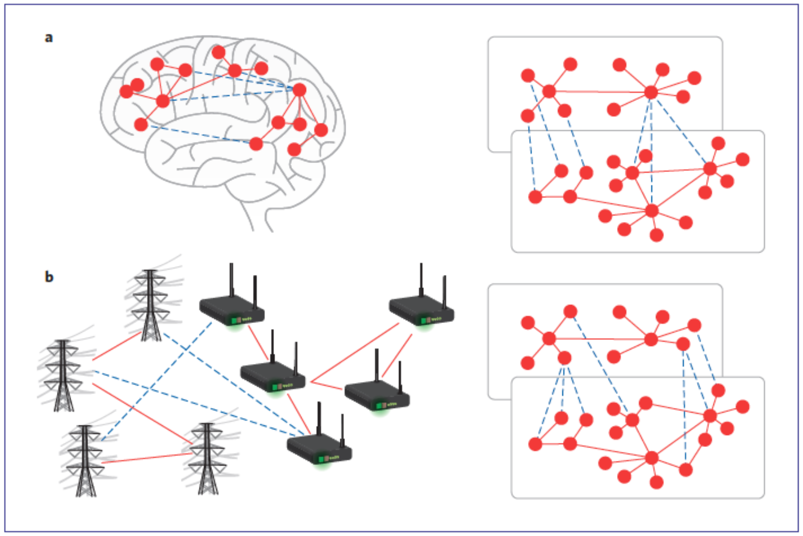
Advances in Discrete Networks
Pittsburgh, 12-14 December 2014



Structure and Dynamics of Multilayer Networks

Ginestra Bianconi

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Queen Mary
University of London

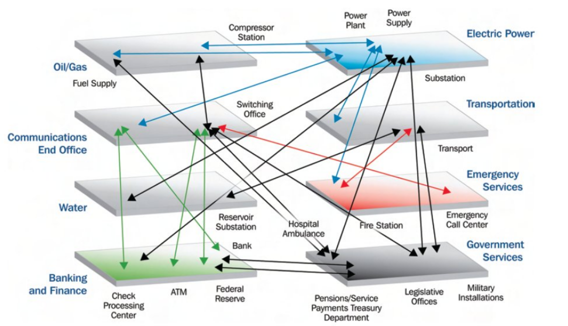
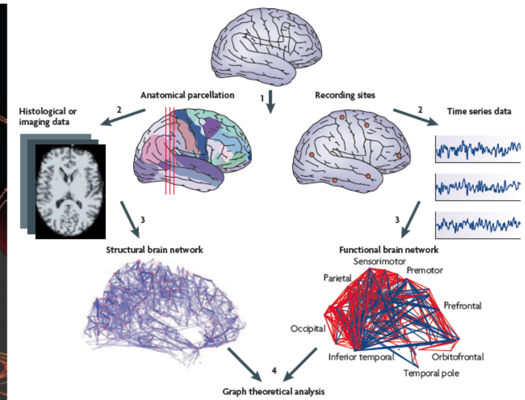


FIGURE 3.1 Connections and interdependencies across the economy. Schematic showing the interconnected infrastructures and their qualitative dependencies and interdependencies. SOURCE: Department of Homeland Security, National Infrastructure Protection Plan, available at http://www.dhs.gov/xprevprot/programs/editorial_0827.shtm.



The function of many complex

technological social and biological

systems

depends on the non-trivial interactions between

different networks

Interacting Transportation networks

Transportation networks are a major example of interacting networks.

Here
blue lines represent short-range commuting flow by car or train
the red lines indicate airline flow for few selected cities

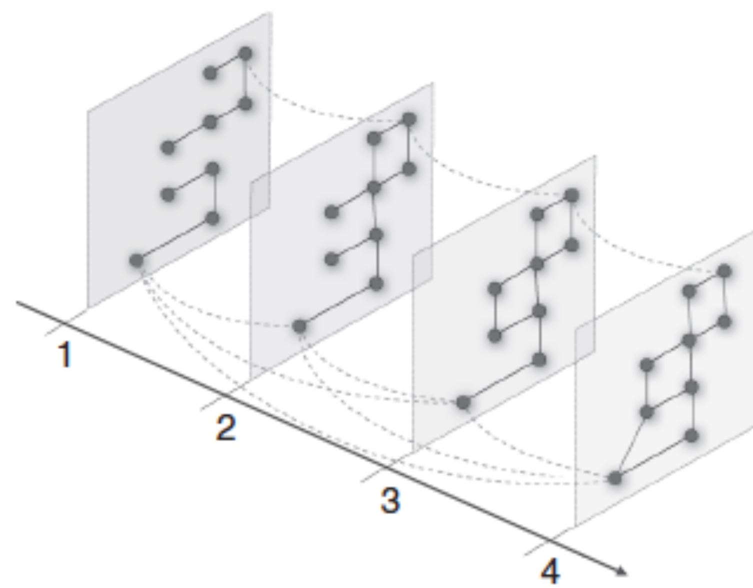
B. GONÇALVES ET AL., INDIANA UNIV.

Vespignani Nature 2010

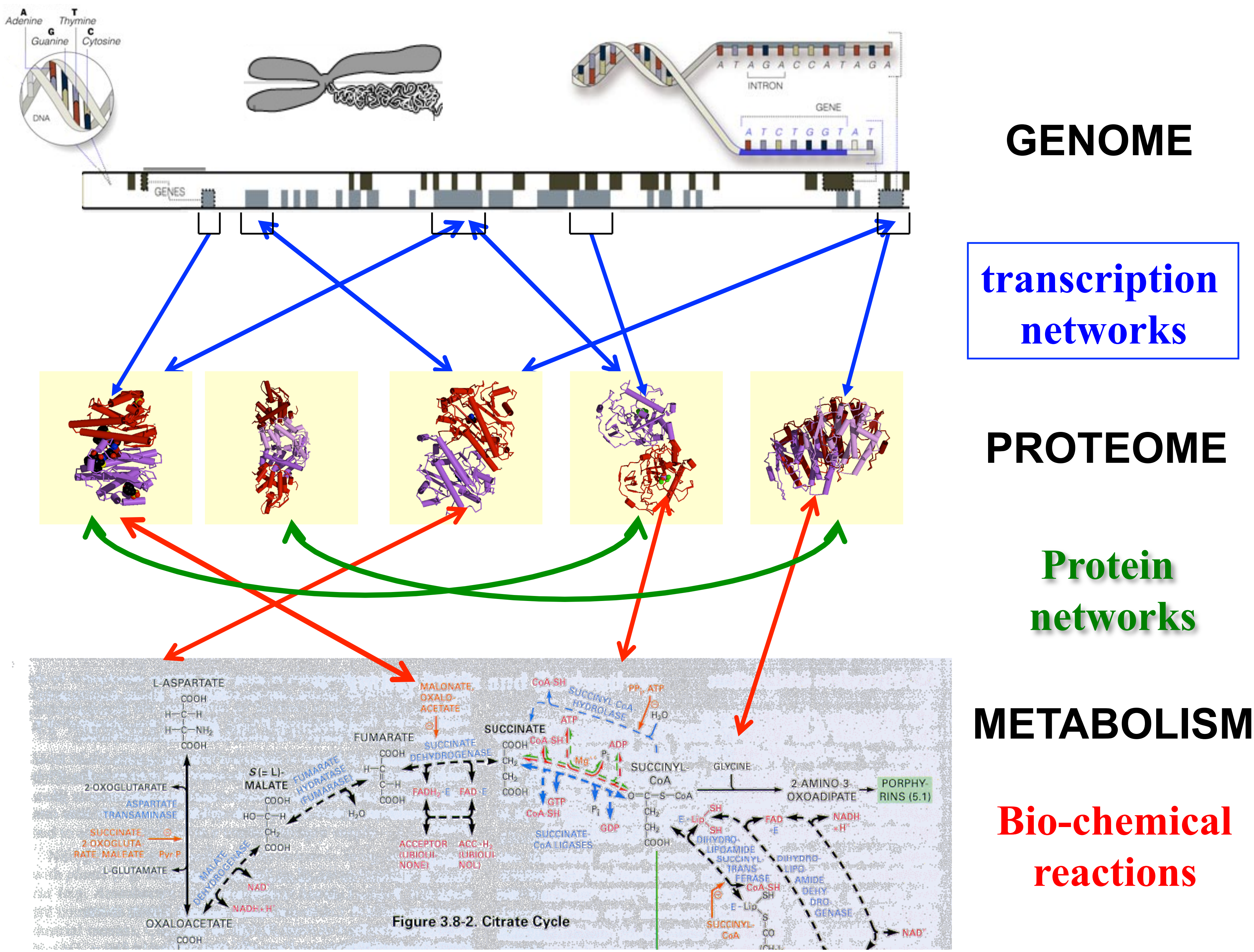
Interacting Social networks

Social networks are interacting and overlapping with profound implications for community detection algorithms

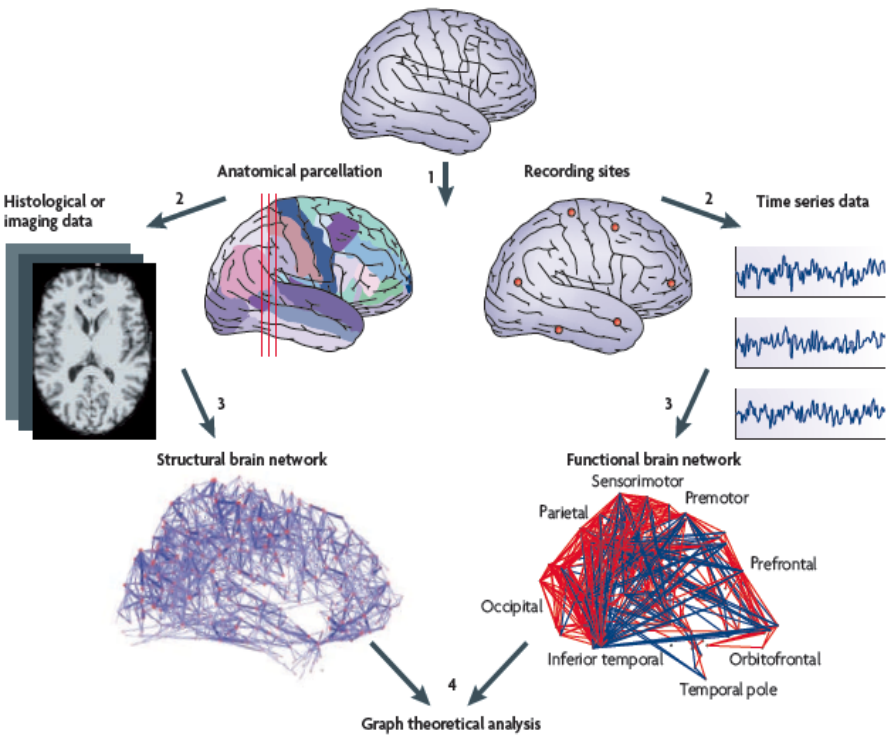
Y.Y. Ahn et al. Nature 2010



P. J. Mucha et al. 2010


A Adenine
T Thymine
G Guanine
C Cytosine
DNA
A T A G A C C A T A G A
INTRON
GENE
A T C T G G T A T
GENES
GENOME
transcription networks
PROTEOME
Protein networks
METABOLISM
Bio-chemical reactions
L-ASPARTATE
MALONATE, OXALO-ACETATE
SUCCINATE
FUMARATE
SUCCINATE DEHYDROGENASE
SUCCINYL-CoA HYDROLASE
SUCCINYL-CoA
GLYCINE
2-AMINO-3-OXOADIPATE
PORPHY-RINS (5.1)
2-OXOGLUTARATE
ASPARTATE TRANSAMINASE
S(= L)-MALATE
FUMARATE HYDRATASE (FUMARASE)
SUCCINATE-CoA LIGASES
L-GLUTAMATE
MALATE DEHYDROGENASE
OXALOACETATE
DIHYDRO-LIPOAMIDE SUCCINYL-TRANSFERASE
DIHYDRO-LIPOAMIDE DEHYDROGENASE
Figure 3.8-2. Citrate Cycle

Interacting and multiplex Brain networks



The brain function is determined

at the same time

by

the structural brain network

and

the functional brain network,

Bullmore Sporns 2009

Multilayers networks

In order to

characterize, model, predict, control

complex systems

we need to characterize

the structure

and the

the function

of

multilayer networks

New review article in Multilayer Networks Physic Reports 544, 1 (2014)

The structure and dynamics of multilayer networks

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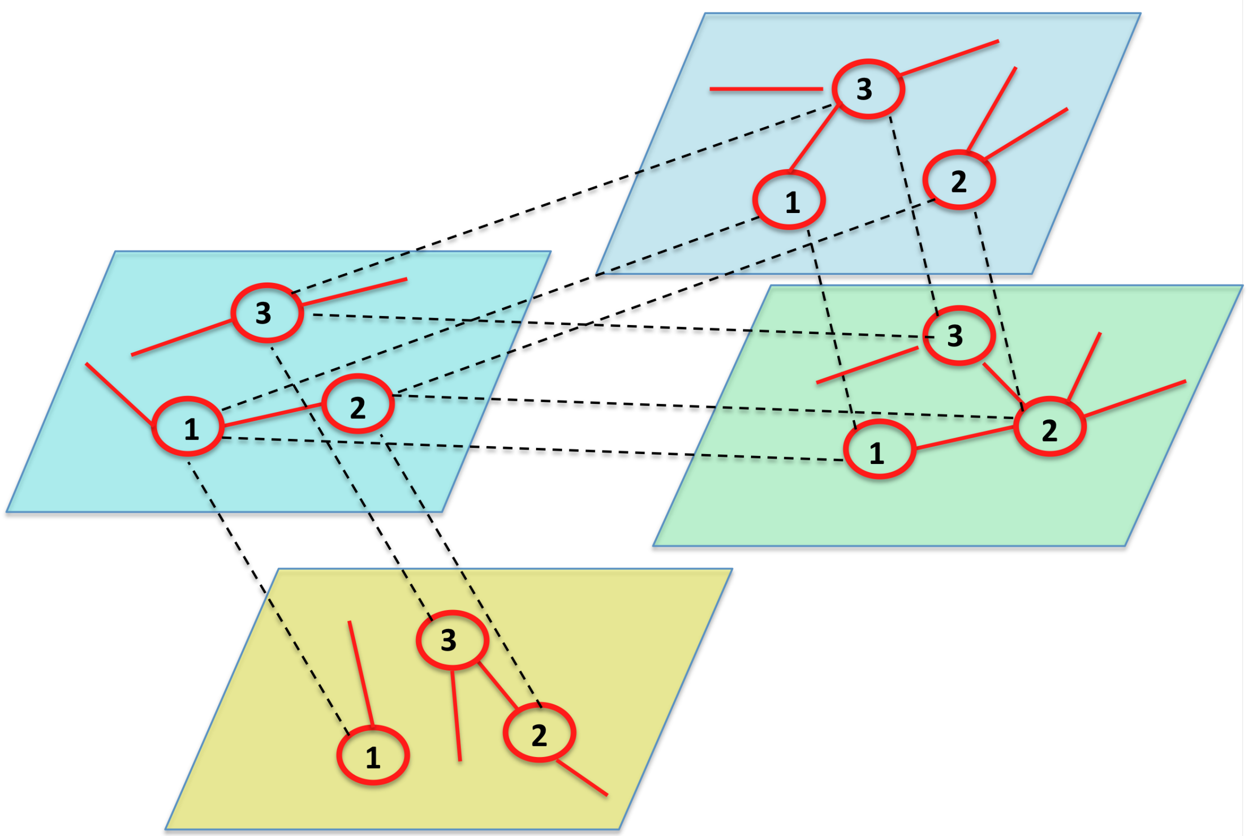
Abstract

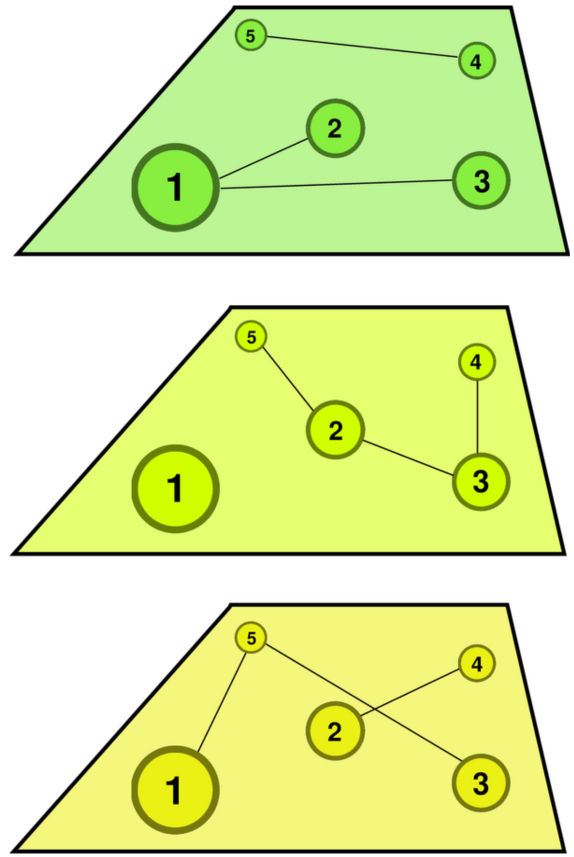
In the past years, network theory has successfully characterized the interaction among the constituents of a variety of complex systems, ranging from biological to technological, and social systems. However, up until recently, attention was almost exclusively given to networks in which all components were treated on equivalent footing, while neglecting all the extra information about the temporal- or context-related properties of the interactions under study. Only in the last years, taking advantage of the enhanced resolution in real

Network of Networks Example



Multiplex

- A multiplex is formed by a set of nodes that are present at the same time on different networks,
- A multiplex is formed by M layers (in the figure M=3)
- Each layer is formed by a network



Representation of a multiplex

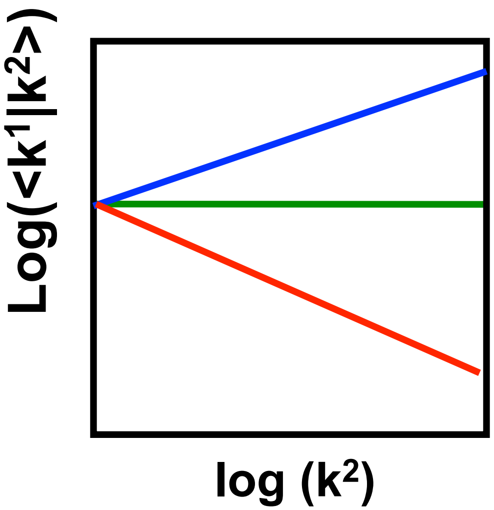
The straightforward representation a multiplex network of N nodes formed by M layers is by means of the set of M adjacency matrices

$$a^{\alpha}$$

with α=1, 2, … M and matrix elements

$$a_{ij}^{\alpha} = \begin{cases} 1 & \text{if node } i \text{ and node } j \text{ are linked in layer } \alpha \\ 0 & \text{otherwise} \end{cases}$$

Conditional average degree of a node in one layer (case of a duplex, i.e. two layers)



Positive degree correlations

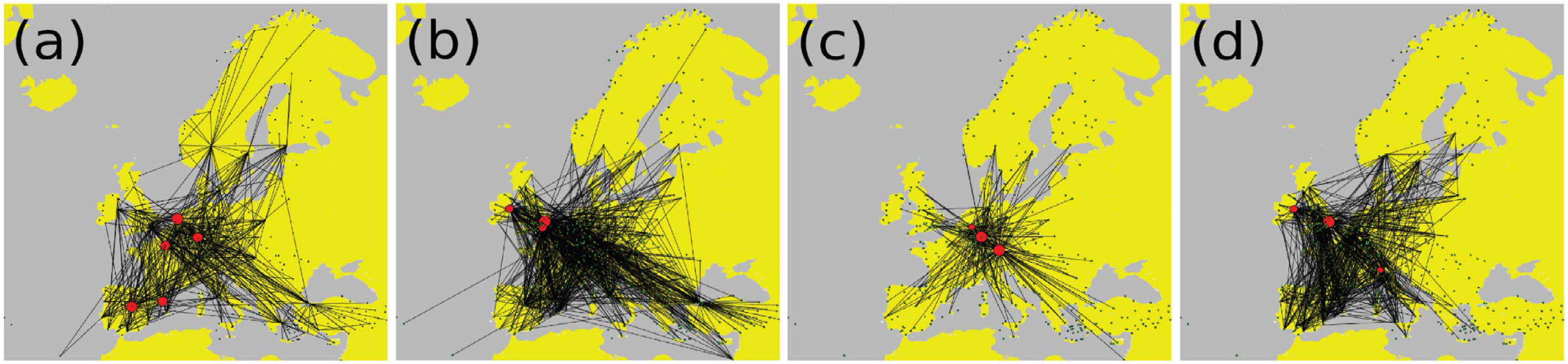
No degree correlations

Negative degree correlations

$$\left\langle k^1 \middle| k^2 \right\rangle = \sum_{k^1} k^1 P(k^1 \mid k^2)$$

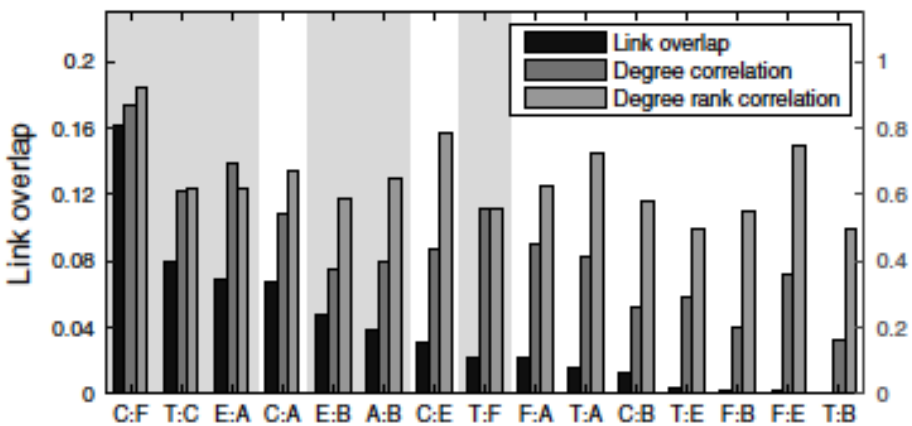
k^1 degree in network 1,k^2 degree in network 2
$P(k^1|k^2)$ probability that a node has degree k^1 in one layer given that
It has degree k^2 in the other layer

Overlap in multiplex networks



- (a) Only links belonging to all airline companies are plotted

Cardillo et al. Scientific Reports (2013).



Social network of online social game

Szell et al . PNAS 2010

Multiplex measures: Overlap

- For two layers α and α' of the multiplex we can define the

  total overlap $O^{\alpha\alpha'}$ as

$$O^{\alpha,\alpha'} = \sum_{i<j} a_{ij}^{\alpha} a_{ij}^{\alpha'}$$

- For a node i of the multiplex, we can define the

  local overlap $o_i^{\alpha,\alpha'}$

$$o_i^{\alpha,\alpha'} = \sum_{j} a_{ij}^{\alpha} a_{ij}^{\alpha'}$$

Class of network models

- **Growing networks:**
  - **Preferential attachment**

    Barabasi & Albert 1999,

    Dorogovtsev & Mendes 2000,

    Bianconi & Barabasi 2001

- Static networks:
  - **Ensembles of networks**

    Bollobas 1979, Chung & Lu 2002,

    Caldarelli et al. 2002, Park & Newman 2003

Growing multiplex (duplex)

- GROWTH

At each time a new node is added to the multiplex.
Every new node has a copy in each layer and has m links in each layer.

- LINEAR PREFERENTIAL ATTACHMENT

The probability that the new link is added to node i in layer α
is given by Π^{α} with

$$\Pi_i^1 \propto a k_i^1 + (1-a) k_i^2$$

$$\Pi_i^2 \propto (1-b) k_i^1 + b k_i^2$$

and a,b $\leq$ 1.

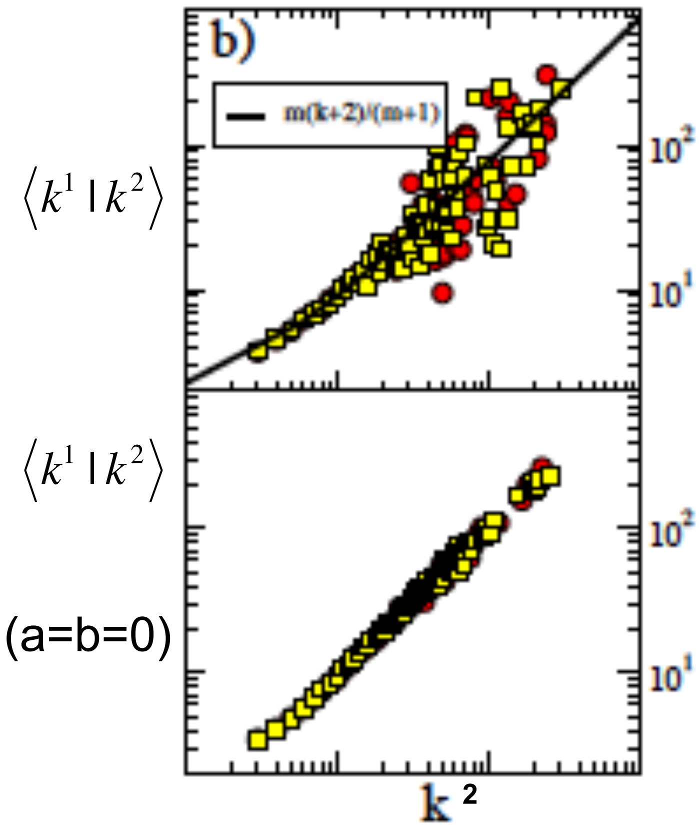
Degree correlations

Nicosia et al PRL 2013

- Case a=b=1   Exact solution

$$P(k^1,k^2)=\frac{2\Gamma(2+2m)\Gamma(k^1)\Gamma(k^2)\Gamma(k^1+k^2-2m+1)}{\Gamma(m)\Gamma(m)\Gamma(k^1-m+1)\Gamma(k^2-m+1)}$$

$$\langle k^1 \mid k^2\rangle=\frac{m}{1+m}(k^2+2)$$



- For general a,b solving in the mean-field approximation it can be obtained

$$\langle k^1 \mid k^2\rangle \propto k^2$$

- From the simulation results it is possible to conclude that the degree correlations are minimal in the a=b=1 case

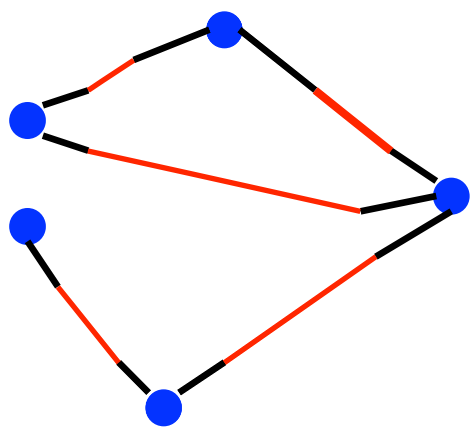
Multilayer networks encode multiple correlations such as the overlap of the links:

Multilinks and their weighted properties

Networks with given degree sequence

Microcanonical ensemble

$$P(G) = \frac{1}{\Sigma_1} \prod_i \delta\left(k_i - \sum_j a_{ij}\right)$$

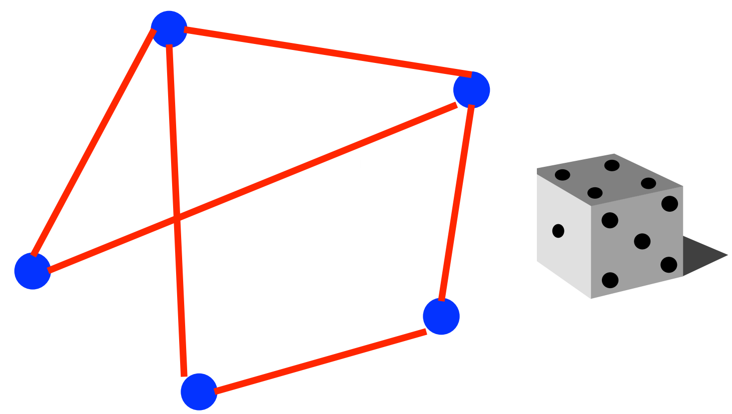


Ensemble of network with exact degree sequence

Configuration model

Canonical ensemble

$$P(G) = \prod_{i<j} p_{ij}^{\,a_{ij}} (1 - p_{ij})^{1-a_{ij}}$$



Ensemble of networks given expected degree sequence

Exponential Random Graph

Entropy of network ensembles

Entropy of a **canonical network ensemble** with linear constraints

$$S = -\left[\sum_{ij} p_{ij}\ln p_{ij} + (1-p_{ij})\ln(1-p_{ij})\right]$$

Entropy of a **microcanonical network ensemble with linear constraints con be found by the cavity method**, in the configuration model for sparse network limit with structural cutoff we get

$$\Sigma = \log[\aleph] = S - \Omega \quad with \quad \Omega = \frac{1}{N}\sum_i \log \pi_{k_i}(k_i)$$

G.Bianconi, A.C. C. Coolen, C.J. Perez-Vicente 2008

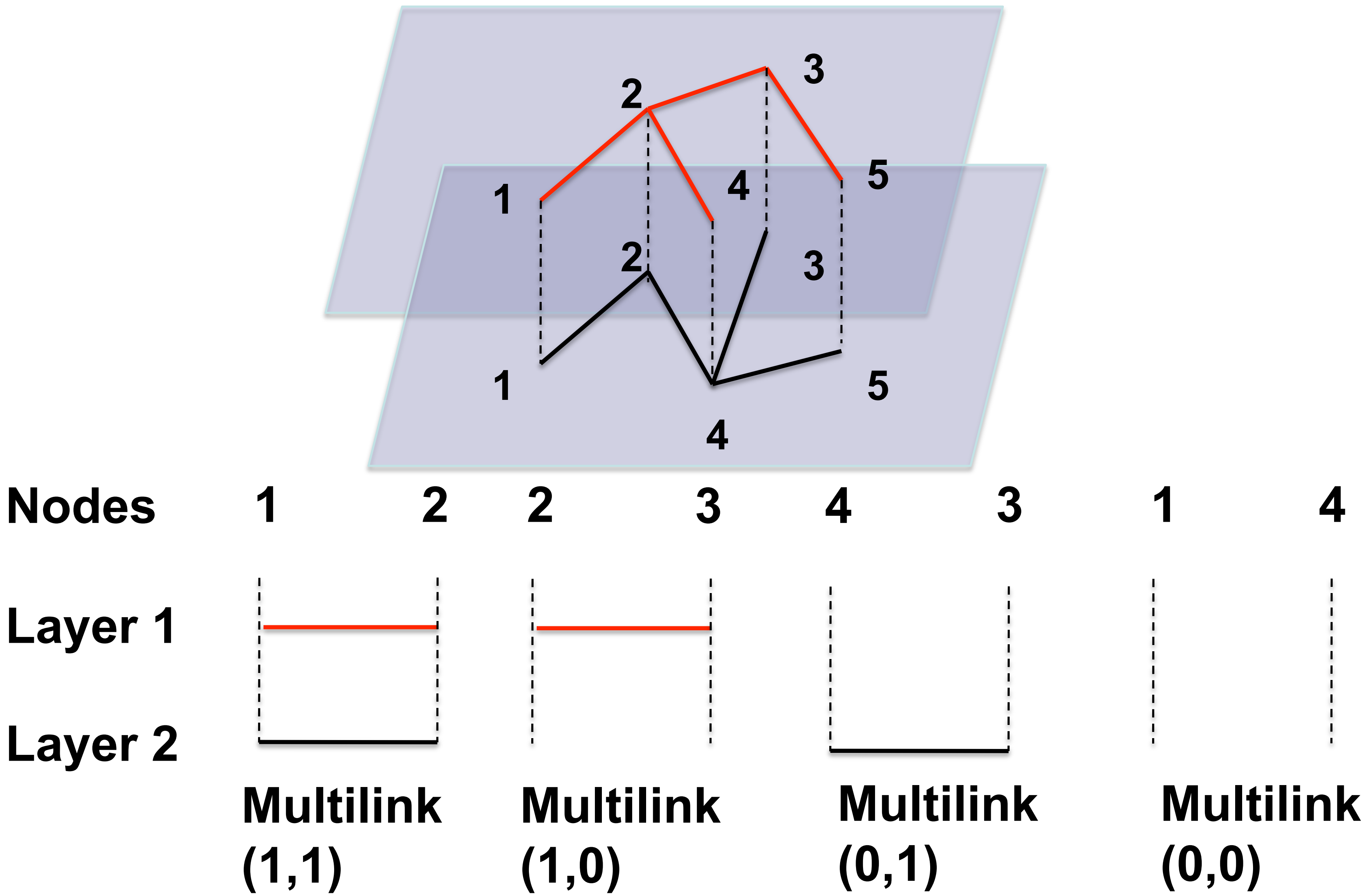
Overlap in the configuration model mutliplex ensembles

In a multiplex network
formed by sparse networks
generated by the configuration model
global and local overlap are
negligible in the large network limit
Therefore we need multilinks.

G. Bianconi PRE (2013)

Multilinks

G. Bianconi
PRE (2013)



Case of two layers

Multiadjacency matrices

$$A_{ij}^{10} = \begin{cases} 1 & if \;\; node \; i \; and \; node \;\; j \; are \; linked \; in \; layer \; 1 \; and \; not \; linked \; in \; layer \; 2 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{01} = \begin{cases} 1 & if \;\; node \; i \; and \; node \;\; j \; are \; linked \; in \; layer \; 2 \; and \; not \; linked \; in \; layer \; 1 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{11} = \begin{cases} 1 & if \;\; node \; i \; and \; node \;\; j \; are \; linked \; in \; layer \; 1 \; and \; in \; layer \; 2 \\ 0 & otherwise \end{cases}$$

$$A_{ij}^{00} = \begin{cases} 1 & if \;\; node \; i \; and \; node \;\; j \; are \; not \; linked \; in \; layer \; 1 \; and \; not \; linked \; in \; layer \; 2 \\ 0 & otherwise \end{cases}$$

Constraints on the multiadjacency matrices

$$A_{ij}^{10} + A_{ij}^{01} + A_{ij}^{11} + A_{ij}^{00} = 1$$

Multidegree

- The multidegree $k_i^{\vec{m}}$ of a node i is defined as

$$k_i^{\vec{m}} = \sum_j A_{ij}^{\vec{m}}$$

- In the case of two layers we have

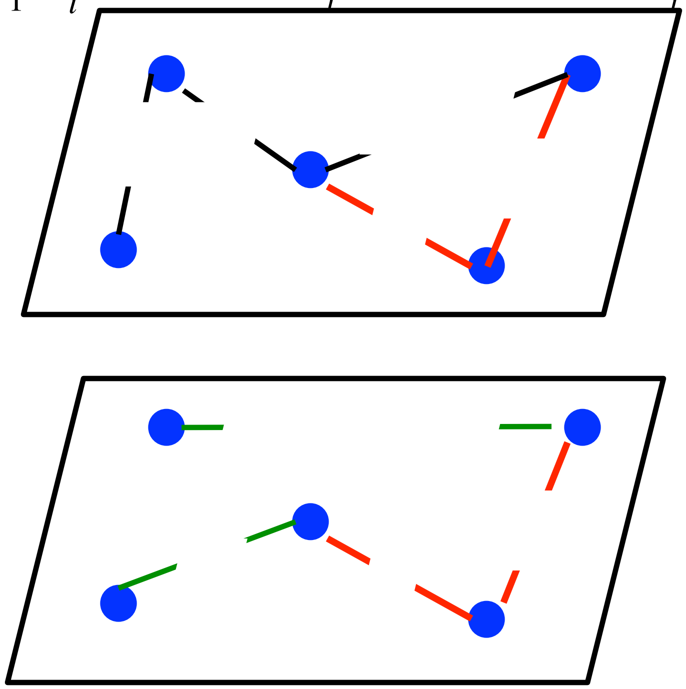
$$k_i^{10} = \sum_j a_{ij}^1 (1 - a_{ij}^2)$$

$$k_i^{01} = \sum_j (1 - a_{ij}^1) a_{ij}^2$$

$$k_i^{11} = \sum_j a_{ij}^1 a_{ij}^2 = o_i$$

Configuration model for the correlated multiplex(microcanonical ensemble)

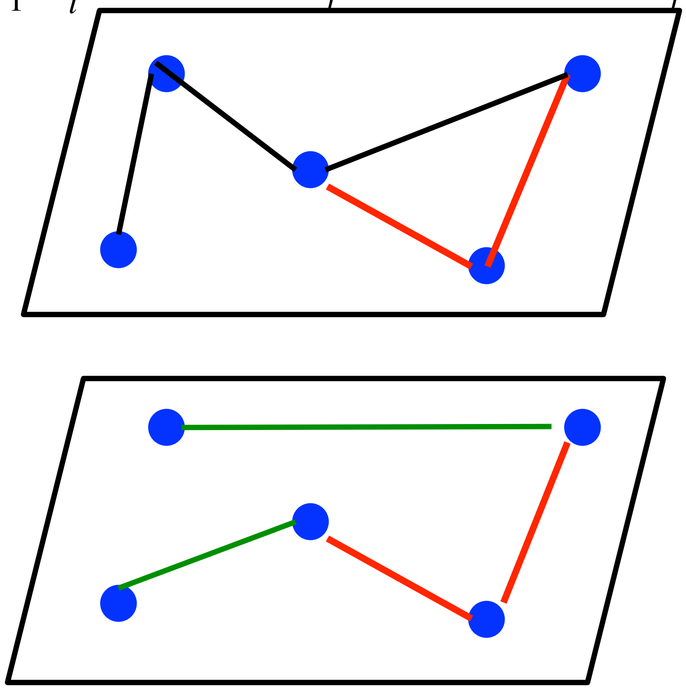
$$P(\vec{G}) = \frac{1}{\Sigma_1} \prod_i \delta(k^{10}{}_i - \sum_j A^{10}_{ij})\delta(k^{01}{}_i - \sum_j A^{01}_{ij})\delta(k^{11}{}_i - \sum_j A^{11}_{ij})$$



Ensemble of multiplex with given multidegree sequence

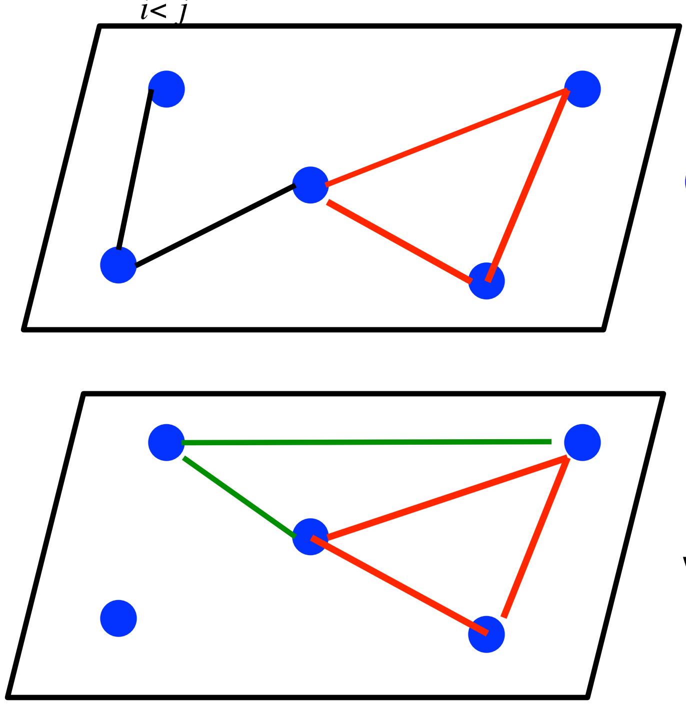
Configuration model for the multiplex with give multidegree sequence

$$P(\vec{G}) = \frac{1}{\Sigma_1}\prod_i \delta(k^{10}{}_i - \sum_j A_{ij}^{10})\delta(k^{01}{}_i - \sum_j A_{ij}^{01})\delta(k^{11}{}_i - \sum_j A_{ij}^{11})$$



Canonical model for the multiplex with given expected multidegree sequence

$$P(\vec{G}) = \prod_{i<j} (p_{ij}^{10} A_{ij}^{10} + p_{ij}^{01} A_{ij}^{01} + p_{ij}^{11} A_{ij}^{11} + p_{ij}^{00} A_{ij}^{00})$$



Constructive algorithm

For every pair of nodes (i,j)

Draw a multilink $\vec{m}$

with probability $p_{ij}^{\vec{m}}$ **,**

i.e. put a link in every layer

where m_α=1.

G. Bianconi PRE (2013)

Multilinks probabilities in a duplex with structural multidegree cutoffs

Probabilities of the multilinks

$$p_{ij}^{10}=\frac{k_i^{10}k_j^{10}}{\langle k^{10}\rangle N}$$

$$p_{ij}^{01}=\frac{k_i^{01}k_j^{01}}{\langle k^{01}\rangle N}$$

$$p_{ij}^{11}=\frac{k_i^{11}k_j^{11}}{\langle k^{11}\rangle N}$$

Structural cutoff

$$k^{10}<\sqrt{\langle k^{10}\rangle N}$$

$$k^{01}<\sqrt{\langle k^{01}\rangle N}$$

$$k^{11}<\sqrt{\langle k^{11}\rangle N}$$

G. Bianconi PRE 2013

Entropy of correlated multiplex ensembles

Entropy of a **canonical multiplex ensemble** with linear constraints

$$S = -\left[\sum_{\vec{m}} \sum_{ij} p_{ij}^{\vec{m}} \ln p_{ij}^{\vec{m}}\right]$$

Entropy of a **microcanonical mutliplex ensemble with linear constraints con be found by the cavity method**, if we fix only the multi degree sequence in the sparse network limit, we get

$$\Sigma = S - \Omega$$

$$\Omega = -\frac{1}{N}\sum_{\vec{m}}\sum_{i} \log \pi_{k^{\vec{m}}}(k^{\vec{m}})$$

G. Bianconi PRE 2013

Spatial Multiplexes

The nodes in a spatial multiplex have a position $\vec{r}$ in their real or hidden embedding space

$$P\left(\vec{G} \mid \left\{\vec{r}_i\right\}\right) = \prod_{\alpha=1,M} P_\alpha\left(G_\alpha \mid \left\{\vec{r}_i\right\}\right)$$

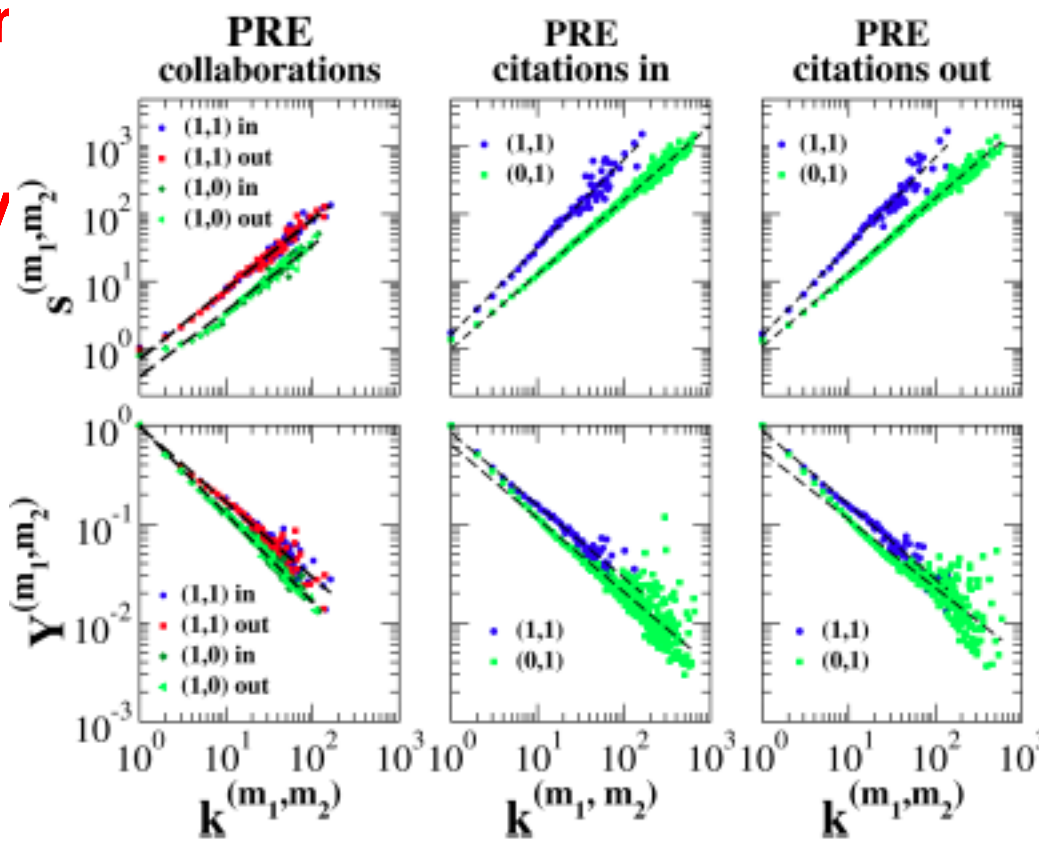
In these ensembles we can observe a significant overlap of the links because nodes that are "close in space" are more likely to be linked in every network

A. Halu, S. Mukherjee and G. Bianconi PRE 2014

Citation-Collaboration Weighted Multiplex Networks

The way you cite your collaborators is different from the way you cite the other scientists.

People tend to cite more the hubs with whom they have collaborated.



G. Menichetti et. al. Plos One (2014)

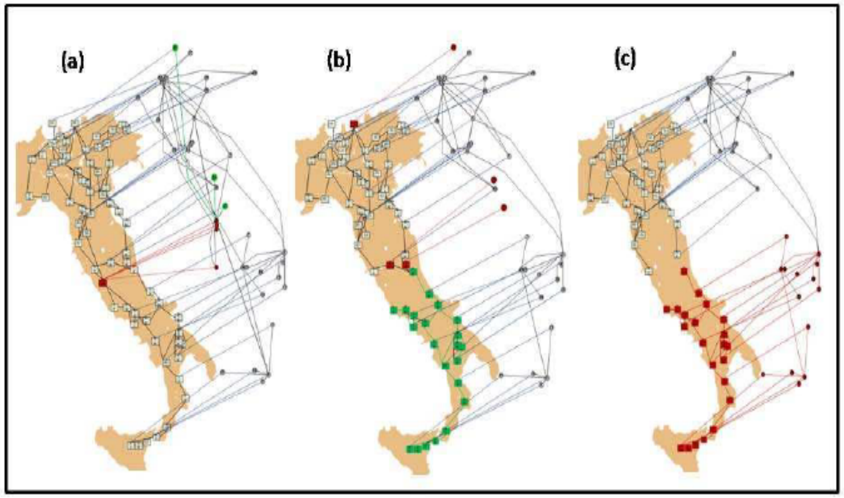
For the robustness

of network of networks

the way the interlinks between networks are placed is crucial:

Single versus multiple percolation phase transitions

Interacting network of networks



- Two or more interacting networks are formed by different nodes but there might be complex interactions and interdependencies between the nodes.
- Interdependencies might increase the fragility of interdependent networks as characterized by the discontinuous emergence of the mutually connected giant component

(Buldyrev et al. Nature 2010)

Giant component in single networks Mutually connected giant component of multiplex networks

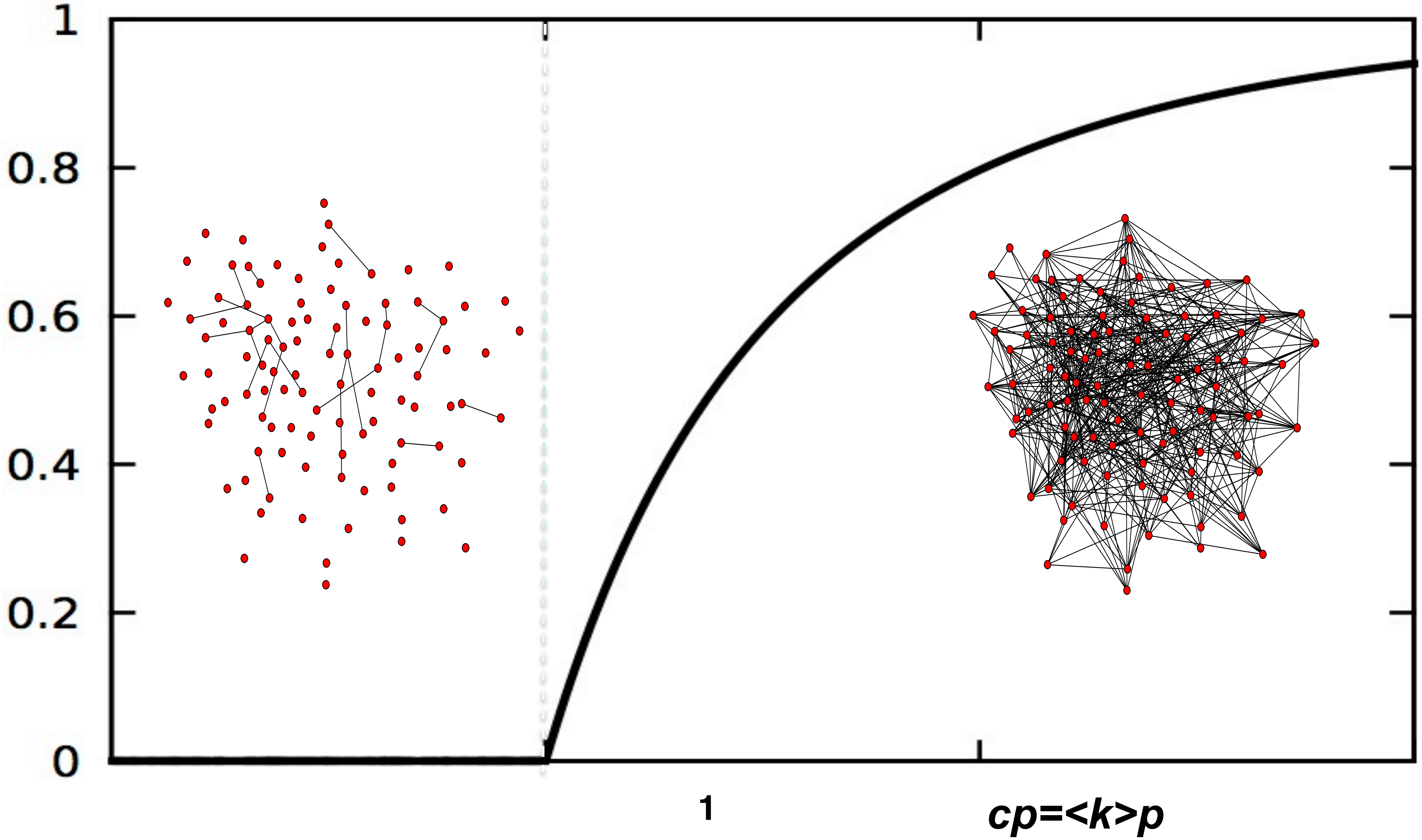
- The size S of the giant component in a Poisson network with average degree c where a fraction 1-p of nodes is damaged is given by

$$S = p\left(1 - e^{-cS}\right)$$

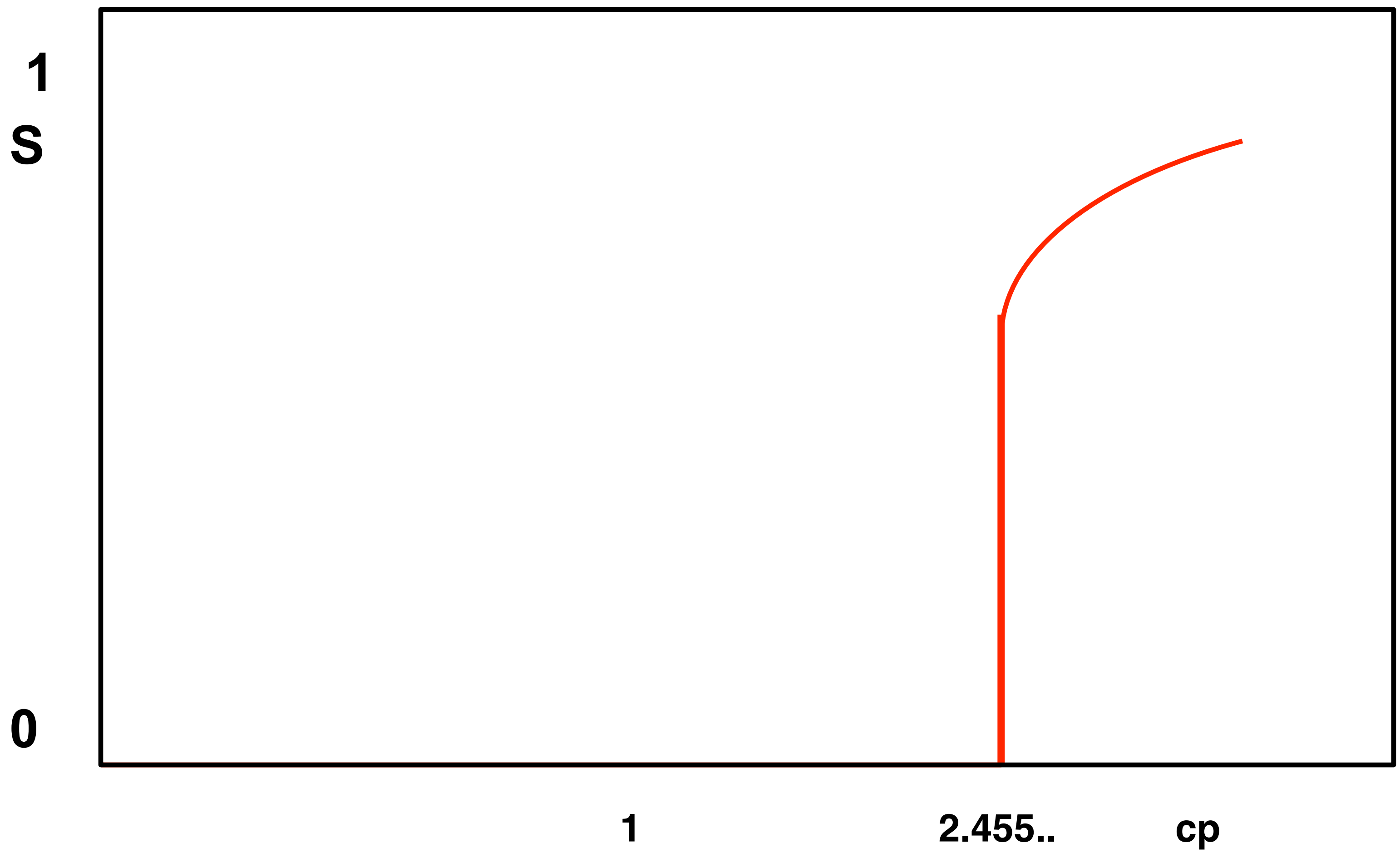
- The size S of the Mutually Connected Giant Component for a multiplex of M>1 Poisson layers with average degree c where a fraction 1-p of nodes is damaged is given by

$$S = p\left(1 - e^{-cS}\right)^{M}$$

Percolation transition of a single Poisson network



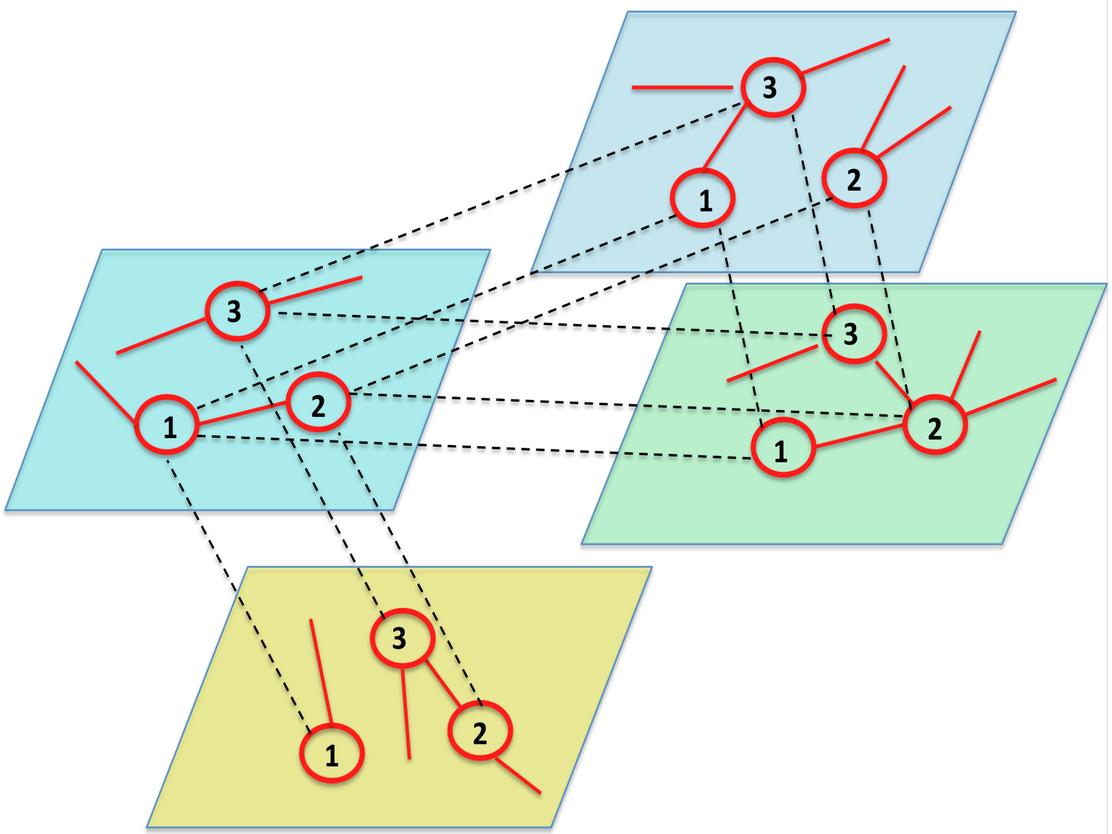
Discontinuous Emergence of the mutually connected giant component in a duplex of Poisson network



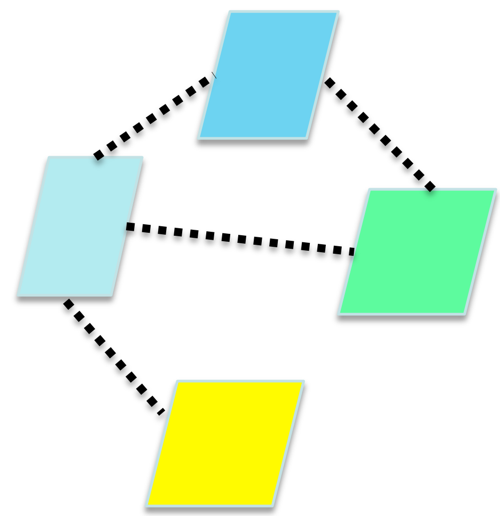
Buldyrev et al Nature 2010

Network of Networks with replica nodes

Network of networks



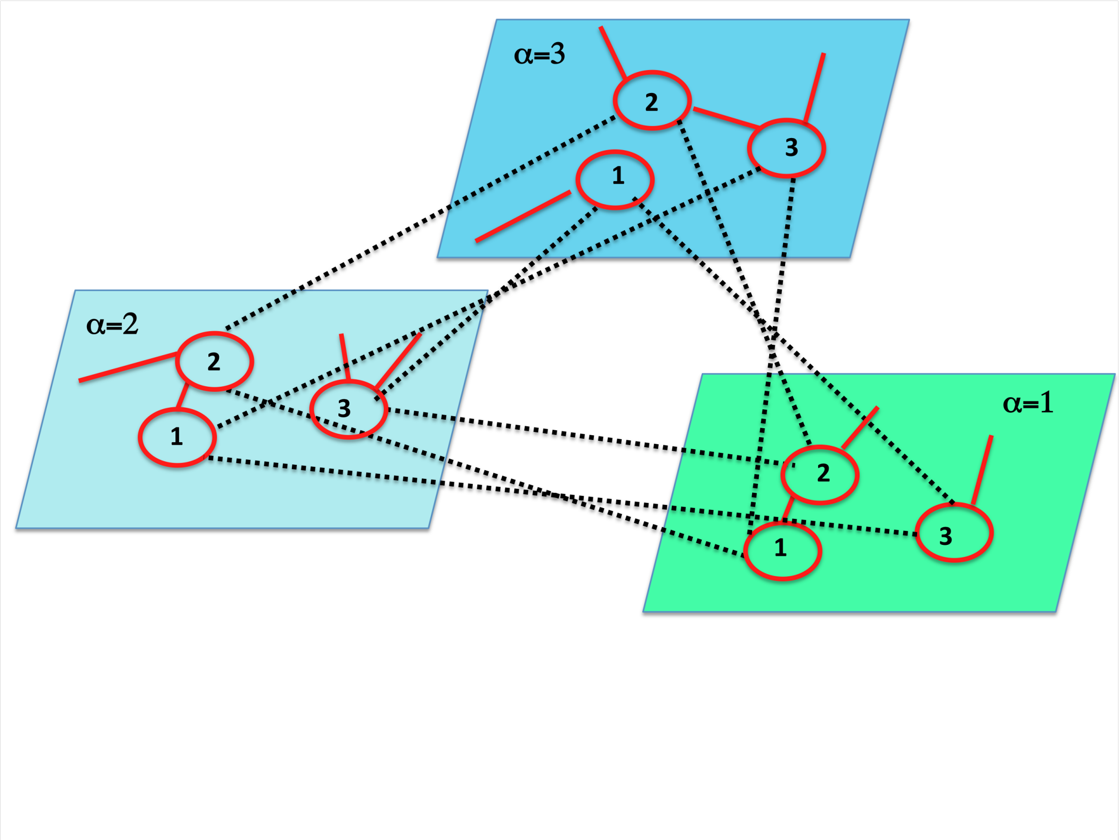
Supernetwork



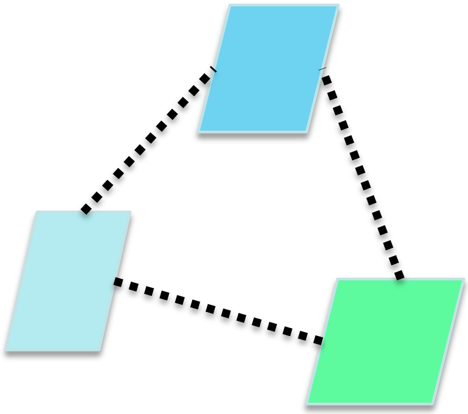
In this case the emergence of the mutually connected giant component is dictated by the same equations valid for a multiplex network.

Network of network without replica nodes

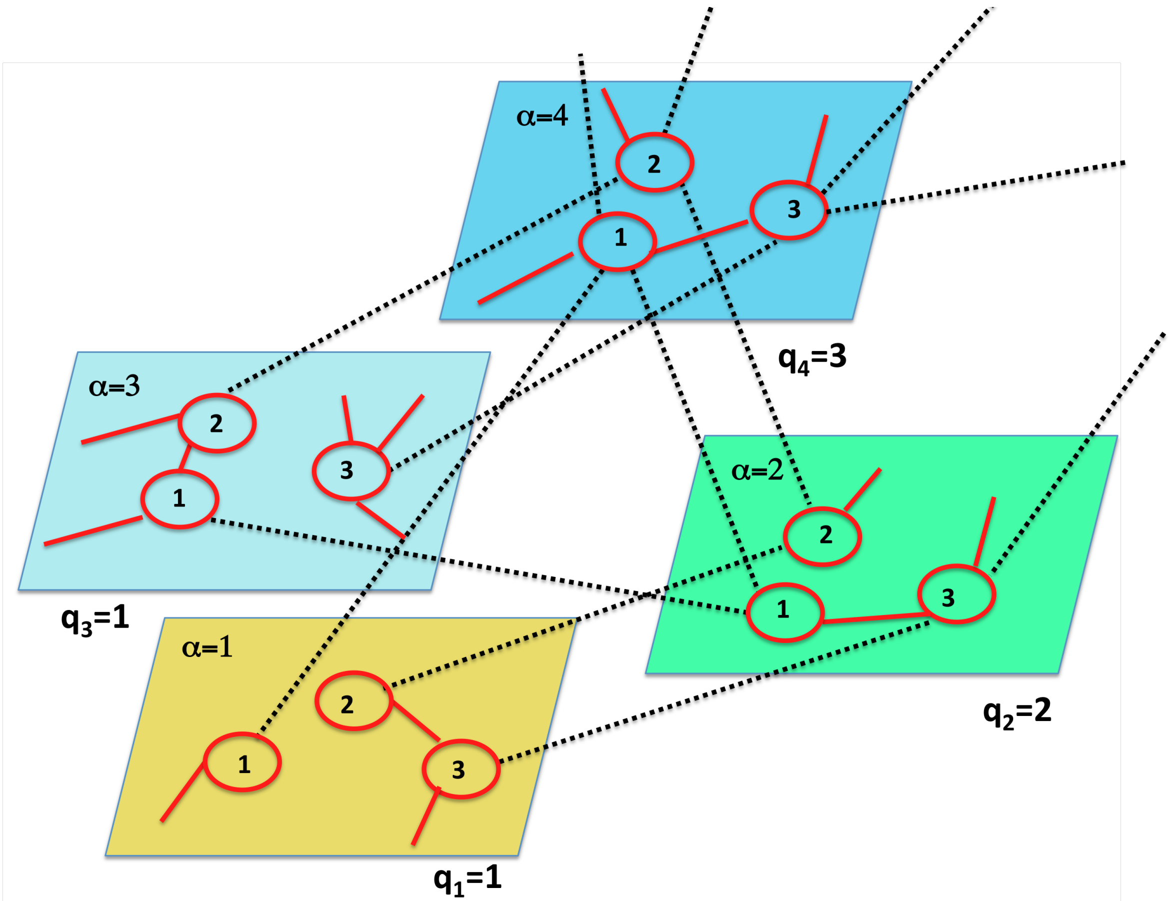
Network of networks



Supernetwork



Configuration model of Network of networks



Every layer α has a supradegree $q_{,\alpha.}$

Therefore every node of layer α has q_α links to q_α replica nodes in some other layer chosen randomly

Case in which each layer is a Poisson network with <k>=c

σ_q is the probability that a node in a layer with superdegree q belongs to the mutually connected component.

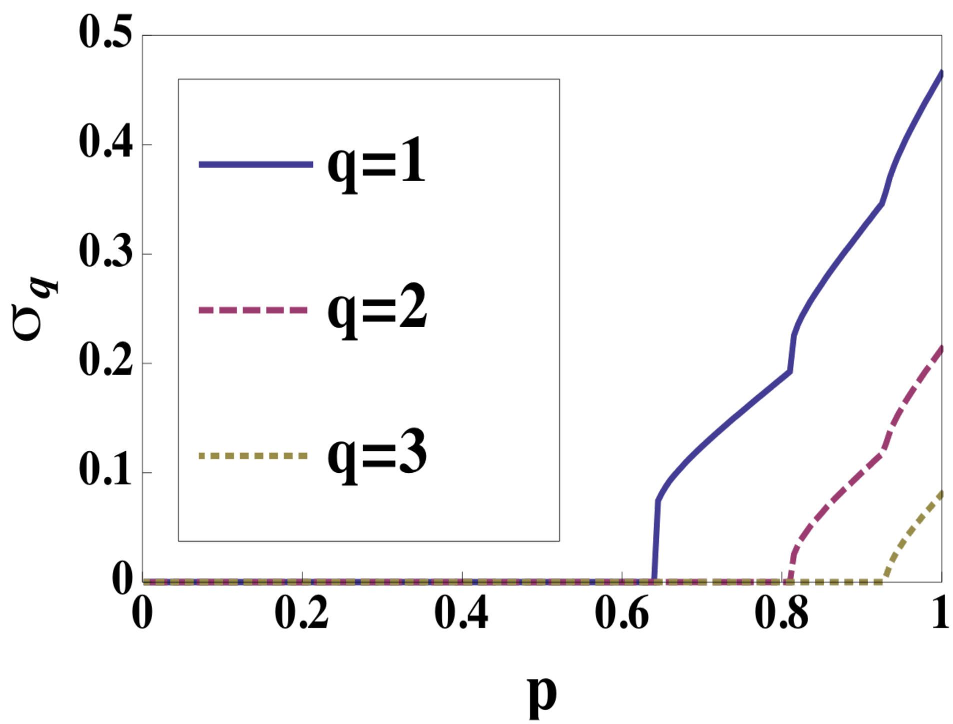
Σ is the order parameter of the percolation problem and is given by the probability that by following a link in the supernetwork we reach a node in the mutually connected component

$$\sigma_q = \frac{1}{c}\left[pc(\Sigma)^q + W\left(-pc(\Sigma)^q e^{-pc(\Sigma)^q}\right)\right]$$

$$\Sigma = G_1^q(\Sigma)\left[p + \sum_q \frac{qP(q)}{\langle q\rangle}\frac{1}{c}(\Sigma)^{-q} W\left(-pc(\Sigma)^q e^{-pc(\Sigma)^q}\right)\right]$$

where W(x) is the principal value of the Lambert function and $G^1_q(x)$ Is the generating function of the superdegree distribution.

Percolation in layers with superdegree q



Case γ=2.8
c=20

Multiple phase transitions!
Layers with larger superdegree are more vulnerable!

G. Bianconi and S.N Dorogovstev (2014)

Nature Physics News & Views

MULTILAYER NETWORKS

Dangerous liaisons?

Many networks interact with one another by forming multilayer networks, but these structures can lead to large cascading failures. The secret that guarantees the robustness of multilayer networks seems to be in their correlations.

Ginestra Bianconi

Natural complex systems evolve according to chance and necessity — trial and error — because they are driven by biological evolution. The expectation is that networks describing natural complex systems, such as the brain and biological networks within the cell, should be robust to random failure. Otherwise, they would have not survived under evolutionary pressure. But many natural networks do not live in isolation; instead they interact with one another to form multilayer networks — and evidence is mounting that random networks of networks are acutely susceptible to failure. Writing in *Nature Physics*, Saulo Reis and colleagues[1] have now identified the key correlations responsible for maintaining robustness within these multilayer networks.

In the past fifteen years, network theory[2,3] has granted solid ground to the expectation that natural networks resist failure. It has also extended the realm of robust systems to man-made self-organized networks that do not obey a centralized design, such as the Internet or the World Wide Web. In fact, it has been shown that many isolated complex biological, technological and social networks are scale free, meaning that their nodes

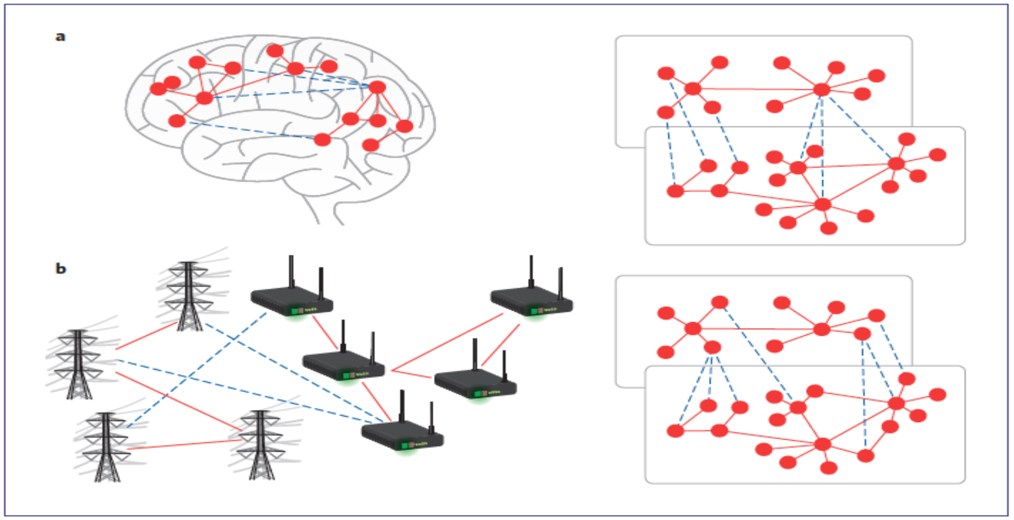


Figure 1 | Reis *et al.*[1] have shown that correlations between intra- (red) and interlayer (blue dotted) interactions influence the robustness of multilayer networks. **a**, In the brain, each network layer has multilayer assortativity and the hubs in each layer are also the nodes with more interlinks, so liaisons between layers are trustworthy. **b**, In complex infrastructures (such as power grids and the Internet), if the interlinks are random, the resulting multilayer network is affected by large cascades of failures[6], and liaisons can be considered dangerous.

Conclusions

- ***Multilayer networks can display new structural properties and correlations in their structure and can allow for new phenomena showing the rich interplay between structure and dynamics.***

- ***The multilinks or the embedding into geometrical space are necessary to obtain network overlap***

- **Weighted multiplex networks display rich correlations between weights and multidegrees encoding information that cannot be inferred from the analysis of single layers taken in isolation.**

- **Percolation on network of networks can display either single or multiple percolation transitions depending on the correlations between the interlinks.**

References and collaborators

- **REVIEW!!!**

  S. Boccaletti, G. Bianconi, R. Criado, C. del Genio, J. Gomez-Gardenes, M. Romance, I. Sendina-Nadal, Z. Wang, M. Zanin,

  Physics Reports 544, 1 (2014).

- **News & Views on robustness of multilayer networks**

  G. Bianconi Nature Physics (2014).

- **Multiplex ensembles**

  G. Bianconi PRE 87, 062806 (2013).

  A. Halu, S. Mukherjiee, G. Bianconi PRE 89, 012806(2014)

- **Weighted Multiplex Networks**

  G. Menichetti, D. Remondini, P. Panzarasa, R. Mondragon and G. Bianconi PloSOne e97857 (2014)

- **Percolation on network of networks**

  G. Bianconi, S. N. Dorogovtsev PRE (2014).

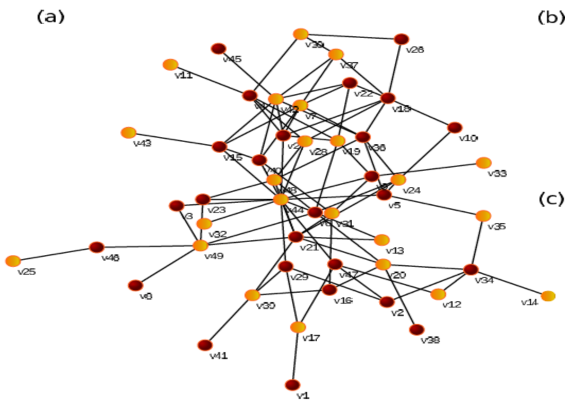
  G. Bianconi, S. N. Dorogovstev, J. F. F. Mendes arXiv:1402.0215 (2014).

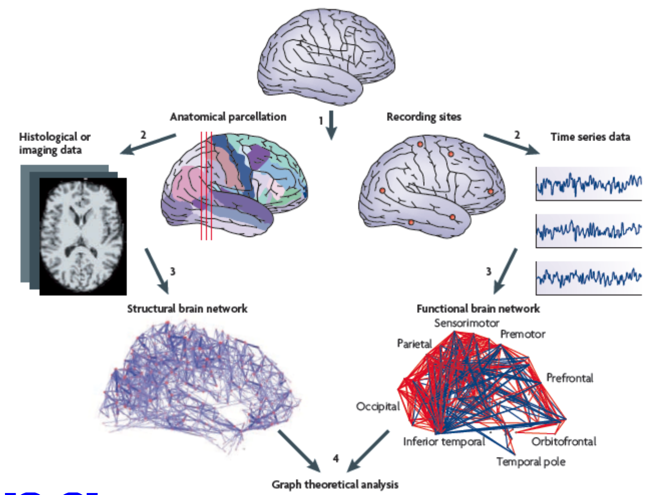
  G. Bianconi, S. N. Dorogovstev, arXiv:1411.4160 (2014).

Other recent results involve topics of interest to this workshop:

controllability

geometry of networks





Of relevance

for understanding

financial networks, and

the brain dynamics

and for network medicine

determining

the controllability of networks

is a central theoretical problem

of network theory

Driver nodes

The driver nodes of a network

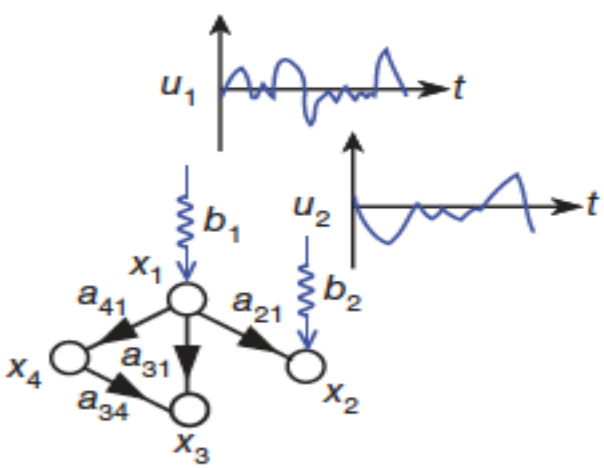
are the nodes that,

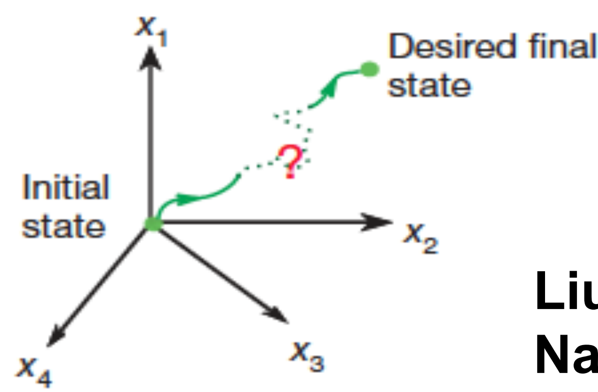
when stimulated by an external signal,

can drive

the dynamical state of a network

to any desired state.



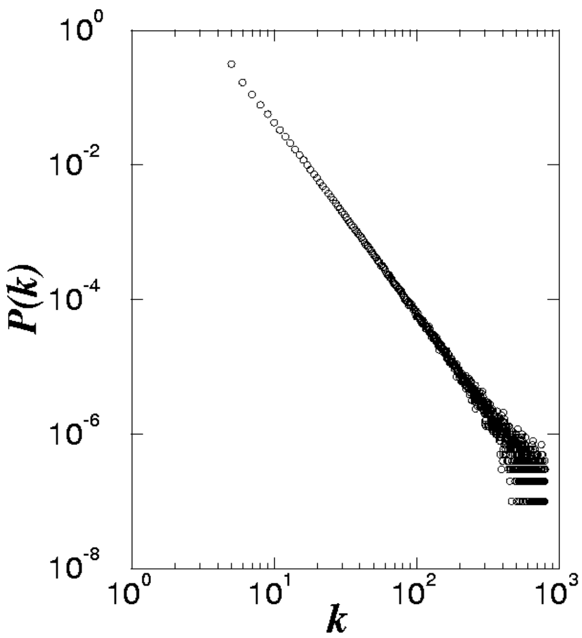


Liu et. al. Nature (2011)

The importance of hubs for the dynamics on complex networks

Hubs and scale-free networks are essential for determining the

- The robustness of complex networks
- The stability of the ferromagnetic phase of the Ising model against thermal fluctuations
- Triggering and orchestrating the synchronization on neuronal networks
- Characterizing the epidemic spreading properties of networks



Which are the key structural properties of networks that determine their structural controllability?

The low In-degree and Out-degree nodes

(*hubs are irrelevant for determining the number of driver nodes in the structural controllability framework developed by Liu, Slotine and Barabasi*)

G. Menichetti, L.Dall'Asta and G. Bianconi
PRL 113, 078701 (2014)

First result:
Sufficient condition for full controllability

For any sparse network

without a finite clustering coefficient,

(where the locally tree-like approximation is valid),

if the minimum in-degree

and the minimum out-degree of the network

are both greater than 2,

the network is fully controllable by an infinitesimal fraction of driver nodes.

Second result: Necessary and sufficient condition for full controllability on a random network with given degree distribution

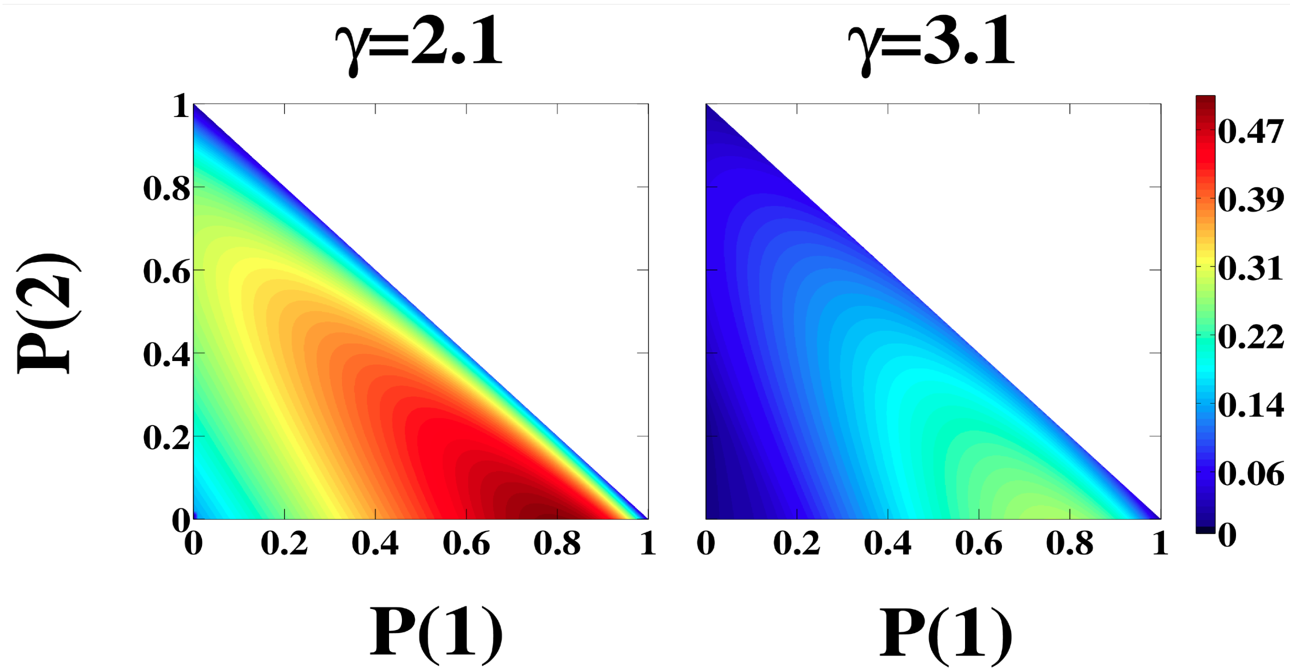
- A random network with given degree distribution is fully controllable iff

$$P^{out/in}(1) = P^{out/in}(2) = 0$$

$$P^{out}(2) < \frac{\langle k \rangle_{in}^{2}}{2\langle k(k-1) \rangle_{in}} \qquad P^{in}(2) < \frac{\langle k \rangle_{out}^{2}}{2\langle k(k-1) \rangle_{out}}$$

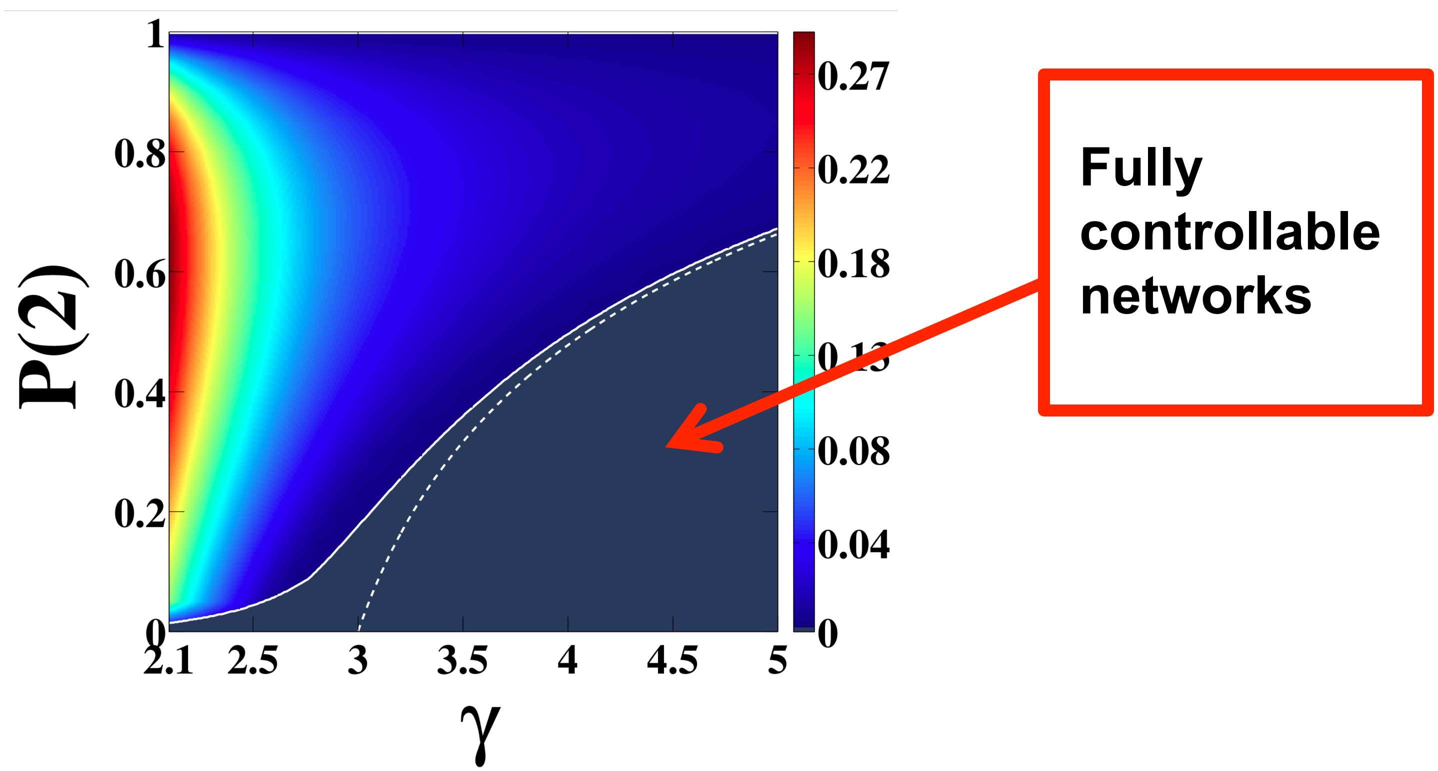
i.e. its minimum in and out-degree are 2 and the nodes with in/out degree 2 are less than a threshold.

Number of driver nodes as a function of the density of low in-degree and out-degree nodes changes smoothly



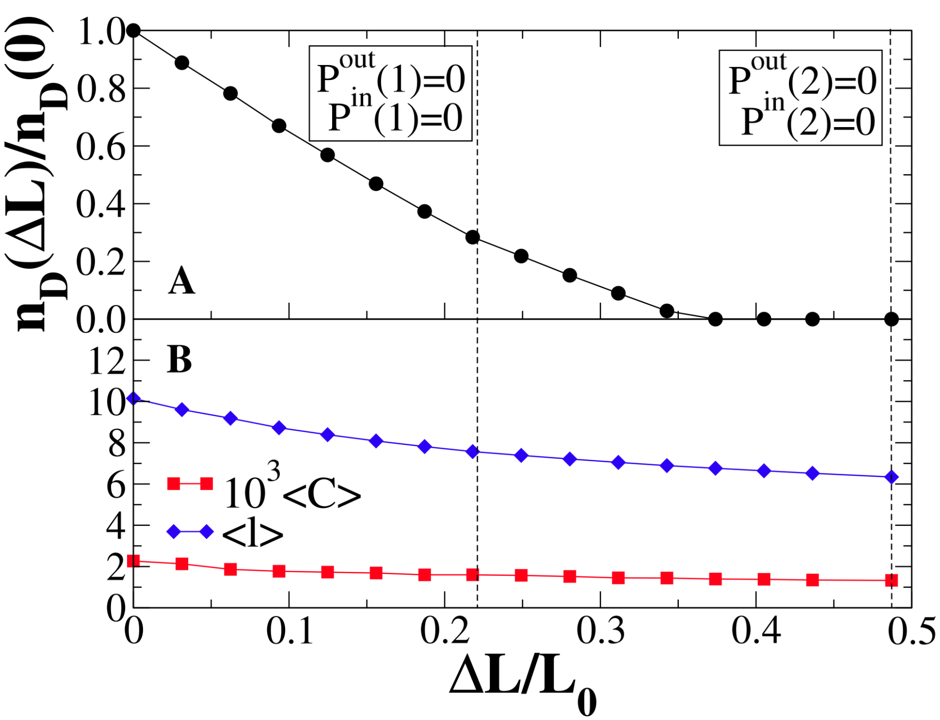
$$P^{in}(k) = P^{out}(k) = \begin{cases} P(1) & for \quad k=1 \\ P(2) & for \quad k=2 \\ Ck^{-\gamma} & for \quad k\in[3,K] \end{cases}$$

Phase diagram



$$P^{in}(k) = P^{out}(k) = \begin{cases} 0 & for \quad k = 0,1 \\ P(2) & for \quad k = 2 \\ Ck^{-\gamma} & for \quad k \in [3, K] \end{cases}$$

Improving the controllability of networks by adding links to low in-degree and low out-degree nodes



Case of the pure scale-free distribution With γ=2.3

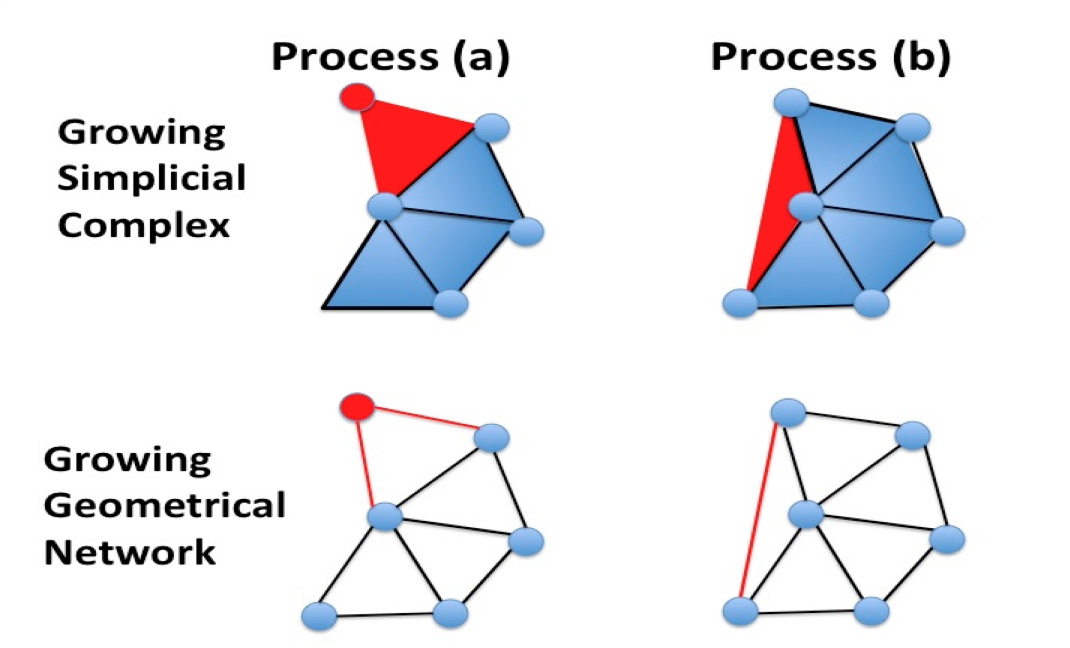
Emergent
Complex Network
Geometry

Many network data appear to have an hidden metric, reflected in their local properties (clustering, graphlets, modules)

but is the hidden metric causing the network dynamics or is it emergent for the dynamics of the network itself?

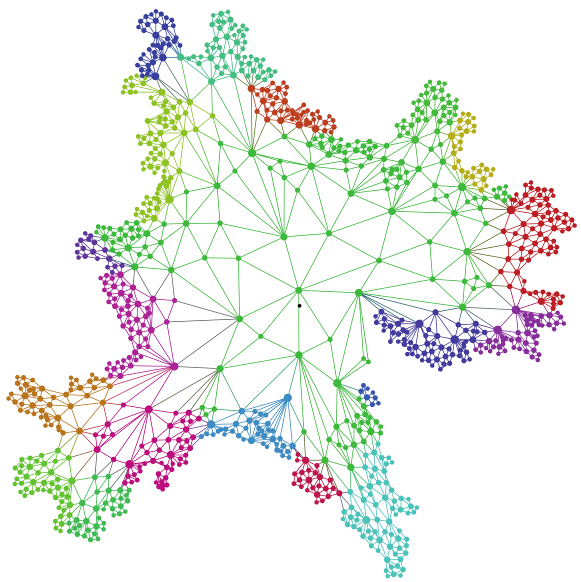
The geometrical growing network

The geometrical growing network is constructed from a simplicial complex formed by triangles where each link can be incident to at most m triangles. There are two processes, process (a) is the addition of triangles and process (b) is "closure of dual loops". Process (b) occurs at each time with probability p.



Random geometries and curvature distribution

$m = 2 \quad p = 0.9$

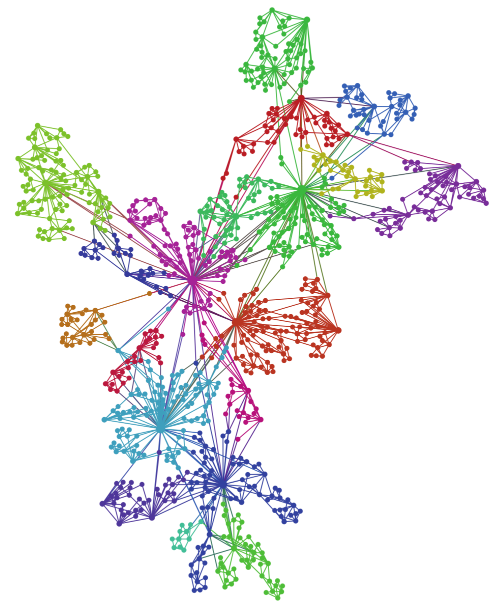


Exponential network

$m = 4 \quad p = 0.9$

Broad degree distribution

$m = \infty \quad p = 0$



Scale-free network

$$\langle R \rangle = \frac{1}{N}$$

$$\langle R^2 \rangle < \infty$$

$$\langle R \rangle = c$$

$$\langle R^2 \rangle = \infty$$

$$\langle R \rangle = \frac{1}{N}$$

$$\langle R^2 \rangle = \infty$$

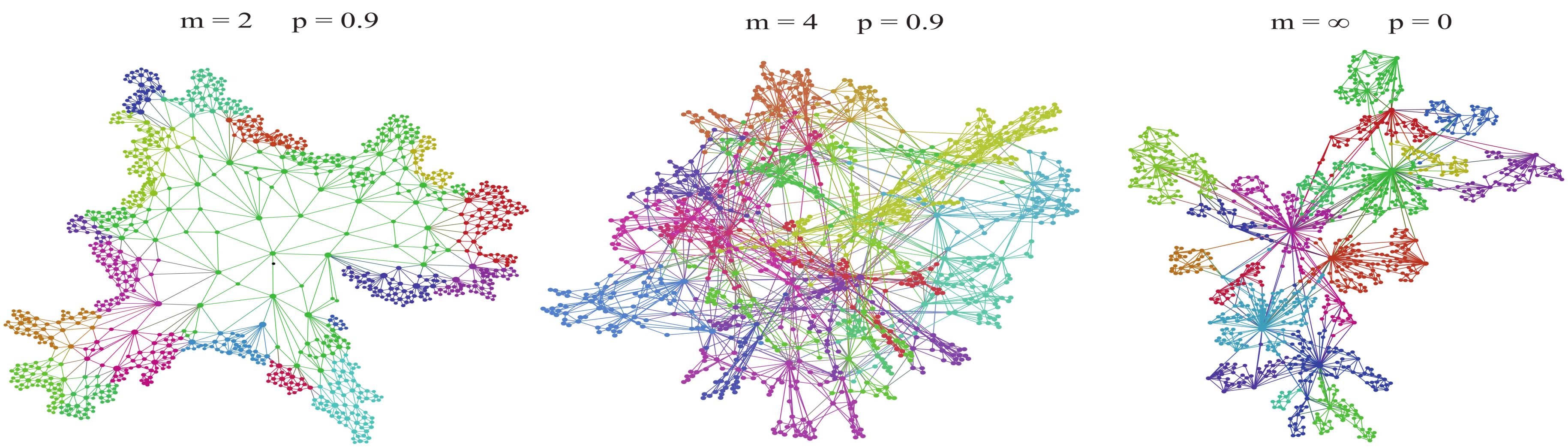
Properties of geometrical growing networks

- **Finite clustering**
- **High modularity**
- **Non trivial k-core**
- **Finite spectral dimension**

Which are properties of many real network datasets.

Open problems: Equilibrium models (ERG), Hidden metric, Characterization of the random geometries

Z. Wu, G. Menichetti, C. Rahmede and G.Bianconi
Emergent Complex Network Geometry
arXiv:1412.3405 (2014).



Thank you for your attention!