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Connection graph stability method for synchronized coupled chaotic systems

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Abstract

This paper elucidates the relation between network dynamics and graph theory. A new general method to determine global stability of total synchronization in networks with different topologies is proposed. This method combines the Lyapunov function approach with graph theoretical reasoning. In this context, the main step is to establish a bound on the total length of all paths passing through an edge on the network connection graph. In particular, the method is applied to the study of synchronization in rings of $2K$-nearest neighbor coupled oscillators. A rigorous bound is given for the minimum coupling strength sufficient for global synchronization of all oscillators. This bound is explicitly linked with the average path length of the coupling graph. Contrary to the master stability function approach developed by Pecora and Carroll, the connection graph stability method leads to global stability of synchronization, and it permits not only constant, but also time-dependent interaction coefficients. In a companion paper ("Blinking model and synchronization in small-world networks with a time-varying coupling," see this issue), this method is extended to the blinking model of small-world networks where, in addition to the fixed $2K$-nearest neighbor interactions, all the remaining links are rapidly switched on and off independently of each other.
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1. Introduction

During the last few decades the study of the dynamics of coupled nonlinear dynamical systems has generated a rapidly growing interest in theoretical physics and other fields of science. The interest was mainly concentrated on the spatiotemporal behavior of coupled systems and on related synchronization phenomena. Traditionally synchronization of two limit-cycle systems means that their time evolution is periodic, with the same period and, perhaps, the same phase. This notion of synchronization is not sufficient when the systems are chaotic. More recently, synchronization of chaotic systems has been discovered [1–3] and since then it has become an important research topic in

mathematics, physics, and engineering. Synchronization of two systems in this case means identical or functionally related solutions, perhaps with a delay. Strong forms of synchrony in chaotic systems include complete [1–25] and cluster [26–31] synchronization. Some weaker forms of synchronization of chaotic systems such as phase and lag synchronization [32], bubbling synchronization [33], and generalized synchronization [34] were also reported. For more references to synchronization in chaotic dynamical systems see [35].

Since early works were concerned with a small number of coupled oscillators, the increasing interest in synchronization phenomena has led many researchers to consider synchronization in large networks of coupled systems with different coupling configurations. One important question about these networks is that of complete synchronization: What are the conditions for the stability of the synchronous state, especially with respect to coupling strengths and coupling configurations of the network? This problem was intensively studied for networks of periodic dynamical systems [36–40] and chaotic systems [4–14,17–22,24,25].

Typically, in networks of continuous time oscillators, the synchronous solution becomes stable when the coupling strength between the oscillators exceeds a critical value. In this context, a central question is to find the bounds on the coupling strengths so that the stability of synchronization is guaranteed. Most methods for determining stability for synchronized chaotic systems developed independently by several research groups are based on the calculation of the eigenvalues of the coupling matrix for different regular coupling schemes and a term depending mainly on the dynamics of the individual oscillators [4,7,8,12,13,17–25]. A general approach to the local synchronization of chaotic systems for any linear coupling scheme, called the master stability function, was developed in [12]. This approach is based on the calculation of the maximum Lyapunov exponent for the least stable transversal mode of the synchronous manifold and the eigenvalues of the connectivity matrix. Stronger global stability results for synchronization in networks of diffusively coupled chaotic systems were obtained in [4,7,29,30]. In [17–19], an analog of the master stability function for global synchronization of chaotic systems was proposed.

However, the eigenvalues of the coupling matrix can often be calculated only for simple coupling schemes such as local coupling, star configuration, global coupling, etc. In more complicated networks, the calculation of the eigenvalues becomes such a difficult task that it is rarely possible to obtain analytical bounds for the synchronization thresholds.

Moreover, for networks of oscillators with a time-varying coupling (where the connectivity matrix is time-dependent), the use of methods based on the eigenvalues of the connectivity matrix and the Lyapunov exponents is often impossible (the linearized system becomes time-dependent and may fail to provide the stability results).

In this paper we have developed a new general method, which we call the connection graph based stability method, to calculate upper bounds for global synchronization in complex networks of mutually coupled chaotic oscillators. Within the framework of this method, we combine the Lyapunov function approach with graph theoretical reasoning. In this context the main step is to establish a bound on the total length of all paths passing through an edge in the network connection graph. This method directly links synchronization with graph theory and allows us to avoid calculating both the Lyapunov exponents and the eigenvalues of the coupling matrix. Furthermore, it guarantees total synchronization from arbitrary initial conditions and not just local stability of the synchronization manifold. Our method is also valid for networks with a time-varying coupling and can be extended to the study of global synchronization in small-world networks with a time-varying coupling. This is the subject of the companion paper [41].

The layout of this paper is as follows. First, in Section 2, we state the problem under consideration. Then, in Section 3, we present the connection graph based stability method for global synchronization in networks of coupled chaotic oscillators. We first develop the general method and then apply it to several examples of concrete networks. In Section 4, we apply our method to global stability of synchronization in rings of $2K$-nearest neighbor coupled chaotic oscillators. A rigorous bound is given for the minimum coupling strength necessary for global synchronization of all the oscillators. This bound is explicitly linked with the average path length of the coupling graph. We also confirm our theoretical results with numerical simulations of the rings of $2K$-nearest neighbor coupled chaotic Lorenz

systems. In Section 5, we compare our method with the master stability function. We show how the eigenvalues of the coupling matrix can also be used in the context of our method for global synchronization. At the same time, we show that the eigenvalue method fails in general for networks with time-dependent coupling coefficients for which the connection graph stability method becomes the ultimate approach. In Section 6, a brief discussion of the obtained results is given. Finally, in Appendix A, we show how global stability of synchronization depends on the parameters of the individual oscillator by considering an example of coupled identical Lorenz systems. In Appendix B, we prove that our method is applicable to the study of global stability of approximate synchronization in networks of slightly non-identical oscillators. We also give an estimate for the corresponding synchronization error and support the general results by an illustrative example of coupled non-identical Lorenz systems.

2. Problem statement

We consider the network

$$\dot{x}_i = F(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t) P x_j, \quad i = 1, \ldots, n, \tag{1}$$

where $x_i = (x_i^1, \ldots, x_i^d)$ is the d-vector containing the coordinates of the ith oscillator. The non-zero elements of the $d \times d$ matrix P determine which variables couple the oscillators. For clarity, we shall consider a vector version of the coupling with the diagonal matrix $P = \text{diag}(p_1, p_2, \ldots, p_d)$, where $p_h = 1, h = 1, 2, \ldots, s$ and $p_h = 0$ for $h = s + 1, \ldots, d$. Note that all the results that obtained in this paper are also valid for other possible cases of scalar and vector couplings between the oscillators.

Let $G = (\varepsilon_{ij}(t))$ be an $n \times n$ symmetric matrix with vanishing row-sums and non-negative off-diagonal elements, i.e. $\varepsilon_{ij} = \varepsilon_{ji}, \varepsilon_{ij} \geq 0$ for $i \neq j$, and $\varepsilon_{ii} = -\sum_{j=1; j \neq i}^{n} \varepsilon_{ij}, i = 1, \ldots, n$.

Therefore, we consider an arbitrary network of mutually coupled systems due to the symmetry of the coupling matrix. The condition for vanishing row-sums is necessary for the existence of the synchronous state.

The matrix G defines a graph with n vertices and m edges. Here, the number of edges m, from all possible $N = n(n-1)/2$ links between the oscillators, equals the number of non-zero above diagonal elements of the matrix $G, \varepsilon_{i_k j_k} > 0$ for $k = 1, 2, \ldots, m$ and $\varepsilon_{i_k j_k} = 0$ for $k = m + 1, \ldots, N$. These m non-zero elements of the matrix G represent both the structure of the coupling and the coupling strengths. The vertices of the graph correspond to the individual oscillators and the edges to the off-diagonal elements of G. Thus, the graph has an edge between node i and node j if $\varepsilon_{ij} = \varepsilon_{ji} > 0$. We suppose that the graph is connected.

We admit an arbitrary time dependence in the coupling matrix even if t is not explicitly stated everywhere. All constraints and criteria for the coupling matrix are understood to hold for all times t.

Our main objective is to obtain conditions of global asymptotic stability of synchronization in the system (1). We want to determine threshold values for the coupling strength required for synchronization, and to reveal their dependence on the number of oscillators, the coupling configuration, and the properties of the individual cell. In particular, the intriguing question is how the synchronization thresholds are related to the average path length of the graph.

3. Connection graph based stability method

The synchronous state of the system (1) is defined by the linear invariant manifold $M = \{x_1 = x_2 = \cdots = x_n\}$. This invariant manifold has dimension d and is often called the synchronization manifold.

3.1. Step I: redundant stability system for the difference variables

Introducing the notation for the differences

$$X_{ij} = x_j - x_i, \quad i, j = 1, \ldots, n, \tag{2}$$

we obtain

$$\dot{X}_{ij} = F(x_j) - F(x_i) + \sum_{k=1}^{n} \{\varepsilon_{jk} PX_{jk} - \varepsilon_{ik} PX_{ik}\}, \quad i, j = 1, \ldots, n. \tag{3}$$

To have the explicit presence of X_{ij} in $F(x_j) - F(x_i)$, we could have expressed the system (3) by its scalar components $X_i = (X_i^{(1)}, \ldots, X_i^{(d)})$ and $F = (F^{(1)}, F^{(2)}, \ldots, F^{(d)})$ and applied the mean value theorem to each scalar variable such that $F^{(h)}(x_j) - F^{(h)}(x_i) = DF^{(h)}(x^*)X_{ij}$, where $h = 1, \ldots, d$ and $DF^{(h)}$ is a d-column Jacobi vector of $F(x^*(t))$, where $x^*(x_i, x_j) \in [x_i, x_j]$. However, we prefer more compact vector notation for the function difference

$$F(x_j) - F(x_i) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\beta} F(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta \right] X_{ij},$$

where DF is a $d \times d$ Jacobi matrix of F.

Hence, we can write

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta \right] X_{ij} + \sum_{k=1}^{n} \{\varepsilon_{jk} PX_{jk} - \varepsilon_{ik} PX_{ik}\}, \tag{4}$$

where $i, j = 1, \ldots, n$. Note that the Jacobian DF can be calculated explicitly via the parameters of the individual subsystem.

The difference equation system (4) has n^2 equations; $n(n - 1)$ of them are responsible for the transversal stability of the corresponding synchronous modes (synchrony in the corresponding pair of oscillators). Clearly, the difference system (4) contains surplus linearly dependent variables and is, strictly speaking, redundant. Taking into account the obvious equalities $X_{ii} \equiv 0$ and $X_{ji} = -X_{ij}$, one can reduce the number of variables to $N = n(n - 1)/2$. Moreover, it is clear that only $n - 1$ difference variables from the remaining $N = n(n - 1)/2$ are linearly independent and can be chosen to correspond to some tree of the coupling graph G. However, we consider all possible differences X_{ij} and this approach will allow us to estimate the synchronization thresholds in complex networks.

Using the technique developed in previous papers [4,29,30] for stability of the difference variable system where only linearly independent combinations were taken into account; we study now the redundant stability system (4).

Adding and subtracting an additional term AX_{ij} from the system (4), we obtain the system

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta - A \right] X_{ij} + AX_{ij} + \sum_{k=1}^{n} \{\varepsilon_{jk} PX_{jk} - \varepsilon_{ik} PX_{ik}\}, \tag{5}$$

where $i, j = 1, \ldots, n$ and the matrix $A = \mathrm{diag}(a_1, a_2, \ldots, a_d)$, where $a_h \geq 0$ for $h = 1, 2, \ldots, s$, and $a_h = 0$ for $h = s+1, \ldots, d$. Note here that the choice of the matrix A is strictly defined by the matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_d)$; we can add (subtract) the additional corresponding terms only in the coupled variables.

Our goal is to obtain conditions under which solutions of the coupled system (5) converge to 0 as $t \to \infty$ and its trivial equilibrium, corresponding to the synchronization manifold of the system (1), is globally asymptotically stable.

The matrix $-A$ is added to damp instabilities caused by eigenvalues with non-negative real parts of the Jacobian DF. At the same time, the instability introduced by the positive definite matrix $+A$ in Eq. (5) can be damped by

the coupling terms. The positive coefficients a_k are put in the matrix A at the place corresponding to the variables by which oscillators are coupled, and therefore the desynchronizing influences of $+a_k$ can be compensated by the negative coupling terms.

To go further, we should make some assumptions and introduce the auxiliary system

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, d\beta - A \right] X_{ij}, \quad i, j = 1, \dots, n. \tag{6}$$

This system is the system (5) for the difference variables where the coupling terms are excluded.

We assume that there exist Lyapunov functions of the form

$$W_{ij} = \tfrac{1}{2} X_{ij}^T H X_{ij}, \quad i, j = 1, \dots, n, \tag{7}$$

where $H = \mathrm{diag}(h_1, h_2, \dots, h_s, H_1)$, $h_1 > 0, \dots, h_s > 0$, and the $(d - s) \times (d - s)$ matrix H_1 is positive definite.

Their derivatives with respect to the system (6) are required to be negative

$$\dot{W}_{ij} = X_{ij}^T H \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, d\beta - A \right] X_{ij} < 0, \quad X_{ij} \neq 0. \tag{8}$$

This is a crucial requirement for the method. It implies that the auxiliary system can be globally stabilized, provided the negative linear part's parameters $a_1, a_2, \dots, a_s$ are sufficiently large. In other words, this implies that oscillators of the system (1) can synchronize when the coupling is made sufficiently large. In general this is not always true. To solve the real problem of synchronization in a network of oscillators, given their individual dynamics and coupling structure, one should start from the question of whether or not the requirement (8) holds. In fact, many examples of linearly coupled dynamical systems satisfy this condition, and global synchronization arises with increasing coupling. However, a few examples of coupled systems for which this is not the case were reported [6,29]. Among them is a lattice of x-coupled Rössler systems in which the stability of the synchronization regime is lost with an increase of coupling such that the requirement (8) cannot be satisfied for this network and global synchronization cannot be achieved. In a recent paper [29], we have linked this phenomenon to the equilibria disappearance bifurcations. Indeed, the nature of *global* synchronization is strongly *dependent* on the dynamical properties of subsystems. Even if the coupled system always stays in a compact set and local synchronization occurs, global synchronization can be absent due to the possible presence of different invariant sets lying outside the synchronization manifold (in an extreme case, this is a multistability effect).

The conditions that guarantee the requirement (8) are based upon the individual node's dynamics and the way the oscillators are coupled (matrix P). The requirement (8) has to be proven for each particular situation (for the concrete individual system and the matrix P). In fact, we have proved it for coupled Lorenz systems [30], double-scrolls [4,25], non-autonomous chaotic pendulums [42], Hodgkin–Huxley-type models and various P matrices.

The details of the proof of assumption (8) for the coupled Lorenz systems can be found in [30], however, as an illustrative example we shall present its main steps in Appendix A. Using a similar argument, one can prove it for many other coupled chaotic systems.

Before starting the study of the transversal stability of the synchronization manifold, we need also one additional assumption on the eventual dissipativeness of the coupled system (1).

Assume that the individual system $\dot{x}_i = F(x_i)$ is eventually dissipative, i.e. there exists a compact set B which attracts all trajectories of the system from the outside. Therefore, there are no trajectories which go to infinity. Note that most chaotic dynamical systems satisfy this assumption. Similar to [30], it can be shown that under this assumption the whole coupled system (1) is eventually dissipative.

Now we can study global stability of the synchronization manifold by constructing the Lyapunov function for the system (5) which has n^2 variables

$$V = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}^{\mathrm{T}} H X_{ij}, \tag{9}$$

where H is the matrix in (7).

The corresponding time derivative has the form

$$\dot{V} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{W}_{ij} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij}^{\mathrm{T}} A X_{ij} - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \{\varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk} + \varepsilon_{ik} X_{ik}^{\mathrm{T}} H P X_{ij}\}. \tag{10}$$

Our goal is to show the negative definiteness of the quadratic form $\dot{V}$. The first sum S_1 is negative definite due to our assumptions (6)–(8), so it remains to study the last two sums. Since the coupling matrix G is symmetric ($X_{ii}^2 = 0$, $X_{ij}^2 = X_{ji}^2$), we can calculate the second sum as follows:

$$S_2 = \sum_{i=1}^{n-1} \sum_{j>i}^{n} A X_{ij}^2. \tag{11}$$

This sum is always positive definite and should be compensated by the third sum

$$S_3 = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \{\varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk} + \varepsilon_{ik} X_{ik}^{\mathrm{T}} H P X_{ij}\}. \tag{12}$$

Renaming in the second term the summation index i by j and vice versa, the second term becomes identical to the first and we obtain

$$S_3 = -\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk}. \tag{13}$$

Using $X_{jj} = 0$, we get

$$S_3 = -\sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j>k}^{n} \varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk} - \sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j<k}^{n} \varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk}. \tag{14}$$

Renaming in the second term j by k and vice versa, and using the symmetry of ε, we get

$$S_3 = -\sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j>k}^{n} \varepsilon_{jk} X_{ji}^{\mathrm{T}} H P X_{jk} - \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{k<j}^{n} \varepsilon_{jk} X_{ki}^{\mathrm{T}} H P X_{kj} = -\sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j>k}^{n} \varepsilon_{jk} (X_{ji}^{\mathrm{T}} + X_{ik}^{\mathrm{T}}) H P X_{jk}. \tag{15}$$

Since $X_{ji}^{\mathrm{T}} + X_{ik}^{\mathrm{T}} = [x_i^{\mathrm{T}} - x_j^{\mathrm{T}} + x_k^{\mathrm{T}} - x_i^{\mathrm{T}}] = X_{jk}^{\mathrm{T}}$, we obtain

$$S_3 = -\sum_{i=1}^{n} \sum_{k=1}^{n-1} \sum_{j>k}^{n} \varepsilon_{jk} X_{jk}^{\mathrm{T}} H P X_{jk} = -\sum_{k=1}^{n-1} \sum_{j>k}^{n} n \varepsilon_{jk} X_{jk}^{\mathrm{T}} H P X_{jk}. \tag{16}$$

Thus we arrive at the conclusion that, in our case, the time derivative $\dot{V} < 0$ if the quadratic form

$$S_2 + S_3 = \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^{\mathrm{T}} H [A - n\varepsilon_{ij} P] X_{ij} \tag{17}$$

is negative definite. This is the case if the following inequality holds.

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} \varepsilon_{ij} X_{ij}^{\mathrm{T}} H P X_{ij} > \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^{\mathrm{T}} H A X_{ij}. \tag{18}$$

Recalling that the matrices $A = \mathrm{diag}(a_1, a_2, \ldots, a_s, 0, \ldots, 0)$ and $P = \mathrm{diag}(p_1, p_2, \ldots, p_s, 0, \ldots, 0)$, and that the vector variables $X_{ij} = \{X_{ij}^{(1)}, \ldots, X_{ij}^{(d)}\}$, we finally come to the following basic assertion.

Theorem 1 (main inequality). *Under the assumption on the eventual dissipativeness of the individual oscillator system and the assumption* (8), *the synchronization manifold of the system* (1) *is globally asymptotically stable if the following inequality holds*

$$\sum_{k=1}^{m} \varepsilon_{i_k j_k} X_{i_k j_k}^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2, \tag{19}$$

where m is the number of non-zero elements in the coupling matrix G and $X_{i_k j_k} k = 1, \ldots, m$ *are defined by links where the coupling coefficients are present.*

Hereafter, the $X_{i_k j_k}$ stand for the scalars $X_{i_k j_k}^{(l)}, l = 1, \ldots, s$, and a stands for the parameters $a_1, \ldots, a_s$ such that in the inequality (19) we have s similar inequalities for the scalars $X_{i_k j_k}^{(l)}$.

Standing for the scalars $X_{ij}^{(l)}$, the X_{ij} are possible difference variables of the system (4) lying above the principal diagonal of the matrix χ, i.e

$$\chi = \begin{pmatrix} 0 & X_{12} & X_{13} & \cdots & \cdots & X_{1n} \\ & 0 & X_{23} & X_{24} & \cdots & X_{1n} \\ & & \ddots & X_{33} & \cdots & X_{3n} \\ & & & \ddots & \ddots & \vdots \\ & & & & 0 & X_{n-1,n} \\ & & & & & 0 \end{pmatrix}.$$

Remark 1. The parameter a is defined by the dynamics of the individual oscillator. For many chaotic systems, a can be found analytically. For some chaotic systems it can be expressed explicitly through the parameters of the individual oscillator (see Appendix A for the calculation of the parameter a for networks (1) of x-coupled Lorenz systems).

Such a calculation gives only sufficient conditions for the required a, and therefore it can give large overestimates. However, such estimates are useful for a rough estimation of the range of coupling strength required for synchronization. They guarantee the stability of the synchronization regime and solve rigorously the problem of whether synchronization occurs in a concrete network with increasing coupling or not.

Obviously, Theorem 1 does not give immediately the bounds for the synchronization thresholds, since the unknown and time-dependent difference variables are still present. The next step is to eliminate them. In the trivial case of global coupling this is straightforward.

Example 1 (Global coupling). Assume that all $\varepsilon_{i_k j_k}(t) \geq \varepsilon > 0$, $k = 1, \ldots, n(n-1)/2$ for all i_k, $j_k \in \{1, \ldots, n\}$ and for all t, i.e the coupling is present in every link. Hence, due to Theorem 1, the bound on the synchronization threshold is

$$\varepsilon(t) > \varepsilon^* = \frac{a}{n} \frac{\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2}{\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2} = \frac{a}{n}.$$

This estimate is well-known for oscillators with mean field coupling.

To get rid of the variables X_{ij} and $X_{i_k j_k}$ in the inequality (19) and, therefore, to obtain the bounds for the synchronization thresholds for different network configurations, one should find a projection of the redundant system of n linearly dependent equations $\{X_{ij}, i, j = 1, \ldots, n\}$ onto the system $\{X_{i_k j_k}, k = 1, \ldots, m\}$. This is the second step in our approach.

3.2. Step II: mapping of the redundant system onto the coupling configuration

We now determine under what conditions on ε, a, and n, inequality (19)

$$\sum_{k=1}^{m} \varepsilon_k(t) \tilde{X}_k^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \tag{20}$$

holds. Here, $\tilde{X}_k = X_{i_k j_k}$ and $\varepsilon_k = \varepsilon_{i_k j_k}$, $m \geq n - 1$.

First we introduce the notion of a basic coupling combination. A maximal linearly independent subset of the variables $\tilde{X}_k$ is called coupling combination. Since the coupling tree is connected and the $X_{i_k j_k}$ satisfy Kirchhoff's voltage laws, the subgraph corresponding to a basic coupling configuration is a tree. In some coupling graphs, the tree is unique, such as in a star configuration or in a 1D nearest neighbor interaction. Usually, there are many different trees.

We shall express all difference variables X_{ij}, $i, j = 1, \ldots, n$ through the coupling configuration variables $\tilde{X}_k$, $k = 1, \ldots, m$, and then to get rid of the presence of the dynamical variables X_{ij} and $\tilde{X}_k$ in the main inequality (20).

For any pair of nodes (i, j), choose a path from node i to node j and denote it by P_{ij}. We define $k \in P_{ij}$ to mean that edge k belongs to the path P_{ij} and we denote by $z(P_{ij})$ the length of P_{ij}, i.e. the number of edges in P_{ij}. Note that if the path P_{ij} passes along the nodes $i, m_1, m_2, \ldots, m_\nu, j$ then $X_{ij} = X_{i,m_1} + X_{m_1,m_2} + \cdots + X_{m_\nu, j}$. Therefore, we can write

$$X_{ij}^2 = \left(\sum_{k \in P_{ij}} \pm \tilde{X}_k \right)^2 \leq z(P_{ij}) \sum_{k \in P_{ij}} \tilde{X}_k^2, \tag{21}$$

where we have applied the general inequality from elementary calculus $(c_1 + c_2 + \cdots c_n)^2 \leq n(c_1^2 + c_2^2 + \cdots + c_n^2)$.

Hence, the right-hand sum in Eq. (20) is finally majorized as follows:

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq \sum_{k=1}^{m} \left(\sum_{i=1}^{n-1} \sum_{j>i;k \in P_{ij}}^{n} z(P_{ij}) \right) \tilde{X}_k^2. \tag{22}$$

Substituting the estimate (22) to the inequality (20), we derive the conditions

$$\varepsilon_k(t) > \frac{a}{n} \sum_{j>i;k \in P_{ij}}^{n} z(P_{ij}) \quad \text{for } k = 1, \ldots, m. \tag{23}$$

This concludes the connection graph stability method for synchronization of coupled systems, bringing us to the following main result of this paper.

Theorem 2. *Under the assumptions of* Theorem 1, *the synchronization manifold of the system* (1) *is globally asymptotically stable if*

$$\varepsilon_k(t) > \frac{a}{n} b_k(n, m) \quad for\, k = 1, \dots, m\, and\, for\, all\, t, \tag{24}$$

where $b_k(n, m) = \sum_{j>i;k \in P_{ij}}^{n} z(P_{ij})$ *is the sum of the lengths of all chosen paths* P_{ij} *which pass through a given edge k that belongs to the coupling configuration.*

The criterion of Theorem 2 is applied as follows.

- *Step* 1: Choose a set of paths $\{P_{ij}|i, j = 1, \dots, n, j > i\}$, one for each pair of vertices i, j. Determine their lengths $z(P_{ij})$, the number of edges comprising each P_{ij}.
- *Step* 2: For each edge k of the connection graph determine the sum $b_k(n, m)$ of the lengths of all P_{ij} passing through k.

Remark 2. For a certain choice of paths P_{ij} we obtain for each ε_k a certain lower bound (24). If, in a given coupled network, all lower bounds on the coupling strengths ε_k are satisfied, Theorem 2 guarantees total synchronization. Often, all coupling strengths in a network are equal, i.e. $\varepsilon_k = \varepsilon$ for all k. In this case the lower bound for ε is $\varepsilon^* = \max_k(a/n)b_k(n, m)$. This amounts to determining the edge k such that the sum of the lengths of all paths through k is maximal.

Remark 3. The number of possible choices of paths is normally huge. However, most of these choices are clearly suboptimal. Usually, one takes for P_{ij} the shortest path from vertex i to vertex j. Sometimes, however, a different choice of paths can lead to smaller lower bounds (24).

We shall now illustrate our criterion by applying it to three different networks. Once again, in each case we first have to choose a path P_{ij} from any node i to any other node $j > i$ and then determine $b_k(n, m)$ for all k. We shall start with the simplest coupling configuration.

Example 2 (Star-configuration). This is a well-studied coupling scheme where the network has a central hub position (this node is marked as the first one) and all other nodes are linked to this node. Fig. 1a illustrates this coupling scheme. In this case, we have the coupled system (1) with the $n \times n$ coupling matrix

$$G = \begin{pmatrix} -g & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1n} \\ \varepsilon_{12} & -\varepsilon_{12} & 0 & \cdots & 0 \\ \varepsilon_{13} & 0 & -\varepsilon_{13} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon_{1n} & 0 & 0 & \cdots & -\varepsilon_{1n} \end{pmatrix},$$

where $g = (\varepsilon_{12} + \varepsilon_{13} + \cdots + \varepsilon_{1n})$.

To find an upper bound for the synchronization thresholds, we shall follow the steps of the above study. Let edge k link the central node 1 and node l, $l = 2, \dots, n$. Then from node 1 to node l there is only one path consisting of
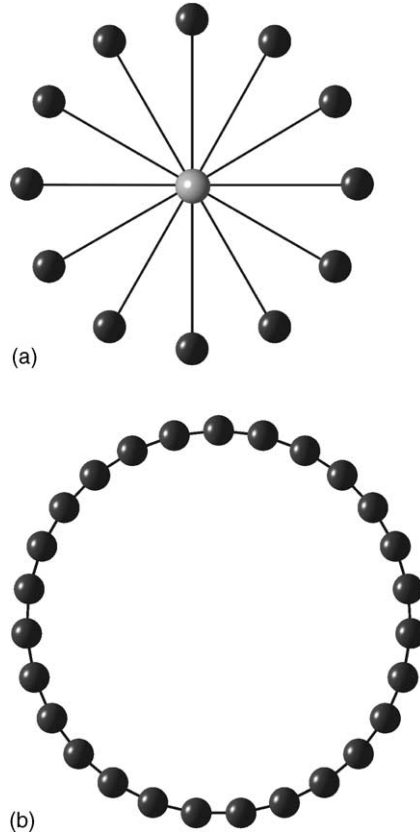
Fig. 1. (a) Star-coupled nodes. (b) Ring of diffusively coupled systems.

the only edge $k \equiv l - 1$ and thus the path length $z(P_{1,l}) = 1$. From node $i > 1$ $(i \neq l)$ to node l there is also only a single path consisting of the edges $i - 1$ and $k \equiv l - 1$ and thus with path length $z(P_{i,l}) = 2$. Therefore,

$$\sum_{j>i; k \in P_{ij}}^{n} z(P_{ij}) = z(P_{1,l}) + \sum_{i=2}^{l-1} z(P_{i,l}) + \sum_{j=l+1}^{n} z(P_{l,j}) = 1 + 2(l - 2) + 2(n - l) = 2n - 3,$$

and we arrive at the following conclusion.

Statement 1. *The synchronization condition* (24) *for the star-coupled network becomes*

$$\varepsilon_{1j}(t) > \varepsilon^* = a\frac{2n - 3}{n} \quad \text{for } j = \overline{2, n} \text{ and for all } t. \tag{25}$$

Obviously, to provide complete global synchronization of all nodes in the star-coupled network, the weakest of all coupling strengths $\varepsilon_{1,j}$ must satisfy the condition (25).

It follows from (25) that if the number of oscillators n is large enough, then the bound for the synchronization threshold is $\varepsilon^* \cong 2a$. Hence, the synchronization threshold in such a large network does not depend on the number of oscillators.

Note that the least eigenvalue of the matrix G for a star matrix with unit entries and $-(n-1)$ in the upper left corner equals $\gamma = -1$ [13]. Therefore, the thresholds obtained by our method for this simple network are similar to those coming from the eigenvalues of the coupling matrix.

As the next example, we consider another well-studied coupling configuration (Fig. 1b).

Let the nodes of the coupling graph be numbered from 1 to n and let the kth edge link the nodes k and $k+1 (\text{mod } n)$.

Example 3 (Ring of diffusively coupled oscillators). The coupling matrix G is

$$
G = \begin{pmatrix}
-(\varepsilon_{12} + \varepsilon_{1n}) & \varepsilon_{12} & 0 & 0 & \varepsilon_{1n} \\
\varepsilon_{12} & -(\varepsilon_{12} + \varepsilon_{23}) & \varepsilon_{23} & 0 & 0 \\
0 & \varepsilon_{23} & \ddots & \ddots & 0 \\
0 & 0 & \ddots & \ddots & \varepsilon_{n-1,n} \\
\varepsilon_{1n} & 0 & 0 & \varepsilon_{n-1,n} & -(\varepsilon_{1n} + \varepsilon_{n-1,n})
\end{pmatrix}. \tag{26}
$$

From any node i to any other node $j > i$ there are exactly two paths on the ring. We take as P_{ij} the shorter path from i to j. When the two paths are of equal length, we take both, with a weighting factor of $1/2$. This is a slight extension of the method. In fact, inequality (21) is replaced by

$$
X_{ij}^2 = \frac{1}{2}\left(\sum_{k \in P_{ij}} \pm \tilde{X}_k\right)^2 + \frac{1}{2}\left(\sum_{k \in \tilde{P}_{ij}} \pm \tilde{X}_k\right)^2 \leq \frac{z(P_{ij})}{2}\sum_{k \in P_{ij}} \tilde{X}_k^2 + \frac{z(\tilde{P}_{ij})}{2}\sum_{k \in \tilde{P}_{ij}} \tilde{X}_k^2,
$$

where P_{ij} and $\tilde{P}_{ij}$ are the two paths from i to j. Thus, the fact to consider two paths from i to j is compensated by dividing the path length by 2. This type of extension will be used more intensively for the $2K$-nearest neighbor coupling later in the paper.

Suppose that a path extends, from a given edge $k m_-$ edges in counterclockwise direction and m_+ edges in clockwise direction.

To apply the general estimate

$$
\varepsilon_k(t) > \frac{a}{n}\sum_{j>i; k \in P_{ij}}^{n} z(P_{ij}) \quad \text{for } k = 1, \ldots, m, \tag{27}
$$

we must calculate the sum length $b_k(n, m)$ of all paths passing through edge k.

The length of the path is $m_- + m_+ + 1$. The constraints are $m_- \geq 0$, $m_+ \geq 0$, and

$$
m_- + m_+ + 1 \leq [\tfrac{1}{2}n].
$$

Here, $[\xi]$ is the integer part of ξ. Hence, we have in the case where n is odd, and thus the two paths from i to j are never of equal length,

$$
\sum_{j>i; k \in P_{ij}}^{n} z(P_{ij}) = \sum_{m_-=0}^{[n/2]-1}\sum_{m_+=0}^{[n/2]-1-m_-}(m_- + m_+ + 1) = \frac{1}{3}\left[\frac{n}{2}\right]^3 + \frac{1}{2}\left[\frac{n}{2}\right]^2 + \frac{1}{6}\left[\frac{n}{2}\right]
$$

$$
= \frac{1}{3}\left(\frac{n-1}{2}\right)^3 + \frac{1}{2}\left(\frac{n-1}{2}\right)^2 + \frac{1}{6}\left(\frac{n-1}{2}\right) = \left(\frac{n^3}{24} - \frac{n}{24}\right).
$$

In the case of even n, we get

$$\sum_{j>i; k\in P_{ij}; |P_{ij}|<n/2}^{n} z(P_{ij}) + \sum_{j=i+n/2; k\in P_{ij}}^{n} \frac{1}{2} z(P_{ij}) + \sum_{j=i+n/2; k\in \tilde{P}_{ij}}^{n} \frac{1}{2} z(\tilde{P}_{ij})$$

$$= \sum_{m_-=0}^{n/2-2} \sum_{m_+=0}^{n/2-2-m_-} (m_- + m_+ + 1) + \frac{n}{2}\frac{1}{2}\frac{n}{2} = \frac{n^3}{24} - \frac{n^2}{8} + \frac{n}{12} + \frac{n^2}{8} = \frac{n^3}{24} + \frac{n}{12},$$

where we have used that every edge k belongs to exactly one of the two length $n/2$ paths from node i to node $i+n/2$, and that i can take all values from 1 to $n/2$.

Thus we come to the following assertion.

Statement 2. *Complete synchronization is globally stable in the ring of n* nearest neighbor coupled oscillators, *if the following inequality holds for all coupling constants ε and all times t*

$$\varepsilon(t) > \varepsilon^* = \begin{cases} a\left(\dfrac{n^2}{24} - \dfrac{1}{24}\right) & \text{for odd } n, \\[3mm] a\left(\dfrac{n^2}{24} + \dfrac{1}{12}\right) & \text{for even } n. \end{cases} \tag{28}$$

Note that ε^* depends quadratically on the number of oscillators n. At the same time, the conventional method for calculating the eigenvalues of the coupling matrix G and the parameter a required for global synchronization gives a similar result for the upper bound $\varepsilon^* = c\,n^2$ under the condition that the number of oscillators n is sufficiently large. Here, $c < a/24$ is a parameter. This law has been proposed and discussed in [25].

Within the framework of our method, the total length of all paths passing through each edge belonging to the connection graph must be calculated. This can be quite a laborious task for networks with complicated coupling schemes. However, once the calculation scheme is constructed, an explicit bound can be obtained (for more complicated cases one can use MAPLE). In following, we will study global synchronization in rings of $2K$-nearest neighbor coupled oscillators (Section 4) and in more complicated networks (companion paper [41]).

4. Example: ring of *2K*-nearest neighbor coupled oscillators

The objective of this section is to apply our method to the study of global synchronization in n oscillators that are coupled in a ring with $2K$-nearest neighbors. Fig. 2 shows this coupling scheme.

The study of global synchronization in such a network is of significant importance for several reasons. First, this lattice represents the main coupling schemes, such as locally, non-locally, and globally coupled networks, for different K. In fact, when $K = 1$ we have a ring of locally coupled oscillators, and when $K = [n/2]$ it becomes a globally coupled network. Second, this network is often used as a pristine world (regular coupled lattice) for small-world systems [24,43–45] such that the knowledge of its synchronization properties is important for the study of small-world systems. In the companion paper, devoted to synchronization in small-world networks, we will also use this network as the pristine world.

Synchronization in such a ring of $2K$-nearest neighbors coupled oscillators was previously studied by Barahona and Pecora [24]. They obtained an estimate of the local synchronization threshold value for such a network (under the assumption $K \ll n$).
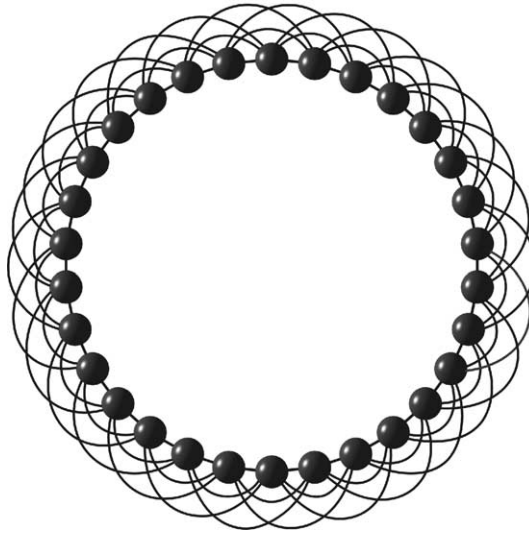
Fig. 2. Ring of $2K$-nearest neighbor coupled oscillators. Each oscillator is connected to its neighbors out to some range K. Here, $n = 30$ and $K = 3$.

In this section we will show the effectiveness of our method for tackling the problem of synchronization in this network. We will analytically obtain good estimates for the threshold coupling values needed for global synchronization for any range of K (the results are also valid for time-dependent couplings). Moreover, we will show explicitly the connection between these synchronization thresholds and the average path length of the coupling graph.

For this network, G is a banded circulant matrix, similar to (26), having non-zero elements only on the main diagonal and on the $2K$ adjacent diagonals: $\varepsilon_{ii} \equiv -2K\varepsilon$ and $\varepsilon_{ij} = \varepsilon_{ji} = \varepsilon$ for $j = i + 1 \bmod n$, $j = i + 2 \bmod n, \ldots, j = i + K \bmod n$ and $1 \leq i \leq n$. This matrix is similar to [24]. In other words, we assume that the interconnection coupling is

$$\varepsilon_{ij} = \begin{cases} \varepsilon & \text{for } 1 \leq |j - i| \bmod n \leq K, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Hence, criterion (19) for complete synchronization becomes

$$\varepsilon(t) \sum_{\gamma=1}^{K} \sum_{q=1}^{n} X_{q,q+\gamma \bmod n}^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2. \tag{30}$$

In this study, we shall follow the steps of the study of the ring of diffusively coupled oscillators (Example 3), which is a particular case of our network for $K = 1$.

In order to apply criterion (30), we derive an inequality of the form

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq b(n, K) \sum_{\gamma=1}^{K} \sum_{q=1}^{n} X_{q,q+\gamma \bmod n}^2. \tag{31}$$

Actually, this inequality will be deduced in K separate, but similar steps that are to be combined in the end.

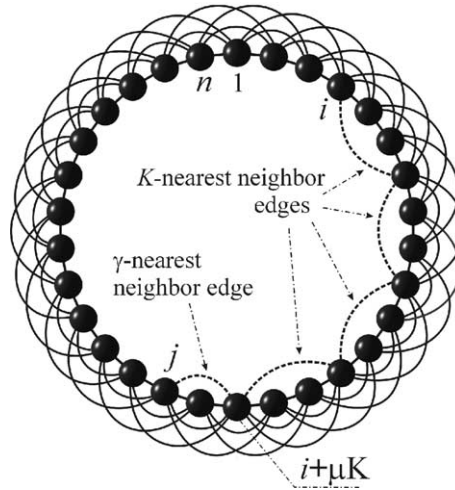In the first step, we choose the following path P_{ij} from vertex i to vertex $j > i$ on the connection graph.

Fig. 3. Path P_{ij} from vertex i to vertex j. It is composed of as many K-nearest neighbor edges as possible, plus one additional γ-nearest neighbor edge. The vertices i and j are such that the path from i to j in clockwise direction is shorter than the path in counter-clockwise direction.

If $j - i \leq [n/2]$, we take successively K-nearest neighbor edges from vertex i to vertex $i + K$, from $i + K$ to $i + 2K$, etc. until the vertex $i + \mu K$, where

$$\mu = \left[\frac{j - i}{K} \right],$$

and $[\xi]$ indicates the integer part of ξ. If $j \neq i + \mu K$, we complete the path P_{ij} by the single γ-nearest neighbor edge, where $\gamma = j - i - \mu K$ (see Fig. 3).

If $j - i > [n/2]$, we take successively K-nearest neighbor edges from vertex j to $j + K \bmod n$, from $j + K \bmod n$ to $j + 2K \bmod n$, etc. until the vertex $j + \mu K \bmod n$, where

$$\mu = \left[\frac{i - j \bmod n}{K} \right].$$

Again, if necessary the single γ-nearest neighbor edge from vertex $j + \mu K \bmod n$ to vertex i completes the path P_{ij} (see Fig. 4).

If $j = i + \mu K$, respectively $i = j + \mu K$, the path length is μ, otherwise $\mu + 1$. We shall always use the bound $\mu + 1$ for the path length.

Having chosen the paths, we now can apply formula (22). Let us now take as edge K a specific K-nearest neighbor edge, from node q to $q + K$.

Suppose that there are $m_- K$-nearest neighbor edges in path P_{ij} preceding edge k and $m_+ K$-nearest neighbor edges in P_{ij} following edge k, if we move in clockwise direction (see Fig. 5).

The constraints for m_- and m_+ are

$$m_- \geq 0, \qquad m_+ \geq 0, \qquad m_- + m_+ + 1 \leq [\tfrac{1}{2}n].$$

For fixed m_- and m_+ there are K different paths P_{ij} because the last edge can be a γ-nearest neighbor edge for $\gamma = 1, \ldots, K - 1$, or the last edge may be absent. In the last case, the path length is
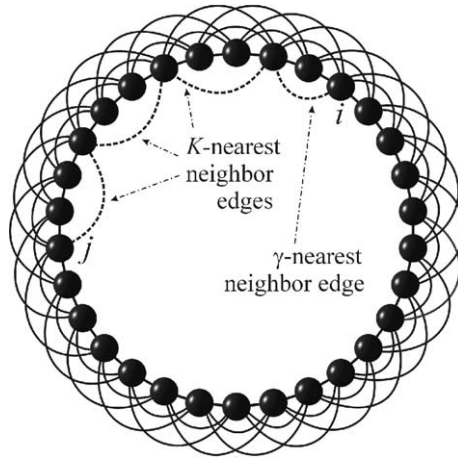
$$z(P_{ij}) = m_- + m_+ + 1,$$

Fig. 4. Path P_{ij} from vertex i to vertex j. It is composed of as many K-nearest neighbor edges as possible, plus one additional γ-nearest neighbor edge. The vertices i and j are such that the path from i to j in counter-clockwise direction is shorter than the path in clockwise direction.

whereas otherwise it is

$$z(P_{ij}) = m_- + m_+ + 2. \tag{32}$$

We shall take always the value (32) as an upper bound. Thus, we obtain

$$\sum_{j>i;k\in P_{ij}}^{n} z(P_{ij}) \leq \sum_{m_-=0}^{[n/2]-1} \sum_{m_+=0}^{[n/2]-1-m_-} K(m_- + m_+ + 2)$$

$$= \frac{K}{3}\left[\frac{n}{2K}\right]^3 + K\left[\frac{n}{2K}\right]^2 + \frac{2K}{3}\left[\frac{n}{2K}\right] \leq \frac{1}{24}\frac{n^3}{K^2} + \frac{1}{4}\frac{n^2}{K} + \frac{n}{3}.$$
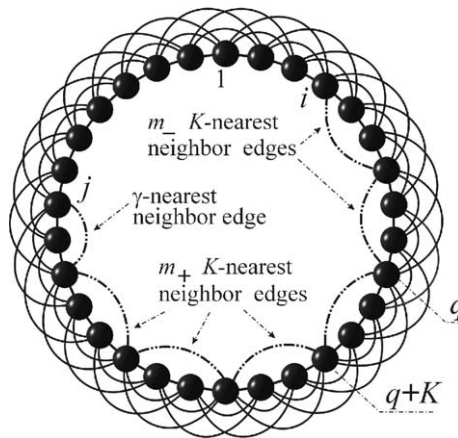

Fig. 5. Path P_{ij} that contains the K-nearest neighbor edge from vertex q to vertex $q + K$. The path leads from i to j in clockwise direction.
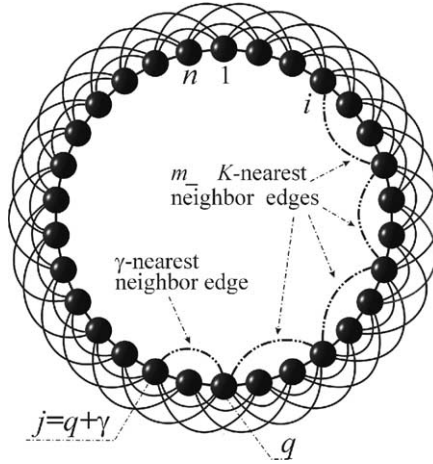
Fig. 6. Path P_{ij} that contains the γ-nearest neighbor edge from vertex $j - \gamma$ to vertex j. The path leads from i to j in clockwise direction.

Since $K \leq [n/2]$ we can write

$$\sum_{j>i; k \in P_{ij}}^{n} z(P_{ij}) \leq \frac{1}{24} \frac{n^3}{K^2} \left(1 + 14\frac{K}{n}\right).$$

This inequality takes care of the contribution of the K-nearest neighbor edges of P_{ij} to $\sum_{j>i} X_{ij}^2$.

Now consider as the edge k the γ-nearest neighbor edge from vertex $j - \gamma$ to vertex j.

Let m_- be the number of K-nearest neighbor edges that precede edge k on path P_{ij} if we move in clockwise direction (see Fig. 6). Then we have the constraints

$$m_- \geq 0, \qquad m_- K \leq [\tfrac{1}{2}n].$$

For a given m_- there is a single path P_{ij} and its length is

$$z(P_{ij}) = m_- + 1.$$

Thus we obtain

$$\sum_{j>i; k \in P_{ij}}^{n} z(P_{ij}) \leq \sum_{m_-=0}^{[n/2]} (m_- + 1) = \frac{1}{2}\left(\left[\frac{n}{2K}\right] + 1\right)\left(\left[\frac{n}{2K}\right] + 2\right) \leq \frac{n^2}{8K^2} + \frac{3n}{4K} + 1.$$

Using again $K \leq [n/2]$, we obtain

$$\sum_{i \neq j; k \in P_{ij}}^{n} z(P_{ij}) \leq \frac{3n^2}{4K^2}.$$

This inequality takes care of the contributions of the γ-nearest neighbor edges of P_{ij} to $\sum_{j>i} X_{ij}^2$ for $\gamma = 1, \ldots, K-1$.

Combining all combinations, we obtain the bound

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq \frac{1}{24} \frac{n^3}{K^2} \left(1 + 14\frac{K}{n}\right) \sum_{i=1}^{n} X_{i,i+K \bmod n}^2 + \sum_{\gamma=1}^{K-1} \frac{3n^2}{4K^2} \sum_{i=1}^{n} X_{i,i+\gamma \bmod n}^2. \tag{33}$$

This inequality is not yet of the required form (31) because the factor $b(n, K)$ is different for the K-nearest edge and the γ-nearest neighbor edge for $\gamma = 1, \ldots, K - 1$. However, we can write similar inequalities, if we compose the paths P_{ij} instead of K-nearest neighbor edges of λ-nearest neighbor edges, for any λ between 1 and $K - 1$. This yields

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq \frac{1}{24} \frac{n^3}{\lambda^2} \left(1 + 14\frac{\lambda}{n}\right) \sum_{i=1}^{n} X_{i,i+\lambda \bmod n}^2 + \sum_{\gamma=1}^{\lambda-1} \frac{3n^2}{4\lambda^2} \sum_{i=1}^{n} X_{i,i+\gamma \bmod n}^2$$

$$\leq \frac{1}{24} \frac{n^3}{\lambda^2} \left(1 + 14\frac{K}{n}\right) \sum_{i=1}^{n} X_{i,i+\lambda \bmod n}^2 + \sum_{\gamma=1}^{\lambda-1} \frac{3n^2}{4\lambda^2} \sum_{i=1}^{n} X_{i,i+\gamma \bmod n}^2. \tag{34}$$

In order to be able to guarantee total synchronization for as low a coupling coefficient as possible, we shall use all of these inequalities simultaneously. At the end, we need to come up with an inequality of the form (31). However, in (34) there is a factor $1/\lambda^2$ that varies from inequality to inequality. Therefore, we multiply each inequality (34) by λ^2, and inequality (33) by K^2, and add them all. Since the coefficients should add up to 1, instead of the factors λ^2 we shall use $\lambda^2 / \sum_{m=1}^{K} m^2$ or, the slightly larger value $\alpha(\lambda) = 3\lambda^2/K^3$.

Indeed, since

$$\sum_{\lambda=1}^{K} \alpha(\lambda) = \sum_{\lambda=1}^{K} \frac{3\lambda^2}{K^3} = \frac{3}{K^3} \left(\frac{K^3}{3} + \frac{K^2}{2} + \frac{K}{6}\right) \geq 1,$$

we can write

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq \sum_{\lambda=1}^{K} \left(\alpha(\lambda) \sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2\right) \leq \sum_{\lambda=1}^{K} \alpha(\lambda) \left[\frac{1}{24} \frac{n^3}{\lambda^2} \left(1 + 14\frac{K}{n}\right) \sum_{i=1}^{n} X_{i,i+\lambda}^2 + \sum_{\gamma=1}^{\lambda-1} \frac{3n^2}{4\lambda^2} \sum_{i=1}^{n} X_{i,i+\gamma}^2\right]$$

$$= \sum_{\lambda=1}^{K} \left[\frac{1}{8} \frac{n^3}{\lambda^3} \left(1 + 14\frac{K}{n}\right) \sum_{i=1}^{n} X_{i,i+\lambda}^2\right] + \sum_{\lambda=1}^{K} \left[\sum_{\gamma=1}^{\lambda-1} \frac{9n^2}{4K^3} \sum_{i=1}^{n} X_{i,i+\gamma}^2\right]$$

$$\leq \sum_{\lambda=1}^{K} \left[\frac{1}{8} \frac{n^3}{\lambda^3} \left(1 + 14\frac{K}{n}\right) \sum_{i=1}^{n} X_{i,i+\lambda}^2\right] + \sum_{\gamma=1}^{K-1} \left[\frac{9n^2}{4K^2} \sum_{i=1}^{n} X_{i,i+\gamma}^2\right].$$

Finally the sum $\sum_{j>i} X_{ij}^2$ is estimated as follows:

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 \leq \frac{1}{8} \left(\frac{n}{K}\right)^3 \left(1 + \frac{65}{4}\frac{K}{n}\right) \sum_{\gamma=1}^{K} \sum_{i=1}^{n} X_{i,i+\gamma \bmod n}^2.$$

This inequality then leads to the sufficient condition for complete synchronization

$$\varepsilon(t) > \varepsilon^* = \frac{a}{n} \left(\frac{n}{2K}\right)^3 \left(1 + \frac{65}{4}\frac{K}{n}\right) \quad \text{for all } t.$$

The leading term in this rigorous bound is $(n/2K)^3$. For large n/K, i.e. when the number of oscillators is large with respect to the range of the interaction only this term matters. This can be expressed as follows.

Statement 3. *Complete synchronization is globally stable in the ring of $2K$-nearest neighbor coupled oscillators if the following inequality holds for all coupling constants ε and all times t*

$$\varepsilon(t) > \varepsilon^* = \frac{a}{n} \left(\frac{n}{2K} \right)^3 \left(1 + \frac{65}{4} \frac{K}{n} \right). \tag{35}$$

In particular, if n/K is large enough, then

$$\varepsilon > \varepsilon^* = \frac{\alpha}{n} \left(\frac{n}{2K} \right)^3 = \frac{\alpha}{n} L^3 \tag{36}$$

is sufficient for complete synchronization.

Here, α is slightly larger than a, and $L = n/2K$ is the average path length of the coupling graph for the ring of $2K$-nearest neighbor coupled oscillators [43–45]. Therefore, the formulas (35) and (36) give an explicit dependence of the synchronization thresholds on the average path length of the coupling graph.

In the general case of an arbitrary $K \leq [n/2]$ and a large n, we can also take into account only the leading term with a correcting coefficient $\alpha = \delta a$, $\delta > 1$ such that the upper bound can be written as follows:

$$\varepsilon > \varepsilon^* = \frac{\alpha}{n} \left(\frac{n}{2K} \right)^3 \tag{37}$$

but in this case δ may be as far off as 9.125.

Consequently, we obtain the following bounds for the coupling thresholds of global synchronization in the ring of $2K$-nearest neighbor coupled oscillators

$$\varepsilon > \varepsilon^* = \frac{\alpha n^2}{8K^3}. \tag{38}$$

One can check the effectiveness and generality of the estimate (38) for different K. For one extreme case where $K = 1$, the network is a ring of diffusively coupled oscillators and the estimate takes the form

$$\varepsilon^* = \tfrac{1}{8} \alpha n^2.$$

One can see that this estimate gives the similar law of the dependence of the synchronization threshold on the number of oscillators that was obtained for the ring of diffusively coupled oscillators (see Example 3).

For another extreme case where $K = [n/2]$, all oscillators of the ensemble are globally coupled and the estimate obeys the known law $\varepsilon^* = \alpha/n$ and shows that the general estimate is effective for any range of K.

Note that between these extremes there is a case with $K_{\text{const}} \cong n^{2/3}$, where the synchronization threshold is constant and does not depend on the number of oscillators.

Once again, the bounds come from sufficient conditions and therefore they give overestimates for the coupling strengths needed for global synchronization. Then the following question arises: if the dependence of the sufficient synchronization coupling values on n and K has the form (38), then what is the dependence between the real threshold coupling values under which global synchronization will arise?

We conjecture that the exact threshold for complete synchronization obeys the same law of dependence on n and K, but with a lower constant β than a which we obtained by stabilizing explicitly the individual oscillators. In support of this claim, we have determined numerically the thresholds for complete synchronization as functions of n for various values of K and we have fitted a curve of the form $\beta n^2/8K^3$ to the data, by letting β vary (Fig. 7). It can be seen that the deviation of the data from the fitted curve is very small, indeed. Recall once again that we consider only the networks of oscillators admitting global synchronization with increasing coupling.

In Fig. 7, we see the synchronization threshold coupling values ε^* calculated numerically for a ring of $2K$-nearest neighbor x-coupled chaotic Lorenz systems for different n and K. These values are depicted by small circles. The corresponding analytical curves $\beta n^2/8K^3$ (solid lines) fit the numerical data in a least-squares sense. The estimates are good even for small networks of oscillators and become more effective while the number of oscillators increases.
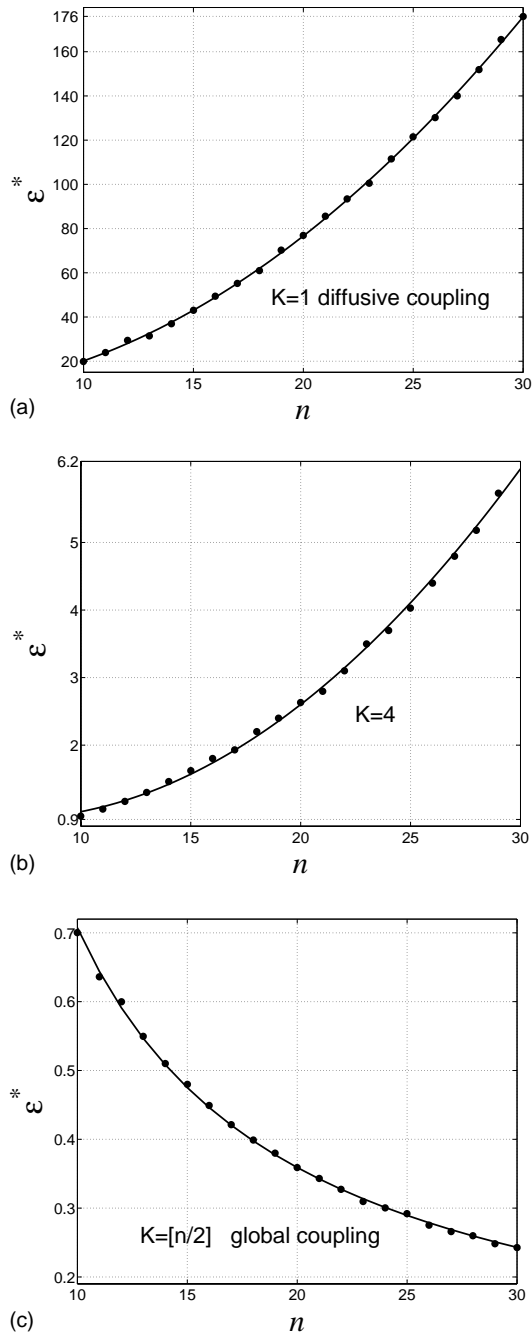
Fig. 7. Dependence of the synchronization thresholds ε^* on the number of oscillators n and on the depth of nearest-neighbor interaction K in the ring of $2K$-nearest neighbor coupled Lorenz systems. The analytical curves $\beta n^2/8K^3$ (solid lines) for different K fit the numerical data (small circles) in a least-squares sense.

5. Connection graph stability method versus the master stability function

As mentioned above, in previous works, the eigenvalues of the connection matrix G are used to establish the local stability of the synchronization threshold. However, they can also be used in our context to obtain a criterion for global synchronization in networks with a *constant time-independent* matrix G. In fact, for our method, the "full" quadratic form defined by the matrix of an all-to-all connection has to be bounded by a factor times the quadratic form defined by the specific connection matrix (cf. Theorem 1). For time-invariant coupling coefficients, the smallest such factor is given by the second largest eigenvalue of the coupling matrix. This can be seen as follows.

Because of zero row sums (providing the existence of the synchronization manifold M), the quadratic form given by the connection matrix, $\sum_{k=1}^{m} \varepsilon_{i_k j_k} X_{i_k j_k}^2 = \sum_{k=1}^{m} \varepsilon_{i_k j_k} (x_{j_k} - x_{i_k})^2$ and the "full" quadratic form $\sum_{i=1}^{n-1} \sum_{j>i}^{n} X_{ij}^2 = \sum_{i=1}^{n-1} \sum_{j>i}^{n} (x_j - x_i)^2$ have the eigenvector $(x_1 = 1, \ldots, x_n = 1)^{\mathrm{T}}$ with eigenvalue 0 (corresponding to the longitudinal direction along the synchronization manifold). In the $n - 1$-dimensional orthogonal subspace to this eigenspace,

$$V = \left\{ (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = 0 \right\},$$

the full quadratic form has $n - 1$ coinciding eigenvalues n and thus

$$\sum_{k=1}^{m} \varepsilon_{i_k j_k} (x_{j_k} - x_{i_k})^2 \geq |\lambda_2| \|\mathbf{x}\|^2 = \frac{|\lambda_2|}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} (x_j - x_i)^2, \tag{39}$$

where λ_2 is the second largest eigenvalue of the connection matrix G. Since for the eigenvalue 0 both sides of the inequality (39) vanish, the inequality holds on the whole space.

The inequality obtained in this way is the best possible in this context of quadratic Lyapunov functions. Coming from the sum of the path lengths, the estimate given in Theorem 2 is somewhat suboptimal, but it has the advantage that it can be obtained not only for regular, but also for quite irregular networks. Also, it allows us to link explicitly the conditions for the stability of synchronization with the average path length of the coupling graph and to show clearly the connection between graph theory and network dynamics.

Besides that, our method has a key advantage over the eigenvalue method in studying networks with time-dependent coupling coefficients. Namely, within the framework of our method, the time-dependent coupling coefficients can be handled without problems (cf. Eq. (2)), whereas inequalities in coupling coefficients do not necessarily result in corresponding inequalities in eigenvalues. This implies that, in general, the eigenvalue method *cannot* be applied to networks with a time-varying coupling structure.

In fact, the master stability function [12] (the eigenvalue method) relies on the argument that the Jacobian J_{coupl} for the whole coupled system (1) can be diagonalized in a coordinate system that isolates the stability of the synchronization manifold from the transverse directions. Namely, the procedure is the following. For clarity, we shall consider the coupled system (1) with uniform coupling coefficients ε ($G = \varepsilon \tilde{G}$). In this case its variational equation on the synchronization manifold has the form

$$\dot{X} = [J(t) + \varepsilon C] X,$$

where $X = (X^1, X^2, \ldots, X^n)$, and $X^i = (\xi_i^{(1)}, \ldots, \xi_i^{(d)})^{\mathrm{T}}$ are the variations on the ith node. $J(t) = I_n \otimes DF(t)$ and $C = \tilde{G} \otimes P$ are $n \cdot d \times n \cdot d$ matrices, and the matrix $\tilde{G}$ has eigenvalues with negative real parts $\lambda_2, \ldots, \lambda_{n-1}$ and one zero eignevalue λ_1.

Let the connectivity matrix $\tilde{G}$ be constant (as assumed for the eigenvalue synchronization method), and let $\hat{L}$ be the $n \times n$ matrix which diagonalizes $\tilde{G}$, i.e. $\hat{L}\tilde{G}\hat{L}^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Let $L = \hat{L} \otimes I_d$. Therefore, applying the linear transformation $Y = LX(X = L^{-1}Y)$, one obtains

$$\dot{Y} = (LJ(t)L^{-1} + \varepsilon LCL^{-1})Y,$$

where LCL^{-1} is a block diagonal matrix with $d \times d$ blocks, and $LJ(t)L^{-1}$ equals $J(t)$ since $J(t) = I \otimes DF(t)$ is the same for each block since it is calculated on the synchronization manifold. Hence, we arrive at the following diagonalized variation equation with each block having the form

$$\dot{Y}^k = [DF(t) + \lambda_k \varepsilon P]Y^k, \tag{40}$$

where λ_k is an eigenvalue of $\tilde{G}$, $k = 1, \ldots, n$. The first eigenvalue $\lambda_1 = 0$ corresponds to the variation equation within the synchronization manifold, and the others have negative real parts and determine the transverse directions. Therefore, if the maximum positive Lyapunov exponent $e_{\max}$ of the system

$$\dot{Y}^k = DF(t)Y^k$$

is smaller than $\varepsilon \min_{k=2,\ldots,n}|\lambda_k|$, then all eigenmodes are stable and the synchronization manifold is locally stable provided the coupling ε is strong enough. This basically completes the eigenvalue approach.

However, if the connectivity matrix $\tilde{G}$ is time-dependent, we have the following variational system

$$\dot{X} = [J(t) + \varepsilon C(t)]X. \tag{41}$$

To diagonalize Eq. (41), we have to take the similarity matrix L that is also dependent on t (eigenvalues of $\tilde{G}$ are time-dependent). Then the transformation $Y = L(t)X$ leaves us with the system

$$\dot{Y} = \dot{L}(t)X + L(t)\dot{X} = [\dot{L}L^{-1} + LJ(t)L^{-1} + \varepsilon LC(t)L^{-1}]Y, \tag{42}$$

where as before $LC(t)L^{-1}$ is the block diagonal matrix and $LJ(t)L^{-1} = J(t)$. However, the additional term $\dot{L}L^{-1}$ is always present and, in general, is not diagonal such that in the general case Eq. (41) cannot be diagonalized and the eigenvalue synchronization method is not applicable.

Indeed, if we transform Eq. (42) as follows:

$$\dot{Y} = J(t)Y + L[L^{-1}\dot{L} + \varepsilon C(t)]L^{-1}Y, \tag{43}$$

and try to apply the eigenvalue method to this general situation, we have to provide two conditions: (i) the matrix $L^{-1}\dot{L} + \varepsilon C(t)$ has $n - 1$ eigenvalues with negative parts (similar to the constant matrix $\tilde{G}$ case); (ii) the matrix $L(L^{-1}\dot{L} + \varepsilon C(t))L^{-1}$ has a diagonal form with negative diagonal elements (except one that equals zero). Obviously, these conditions do not always hold.

This also shows that one cannot use an intuitive but misleading conception that if the corresponding $n - 1$ eigenvalues of the whole coupled system Jacobian $M(t) = J(t) + \varepsilon C(t)$ have negative real parts for all times t, then the variational system is stable. Indeed, this statement is not, in general, true, by the same reasoning ($\dot{Y} = [\dot{L}L^{-1} + LM(t)L^{-1}]Y$).

At the same time, our method based on graph theoretical reasoning can provide synchronization bounds for the time-dependent coupling coefficients, and it becomes the ultimate tool for the study of global synchronization in such networks.

As a trivial illustrative example, we consider the ring of $2K$-nearest neighbor coupled oscillators, considered in Section 4, where the coupling coefficients ε in Eq. (29) are time-dependent. Namely, ε stands for $\varepsilon(2 + n_{ij}(t))$, where $n_{ij}(t)$ is bounded noise, $|n_{ij}(t)| < 1$. For this time-dependent coupling, our method gives the same synchronization bound (29), whereas the eigenvalue method fails to provide an analytically derived bound.

Another prominent example of networks with time-dependent coupling coefficients is networks of pulse-coupled bursting neurons. The coupling here is nonlinear and dependent on the arrival of the spikes. Although, this type of coupling does not belong to the class of linearly coupled networks (1) considered in this paper, our method can be extended to the study of the stability of bursting synchronization (this will be reported elsewhere). Once again, there is no simple way to apply the eigenvalue method to such pulse-coupled networks.

6. Conclusions

We have developed a novel method for proving complete synchronization in networks of mutually coupled cells of periodic or chaotic oscillators with arbitrary connection graphs. The method reveals a clear connection between synchronization and graph theory. Criteria are developed that allow us to establish upper bounds on the coupling strength necessary to achieve complete synchronization. The coupling strength may vary from pair to pair of interacting cells and it may even depend on time. The bounds on the coupling strengths depend on the coupling graph in general and on the number of cells in particular. The dependence of the bounds on the number of cells is to a high degree of precision the same as the dependence of the real limit of complete synchronization that is determined by numerical simulation. Only the multiplicative factor that is related to the dynamics of the individual cells is higher in our rigorous bonds with respect to the factor obtained by numerical simulations.

In order to derive our results, we have used a quadratic form in the difference variables of all possible pairs of cells. In order to show that this is a Lyapunov function for the difference variables, we have to calculate the following graph theoretical quantity. We have to choose a path on the connection graph between any two vertices of the graph. Then, for each edge of the connection graph, the sum of the path lengths of all paths containing this edge must be determined. The coupling constant that guarantees complete synchronization is proportional to this sum.

We have showed how to determine this quantity for four examples, namely for all-to-all coupling, star coupling, next nearest neighbor coupling in a ring and $2K$-nearest neighbor coupling in a ring. Especially the last example is highly non-trivial, but our method allows one to achieve an excellent result with only moderate effort.

It should be emphasized that our results guarantee global synchronization of all cells based on a method that is completely rigorous. This contrasts with the recently published results that prove local stability of the synchronization manifold by calculating analytically the eigenvalues of the coupling matrix and numerically the transversal Lyapunov exponents. Being based partially on numerical calculations, these results usually provide a bound that is closer than ours to the "real" limit of total synchronization. However, it is remarkable that both approaches, and the purely numerical results as well, lead to practically the same dependence of the bounds for complete synchronization on the number of cells and on the depth of nearest neighbor interactions. The main difference is a higher multiplicative constant for our stronger, but more conservative Lyapunov function approach.

Note that we can also use the eigenvalues of the connectivity matrix for the Lyapunov function approach (cf. Section 5). In this way we may obtain, in the case of constant connection matrices, a better bound for the global synchronization threshold than with the connection graph stability method. However, this eigenvalue based method may be difficult to apply for irregular graphs, it gives a less direct relation to graph theoretical quantities and in general it fails for time-dependent coupling coefficients.

We are convinced that our approach will lead to more non-trivial results. In a companion paper we will give such results for small-world networks.

We have assumed from the beginning that the connection matrix is symmetric, i.e. that the coupling from cell j to cell i has the same strength as the coupling from cell i to cell j. While total synchronization in networks with non-symmetric coupling is still possible in many cases, our method is not directly applicable in this context. This problem remains a subject for future work.

Finally, one should remark that the method is valid for networks of slightly non-identical oscillators. In this case, perfect synchronization cannot exist anymore, but approximate synchronization is still possible and therefore similar global stability conditions of approximate synchronization can be derived by means of the presented method and the technique developed in [30]. In Appendix B, we present the details of how the method can be applied to networks of slightly non-identical oscillators.

Our approach can also be extended to the study of synchronization in coupled map lattices with linear and nonlinear coupling, including, in particular, the case of the Kaneko-type non-local coupling. The method can also be applied to the study of synchronization in networks of strongly pulse-coupled bursting cells. Obviously, all the above mentioned cases are subjects for future study.

Acknowledgements

Appendix A

In this appendix, we give the details of the proof of the assumption (8) for the network (1) of x-coupled Lorenz systems [30] (we have knowingly chosen a scalar version of the coupling which is the most difficult case from the stabilization point of view). The coupled system reads

$$\dot{x}_i = \sigma(y_i - x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t)x_j, \qquad \dot{y}_i = rx_i - y_i - x_iz_i, \qquad \dot{z}_i = -bz_i + x_iy_i, \quad i = 1, \ldots, n \qquad (A.1)$$

for which the matrix $P = \mathrm{diag}\{1, 0, 0\}$ and the vector (x_i, y_i, z_i) stands for the vector x_i from (1). σ, r, and b are standard parameters. All other notations are similar to those of the system (1). Recall that $\varepsilon_{ii} = -\sum_{j=1; j\neq i}^{n} \varepsilon_{ij}$, $i = 1, \ldots, n$.

To prove that the condition (8) is true for the coupled system (A.1), we shall follow the steps of our previous study [30].

(a) The individual non-perturbed Lorenz system ($\varepsilon_{ij} \equiv 0$) is eventually dissipative and has an absorbing domain

$$B = \left\{ x^2 + y^2 + (z - r - \sigma)^2 < \frac{b^2(r + \sigma)^2}{4(b - 1)} \right\}.$$

Hence, the coordinates of the attractor of the individual Lorenz system are estimated to be bounded by

$$|\varphi| < \frac{b(r + \sigma)}{2\sqrt{b - 1}}, \qquad \varphi = x, y, (z - r - \sigma). \qquad (A.2)$$

It can be easily shown [30] that the estimates (A.2) are valid for the coordinates of each oscillator of the coupled system (A.1).

(b) The auxiliary system (6) written for the difference variables $X_{ij} = x_j - x_i$, $Y_{ij} = y_j - y_i$, and $Z_{ij} = z_j - z_i$ of the coupled system (A.1) and having the matrix $A = \mathrm{diag}\{a, 0, 0\}$, takes the form

$$\dot{X}_{ij} = \sigma(Y_{ij} - X_{ij}) - aX_{ij}, \qquad \dot{Y}_{ij} = (r - U_{ij}^{(z)})X_{ij} - Y_{ij} - U_{ij}^{(x)}Z_{ij},$$

$$\dot{Z}_{ij} = U_{ij}^{(y)}X_{ij} + U_{ij}^{(x)}Y_{ij} - bZ_{ij}, \quad i, j = 1, \ldots, n, \tag{A.3}$$

where $U_{ij}^{(\xi)} = (\xi_i + \xi_j)/2$ for $\xi = x, y, z$ are the corresponding sum variables, and the extra auxiliary system's term $-aX_{ij}$ stands for the contribution of the coupling term in the original system for the differences. In the system (A.3), we got rid of the cross terms with the help of the formula

$$\xi_j\eta_j - \xi_i\eta_i = U^{(\eta)}(\xi_j - \xi_i) + U^{(\xi)}(\eta_j - \eta_i).$$

To show that the trivial equilibrium of the auxiliary system (A.3) becomes globally stable provided the parameter a is sufficiently large, we consider the Lyapunov function candidates (7) with the unit matrix $H = I$,

$$W_{ij} = \tfrac{1}{2}X_{ij}^2 + \tfrac{1}{2}Y_{ij}^2 + \tfrac{1}{2}Z_{ij}^2, \quad i, j = 1, \ldots, n. \tag{A.4}$$

Their derivatives with respect to the system (A.3) are calculated as follows:

$$\dot{W}_{ij} = -[(a + \sigma)X_{ij}^2 + (U^{(z)} - r - \sigma)X_{ij}Y_{ij} + Y_{ij}^2 - U^{(y)}X_{ij}Z_{ij} + bZ_{ij}^2]. \tag{A.5}$$

Therefore, applying the Silvester criterion for negative definiteness of the quadratic forms, we obtain the conditions $a + \sigma > 0$,

$$\begin{vmatrix} a + \sigma & \tfrac{1}{2}(U^{(z)} - r - \sigma) \\ \tfrac{1}{2}(U^{(z)} - r - \sigma) & 1 \end{vmatrix} > 0, \quad \text{and} \quad \begin{vmatrix} a + \sigma & \tfrac{1}{2}(U^{(z)} - r - \sigma) & -\tfrac{1}{2}(U^{(y)}) \\ \tfrac{1}{2}(U^{(z)} - r - \sigma) & 1 & 0 \\ -\tfrac{1}{2}(U^{(y)}) & 0 & b \end{vmatrix} > 0. \tag{A.6}$$

Taking into account the estimate (A.2) for the coordinates $U^{(y)}$ and $U^{(z)}$, we finally obtain the following sufficient condition for negative definiteness of the quadratic forms:

$$a > a^* = \frac{b(b + 1)(r + \sigma)^2}{16(b - 1)} - \sigma. \tag{A.7}$$

Therefore, under this condition, the trivial solution of the auxiliary system (6) is globally asymptotically stable and the condition (8) is true for networks of x-coupled Lorenz systems.

This completes the proof.

Appendix B. Synchronization in networks of non-identical oscillators

Using the theory developed for lattices of locally coupled oscillators with parameter mismatch [30], we show here how the method can be applied to the study of global synchronization in a complex network of mutually coupled slightly non-identical systems.

Let us consider such a network that can be modeled by the general coupled system (1) with an additional mismatch term

$$\dot{x}_i = F(x_i) + \mu f_i(x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t)Px_j, \quad i = 1, \ldots, n, \tag{B.1}$$

where μ is a scalar parameter, and $f_i : R^d \to R^d$ is a smooth mismatch function. Recall that $x_i = (x_i^1, \ldots, x_i^d)$ is a vector. All other notations are similar to those of the system (1).

Obviously, perfect synchronization is not possible in the network (B.1) and only approximate synchronization can be observed. We say that global δ-synchronization arises in the system if the δ-neighborhood of the synchronous manifold $M = \{x_1(t) = x_2(t) = \cdots = x_n(t)\}$, that is invariant when the coupled systems are identical,

$$\|x_i(\mu, t) - x_j(\mu, t)\| < \delta(\mu), \quad i \neq j, \ i, j = 1, \ldots, n, \qquad \lim_{\mu \to 0} \delta(\mu) = 0$$

becomes globally stable. Now we should prove that the main statements of the connection graph stability method are directly applicable to the study of global stability of δ-synchronization.

We assume that each individual subsystem of the network (B.1) is eventually dissipative and has an absorbing domain $B_i(\mu)$ for some region of the parameter μ. Similar to the identical system case, one can show that the entire coupled system (B.1) has the absorbing domain $B(\mu)$ that is a topological product of $B_i(\mu)$, $i = 1, \ldots, n$.

The redundant stability system for the difference variables (4) written for the system (B.1) has an extra mismatch term and takes the form

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta \right] X_{ij} + \mu \bar{e}_{ij} + \sum_{k=1}^{n} \{\varepsilon_{jk} P X_{jk} - \varepsilon_{ik} P X_{ik}\}, \quad i, j = 1, \ldots, n, \tag{B.2}$$

where $\bar{e}_{ij} = [f_j(x_j) - f_i(x_i)]|_{B(\mu)}$ is a mismatch difference calculated within the absorbing domain $B(\mu)$.

Here we follow the steps of the identical oscillator study, except that we add and subtract two additional terms AX_{ij} and $H^{-1}CX_{ij}$ from the system (B.2)

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta - A + H^{-1}C \right] X_{ij} + \{\mu \bar{e}_{ij} - H^{-1}CX_{ij}\} + AX_{ij}$$

$$+ \sum_{k=1}^{n} \{\varepsilon_{jk} P X_{jk} - \varepsilon_{ik} P X_{ik}\}, \tag{B.3}$$

where the matrices A, H are identical to those of the systems (5) and (8). The matrix $C = \mathrm{diag}(c_1, \ldots, c_d)$. The linear term $H^{-1}CX_{ij}$, with negative sign, is added to compensate the contribution of the mismatch term $\mu \bar{e}_{ij}$ for X_{ij} not too small. At the same time, the positive term $+H^{-1}CX_{ij}$ can be damped by the appropriate choice of the values of the matrix A in the negative term $-AX_{ij}$. Finally, the contribution of the positive term $+AX_{ij}$ can be compensated by making the coupling sufficiently strong. This implies that to provide global approximate synchronization of non-identical systems, the coupling strength should be made stronger than for identical synchronization.

The auxiliary system for Eq. (B.3) takes the form

$$\dot{X}_{ij} = \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta - A + H^{-1}C \right] X_{ij}, \quad i, j = 1, \ldots, n. \tag{B.4}$$

Now we should show that the system (B.4) is globally stable. Applying the Lyapunov functions W_{ij} (7) for Eq. (B.4), we obtain

$$\dot{W}_{ij} = X_{ij}^{\mathrm{T}} H \left[\int_0^1 DF(\beta x_j + (1 - \beta)x_i) \, \mathrm{d}\beta - A \right] X_{ij} + X_{ij}^{\mathrm{T}} CX_{ij}. \tag{B.5}$$

We require that

$$\dot{W}_{ij} < 0, \quad i, j = 1, \ldots, n. \tag{B.6}$$

Similar to Eq. (8), one has to prove this statement for each particular situation that is defined by the choice of the concrete individual oscillator and by which variables the oscillators are coupled (through the matrix P). In most cases this statement directly follows from the principal requirement (8).

Applying the Lyapunov function V, defined in Eq. (9), for the redundant system (B.3), we obtain

$$\dot{V} = S_1 + S_2 + S_3 + S_{\text{diff}}, \tag{B.7}$$

where $S_1 = (1/2) \sum_{i=1}^{n} \sum_{j=1}^{n} \dot{W}_{ij}$, S_2 and S_3 are the sums (11) and (12), respectively. The sum having the mismatch difference term is $S_{\text{diff}} = \sum_{i=1}^{n} \sum_{j=1}^{n} \{\mu X_{ij}^{\text{T}} H \bar{e}_{ij} - X_{ij}^{\text{T}} C X_{ij}\}$.

The sum S_1 is negative definite due to the assumption (B.6). The sums S_2 and S_3 coming from the identical oscillator case are negative definite under the conditions of Theorem 2, except that a is now dependent on (C, μ). Therefore, to obtain the conditions on the region of negative definiteness of the quadratic form (B.7), it remains to study the quadratic sum S_{diff}.

The values $\bar{e}_{ij}^{(l)}$, $l = 1, \ldots, d$ are bounded in the absorbing domain $B(\mu)$, i.e. $|\bar{e}_{ij}^{(l)}| < \bar{e}^{(l)}$. Denote $M^{(l)} = |\sum_{i=1}^{d} h_{li} \bar{e}^{(i)}|$. Then the sum S_{diff} can be bounded as follows:

$$S_{\text{diff}} < \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{d} \{[\mu M^{(l)} - c_l |X_{ij}^{(l)}|] |X_{ij}^{(l)}|\}.$$

Therefore, $S_{\text{diff}} < 0$, and hence the entire sum $\dot{V} < 0$, for $|X_{ij}^{(l)}| > \mu M^{(l)}/c_l$, $l = 1, \ldots, d$.

To obtain an estimate for the synchronization error $\delta = \max_{l \in [1,d]} X_{ij}^{l}(\mu, t)$, $l = 1, \ldots, d$, we should enclose the domain $\{\dot{V} < 0\}$ into some region bounded by a certain level V_0 of the Lyapunov function (9).

The enclosure $\{|X_{ij}^{(l)}| < \mu M^{(l)}/c_l, l = 1, \ldots, d\} \subset \{V < V_0\}$ determines that $\dot{V}$ is negative outside of the region

$$|X_{ij}^{(l)}| < \frac{v_l \mu M^{(l)}}{c_l}, \quad l = 1, \ldots, d, \tag{B.8}$$

where the constants v_l are defined by the level V_0.

Therefore, we arrive at the following assertion.

Statement B.1. *If the assumption* (B.6) *holds* (*the corresponding parameter vector $A(\mu, C)$ can be calculated*) *then under the conditions of* Theorem 2, *approximate δ-synchronization in the coupled non-identical oscillator system* (B.1) *is globally stable*, *where*

$$\delta = \mu \left[\max_{l \in [1,d]} v_l \frac{M^{(l)}}{c_l} \right], \quad l = 1, \ldots, d. \tag{B.9}$$

The law of the (δ, ε) dependence is implicitly expressed via the dependence on $(c_1, \ldots, c_d)$. Indeed, while the auxiliary parameters c_l are increasing, the synchronization error $\delta(c_1, \ldots, c_d)$ decreases. At the same time, the stability parameter vector A, and therefore the synchronization threshold $\varepsilon^*(c_1, \ldots, c_d)$, should be augmented. This is a trade-off between the synchronization precision and the lowest bound of the synchronization threshold. Since we deal with sufficient conditions, it is often possible to choose some optimal values $c_1^{\text{o}}, \ldots, c_d^{\text{o}}$.

We shall now make this general approach more concrete by considering an example of the network (A.1)–(B.1) of x-coupled non-identical Lorenz systems:

$$\dot{x}_i = (\sigma + \sigma_i)(y_i - x_i) + \sum_{j=1}^{n} \varepsilon_{ij}(t)x_j, \qquad \dot{y}_i = (r + r_i)x_i - y_i - x_i z_i,$$

$$\dot{z}_i = -(b + b_i)z_i + x_i y_i, \quad i = 1, \ldots, n, \tag{B.10}$$

where σ_i, r_i, b_i are mismatch parameters that are assumed to be uniformly bounded $|\sigma_i| < \mu$, $|r_i| < \mu$, and $|b_i| < \mu$. All other notations are similar to those of the system (A.1).

Once again, we have chosen the scalar version of the coupling and mismatch parameters that are present in *all three* equations of the Lorenz system. This is the most difficult case to prove the stability of synchronization and to compensate mismatch instabilities arising in the three equations by increasing the coupling through a single variable.

The coordinates of the coupled identical oscillator system (B.10) without mismatch $\mu = 0$ are bounded due to Eq. (A.2) (Appendix A).

Being similar to Eq. (A.3) (except that three extra terms are present), the auxiliary system (B.4)–(B.10) with the matrices $H = I$, $A = \mathrm{diag}(a, 0, 0)$, and $C = \mathrm{diag}(c, c, c)$ reads

$$\dot{X}_{ij} = \sigma(Y_{ij} - X_{ij}) - aX_{ij} + cX_{ij}, \qquad \dot{Y}_{ij} = (r - U_{ij}^{(z)})X_{ij} - Y_{ij} - U_{ij}^{(x)}Z_{ij} + cY_{ij},$$

$$\dot{Z}_{ij} = U_{ij}^{(y)})X_{ij} + U_{ij}^{(x)}Y_{ij} - bZ_{ij} + cZ_{ij}, \quad i, j = 1, \ldots, n. \tag{B.11}$$

Similar to the identical Lorenz system study (see Appendix A), one can show that the auxiliary system (B.11) is globally stable if

$$a > a^*(c) = \frac{b^2(r + \sigma)^2(b - 2c + 1)}{16(b - c)(b - 1)(1 - c)} + c - \sigma, \tag{B.12}$$

where $0 < c < 1, 0 < c < b$. Since the parameter b is assumed to be greater than 1 (in the original Lorenz system $b = 8/3$) therefore the auxiliary parameter c must be chosen from the interval $(0, 1)$.

Now we show how to choose the parameter c and estimate the synchronization error δ.

The error δ depends on the mismatch functions

$$\mu f_i^{(1)}(x_i, y_i, z_i) = \sigma_{ij}(y_i - x_i), \qquad \mu f_i^{(2)}(x_i, y_i, z_i) = r_i x_i, \qquad \mu f_i^{(3)}(x_i, y_i, z_i) = -b_i z_i \tag{B.13}$$

that are linear. Hence, the differences $\bar{e}_{ij}^{(l)} = [f_j^{(l)}(x_j, y_j, z_j) - f_i^{(l)}(x_i, y_i, z_i)], l = 1, 2, 3$ are also linear functions of the coordinates of the coupled system (B.10), and they can be bounded via the absorbing domain (A.2). Therefore, taking into account Eq. (B.13) and using $H = I$, we obtain the estimate on the maximal mismatch function difference,

$$M = \max\{\bar{e}_{ij}^{(l)} l = 1, 2, 3\} = \bar{e}^{(1)} = \frac{2(b + \mu)(\sigma + r + 2\mu)}{\sqrt{b + \mu - 1}}. \tag{B.14}$$

Thus the synchronization error δ is estimated as follows:

$$\delta = 4\mu \left[\frac{(b + \mu)(\alpha + 2\mu)}{\sqrt{b + \mu - 1}} \right] \tag{B.15}$$

for the chosen auxiliary parameter $c = 1/2$. The constants $w_k, k = 1, 2, 3$ from Eq. (B.8) are equal here to $\sqrt{2}$. Now we should also use $c = 1/2$ in Eq. (B.12).

While the mismatch parameter μ is small, the estimate (B.15) presents a quasi-linear law of the dependence of the synchronization error δ on the mismatch coefficient μ.

The relative synchronization error δ^{rel}, expressed as the ratio of the maximal amplitude $A_{\max} = \max\{x_i, y_i, z_i, i = 1, \ldots, n\}$ of the oscillators can be estimated as follows:

$$\delta^{\text{rel}} = \frac{2\sqrt{2}\mu M}{A_{\max}}. \tag{B.16}$$

Recall that we have also bounded the maximal value M of the mismatch function difference by $A_{\max}$, therefore the estimate (B.16) takes the form

$$\delta^{\text{rel}} = 2\sqrt{2}\mu. \tag{B.17}$$

Note that we do not require $\delta(\mu)$ to be small. The limit value of the error can be chosen as large as 30%. This gives the limit of the maximal mismatch rate μ. In our sufficient conditions (B.17), this corresponds to $\mu = 0.16$ that is equivalent to 5% parameter mismatch (b_{ij}/b).

Concluding the proof of global stability of δ-synchronization in the network (B.10), we come to the following statement.

Statement 5. *Approximate synchronization with the synchronization error $\delta^{\text{rel}} = 2\sqrt{2}\mu$ is globally stable in a network of coupled non-identical Lorenz systems (1)–(43) and (A.1)–(B.10) if*

$$\varepsilon_k > \frac{a}{n} b_k(n, m) \quad \text{for } k = 1, \ldots, m,$$

where $a = b^3(r + \sigma)^2 / 8(b - 1/2)(b - 1) + 1/2 - \sigma$, and $b_k(n, m)$ is defined by the coupling configuration and determined in the conditions of Theorem 2.

Using similar arguments, one can also derive the stability conditions for synchronization in networks of other slightly non-identical limit-cycle and chaotic systems.

References

[1] H. Fujisaka, T. Yamada, Prog. Theor. Phys. 69 (1983) 32;
 H. Fujisaka, T. Yamada, Prog. Theor. Phys. 72 (1984) 885.
[2] V.S. Afraimovich, N.N. Verichev, M.I. Rabinovich, Radiophys. Quant. Electron. 29 (1986) 795.
[3] L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. 64 (1990) 821.
[4] V.N. Belykh, N.N. Verichev, L.J. Kocarev, L.O. Chua, in: R.N. Madan (Ed.), Chua's Circuit: A Paradigm for Chaos, World Scientific, Singapore, 1993, 325.
[5] J.F. Heagy, T.L. Carroll, L.M. Pecora, Phys. Rev. E 50 (1994) 1874.
[6] J.F. Heagy, L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. 74 (1994) 4185.
[7] C.W. Wu, L.O. Chua, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 43 (1996) 161.
[8] L.M. Pecora, T.L. Carroll, G.A. Johnson, D.J. Mar, J.F. Heagy, Chaos 7 (1997) 520, and references therein.
[9] V.S. Afraimovich, S.N. Chow, J.K. Hale, Physica D 103 (1997) 442.
[10] J.K. Hale, J. Dyn. Diff. Eq. 9 (1997) 1.
[11] V.S. Afraimovich, W.W. Lin, Dyn. Stab. Syst. 13 (1998) 237.
[12] L.M. Pecora, T.L. Carroll, Phys. Rev. Lett. 80 (1998) 2109.
[13] L.M. Pecora, Phys. Rev. E. 58 (1998) 347.
[14] K. Josić, Nonlinearity 13 (2000) 1321.
[15] D.H. Zanette, A.S. Mikhailov, Phys. Rev. E 57 (1998) 276.
[16] S.C. Manrubia, A.S. Mikhailov, Phys. Rev. E 60 (1999) 1579.
[17] A.Yu. Pogromsky, H. Nijmeijer, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 48 (2001) 152.
[18] C.W. Wu, in: L.O. Chua (Ed.), Synchronization in Coupled Chaotic Circuits and Systems, vol. 41, World Scientific Series on Nonlinear Science, Series A, World Scientific, Singapore, 2002.
[19] C.W. Wu, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 50 (2003) 294.

[20] J. Jost, M.P. Joy, Phys. Rev. E 65 (2001) 016201.
[21] G. Rangarajan, M. Ding, Phys. Lett. A 296 (2002) 204.
[22] Y. Chen, G. Rangarajan, M. Ding, Phys. Rev. E. 67 (2003) 026209.
[23] X.F. Wang, G. Chen, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 49 (2002) 54.
[24] M. Barahona, L.M. Pecora, Phys. Rev. Lett. 89 (2002) 054101.
[25] V.N. Belykh, I.V. Belykh, M. Hasler, K.V. Nevidin, Int. J. Bifurc. Chaos 13 (2003) 756.
[26] K. Kaneko, Physica D 41 (1990) 137;
    K. Kaneko, Physica D 54 (1991) 5;
    K. Kaneko, Physica D 55 (1992) 368;
    K. Kaneko, Physica D 75 (1994) 55.
[27] A. Sherman, Bull. Math. Biol. 56 (1994) 811.
[28] J.R. Terry, K.S. Thornburg Jr., D.J. DeShazer, G.D. Vanwiggeren, S. Zhu, P. Ashwin, R. Roy, Phys. Rev. E 59 (1999) 4036.
[29] V.N. Belykh, I.V. Belykh, M. Hasler, Phys. Rev. E 62 (2000) 6332.
[30] I.V. Belykh, V.N. Belykh, K.V. Nevidin, M. Hasler, Chaos 13 (2003) 165.
[31] A.Yu. Pogromsky, G. Santoboni, H. Nijmeijer, Physica D 172 (2002) 65.
[32] A. Pikovsky, M. Rosenblum, J. Kurths, Synchronization: A Universal Concept in Nonlinear Science, Cambridge University Press, Cambridge, 2001.
[33] P. Ashwin, J. Buescu, I. Stewart, Phys. Lett. A 193 (1994) 126;
    P. Ashwin, J. Buescu, I. Stewart, Nonlinearity 9 (1996) 703;
    M. Hasler, Yu. Maistrenko, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 44 (1997) 856.
[34] N.F. Rulkov, M.M. Suschik, L.S. Tsimring, H.D.I. Abarbanel, Phys. Rev. E 51 (1995) 980;
    H.D. Abarbanel, N.F. Rulkov, M.M. Sushik, Phys. Rev. E 53 (1996) 4528;
    L. Kocarev, U. Parlitz, Phys. Rev. Lett. 76 (1996) 1816.
[35] J. Kurths, S. Boccaletti, C. Grebogi, Y.-C. Lai (Organizers), Focus Issue: Control and Synchronization in Chaotic Dynamical Systems, Chaos 13 (2003) 126.
[36] Y. Kuramoto, in: H. Araki (Ed.), International Symposium on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics 39, Springer-Verlag, Berlin, 1975, p. 420.
[37] N. Kopell, G.B. Ermentrout, Math. Biosci. 90 (1988) 87.
[38] S. Watanabe, S.H. Strogatz, Phys. Rev. Lett. 70 (1993) 2391.
[39] S.H. Strogatz, R.E. Mirollo, Physica D 31 (1988) 143–168.
[40] D. Somers, N. Kopell, Physica D 89 (1995) 169.
[41] I.V. Belykh, V.N. Belykh, M. Hasler, Physica D, this issue.
[42] I.V. Belykh, Radiophys. Quant. Electron. 38 (1995) 69.
[43] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440.
[44] S.H. Strogatz, Nature 410 (2001) 268.
[45] M.E.J. Newman, C. Moore, D.J. Watts, Phys. Rev. Lett. 84 (2000) 3201.