2014 Fall Theme Semester on Discrete Networks: Geometry, Dynamics and Applications

Basic Rigidity in three flavors

Robert Connelly
Department of Mathematics Cornell University

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Given a graph that is realized in some Euclidean space, with edges of fixed lengths joining the vertices, when does there exist other configurations with the same edge lengths? There are three flavors of this question:

(a) Local rigidity – infinitesimal rigidity: There is no continuous motion of the vertices other than the "trivial ones" that are restrictions of rigid motions of the whole space.

(b) Global rigidity: There are no other non-congruent configurations with the same corresponding edge lengths in the same Euclidean space.

(c) Universal rigidity: There are no other non-congruent configurations with the same corresponding edge lengths in any higher dimensional Euclidean space.

Each of the rigidity flavors above have their own techniques, refinements, and examples to be explained later.

Rigidity of surfaces. Cauchy had a very pleasant method for proving the rigidity of convex polyhedral surfaces. Each face of the polyhedron is a rigid plate, and the plates are hinged and allowed to rotate along their edges.

Nevertheless, within the category of convex polyhedra, they are globally rigid, and, originally shown by Max Dehn, infinitesimal rigidity follows naturally. In a practical sense, from a structural engineering point of view, though, if a convex surface is subdivided arbitrarily, it is not infinitesimal rigid but remains prestress stable, which is weaker than infinitesimal rigidity, but still implies local rigidity.

It was long thought that any embedded polyhedral surface, convex or not, was at least locally rigid. This is false. There do exist easy constructions of flexible embedded polyhedral surfaces. The analogous question for smooth surfaces is still not known. The role of the differentiability constraints on the surface is subtle and boringly hard. I. Sabitov has shown, also, that if a polyhedral surface flexes continuously, the volume it bounds is constant. There is no mathematical bellows.

Generic rigidity. A finite configuration of points in generic if the set of coordinates are algebraically independent over the rationals. If one is given a discrete structure, it is often taken as a matter of faith that, since no one "really" knows where the vertices are, one might as well assume that the configuration is generic, since almost all configurations are generic. For the local and global flavors, assuming the configuration is generic has the effect of making the properties depend only on the graph, and it allows efficient combinatorial techniques to come to bear. For generic local and global rigidity in the plane, and for special classes of frameworks in higher dimensions, there are very efficient polynomial-time combinatorial algorithms that decide their rigidity.

On the other hand, in any dimension, there are numerical calculations that decide local and global generic rigidity, at least most of the time. But for global rigidity, if the configuration is given exactly, it can be quite difficult to tell if its framework is globally rigid. Almost any given specification of the configuration, defined by humans, will not be generic.

Universal rigidity, tensegrities, and the existence of realizations. The sculptor Kenneth Snelson has built large structures composed of rigid struts suspended in midair with cables in tension. R. Buckminster Fuller called these works of art tensegrities for their "tensional integrity". Most structures that are stable can be understood as being "prestress stable" when analyzed properly. The internal stress in the structure is the critical component of its stability, and when it dominates the equilibrium constraints it has the effect of being rigid not just in the dimension that we know and love, but in all higher dimensions. It is universally rigid. It then turns out that if a tensegrity is universally rigid, there is a certificate, that can be numerically calculated, that verifies universal rigidity. There is also another benefit. Suppose that a configuration is known to exist, but only some of the pairwise distances between its vertices are known. This situation comes up with nuclear magnetic resonance (NMR) spectroscopy, for example. If the target framework is universally rigid, a standard algorithm, using semi-definite programming, will find a sequence of configurations converging to the one satisfying the given constraints.