, 2004.