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A lower bound for algebraic connectivity based on the connection-graph-stability method

Ali Ajdari Rad[a,*], Mahdi Jalili[b], Martin Hasler[a]

[a] *École Polytechnique Fédéral de Lausanne, Laboratory of Nonlinear Systems, School of Computer and Communication Science, 1015 Lausanne, Switzerland*

[b] *Department of Computer Engineering, Sharif University of Technology, Tehran, Iran*

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ABSTRACT

This paper introduces the connection-graph-stability method and uses it to establish a new lower bound on the algebraic connectivity of graphs (the second smallest eigenvalue of the Laplacian matrix of the graph) that is sharper than the previously published bounds. The connection-graph-stability score for each edge is defined as the sum of the lengths of the shortest paths making use of that edge. We prove that the algebraic connectivity of the graph is bounded below by the size of the graph divided by the maximum connection-graph-stability score assigned to the edges.

1. Introduction

Let $G = (V, E)$ be a connected simple graph with $n = |V|$ vertices and $|E|$ edges. For vertices u and v of G, a path of minimum length from u to v is called a *shortest path* from u to v. Such path is denoted by P_{uv}. The length of the longest path in the collection of shortest paths P_{uv} when (u, v) varies over all vertices is called *diameter* of the graph, denoted by d_{max}. The Laplacian matrix of G is defined as $L = D - A$, where A is the binary adjacency matrix and $D = diag(d_u; u \in V)$ is a diagonal matrix that records the degrees of the vertices of G. The matrix L is a positive semidefinite, symmetric and singular matrix whose eigenvalues are in the form of $\lambda_n(G) \geqslant \lambda_{n-1}(G) \geqslant \cdots \geqslant \lambda_2(G) \geqslant \lambda_1(G) = 0$. These eigenvalues are important in graph theory and have close relations to numerous graph invariants. Among them, the second smallest eigenvalue, λ_2, called the algebraic connectivity, has attracted more attention. There are several lower bounds on λ_2 based on simple properties of the graph such

* Corresponding author.
E-mail addresses: ali.ajdarirad@epfl.ch (A.A. Rad), mjalili@sharif.edu (M. Jalili), martin.hasler@epfl.ch (M. Hasler).

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as diameter, order, and number of edges (see [1] for a comprehensive review). For example, Mohar [5] showed that $\lambda_2 \geqslant \frac{4}{nd_{max}}$ and recently Lu [4] proved that $\lambda_2 \geqslant \frac{2n}{2+(n-1)nd_{max}-2|E|d_{max}}$. In this paper, we present a new lower bound for λ_2 based on the connection-graph-stability scores associated to the edges, which are defined for each edge as the sum of the length of all the shortest paths making use of that edge. We also prove that the proposed lower bound is always sharper than the previously mentioned bounds of Mohar and Lu.

2. Lower bound based on connection-graph-stability method

The connection-graph-stability method was proposed by Belykh et al. [6] to establish a criterion for the global stability of synchronization manifold of a network of coupled dynamical systems. Here, we use the concept to obtain a lower bound for algebraic connectivity of a graph.

Definition 1 (*Connection-graph-stability score*)**.** For each pair of vertices u and v, let us choose a path P_{uv} from u to v (not necessarily the shortest path). The Connection-graph-stability score for each edge k of G with respect to this collection of paths is denoted by C_k and defined as the sum of the length of the paths P_{uv} that contains edge k, i.e.

$$C_k = \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n}\varphi_{uv}(k)|P_{uv}|, \tag{1}$$

where

$$\varphi_{uv}(k) = \begin{cases} 1 & \text{if} \quad k \in P_{uv} \\ 0 & \text{if otherwise.} \end{cases}$$

The maximum connection-graph-stability score assigned to the edges of G is denoted by C_{max}, i.e. $C_{max} = \max_{k\in E} C_k$.

Theorem 1. *Let G be a connected simple graph with n nodes. Then,*

$$\lambda_2 \geqslant \frac{n}{C_{max}}, \tag{2}$$

Proof. Let G be a simple graph. Fiedler [3] showed that

$$\lambda_2 = \min \frac{2n\sum_{uv\in E}(x_u - x_v)^2}{\sum_{u\in V}\sum_{v\in V}(x_u - x_v)^2}, \tag{3}$$

where the minimum is taken over all non-constant vectors $x = (x_v)_{v\in V(G)}$ with $\|x\| = 1$. Denoting $X_{uv} = x_u - x_v$, Eq. (3) can be rewritten as

$$\lambda_2 = \min \frac{2n\sum_{uv\in E} X_{uv}^2}{\sum_{u\in V}\sum_{v\in V} X_{uv}^2}. \tag{4}$$

Let P_{uv} be the path $u - m_1 - m_2 - \cdots - m_k - v$ with $m_i \in V$ being the path connecting u to v and $|P_{uv}|$ its length, then X_{uv} can be expressed as

$$X_{uv} = X_{um_1} + X_{m_1m_2} + \cdots + X_{m_kv} = \sum_{e\in P_{uv}} X_e.$$

Applying the Cauchy–Schwartz inequality, one obtains

$$X_{uv}^2 = \left(\sum_{e\in P_{uv}} X_e\right)^2 \leqslant |P_{uv}| \sum_{e\in P_{uv}} X_e^2.$$

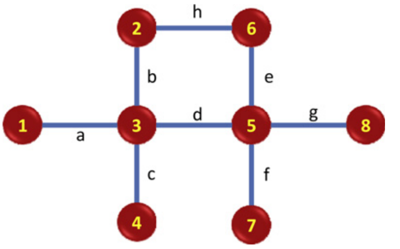

Fig. 1. An example graph with 8 nodes and 8 edges. As it is mentioned in the text, the set of paths that include the shortest path connecting vertices 1 and 8, i.e. *adg*, gives higher maximum connection-graph-stability comparing to the set that a longer path *abheg* has been selected instead.

Then,

$$\sum_{u=1}^{n}\sum_{v=1}^{n} X_{uv}^2 \leqslant \sum_{u=1}^{n}\sum_{v=1}^{n}\left(|P_{uv}|\sum_{e\in E}\varphi_{uv}(e)X_e^2\right) = \sum_{e\in E} 2C_e X_e^2 \leqslant 2C_{max}\sum_{e\in E} X_e^2. \tag{5}$$

Substituting (5) in (4)

$$\lambda_2 = \min \frac{2n\sum_{uv\in E}(x_u-x_v)^2}{\sum_{u\in V}\sum_{v\in V}(x_u-x_v)^2} \geqslant \min \frac{2n\sum_{uv\in E}(x_u-x_v)^2}{2C_{max}\sum_{uv\in E}(x_u-x_v)^2},$$

and finally $\lambda_2 \geqslant \frac{n}{C_{max}}$. $\square$

The above result is valid for any set of paths, not necessarily shortest paths. While shortest paths are well defined and intuitively good choices, in some cases replacing a shortest path with a longer path gives a better, i.e. lower, maximum connection-graph-stability score. For example, in the graph shown in Fig. 1, let us first choose a shortest path for each pair of vertices. Since for almost all of the pairs there is exactly one shortest path, here, we just list the shortest paths that have an alternative choice namely: $P_{16} = abh$, $P_{25} = bd$, $P_{27} = bdf$, $P_{36} = de$, and $P_{46} = cde$. According to this set of paths, the connection-graph-stability scores become: $C_a = 16$, $C_b = 13$, $C_c = 16$, $C_d = 31$, $C_e = 13$, $C_f = 16$, $C_g = 16$, and $C_h = 7$, where the maximum connection-graph-stability score is $C_{max} = C_d = 31$. If the shortest path connecting vertices 1 and 8, i.e. *adg*, is replaced by a longer path *abheg*, then the connection-graph-stability scores change to: $C_a = 18$, $C_b = 18$, $C_c = 16$, $C_d = 28$, $C_e = 18$, $C_f = 16$, $C_g = 18$, and $C_h = 12$. In this new configuration, the maximum connection-graph-stability score decreases and becomes $C_{max} = 28$.

Definition 2 (*Path weighting strategy*)**.** For any pair $u, v \in V$ let us consider an arbitrary non-empty set of paths connecting u and v denoted by $\mathbb{P}_{uv}$, i.e. $\mathbb{P}_{uv} = \left\{P_{uv}^{(1)}, P_{uv}^{(2)}, \ldots, P_{uv}^{(n_{uv})}\right\}$ with $n_{uv} \geqslant 1$. Then, for any pair $u, v \in V$ and their corresponding set of paths $\mathbb{P}_{uv}$, let us choose a vector $\alpha_{uv} = \left(\alpha_{uv}^{(1)}, \cdots, \alpha_{uv}^{(n_{uv})}\right)$, where $n_{uv} = |\mathbb{P}_{uv}|$, such that $\alpha_{uv}^{(q)} \geqslant 0$ and $\sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} = 1$. The corresponding path weighting strategy is denoted by α, the set of all these vectors. If just the shortest paths between pairs of vertices u and v are considered and the corresponding weighting strategy is to set all $\alpha_{uv}^{(q)}$ equal, then this specific weighting strategy is called the *normal path weighting strategy* and denoted by $\bar{\alpha}$.

Definition 3 (*Extended connection-graph-stability score*)**.** The extended connection-graph-stability score $C_k(\alpha)$ for edge k is defined as follows

$$C_k(\alpha) = \frac{1}{2}\sum_{u=1}^{n}\sum_{v=1}^{n}\sum_{q=1}^{n_{uv}} \varphi_{uv}^{(q)}(k)|P_{uv}^{(q)}|\alpha_{uv}^{(q)}, \tag{6}$$

where

$$\varphi_{uv}^{(q)}(k) = \begin{cases} 1 & \text{if} \quad k \in P_{uv}^{(q)} \\ 0 & \text{otherwise} \end{cases}$$

and α is the corresponding path weighting strategy. Using the path weighting strategy α, the maximum extended connection-graph-stability score assigned to the edges of G is denoted by $C_{max}(\alpha)$.

Theorem 2. *Let G be a connected simple graph with n nodes. Then, for any path weighting strategy α we have*

$$\lambda_2 = a(G) \geqslant \frac{n}{C_{max}(\alpha)}, \tag{7}$$

Proof. Consider $\mathbb{P}_{uv}$ and n_{uv} for two arbitrary vertices. Let $P_{uv}^{(q)} = um_1m_2 \cdots m_k v$ (with $m_i \in V(G)$) be the qth path that connects u to v with length $|P_{uv}^{(q)}|$. Using the qth shortest path, X_{uv}, can be expressed as

$$X_{uv} = x_u - x_v = X_{um_1} + X_{m_1m_2} + \cdots + X_{m_k v} = \sum_{e \in P_{uv}^{(q)}} X_e.$$

Using the Cauchy–Schwartz inequality

$$X_{uv}^2 = \left(\sum_{e \in P_{uv}^{(q)}} X_e \right)^2 \leqslant |P_{uv}^{(q)}| \sum_{e \in P_{uv}^{(q)}} X_e^2 = |P_{uv}^{(q)}| \sum_{e \in E} \varphi_{uv}^{(q)}(e) X_e^2. \tag{8}$$

On the other hand, one can express X_{uv} as weighted average of its alternative expansions as follows

$$X_{uv}^2 = \sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} X_{uv}^2 \leqslant \sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} |P_{uv}^{(q)}| \sum_{e \in E} \varphi_{uv}^{(q)}(e) X_e^2, \tag{9}$$

Hence,

$$\begin{aligned} \sum_{u=1}^{n} \sum_{v=1}^{n} X_{uv}^2 &\leqslant \sum_{u=1}^{n} \sum_{v=1}^{n} \sum_{q=1}^{n_{uv}} \alpha_{uv}^{(q)} |P_{uv}^{(q)}| \sum_{e \in E} \varphi_{uv}^{(q)}(e) X_e^2 \\ &= \sum_{e \in E} 2C_e(\alpha) X_e^2 \leqslant 2C_{max}(\alpha) \sum_{e \in E} X_e^2. \end{aligned} \tag{10}$$

Substituting (10) in (4), we obtain

$$\lambda_2 = \min \frac{2n \sum_{uv \in E} X_{uv}^2}{\sum_{u \in V} \sum_{v \in V} X_{uv}^2} \geqslant \min \frac{2n \sum_{uv \in E} X_{uv}^2}{2C_{max}(\alpha) \sum_{uv \in E} X_{uv}^2},$$

and finally $\lambda_2 \geqslant \frac{n}{C_{max}(\alpha)}$. □

3. Comparison with the other lower bounds

3.1. Mohar's lower bound

Here, we show that the lower bound obtained in Theorem 1 is always stronger than the one previously proposed by Mohar.

Theorem 3. *For any connected graph G with n vertices, diameter d_{max}, and maximum connection-graph-stability number C_{max}, we have*

$$\frac{n}{C_{max}} \geqslant \frac{4}{nd_{max}}.$$

Proof. Consider the set of all shortest paths passing through an edge v_1v_2. Define two subsets of vertices as follows: $v \in V_1$ if there is a shortest path from v_2 to v containing v_1 and $v \in V_2$ if there is a shortest path from v_1 to v containing v_2. Note that $v_1 \in V_1$ and $v_2 \in V_2$. Then, $V_1 \cap V_2 = \emptyset$.

If this is not the case, there is a vertex $v \in V_1 \cap V_2$. Then, there is shortest path $vP_1v_2v_1$ from v to v_1 and a shortest path $vP_2v_1v_2$ from v to v_2. If $|P_1| \leqslant |P_2|$, then the path vP_1v_2 is a shorter path from v to v_2 than $vP_2v_1v_2$, which is contradictory. If $|P_2| \leqslant |P_1|$, we come to the same contradiction.

Now let $m_1 = |V_1|$ and $m_2 = |V_2|$, then there are at most m_1m_2 shortest paths between V_1 and V_2 passing through e, hence $C_e \leqslant m_1m_2d_{max}$, where C_e denotes the connection-graph-stability score of edge e. In addition, $m_2 \leqslant n - m_1$, thus $C_e \leqslant m_1(n - m_1)d_{max}$, which is maximized for $m_1 = \frac{n}{2}$. Therefore, for any edge e of the graph, and hence for the edge that has the maximum connection-graph-stability score, we have

$$C_e \leqslant \left(\frac{n}{2}\right)^2 d_{max}$$

and hence,

$$\frac{n}{C_e} \leqslant \frac{n}{\left(\frac{n}{2}\right)^2 d_{max}} = \frac{4}{nd_{max}}. \qquad \square$$

3.2. Lu's lower bound

Here, we show that the lower bound obtained in Theorem 1 is also always stronger than Lu's lower bound.

Theorem 4. *For any connected graph G with n vertices, number of edges $|E|$, diameter d_{max} and maximum connection-graph-stability score C_{max}, we have*

$$\frac{n}{C_{max}} \geqslant \frac{2n}{2 + (n-1)nd_{max} - 2|E|d_{max}}.$$

Proof. For each pair of distinct vertices u and v, consider a shortest path, i.e. P_{uv}. Among these $\frac{n(n-1)}{2}$ shortest paths, there are $|E|$ of these of length one. Suppose that e is the edge corresponding to C_{max}. There is only one path of length one passing through this edge (the path connecting adjacent vertices of e). At the same time, at most $\frac{n(n-1)}{2} - |E|$ paths with length more than one can make use of e. According to the definition of the C_{max}, the connection-graph-stability score of e is equal to the sum of the length of these paths plus the length of the path connecting adjacent vertices of e, which is one. Recall the maximum path length, i.e. diameter d_{max}, thus,

$$C_{max} \leqslant 1 + \left(\frac{n(n-1)}{2} - |E|\right) d_{max}$$

and therefore:

$$\frac{n}{C_{max}} \geqslant \frac{2n}{2 + (n-1)nd_{max} - 2|E|d_{max}}. \qquad \square$$

4. Applications

The dependence of the maximum connection-graph-stability score on the number of vertices of some well-known families of graphs can be calculated analytically [6], and thus the proposed bound can also be explicitly calculated as a function of the size of the graph. Table 1 summarizes the results on Complete, Path, Cycle, Star, and Peterson graphs.

It should be mentioned that in general, one should know the set of paths and the weighting strategy used for calculating the connection-graph-stability scores. In the examples mentioned in Table 1, we used only shortest paths. In addition, except for the Peterson graph, there is just one shortest path

Table 1
Algebraic connectivity, λ_2, in some well-known graphs and different lower bounds, i.e. Mohar's, Lu's and the proposed one.

Graph	λ_2	Mohar's	Lu's	The lower bound (2)
Complete graph	n	$\frac{4}{n}$	n	n
Path (n is even)	$2\left(1-\cos\left(\frac{\pi}{n}\right)\right)$	$\frac{4}{n(n-1)}$	$\frac{2n}{2+(n-2)(n-1)^2}$	$\frac{8}{n^2}$
Cycle (n is odd)	$2\left(1-\cos\left(\frac{2\pi}{n}\right)\right)$	$\frac{8}{n(n-1)}$	$\frac{2n}{2+n(n-1)(\frac{n-3}{2})}$	$\frac{24}{(n^2-1)}$
Star	1	$\frac{2}{n}$	$\frac{n}{1+(n-2)(n-1)}$	$\frac{n}{2n-3}$
Peterson graph	2	0.2	0.164	1.11

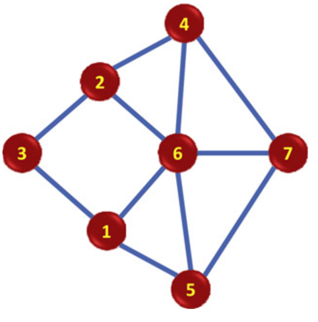

Fig. 2. An example graph with 7 nodes and 11 edges. The algebraic connectivity of this graph is 1.58 and the calculated lower bounds are 1.58, 0.19, 0.22, 0.87 and 1.04 for Lu's, Mohar's, lower bound (1) and (2), respectively.

between any two distinct pairs of vertices in these examples. Therefore, the problem of tuning the weighting strategy does not exist. In the case of the Peterson graph, for the sake of simplicity, the scores were calculated based on the *normal path weighting strategy*. As another example, let us consider the graph shown in Fig. 2. The algebraic connectivity of this graph is 1.58. The calculated lower bounds are 0.19, 0.22, 0.87 and 1.04 for Lu's, Mohar's, lower bound (1) and (2), respectively. It should be mentioned that for calculation of the lower bound (2), the *normal path weighting strategy* is used. In addition, for calculation of the lower bound (1), just shortest paths were used and if there were more than one shortest path between two vertices the one which passes through the neighbor with lower label was considered, e.g. $P_{36} = v_3 - v_1 - v_6$.

5. Discussion and conclusion

In this paper a novel lower bound for algebraic connectivity of graphs based on the connection-graph-stability method was proposed. Furthermore, it was proved that the new lower bound dominates those given by Mohar [5] and Lu [4]. From complexity point of view, the connection-graph-stability scores can be calculated in $O(|V|^2)$. The extended versions of the connection-graph-stability method needs a path weighting strategy and finding the best weighting strategy is not trivial. To make it simple, the *normal path weighting strategy*, i.e. considering just shortest paths instead of any-path and uniform α strategy, can be considered. In this case the extended connection-graph-stability score can be interpreted as a weighted edge-betweenness-centrality measure, where each path is weighted by its length divided by the number of shortest paths between the same vertices. By this simplification, the extended connection-graph-stability scores can be calculated in polynomial time for each edge using a slightly modified version of Brandes [2] algorithm which has $O(|V||E|)$ computational complexity.

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