SPECTRAL THEORY FOR DYNAMICS ON GRAPHS CONTAINING ATTRACTIVE AND REPULSIVE INTERACTIONS[*]

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Abstract. Many applied problems can be posed as a dynamical system defined on a network with attractive and repulsive interactions. Examples include synchronization of nonlinear oscillator networks; the behavior of groups, or cliques, in social networks; and the study of optimal convergence for consensus algorithm. It is important to determine the index of a matrix, i.e., the number of positive and negative eigenvalues, and the dimension of the kernel. In this paper we consider the common examples where the matrix takes the form of a signed graph Laplacian. We show that the there are topological constraints on the index of the Laplacian matrix related to the dimension of a certain homology group. When the homology group is trivial, the index of the operator is determined only by the topology of the network and is independent of the strengths of the interactions. In general, these constraints give bounds on the number of positive and negative eigenvalues, with the dimension of the homology group counting the number of eigenvalue crossings. The homology group also gives a natural decomposition of the dynamics into "fixed" degrees of freedom, whose index does not depend on the edge weights, and an orthogonal set of "free" degrees of freedom, whose index changes as the edge weights change. We also explore the spectrum of the Laplacians of signed random matrices.

Key words. spectral graph theory, dynamics on networks, graph Laplacian, social networks

AMS subject classifications. 34D06, 34D20, 37G35, 05C31

DOI. 10.1137/130913973

1. Introduction. There are many applied problems that can ultimately be reduced to the question of understanding the evolution of a dynamical system living on a network or graph. In these applications it is often important to understand the dynamics in terms of the topological properties of the graph. In this paper we consider a simple, loop-free, connected, undirected edge-weighted graph $\Gamma = (V(\Gamma), E(\Gamma))$ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. For each edge in $E(\Gamma)$ connecting vertex i with vertex j we associate a weight γ_{ij}, which is assumed to be nonzero but may take either sign. If there is no edge connecting vertices i and j, the weight γ_{ij} is understood to be zero. For such a graph we define the *signed Laplacian matrix* $\mathcal{L}(\Gamma)$ by

$$(1.1) \qquad \mathcal{L}(\Gamma)_{ij} = \begin{cases} \gamma_{ij}, & i \neq j, \\ -\sum_{k \neq i} \gamma_{ik}, & i = j. \end{cases}$$

Note that $\mathcal{L}(\Gamma)$ is symmetric, so all eigenvalues of $\mathcal{L}(\Gamma)$ are real. If the weights are all positive, $\gamma_{ij} > 0$, then $\mathcal{L}$ is the (combinatorial) graph Laplacian: it is a negative semidefinite matrix, with the dimension of the kernel equal to the number of connected components of the graph Γ. In many applications the weights are not guaranteed to be positive. In this case the matrix is no longer semidefinite, and we are interested in determining the number of positive, zero, and negative eigenvalues of $\mathcal{L}(\Gamma)$, which we denote as $n_+(\Gamma), n_0(\Gamma), n_-(\Gamma)$, respectively. We note that there are several other

definitions of the graph Laplacian in the literature, most of which can be related to $\mathcal{L}$ above via a change of basis $S^{\mathsf{T}}\mathcal{L}S$. Since (by the Sylvester theorem) such a basis change for a nonsingular matrix S leaves $n_{+/-/0}(\Gamma)$ invariant, the present paper applies to these other variants of the graph Laplacian as well.

We give the best possible bounds on $n_-(\Gamma), n_0(\Gamma), n_+(\Gamma)$ involving only topological information, i.e., connectivity of the graph and the sign information on the edge weights. For a graph with N vertices, the difference between the upper and lower bounds is an integer that can vary between 0 and $N-1$, depending on the topology of the graph. This integer represents the dimension of a certain homology group and counts the number of possible eigenvalue crossings from the left to the right half-plane. This homological construction gives a natural splitting of the vector space into a "fixed" subspace, where there cannot be an eigenvalue crossing, and a "free" subspace, where all of the eigenvalue crossings occur. We also show that these bounds are strictly better than those implied by the Gershgorin theorem. Finally we will conclude with some numerical experiments and examples.

1.1. Applications. Several applied problems motivate this question.

(1) Given a graph $\Gamma = (V, E)$ and symmetric coupling functions $\varphi_{ij}(\cdot) = \varphi_{ji}(\cdot)$, define

$$(1.2) \qquad \frac{d}{dt}x_i = F_i(\mathbf{x}) := \omega_i + \sum_{(i,j)\in E} \varphi_{ij}(x_j - x_i).$$

A well-studied example of this type of dynamical system is the Kuramoto oscillator network [1, 2, 3, 4], where $\varphi_{ij}(\cdot) = \gamma_{ij}\sin(\cdot)$. Assume $\mathbf{x}^*$ is a fixed point for (1.2), i.e., $F_i(\mathbf{x}^*) = 0$ for all i. The stability of this point is determined by the index of the Jacobian J, where

$$J_{ij} = \begin{cases} \varphi'_{ij}(x_j^* - x_i^*), & i \neq j, \\ -\sum_k \varphi'_{ik}(x_k^* - x_i^*), & i = j. \end{cases}$$

The Jacobian J is a graph Laplacian of the form (1.1); thus, determining the stability indices for fixed points of (1.2) is equivalent to the problem studied here [5, 6, 7, 8, 9, 10]. When studying this dynamical system, the first object of study is always the stable points. But, for example, if we consider such a system perturbed by small white noise, then understanding the dynamics requires that we identify all of the 1-saddles, i.e., those points which are unstable but with exactly one unstable direction (e.g., see, [11] for the general theory, and [10] for a specific application to this problem). The off-diagonal terms in J can be of either sign, so we need to consider both signs on the weights. For a generic choice of ω_i, $\varphi'_{ij}(x_i^* - x_j^*)$ is nonzero for all $(i,j) \in E$, implying that the graph determining J and the graph defined by the original interactions in (1.2) have the same underlying topology.

(2) Another example is the stability of a fixed point for a neuronal network. For any network system with both "positive" and "negative" feedbacks, the linear stability analysis reduces to the eigenvalue problem (1.1). In practice, synaptic strengths are difficult to measure experimentally and change significantly due to neural plasticity; however, the nature of the interaction (excitatory vs. inhibitory) is anatomical and will not change. In short, we have reliable data on the signs, but not the magnitudes, of synaptic connections. This leads naturally to considering topological bounds, i.e., bounds involving only the signs of the entries of the matrix.

(3) The optimal choice of weights for a consensus algorithm can sometimes have negative weights, as was first pointed out in [12]. Specifically, given $\Gamma = \{\gamma_{ij}\}$, the *linear discrete-time consensus algorithm* is the iterated map $x_i(t+1) := \sum_j \gamma_{ij} x_j(t)$. It is natural to seek to optimize the choice of γ_{ij} (given the underlying graph topology and the constraint that $\mathbf{1}$ be a stable fixed point of the map) to obtain the most rapid convergence to consensus (i.e., make the Lyapunov exponent of the dynamical system as small as possible). If the weights are assumed positive, then this is equivalent to finding the most rapidly mixing Markov chain [13], but it was discovered in [12] that there are network topologies where the optimal choice involves negative weights, i.e., the system converges more rapidly than any Markov chain. This observation inspired a deluge of work (examples include [14, 15, 16, 17]) on this fast convergence problem, and it has been observed that the need for negative weights is typical in many contexts. Of course, one obvious constraint on determining the optimal choice of weights is that $n_+(\Gamma) = 0$ so that we obtain convergence at all. In the work mentioned above, this was always obtained by solving a semidefinite programming problem over the set of all weights associated to a particular unweighted graph Γ. The results in the current work give topological (i.e., weight-independent) bounds for $n_+(\Gamma)$.

(4) In the classic work of [18], matrices of the form (1.1) modeled the interaction of tribal groups within an alliance in Papua New Guinea.[1] In this work the underlying graph has sixteen vertices, representing the different tribal units; edges represent relations between different tribal groups. These relations can be friendly ("rova") or antagonistic ("hina") corresponding to $\gamma_{ij} > 0$ and $\gamma_{ij} < 0$, respectively. Another anthropological example is the SlashDot Zoo, a social network associated with the website SlashDot [19]. On this website participants can label each other as friend or foe, leading to signed edges. In the context of social network models, the index $n_+(\Gamma)$ indicates the tendency of the network to separate into mutually antagonistic subgroups.

(5) Signed graphs and signed Laplacians arise very naturally in knot theory. There is a classical procedure associating a quadratic form to a knot, originally due to Goeritz (see [20, 21]). In Goeritz's construction one considers a planar projection of the knot, with regions of the plane alternately colored black and white. Two regions of given color are connected by an edge if they share a crossing. The weight attached to the edge is $+1$ if the crossing is left-handed and -1 if the crossing is right-handed. The reduced determinant—the product over the nonzero eigenvalues of the associated graph Laplacian—can be shown to be a knot invariant. Related results include a construction by Kauffman of a Tutte polynomial for signed graphs [22] which specializes to known invariants like the Jones [23] and Kauffman bracket polynomials [24]. This construction was further generalized to matroids by Zaslavsky [25]. Signed graphs and their Laplacians have also been studied in the graph theory community independent of their connection to knot theory (see [26, 27, 28] and following also [29]), but the reader should be aware that there are several different generalizations of the Laplacian to the case where negatively weighted edges are allowed. In one variation definition the diagonal entries are taken to be minus the sum of the absolute values of the edge weights. In this case the matrix is typically not zero-sum, and is negative semidefinite. Clearly the spectral questions are different and there is no obvious correspondence between the two. Other authors have also used statistical mechanics models on signed networks for the purposes of community detection, ranking, and social dynamics [30, 31, 32].

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[1]It should be noted that Hage and Harary were concerned with the question of balance in signed graphs, a very different question from the ones we consider here.
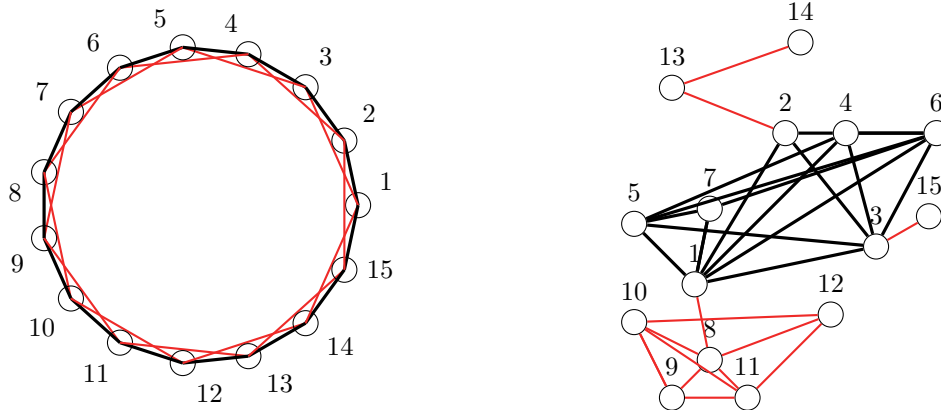
FIG. 1.1. *Two graphs on* 15 *nodes. Thick black lines represent positively weighted edges, and thin red lines represent negatively weighted edges (color available online).*

1.2. Example. To motivate our main questions, we begin with an example of two networks, shown in Figure 1.1. Thin lines represent negative (excitatory) interactions, and thick lines represent positive (inhibitory) interactions. Each of these graphs has fifteen vertices, and fifteen positive and fifteen negative edges. The dynamics, however, are typically very different. Our assertion is that these two examples lay out the marked difference between the way mathematicians tend to view these problems and the way other network scientists view them.

Since symmetry reduction is one of the most common and most powerful techniques in applied mathematics, the mathematical view would be that the dynamics of the first network should be easier to understand, since it has a large group of symmetries (the dihedral group D_{15}). The network science view, however, would claim that the second network is simpler and easier to understand, in that this network can clearly be separated into four functional units.

We show that for the questions asked in this paper, the second point of view is more natural. The first network has the property that a certain homology group has maximal dimension. This implies that the spectrum of the Laplacian is maximally arbitrary: it must have one zero eigenvalue, but the signs of the other fourteen eigenvalues can be chosen arbitrarily with an appropriate choice of weights. In the second network, the analogous homology group is trivial. This implies that the spectrum is rigid: the Laplacian always has eight negative eigenvalues, one zero eigenvalue, and six positive eigenvalues regardless of the choice of weights on the edges.

2. Main theorem. The main theorem, Theorem 2.10, gives tight upper and lower bounds on the number of positive, negative, and zero eigenvalues. In section 2.1 we present some preliminary definitions and lemmas; in section 2.2, we state and prove the main result of the paper giving bounds on the index; in section 2.3 we give a topological description of the main theorem.

2.1. Preliminaries. To state the main theorem of the paper, we first present a few definitions.

DEFINITION 2.1. *Given a graph* Γ, *we define the subgraph* Γ_+ *(resp.,* Γ_-*) to be the subgraph with the same vertex set as* Γ, *together with the edges of positive (resp., negative) weights, i.e.,* $(\Gamma_+)_{ij} = \max(\gamma_{ij}, 0), (\Gamma_-)_{ij} = \min(\gamma_{ij}, 0).$

We also use $\mathcal{L}_+$ (resp., $\mathcal{L}_-$) as shorthand for $\mathcal{L}(\Gamma_+)$ (resp., $\mathcal{L}(\Gamma_-)$) below when the graph Γ is understood.

DEFINITION 2.2. *Given a graph Γ, $\mathcal{ST}(\Gamma)$ denotes the set of all spanning trees of Γ. Also, $\mathcal{ST}_k(\Gamma)$ denotes the set of all spanning trees of Γ having exactly k edges in Γ_-. Note that $\cup_{k=0}^{N-1} ST_k = ST$ and $ST_k \cap ST_{k'} = \emptyset$ if $k \neq k'$.*

NOTATION 2.3. *We let $c(\Gamma)$ denote the number of connected components of a graph Γ. We let $\Gamma_+^{(i)}$, $i \in (1 \ldots c(\Gamma_+))$, denote the ith component of Γ_+, and similarly for $\Gamma_-^{(i)}$. For any weighted graph Γ, we define the three indices $n_0(\Gamma), n_-(\Gamma), n_+(\Gamma)$ as the number of zero, negative, and positive eigenvalues of $\mathcal{L}(\Gamma)$.*

DEFINITION 2.4. *We define the* flexibility *of a weighted graph as the number*

$$\tau(\Gamma) := |V(\Gamma)| - c(\Gamma_-) - c(\Gamma_+) + 1.$$

If $\tau(\Gamma) = 0$, then we say that Γ is rigid. *We show below that the flexibility is always a nonnegative number.*

Remark 2.5. Any graph Laplacian has all row sums equal to zero, so that $\mathcal{L}(\Gamma)\mathbf{1} = \mathbf{0}$, and one necessarily has $n_0(\Gamma) \geq 1$. It is well known [33, 34] that if all the weights $\gamma_{ij} \geq 0$, then the graph Laplacian is negative semidefinite, with $n_0(\Gamma) = c(\Gamma)$ and thus $n_-(\Gamma) = |V(\Gamma)| - c(\Gamma)$. In particular, if Γ is connected with positive weights,

$$n_+(\Gamma) = 0, \quad n_0(\Gamma) = 1, \quad n_-(\Gamma) = |V(\Gamma)| - 1.$$

This is no longer true when the edge weights are allowed to be negative—the Laplacian matrix of a connected graph can have multiple zero and positive eigenvalues.

The following fact will follow from the topological interpretation of the flexibility, but it is convenient to have a self-contained graph-theoretic proof at this time.

LEMMA 2.6. *Every connected signed graph Γ satisfies the inequality*

$$(2.1) \qquad\qquad c(\Gamma_+) + c(\Gamma_-) \leq |V(\Gamma)| + 1.$$

It follows that the flexibility $\tau(\Gamma)$ of any graph is a nonnegative integer.

Proof. First note that if Γ_- contains, as a subgraph, a forest[2] F with ℓ edges, then $c(\Gamma_-) \leq |V| - \ell$. Define a new graph $\widetilde{\Gamma} = (\widetilde{V}, \widetilde{E})$, where $\widetilde{V}$ are the connected components of Γ_+ and $(a,b) \in \widetilde{E}$ iff there is at least one edge in Γ_- from a vertex in component a to a vertex in component b. Since Γ is connected, so is $\widetilde{\Gamma}$. Consider any spanning tree $\mathcal{T}$ of $\widetilde{\Gamma}$. By definition, this contains $c(\Gamma_+) - 1$ edges, since it is a tree on $c(\Gamma_+)$ vertices. Now consider this tree "lifted" into Γ, where, for every edge in $\widetilde{\Gamma}$ of the form $a \leftrightarrow b$, choose one edge in E that connects component a to component b. This is a forest, since by construction it contains no cycles. Therefore $c(\Gamma_-) \leq |V(\Gamma)| - (c(\Gamma_+) - 1)$. $\square$

The main machinery that we will use in this paper is the celebrated Kirchhoff matrix tree theorem [35, 36]. To state it, we first present some notation, which essentially follows that of Tutte [37].

DEFINITION 2.7. *If T is a tree, define $\pi(T)$ to be the product over the edge weights in the tree*

$$(2.2) \qquad\qquad \pi(T) := \prod_{i<j, (i,j)\in E} \gamma_{ij}.$$

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[2] A forest is a graph with no loops, just as tree is a connected graph with no loops—thus a forest is a disjoint collection of trees.

Let Γ *be a weighted graph with* $|V(\Gamma)| = N$, *and let* $\mathcal{L}(\Gamma)$ *be its graph Laplacian. We know that* $\mathcal{L}(\Gamma)$ *has a zero eigenvalue, and therefore* $\det(\mathcal{L}(\Gamma)) = 0$. *Order the* n *eigenvalues of* $\mathcal{L}(\Gamma)$ *so that* $\lambda_1 = 0$; *then define*

$$(2.3) \qquad \mathcal{M}(\Gamma) = \frac{(-1)^{N-1}}{N} \prod_{i=2}^{N} \lambda_i.$$

In other words $\mathcal{M}(\Gamma)$ *is (up to the multiplicative prefactor) the linear term in the characteristic polynomial of the Laplace matrix. Note that* $\mathcal{M}(\Gamma) \neq 0$ *iff* 0 *is a simple eigenvalue of* $\mathcal{L}(\Gamma)$.

With this notation the Kirchhoff matrix tree theorem can be stated as follows.

LEMMA 2.8 (weighted matrix tree theorem). *Let* Γ *be a connected, weighted graph, and let* $\mathcal{ST}(\Gamma)$ *be the set of all spanning trees of* Γ. *Then*

$$(2.4) \qquad \mathcal{M}(\Gamma) = \sum_{T \in \mathcal{ST}(\Gamma)} \pi(T).$$

Remark 2.9. This is Theorem VI.29 in the text of Tutte [37], where a proof is provided. Notice that if all of the edge weights are nonnegative, then the sum in (2.4) is a sum of positive terms. This is an alternate proof that the kernel of a graph Laplacian with positive weights is simple for a connected graph. However, once we allow negative weights, the sum on the right-hand side can have cancellations and will not be sign-definite. This is the major difficulty in understanding the spectral properties of graphs with negative weights.

2.2. Main theorem.

THEOREM 2.10. *Let* Γ *be a connected signed graph, and let* $n_-(\Gamma), n_0(\Gamma), n_+(\Gamma)$ *be the number of negative, zero, and positive eigenvalues, respectively. Then for any choice of weights one has the following inequalities:*

$$(2.5) \qquad \begin{aligned} c(\Gamma_+) - 1 &\le n_+(\Gamma) \le N - c(\Gamma_-), \\ c(\Gamma_-) - 1 &\le n_-(\Gamma) \le N - c(\Gamma_+), \\ 1 &\le n_0(\Gamma) \le N + 2 - c(\Gamma_-) - c(\Gamma_+). \end{aligned}$$

Further, these bounds are tight: for any given graph there exist open sets of weights giving the maximal number of negative eigenvalues

$$n_+(\Gamma) = c(\Gamma_+) - 1, \quad n_-(\Gamma) = N - c(\Gamma_+), \quad n_0(\Gamma) = 1,$$

as well as open sets of weights giving the maximal number of positive eigenvalues

$$n_+(\Gamma) = N - c(\Gamma_-), \quad n_-(\Gamma) = c(\Gamma_-) - 1, \quad n_0(\Gamma) = 1.$$

Remarks 2.11. In each inequality in (2.5), the difference between the upper and lower bounds is the flexibility of the graph $\tau(\Gamma)$. This shows that $\tau(\Gamma)$ counts the number of eigenvalue crossings: there are $c(\Gamma_+) - 1$ eigenvalues which are always negative, $c(\Gamma_-) - 1$ eigenvalues which are always positive, and $\tau(\Gamma) = N + 1 - c(\Gamma_-) - c(\Gamma_+)$ eigenvalues whose signs depend on the choice of weights. For rigid graphs there are no eigenvalue crossings and the index is independent of the choice of weights.

For network models such as (1.2), the theorem implies that a necessary condition for stability of a phase-locked state is the existence of a path in Γ between any two oscillators with $\varphi_{ij}'(x_i - x_j) > 0$ on all edges in the path, and, moreover, that this

condition is sufficient when the negative edges are sufficiently weak. Stated this way, this is precisely Theorem 3.1 of [38]; that theorem can be thought of as the "zero index" case of Theorem 2.10.

The main idea of the proof is to consider a one-parameter family of weighted Laplacian matrices and an associated polynomial, the zeros of which detect eigenvalue crossings. We then show that this polynomial has exactly τ roots in the positive half-line, counting multiplicities, which gives the theorem. We prove this below, but present some definitions and lemmas first.

DEFINITION 2.12. *Given Γ, define a one-parameter family of weighted graphs $\Gamma(t)$ as follows: the weights of $\Gamma(t)$ are related to those of Γ by*

$$\gamma_{ij}(t) := \begin{cases} \gamma_{ij}, & \gamma_{ij} > 0, \\ t \cdot \gamma_{ij}, & \gamma_{ij} < 0, \end{cases}$$

or, more compactly, $\Gamma(t) = \Gamma_+ + t\Gamma_-$. Obviously $\Gamma = \Gamma(1)$, and by linearity $\mathcal{L}(\Gamma(t)) = \mathcal{L}(\Gamma_+) + t\mathcal{L}(\Gamma_-) = \mathcal{L}_+ + t\mathcal{L}_-$.

Remark 2.13. We will use an unusual convention in numbering the eigenvalues of $\mathcal{L}(\Gamma(t))$: define $\lambda_1(t) = 0$ for all t, then number the remaining eigenvalues in decreasing fashion, i.e., $\lambda_2(t) \geq \lambda_3(t) \geq \cdots \geq \lambda_n(t)$, so that $\lambda_2(t)$ is the largest eigenvalue only if there is a positive eigenvalue, etc. (The $\lambda_i(t)$ can of course be chosen to be continuous.) This convention is a bit unusual in that eigenvalues 2 through n are ordered, but 1 can be anywhere in the sequence, though it has the advantage for stating the generalization of (2.3):

$$(2.6) \qquad \mathcal{M}(\Gamma(t)) = \frac{(-1)^{N-1}}{N} \prod_{i=2}^{N} \lambda_i(t).$$

Moreover, when we pair off eigenvalues and eigenvectors below, we will always use the convention that $\lambda_1(t)$ always corresponds to the eigenvalue $\mathbf{1}$; if there are multiple zero eigenvalues, then $\lambda_i(t)$ with $i > 1$ correspond to eigenvectors orthogonal to $\mathbf{1}$.

LEMMA 2.14. *$\mathcal{M}(\Gamma(t))$ is a polynomial in t which takes the form*

$$(2.7) \qquad \mathcal{M}(\Gamma(t)) = \sum_{k=c(\Gamma_+)-1}^{N-c(\Gamma_-)} a_k(-t)^k,$$

where the coefficients a_k are given by

$$a_k = \sum_{T \in \mathcal{ST}_k(\Gamma)} |\pi(T)|.$$

All of the a_k appearing in (2.7) are nonnegative; moreover, the first and last coefficients, $a_{c(\Gamma_+)-1}$ and $a_{N-c(\Gamma_-)}$, are strictly positive.

Proof. Clearly $\mathcal{M}(\Gamma(t))$ is a polynomial in t, since it is given by sums and products of terms, each affine in t. Since there is one power of t associated with each negatively weighted edge and $\mathcal{ST} = \cup_{k=0}^{N-1}\mathcal{ST}_k$, it follows that

$$\mathcal{M}(\Gamma(t)) = \sum_{k=0}^{N-1} a_k(-t)^k,$$

with a_k defined as above. Note that $\mathcal{ST}_k$ is empty for $k < c(\Gamma_+) - 1$ and nonempty for $k = c(\Gamma_+) - 1$. To see this, since Γ_+ has $c(\Gamma_+)$ components, we need at least

$c(\Gamma_+) - 1$ negative edges to connect them. Moreover, there is at least one way to construct a spanning tree with $c(\Gamma_+) - 1$ negative edges: construct a spanning tree on each of the $c(\Gamma_+)$ components of Γ_+ using only positive edges, and then connect them with $c(\Gamma_+) - 1$ negative edges. This can always be done since the graph is assumed to be connected. Since $a_{c(\Gamma_+)-1}$ is a sum over a nonempty set of positive terms, it is positive. The upper bounds follow from the dual argument: reversing the roles of Γ_+ and Γ_- shows that any spanning tree must have at least $c(\Gamma_-) - 1$ positive edges. Since any spanning tree has exactly $N - 1$ edges, there are at most $N - 1 - (c(\Gamma_-) - 1) = N - c(\Gamma_-)$ negative edges.   □

Remark 2.15. $\mathcal{M}(\Gamma(t))$ is strongly reminiscent of other graph polynomials such as the chromatic, rank, and Tutte polynomials, which have definitions in terms of sums over spanning trees [39, 40]. We will show below that $\mathcal{M}(\Gamma(t))$ satisfies a contraction-deletion relation similar to that satisfied by other graph polynomials.

LEMMA 2.16. *The roots of the polynomial $\mathcal{M}(\Gamma(t))$ are real and nonnegative.*

Proof. By definition, a root of the polynomial $\mathcal{M}(\Gamma(t))$ corresponds to a solution of $\mathcal{L}_+ v = -t\mathcal{L}_- v$, where v can be assumed to be orthogonal to $\mathbf{1}$. A standard result [41] in the theory of the generalized symmetric eigenvalue problem (GSEP) is that a sufficient condition for the problem to have all real eigenvalues is that there exist a linear combination of $\mathcal{L}_+$ and $\mathcal{L}_-$ that is strictly positive definite. Since Γ is connected, $\mathcal{L}_- - \mathcal{L}_+$ is strictly positive definite on $\mathbf{1}^\perp$, as it is the negative of the graph Laplacian for a connected graph with positive weights. Therefore $\mathcal{L}_+ v = -t\mathcal{L}_- v$ has only real roots.[3] Finally, since the coefficients alternate in sign, $p(t)$ is obviously nonzero for $t < 0$. Therefore all roots are nonnegative.   □

Remark 2.17. The fact that $\mathcal{M}(\Gamma(t))$ has only real roots implies, via Newton's inequality, that the sequence $\{a_k\}_{k=0}^N$ is log-concave, i.e., $a_{k+1}a_{k-1} \le a_k^2$. In the case with edge weights all ± 1, the coefficients a_k are integers which count[4] the number of spanning trees of Γ having exactly k edges in Γ_-.

LEMMA 2.18. *For $i > 1$, the eigenvalues $\lambda_i(t)$ are nondecreasing functions of t that cross zero transversely, i.e., if $\lambda_i(t) = 0$, then $\lambda_i'(t) > 0$.*

Proof. Since $\mathcal{L}_-$ is positive semidefinite, it follows from the Courant minimax theorem [46, 47] that the $\lambda_i(t)$ are nondecreasing, i.e., $\lambda_i'(t) \ge 0$.

If $\lambda_i(t)$ vanishes at $t = t^*$, then from degenerate perturbation theory [48], we have that $\lambda_i'(t^*)$ is equal to one of the eigenvalues of the matrix $\mathcal{L}_+|_{\ker(\mathcal{L}(t^*))}$. In order for $\mathcal{L}_+|_{\ker(\mathcal{L}(t^*))}$ to have a zero eigenvalue there is necessarily a vector in $\ker(\mathcal{L}_+) \cap \ker(\mathcal{L}_-)$. Such a vector would be constant on components of Γ_+ and Γ_-, and thus, by the connectedness assumption, on all of Γ. The only vectors in $\ker(\mathcal{L}_+) \cap \ker(\mathcal{L}_-)$ are thus multiples of $\mathbf{1}$, and so any other zero eigenvalue must cross through the origin transversely.   □

LEMMA 2.19. *The dimension of the kernel of $\mathcal{L}(\Gamma(t^*))|_{\mathbf{1}^\perp}$ is equal to the multiplicity of t^* as a root of $\mathcal{M}(\Gamma(t))$.*

Proof. The polynomial $\mathcal{M}(\Gamma(t))$ is proportional to $\prod_{i=2}^N \lambda_i(t)$, where, from the above, the $\lambda_i(t)$ have only simple roots. Thus the multiplicity of a root of $\mathcal{M}(\Gamma(t))$ is equal to the number of $\lambda_i(t)$ that vanish there.   □

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[3]For more information on the GSEP, see the review paper of Parlett [41] or Theorem 1 in the paper of Crawford [42].

[4]A large number of other combinatorial sequences share this property. See the review papers of Stanley [43] or Brenti [44] for details. The analogous problem of the log-concavity of the coefficients of the chromatic polynomial, a much more difficult problem, was a long-standing conjecture that has recently been established by Huh [45].

Remark 2.20. Lemma 2.14 shows that $\mathcal{M}(\Gamma(t))$ is a polynomial. It follows from (2.3) that any values of t for which $\lambda_i(t) = 0$ with $i > 1$ must be roots of this polynomial. Lemmas 2.16 and 2.18 state that there is a one-to-one correspondence between the roots of $\mathcal{M}(\Gamma(t))$ and the values of t at which the eigenvalues $\lambda_i(t)$ cross the origin. Because of this, we refer to $\mathcal{M}(\Gamma(t))$ as the *crossing polynomial of* Γ below.

We are now in a position to prove Theorem 2.10.

Proof of Theorem 2.10. As shown above, $\lambda_i(t)$ are nondecreasing functions of t. For $t = 0$ the matrix $\mathcal{L}(t)$ is a graph Laplacian with $c(\Gamma_+)$ components, so it has a $c(\Gamma_+)$-dimensional kernel and an $N - c(\Gamma_+)$-dimensional negative definite subspace. By Lemma 2.18, $c(\Gamma_+) - 1$ of these zero eigenvalues cross transversely into the positive half-line, so for t small and positive the index of $\mathcal{L}(\Gamma(t))$ is $(N - c(\Gamma_+), 1, c(\Gamma_+) - 1)$. By Lemma 2.16, the crossing polynomial has exactly τ roots on the open positive half-line, and each root of the crossing polynomial corresponds to an eigenvalue crossing from the left half-line to the right, so one has exactly τ eigenvalue crossings. This gives

$$c(\Gamma_+) - 1 \leq n_+(\Gamma) \leq N - c(\Gamma_-),$$
$$c(\Gamma_-) - 1 \leq n_-(\Gamma) \leq N - c(\Gamma_+).$$

When t is small and positive the lower holds for $n_+(\Gamma)$ and the upper bound for $n_-(\Gamma)$, and vice versa when t is large. $\square$

2.3. Topological characterization of the flexibility. The quantity τ is a measure of the flexibility of the network dynamics, as it measures the number of eigenvalues that can cross the imaginary axis as the weights of the connections are varied. In this section we give a topological interpretation of τ that allows one to identify important structures in the network.

DEFINITION 2.21. *Let $H_1(\Gamma)$ be the space of cycles in the graph Γ, and consider the map $\partial\colon H_1(\Gamma) \to \mathbb{R}^N$, defined as follows: to $\gamma \in H_1(\Gamma)$ we associate $v^\gamma = \partial(\gamma) \in \mathbb{R}^N$:*

- *Every time the cycle enters vertex i from a negatively weighted edge and exits through a positively weighted edge, v_i^γ increases by one.*
- *Every time the cycle enters vertex i from a positively weighted edge and exits through a negatively weighted edge, v_i^γ decreases by one.*
- *For vertices not on the cycle, or for vertices where the cycle enters and exits through edges of like weights, v_i^γ is zero.*

It is clear that this map is linear, and that the image is an additive group. The kernel of this map is given by (sums of) cycles supported entirely on edges of one type; only cycles with both types of edges give rise to nonzero v^γ—these are called "cycles of mixed type."

LEMMA 2.22. *The flexibility*

$$\tau = N + 1 - c(\Gamma_+) - c(\Gamma_-)$$

is equal to the dimension of $\mathrm{im}(\partial)$.

Proof. This is a straightforward application of the Mayer–Vietoris sequence [49, section 25]. We have the exact sequence

$$H_1(\Gamma) \xrightarrow{\partial} H_0(\Gamma_+ \cap \Gamma_-) \xrightarrow{\alpha} H_0(\Gamma_+) \oplus H_0(\Gamma_-) \xrightarrow{\beta} H_0(\Gamma).$$

Here ∂ is effectively the map defined in Definition 2.21 and the map β is a surjection. Since Γ is connected, $\dim(H_0(\Gamma)) = 1$, $\dim(H_0(\Gamma_+) \oplus H_0(\Gamma_-)) = c(\Gamma_+) + c(\Gamma_-)$, and, since Γ_+ and Γ_- have no common edges, $\dim(H_0(\Gamma_+ \cap \Gamma_-)) = N$. The exactness implies that $\dim(\ker(\beta)) = \dim(\operatorname{im}(\alpha)) = c(\Gamma_+) + c(\Gamma_-) - 1$ and $\dim(\operatorname{im}(\partial)) = \dim(\ker(\alpha)) = N - (c(\Gamma_+) + c(\Gamma_-) - 1) = N + 1 - c(\Gamma_+) - c(\Gamma_-) = \tau$. $\qquad\square$

Remark 2.23. There are two equivalent ways to characterize τ: it is the dimension of $\operatorname{im}(\partial)$ or, equivalently, it is the dimension of $H_1(\Gamma)/\ker(\partial)$, the quotient of the group of cycles by the kernel of ∂. Since $\ker(\partial)$ is the subgroup generated by cycles in $H_1(\Gamma_+)$ and $H_1(\Gamma_-)$ (cycles of one color), what τ counts is the number of fundamentally mixed cycles: cycles in Γ that cannot be decomposed into a sum of cycles, each of which is in Γ_+ and Γ_-.
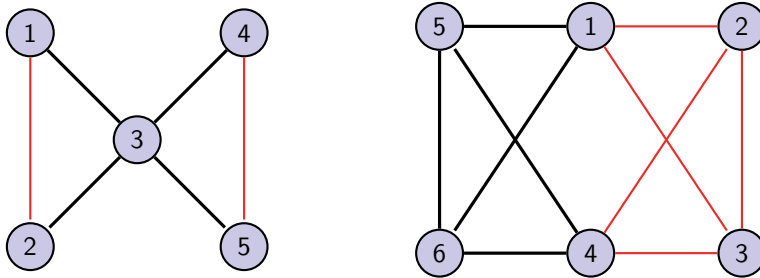

FIG. 2.1. *The two graphs referenced in Example* 2.24.

Example 2.24. The following illustrates the map ∂ for two different graphs. The first graph in Figure 2.1 has two different cycles of mixed type. The first (γ_1) is $3 \to 1 \to 2 \to 3$ and the second (γ_2) is $3 \to 5 \to 4 \to 3$. Thus $\operatorname{im}(\partial)$ consists of all integer linear combinations of $v^{\gamma_1} = (-1, 1, 0, 0, 0)$ and $v^{\gamma_2} = (0, 0, 0, 1, -1)$: all vectors of the form $(-j, j, 0, k, -k)$.

The second graph, however, has effectively only one cycle of mixed type. The equivalence class of mixed cycles consists of a path from vertex one to vertex four through the thick edges plus a path from vertex four to vertex one through the thin edges. The difference between any two cycles in the equivalence class is a sum of cycles through one type of edge. Thus $\operatorname{im}(\partial)$ consists of all integer multiples of $(1, 0, 0, -1, 0, 0)$.

It would be more satisfying to give an explicit bijection between mixed cycles in the graph and the eigenspaces which cross over. This is too much to expect, since the eigenspace depends on the edge weights, while the mixed cycles do not. Nevertheless, we will see that there is a sort of topological stand-in for the eigenspace which nicely characterizes the modes which do not cross over.

DEFINITION 2.25. *Let S be a subspace of R^N (chosen independently of the weights γ_{ij}), and let P_S be the orthogonal projection onto this subspace. S is said to be a subspace of fixed index if the matrix $P_S\mathcal{L}: S \mapsto S$ has the same index regardless of the choice of weights, and is maximal if $\dim(S) = c(\Gamma_+) + c(\Gamma_-) - 1$.*

It follows from the Courant minimax principle that the projection of a symmetric matrix onto a subspace cannot have more positive or more negative eigenvalues than the original matrix. Theorem 2.10 implies that appropriate choices of the weights give as few as $c(\Gamma_+) - 1$ negative eigenvalues and (for a different choice of weights) $c(\Gamma_-) - 1$ positive eigenvalues. It follows that the maximal dimension of any fixed subspace is $c(\Gamma_+) + c(\Gamma_-) - 1$ (since there is always one zero eigenvalue). We conclude this section by showing that there is always a maximal subspace of fixed index that has a natural

topological construction. Effectively, this subspace gives a natural decomposition into modes which do not have an eigenvalue crossing, and modes which do. We now define two complementary subspaces, one of which will be a maximal subspace of fixed index.[5]

DEFINITION 2.26. *Let B be a basis for $\mathrm{im}\,\partial$, and let S_{free} be the subspace of $\mathbb{R}^N$ spanned by B. Let $\Gamma_{+,i}$ be the ith component of Γ_+, and let $v^{i,+}$ be the characteristic vector of $\Gamma_{+,i}$ (i.e., $v_j^{i,+} = 1$ iff $j \in \Gamma_{+,i}$). Similarly, let $v^{i,-}$ be the characteristic vector of Γ_{-i}. Then*

$$S_{\mathrm{fixed}} = \mathrm{Span}(\{v^{i,+}\}_{i=1}^{c(\Gamma_+)}, \{v^{i,-}\}_{i=1}^{c(\Gamma_-)}).$$

LEMMA 2.27. *The subspaces S_{free} and S_{fixed} are orthogonal complements.*

Proof. First we check that $\dim(S_{\mathrm{fixed}}) = c(\Gamma_+) + c(\Gamma_-) - 1$. Note that the set of vectors $\{v^{i,+}\}$ is linearly independent, as is the set of vectors $\{v^{i,-}\}$. However $\{v^{i,+}\} \cup \{v^{i,-}\}$ is not a linearly independent set, since

$$(2.8) \qquad \sum_{i=1}^{c(\Gamma_+)} v^{i,+} = \sum_{i=1}^{c(\Gamma_-)} v^{i,-} = (1,1,1,\ldots,1).$$

The space $\mathrm{Span}(v^{i,+}) \cap \mathrm{Span}(v^{i,-})$ consists of those vectors that are constant on components of Γ_+ and constant on components of Γ_-. Since Γ is connected this means that these vectors must be constant on all of Γ, and are thus proportional to $(1,1,1,\ldots,1)$, and thus (2.8) is the only relation in $\{v^{i,+}\} \cup \{v^{i,-}\}$, and $\dim(S_{\mathrm{fixed}}) = c(\Gamma_+) + c(\Gamma_-) - 1$.

If γ is a mixed cycle, with v^γ the corresponding vector, and v is constant on a component $\Gamma_{+,i}$, then $\langle v^\gamma, v \rangle = 0$. To see this, note that the number of times the cycle γ enters $\Gamma_{+,i}$ equals the number of times it leaves $\Gamma_{+,i}$. When it enters and leaves $\Gamma_{+,i}$ it must do so through a negative edge, giving $+1$ in some entry of v^γ on entering and -1 on exiting. Thus $\langle v^\gamma, v \rangle$ is the sum of an equal number of $+1$ and -1 entries and is zero.

By Lemma 2.22, $\dim(S_{\mathrm{free}}) = \tau$. Since S_{fixed} and S_{free} are orthogonal and have complementary dimensions they are orthogonal complements of one another. □

DEFINITION 2.28. *We define the following subspaces of S_{fixed}:*
- *$S_{\mathrm{fixed}}^+ = \{w | w \in \{\mathrm{Span}(\{v^{i,+}\}_{i=1}^{c(\Gamma_+)}) \text{ and } \langle w, (1,1,\ldots,1)\rangle = 0\}$;*
- *$S_{\mathrm{fixed}}^- = \{w | w \in \{\mathrm{Span}(\{v^{i,-}\}_{i=1}^{c(\Gamma_-)}) \text{ and } \langle w, (1,1,\ldots,1)\rangle = 0\}$;*
- *$S_{\mathrm{fixed}}^0 = \mathrm{Span}((1,1,1,\ldots,1))$.*

LEMMA 2.29. *The subspaces $S_{\mathrm{fixed}}^{+/-/0}$ are $\mathcal{L}$-orthogonal, i.e., if v and w are chosen from two different subspaces of S_{fixed}^+, S_{fixed}^-, S_{fixed}^0, then $\langle v, \mathcal{L}w \rangle = \langle \mathcal{L}v, w \rangle = 0$.*

Proof. $S_{\mathrm{fixed}}^0 \subset \ker(\mathcal{L})$, so if v or $w \in S_{\mathrm{fixed}}^0$, then we are done. Thus assume $v, w \in S_{\mathrm{fixed}}^\pm$. By definition $v^{i',-}$ is constant on the component $\Gamma_{i',-}$ and zero off of this component. Let $\partial\Gamma_{i',-}$ denote the set of vertices which are not in $\Gamma_{i',-}$ but which are connected to it by a positive edge. By direct computation it is easy to see that

$$(\mathcal{L}v^{i',-})_j = \begin{cases} -\sum_{k \in \partial\Gamma_{i',-}} \gamma_{k,j}, & j \in \Gamma_{i',-}, \\ \sum_{k \in \Gamma_{i',-}} \gamma_{k,j}, & j \in \partial\Gamma_{i',-}. \end{cases}$$

---

[5]The construction of these subspaces is similar in spirit to the construction of the cut-space and cycle-space from algebraic graph theory, although the cut-space and cycle-space are usually defined as subspaces of the vector space over the edge set, not the vector space over the vertices. For a nice description of the cut- and cycle-spaces and some applications to the theory of electrical networks, see the paper of Bryant [50].

Thus $\mathcal{L}v^{i',-}$ is a linear combination of vectors that are 1 on some vertex in $\Gamma_{i',-}$ and -1 on some vertex in $\partial\Gamma_{i',-}$. Each of these vectors is necessarily orthogonal to $v^{i,+}$ since $v^{i,+}$ is constant on components of Γ_+. $\quad\square$

We need one more lemma, the well-known Sylvester theorem [51].

LEMMA 2.30 (Sylvester's law of inertia). *If A is a square matrix and S a square nonsingular matrix, then A and $B = S^{\intercal}AS$ have the same index. The matrices A and B are said to be Sylvester equivalent.*

PROPOSITION 2.31. *The subspace S_{fixed} is a maximal subspace of fixed index: The operator $P_{S_{\text{fixed}}}\mathcal{L}$ (considered as an operator from S_{fixed} to S_{fixed}) has index*

$$\text{ind}(P_{S_{\text{fixed}}}\mathcal{L}) = (c(\Gamma_+) - 1, 1, c(\Gamma_-) - 1),$$

independent of the choice of edge weights.

Proof. Using Lemma 2.29 it follows that $P_{S_{\text{fixed}}}\mathcal{L}$ has the following block structure:

$$P_{S_{\text{fixed}}}\mathcal{L}P_{S_{\text{fixed}}} = \mathcal{L}_- \oplus \mathcal{L}_0 \oplus \mathcal{L}_+,$$

where $\mathcal{L}_+ = P_{S_{\text{fixed}}^+}\mathcal{L}$, $\mathcal{L}_- = P_{S_{\text{fixed}}^-}\mathcal{L}$, and $\mathcal{L}_0 = P_{S_{\text{fixed}}^0}\mathcal{L} = 0$. Consider the block $\mathcal{L}_- \oplus \mathcal{L}_0$. This arises by orthogonal projection of $\mathcal{L}$ onto the set of vectors constant on Γ_-, and thus this matrix is Sylvester equivalent to the following graph Laplacian: one contracts on the negative edges, giving a graph with vertices corresponding to the components of Γ_-. This is a standard graph Laplacian on a connected graph with $c(\Gamma_-)$ vertices and thus has $c(\Gamma_-) - 1$ negative eigenvalues and one zero eigenvalue. Similarly $\mathcal{L}_0 \oplus \mathcal{L}_+$ is Sylvester equivalent to the negative of the graph Laplacian given by contracting on the positive edges. This has, by the same argument, $c(\Gamma_+) - 1$ positive eigenvalues and one zero eigenvalue. The zero eigenvalue is obviously counted twice in this argument, giving the result. $\quad\square$

3. Refinements. In this section, we present some refinements of Theorem 2.10. In section 3.1 we remind the readers of the deletion–contraction theorem and specialize its statement to this context. In section 3.2 we define the notion of bifurcation for a signed graph and present a computation of the bifurcation value for a family of graphs. In section 3.3 we discuss the asymptotics of the individual eigenvalues of $\mathcal{L}(\Gamma(t))$ in the limits $t \to 0, \infty$.

3.1. Deletion–contraction theorem.

DEFINITION 3.1. *Let $\Gamma = (V, E)$ be a weighted multigraph (loops and multiple edges allowed), and let $e \in E(\Gamma)$ be an edge.*
  - *We denote by $\Gamma \setminus e$ the graph obtained by removing edge e.*
  - *If $e = (v_1, v_2)$ is an edge with $v_1 \neq v_2$, we define $\Gamma.e$ as follows: identify the two vertices v_1 and v_2 as a single vertex v^*; for any vertex w connected to v_1 or v_2, we define the new edge weight $\gamma_{v^*,w} = \gamma_{v_1,w} + \gamma_{v_2,w}$.*

THEOREM 3.2 (deletion–contraction theorem). *Let Γ be a weighted multigraph with $e \in E(\Gamma)$. Then $\mathcal{M}(\Gamma)$ can be computed by applying the following rules:*
  - *If e is not a loop, then $\mathcal{M}(\Gamma) = \mathcal{M}(\Gamma \setminus e) + \gamma_e \mathcal{M}(\Gamma.e)$.*
  - *If e is a loop, then $\mathcal{M}(\Gamma) = \mathcal{M}(\Gamma \setminus e)$.*
  - *If Γ is disconnected, then $\mathcal{M}(\Gamma) = 0$.*

Proof. The deletion-contraction recursion is well known; see Chapter 13.2 of the text of Godsil and Royle [52] for one proof. $\quad\square$

Example 3.3. As an example, let us consider the graphs given in Figure 3.1 (here, think of the t as a symbol to separate out the terms containing the special edge). First, notice that Γ has three spanning trees not containing e, of weights $8, 12, 24$,
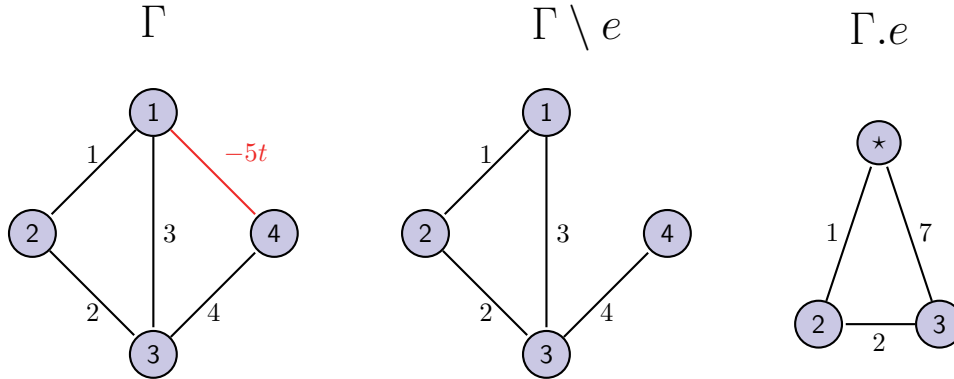
FIG. 3.1. *The graphs* Γ, $\Gamma \setminus e$, *and* $\Gamma.e$, *where* e *is chosen to be the edge with negative weight.*

and five spanning trees containing e, of weights $-10t, -15t, -20t, -30t, -40t$. By Lemma 2.8, $\mathcal{M}(\Gamma) = 44 - 115t$. Every spanning tree of Γ that does not contain e is also a spanning tree of $\Gamma \setminus e$, and thus $\mathcal{M}(\Gamma \setminus e) = 44$. On the other hand, $\Gamma.e$ has three spanning trees of weights $2, 7, 14$. Therefore $\mathcal{M}(\Gamma.e) = 23$. We see that $\mathcal{M}(\Gamma) = 44 - 115t = 44 + (-5t)(23) = \mathcal{M}(\Gamma \setminus e) - \gamma_e \mathcal{M}(\Gamma.e)$.

3.2. Bifurcations and loss of stability. In the context of dynamical systems, an important distinction to be made is that between graphs for which $n_+(\mathcal{L}(\Gamma)) = 0$ and those for which $n_+(\mathcal{L}(\Gamma)) > 0$; the former are called *stable* and the latter *unstable*. This notion comes from the fact that if we consider the ordinary differential equation $\dot{x} = \mathcal{L}(G)x$, then the origin is stable to perturbations only if $n_+(\mathcal{L}(G)) = 0$; if not, then perturbations move away from the origin at an exponential rate.

LEMMA 3.4. *Define* $t^\star(\Gamma) \in [0, \infty]$ *by*

(3.1)
$$t^\star(\Gamma) := \sup_{t \geq 0} n_+(\mathcal{L}(\Gamma(t))) = 0.$$

For $t \leq t^\star(\Gamma)$, *the Laplacian is stable, and for* $t > t^\star(\Gamma)$, *the Laplacian is unstable—in short, it undergoes a* dynamical bifurcation *at* $t^*(\Gamma)$. *Moreover, there is a nontrivial bifurcation—i.e.,* $t^\star(\Gamma) \in (0, \infty)$—*iff* Γ_+ *is connected and* Γ_- *nonempty.*

Proof. From Lemma 2.18, $n_+(\mathcal{L}(\Gamma(t)))$ is a nondecreasing function of t, and this establishes the bifurcation statement. Further, it follows from Theorem 2.10 that $t^\star(\Gamma) > 0$ iff Γ_+ is connected, and $t^*(\Gamma) < \infty$ iff $\Gamma_- \neq \emptyset$.   □

THEOREM 3.5. $t^\star(\cdot)$ *is a monotone function on graphs: if we add a positive edge to* Γ, *or increase the weight on any positive edge of* Γ, *then* $t^\star$ *increases or stays the same. Conversely, if we add a negative edge to* Γ, *or increase the magnitude of the weight of any negative edge of* Γ, *then* $t^\star$ *decreases or stays the same.*

Proof. This proof uses a relatively standard technique of spectral graph theory [34, section 1.4], but retooled for the current context. If x is any vector defined on the vertices of the graph, then $\langle x, \mathcal{L}(\Gamma)x \rangle = -\frac{1}{2} \sum_{i,j} \gamma_{ij}(x_i - x_j)^2$. Let $\widetilde{\Gamma}$ be a graph obtained from Γ by increasing one or more positive edges, or by adding a positive edge. Then $\langle x, \mathcal{L}(\Gamma)x \rangle \geq \langle x, \mathcal{L}(\widetilde{\Gamma})x \rangle$, and thus by the Courant minimax theorem [46, 47], the eigenvalues of $\mathcal{L}(\widetilde{\Gamma})$ are less than or equal to those of $\mathcal{L}(\Gamma)$. A similar statement holds, mutatis mutandis, for $\mathcal{L}(\widetilde{\Gamma}(t))$ and $\mathcal{L}(\Gamma(t))$ for any $t \geq 0$. From this it follows that $t^\star(\widetilde{\Gamma}) \geq t^\star(\Gamma)$. The same argument holds, in reverse, if we make edges more negative.   □

In short, Theorem 3.5 says that the bifurcation value respects the canonical partial ordering on graphs, i.e., is a monotone function of the graph in the standard sense used in graph theory [53, 54]. As such we can obtain bounds in a straightforward manner. In particular, we can easily get global bounds for all graphs with one negative edge and with unit magnitude weights.

PROPOSITION 3.6. *Let Γ be a graph with $\gamma_{ij} = \pm 1$, a single negative edge, and assume that Γ_+ is connected. Then $t^\star(\Gamma) \in \left[\frac{1}{N-1}, \frac{N-2}{2}\right]$. These extremes are attained by the ring graph and the complete graph, respectively.*

Proof. Every graph with one negative edge is a subgraph of the complete graph K_N with one negative edge and contains a subgraph with N edges, one of which is negative. From this and Theorem 3.5, we need consider only these extreme cases. We will denote the negative edge by e. Since Γ_+ is connected, and $|\Gamma_-| = 1$, Theorem 3.2 implies that $t^\star = \mathcal{M}(\Gamma \setminus e)/\mathcal{M}(\Gamma.e) = \mathcal{M}(\Gamma_+)/\mathcal{M}(\Gamma.e)$.

We first consider a graph with N edges, one of which is negative. Thus Γ_+ is a tree and has exactly one spanning tree, so $\mathcal{M}(\Gamma \setminus e) = \mathcal{M}(\Gamma_+) = 1$. Contracting on e gives a graph with $N - 1$ vertices and either $N - 1$ or $N - 2$ edges (the latter occurs if the negative edge connects two leaves with a common parent). It is not hard to see that in the latter case we obtain a tree with one edge with weight 2, and in this case $\mathcal{M}(\Gamma.e) = 2$. In the former case, we obtain a graph with a cycle of length $3 \leq m \leq N - 1$ and unit edge weights, and it is not hard to see that in this case $\mathcal{M}(\Gamma.e) = m$. In this case, $t^\star = 1/m$, and minimizing over all such graphs gives us that $t^\star \geq 1/(N-1)$.

Now consider K_N with one negative edge. Deleting and contracting gives OG_N, the complete graph minus one edge, and DR_{N-1}, the complete graph on $N-1$ nodes with a single distinguished vertex, all of whose edges have weight two. Using standard counting arguments, $\mathcal{M}(OG_N) = (N-2)N^{N-3}$ and $\mathcal{M}(DR_N) = 2(N+1)^{N-2}$, giving $t^\star = \mathcal{M}(OG_N)/\mathcal{M}(DR_{N-1}) = ((N-2)N^{N-3})/(2N^{N-3}) = \frac{N-2}{2}$. (This also follows from a computation of the eigenvalues of $K_N(t)$: $(0, \{-N\}^{(N-2)}, 2t - (N-2))$.) $\quad\square$

One could perform a similar analysis of bounds on the bifurcation parameter for graphs with more than one negative edge, or with various weights, using similar techniques. One notes that the bifurcation parameter scales quite differently for sparse and dense graphs, in a manner very similar to the way the spectral gap scales in the standard theory [34].

3.3. Detailed eigenvalue asymptotics. We now consider the spectrum in the limits where the strength of the negative edges is much weaker or much stronger than the strength of the positive edges. Specifically we consider the one-parameter family of graph Laplacians $L(\Gamma(t))$ in the limits $t \to 0^+$ and $t \to \infty$. The spectrum splits naturally into two parts, which correspond to the eigenvalues of graph Laplacians on the deleted and contracted graphs. This is an analogue on the level of the spectrum of the contraction-deletion algorithm for computing the crossing polynomial: the crossing polynomial is given by the sum of the crossing polynomials for the contracted and deleted graphs, while the spectrum is given (asymptotically!) by the union of the deleted and contracted graphs.

THEOREM 3.7. *If t is sufficiently large, $\mathcal{L}(t)$ has exactly $N - c(\Gamma_-)$ positive eigenvalues, $c(\Gamma_-) - 1$ negative eigenvalues, and one zero eigenvalue. Numbering the eigenvalues in decreasing order, the $N - c(\Gamma_-)$ positive eigenvalues are given by*

$$\lambda_i(\mathcal{L}_\Gamma) = t \cdot \lambda_i(\mathcal{L}_{\Gamma.-}) + O(1), \qquad i \in \{1, \ldots, N - c(\Gamma_-)\}.$$

The $N - c(\Gamma_-)$ negative eigenvalues are given to leading order by

$$\lambda_i(\mathcal{L}_\Gamma) = \tilde{\lambda}_i(\mathcal{L}_{\Gamma.-}) + o(1), \qquad i \in \{N - c(\Gamma_-) + 2, \ldots, N\},$$

where $\tilde{\lambda}_j(\mathcal{L}_{\Gamma.-})$ are solutions to $\mathcal{L}_{\Gamma.-}v = \lambda S v$, where $L_{\Gamma.-}$ is the graph Laplacian formed by contracting on the negative edges and S the contracted inner product: the diagonal matrix with entries $S_{ii} = |V(\Gamma_{-,i})|$.

Proof. The full graph Laplacian can be written $\mathcal{L}(t) = t\left(\mathcal{L}(\Gamma_-) + t^{-1}\mathcal{L}(\Gamma_+)\right)$, so it suffices to understand the eigenvalues of $\mathcal{L}(\Gamma_-) + t^{-1}\mathcal{L}(\Gamma_+)$ for t large. The spectrum of $\mathcal{L}(\Gamma_-)$ consists of $c(\Gamma_-)$ zero eigenvalues and $N - c(\Gamma_-)$ positive eigenvalues. The positive eigenvalues give the linearly growing eigenvalues in the asymptotic above.

To understand how the $c(\Gamma_-)$-dimensional kernel breaks under perturbation, we must do a degenerate perturbation theory calculation. Following [48], to leading order the eigenvalues are given by the eigenvalues of $P_{\ker(\mathcal{L}_-)}L_{\Gamma_+}P_{\ker(\mathcal{L}_-)}$, where $P_{\ker(\mathcal{L}_-)}$ is the orthogonal projection onto the kernel of $\mathcal{L}_-$. $\ker(\mathcal{L}_-)$ consists of vectors that are constant on components of $\mathcal{L}_-$. We identify a vector $w \in \ker(\mathcal{L}_-)$ with $\tilde{w} \in \mathbb{R}^{c(\Gamma_-)}$ by the following rule: if $w_i = \alpha$ for all vertices in component $\Gamma_{-,j}$, then $\tilde{w}_j = \alpha$. Under this identification the natural inner product on $\mathbb{R}^N$ maps to the inner product

$$\langle \tilde{v}, \tilde{w} \rangle = \sum_{i=1}^{c(\Gamma_-)} |V(\Gamma_{-,i})|\tilde{v}_i\tilde{w}_j.$$

There is one entry per component of Γ_-, and the inner product is diagonal with weights given by the number of vertices in the corresponding component of Γ_i. It is straightforward to see that $P_{\ker(\mathcal{L}_-)}L_{\Gamma_+}P_{\ker(\mathcal{L}_-)}$ is exactly the Laplace matrix obtained by contracting on the negative edges of the graph, completing the proof. $\quad\square$

Remark 3.8. The previous theorem can be written in the following compact way:

$$\mathrm{Spec}^\star(\mathcal{L}(t)) \approx \begin{cases} \mathrm{Spec}^\star(\Gamma_-) \cup \mathrm{Spec}^\star(\Gamma.-), & t \to +\infty, \\ \mathrm{Spec}^\star(\Gamma_+) \cup \mathrm{Spec}^\star(\Gamma.+), & t \to 0^+, \end{cases}$$

where $\mathrm{Spec}^\star(A)$ is defined to be the eigenvalues of the matrix A restricted to the subspace $(1,1,1,\ldots,1)^\perp$ of mean zero vectors, and with the understanding that the eigenvalues for the contracted graph are taken with respect to the natural inner product S. Thus there is an (approximate) contraction–deletion relation for the spectrum analogous to the contraction–deletion relation satisfied by the crossing polynomial.

Example 3.9. We consider the graph in Figure 3.2, where all positive (thick) edges are weighted $+1$ and all negative (thin) edges weighted -1.

The subgraph Γ_- consisting of only the negative links has three components. Component A consists of vertices 1, 2, 3; component B consists of vertices 4 and 5; component C consists of vertices 6, 7, 8, 9. There are $N - c(\Gamma_-) = 9 - 3 = 6$ nonzero eigenvalues corresponding to the graph Laplacian associated with the negative edges. The nonzero eigenvalues associated with component A are 1 and 3, with component B is 2, and with component C are 4, 2, and 2. This gives six eigenvalues that grow linearly:

$$\lambda_9 \approx 4t + O(1), \quad \lambda_8 \approx 3t + O(1), \quad \lambda_7 \approx 2t + O(1),$$
$$\lambda_6 \approx 2t + O(1), \quad \lambda_5 \approx 2t + O(1), \quad \lambda_4 \approx 1t + O(1).$$

Contracting the thin edges gives the three cycle, with one vertex corresponding to each component of Γ_-. There are two edges between each component in the original
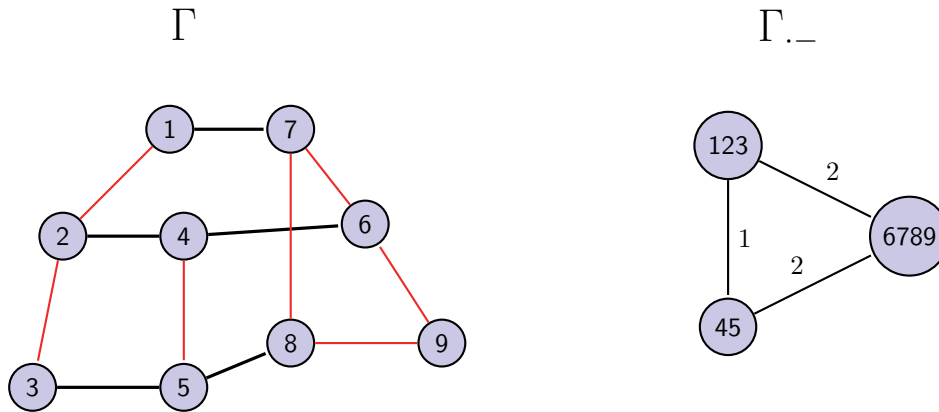
FIG. 3.2. *The graph Γ and the graph $\Gamma._-$. In Γ we are taking all weights to be ± 1.*
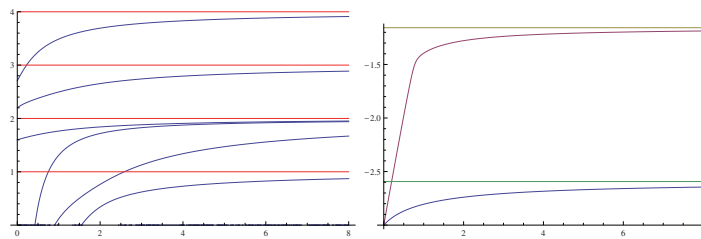

FIG. 3.3. *Eigenvalues of $\Gamma(t)$ as a function of t, where $\Gamma(t)$ is defined in Example* 3.9.

graph, so these edges are reweighted accordingly. The norm is contracted as well, and can be written as $\|v\|^2 = v^\mathsf{T} S v$, where S is the matrix

$$S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

The diagonal entries reflect the fact that the components have three, two, and four vertices, respectively. Thus the eigenvalue problem becomes

$$\begin{pmatrix} -3 & 2 & 1 \\ 2 & -4 & 2 \\ 1 & 2 & -3 \end{pmatrix} \vec{v} = \lambda \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \vec{v},$$

giving the negative eigenvalues as

$$\lambda_2(t) \approx \frac{1}{8}\left(\sqrt{33} - 15\right) + O(1/t), \quad \lambda_1(t) \approx \frac{1}{8}\left(-\sqrt{33} - 15\right) + O(1/t).$$

The flexibility is equal to $\tau(\Gamma) = 10 - 3 - 4 = 3$, so there are three eigenvalue crossings. One can compute that the crossing polynomial is given by $P_\Gamma(t) = 171t^3 - 702t^4 + 828t^5 - 288t^6$. The nonzero roots occur at $t \approx .43, t \approx .90, t \approx 1.55$.

Numerical results are shown in Figure 3.3. The first plot shows a graph of $\lambda_i(t)/t$ for $i = 3, \ldots, 9$ and $t \in (0, 8)$. Clearly the scaled eigenvalues converge to the correct values. The second plot shows $\lambda_i(t)$ for $i = 1, 2$ and $t \in (0, 8)$. Again it is clear that they are converging to $-\frac{1}{8}(\sqrt{33} + 15)$ and $-\frac{1}{8}(15 - \sqrt{33})$, respectively. One can also see that there are three eigenvalue crossings at the correct t values.

3.4. Comparison to the Gershgorin theorem. A widely used tool for estimating the locations of the eigenvalues of a matrix is the Gershgorin circle theorem.

THEOREM 3.10 (Gershgorin). *Given a matrix $M = \{M_{ij}\}_{i,j=1}^{N}$, we define*

$$D_i = \left\{ z : |z - M_{ii}| \leq \sum_{j \neq i} |M_{ij}| \right\}, \qquad i = 1, \ldots, N.$$

Then the eigenvalues of M lie in the union of the disks, $\mathrm{Spec}(M) \subseteq \cup_{i=1}^{N} D_i$.

Since the disks are defined using edge weights, one might expect the Gershgorin theorem to give more information on the signs of the eigenvalues than the purely topological considerations. In fact, we will show that for a nontrivial signed Laplacian, the Gershgorin theorem always gives results that are strictly worse than those given by Theorem 2.10, in terms of counting signs. (Of course, the Gershgorin theorem gives more than just sign information.) If all of the edges emanating from a vertex are of the same sign, then the Gershgorin disc lies entirely in one closed half-plane; if they are all positive, in the closed left half-plane; and if negative, the closed right half-plane. The origin lies in the interior of the disk and the eigenvalue has undermined sign iff a vertex has edges of both signs. The main observation in this section is that the number of such discs is always strictly larger than τ.

PROPOSITION 3.11. *Suppose Γ is connected and contains edges of both signs. If the origin lies in the interior of n Gershgorin discs, then $n \geq \tau + 1$.*

Proof. By the construction of section 2.3, to each cycle we associate a vector that has nonzero entries only on those vertices where the cycle enters a vertex on an edge of one sign and leaves on an edge of a different sign. These vectors can only have nonzero entries in vertices which have both types of edge, so they obviously lie in a subspace isomorphic to $\mathbb{R}^n$. These vectors are necessarily orthogonal to $(1, 1, 1, \ldots, 1)$, so there can be at most $n - 1$ linearly independent ones and $\tau \leq n - 1$. $\square$

Example 3.12. Note that $n = \tau + 1$ can be achieved—one example is when $c(\Gamma_+) = 1$ and $c(\Gamma_-) = 1$, when $\tau = N - 1$ and $n = N$. It can also happen that n is much larger than τ. Consider, for example, a cyclic graph with an even number of edges of alternating sign. In this case $n = N$, since all vertices have edges of both type; $\tau = 1$, since there is only one linearly independent loop.

The graph depicted in Figure 3.2 has a flexibility of $\tau = 9 + 1 - 4 - 3 = 3$. This graph has eight vertices which have edges of both signs, and thus eight Gershgorin discs that contain the origin in the interior. In the second graph in Figure 1.1 there are three vertices that have edges of both types, and thus three Gershgorin discs that contain the origin as an interior point. The flexibility of this graph, however, is zero so that the number of positive, negative, and zero eigenvalues is fixed and does not vary with the edge weights.

4. Applications and numerical computations.

4.1. Random graphs and bifurcations. Recall the definition of $t^\star$ from (3.1); it is a map from the set of finite graphs to $[0, \infty]$. For any ensemble $\mathcal{G}$ of graphs and a probability measure $\mathbb{P}$ on $\mathcal{G}$, this induces a random variable $T : \mathcal{G} \to [0, \infty]$. We will consider three signed generalizations of standard random graph models. The first two are signed generalizations of the uniform graph models of Erdős and Rényi [55, 56], and the third is a generalization of the small-world model of Watts and Strogatz [57].

1. $\mathcal{G}_{\mathsf{UP}}(N, p_+, p_-)$: Define $Z \sim \mathcal{G}_{\mathsf{UP}}(N, p_+, p_-)$ as follows: choose two unweighted random graphs X, Y with independent edges, and $\mathbb{P}(X_{ij} = 1) = p_+$ and $\mathbb{P}(Y_{ij} = 1) = p_-$. Then $Z = X - Y$.

2. $\mathcal{G}_{\mathsf{FE}}(N, M, p_+)$: Choose M edges uniformly in the set of possible edges (more specifically, enumerate all of the edges in the list $[N(N-1)/2]$ in some manner, then choose a uniform random subset of length M inside this list). Each edge is then given the weight $+1$ with probability p_+ and -1 with probability $1 - p_+$, independently.

3. $\mathcal{G}_{\mathsf{SW}}(N, K, p_+, p_{\mathsf{rewire}})$: Start with a locally connected ring where $(i, i+j) \in E$ for all $i \in [N], j \in [K]$, i.e., each vertex is connected to the next K vertices in the enumeration. We then "rewire" each edge with probability p_{rewire}, by which we mean with probability $1 - p_{\mathsf{rewire}}$ we leave the edge where it is, and with probability p_{rewire} we move the edge to (k, l) chosen uniformly in $[N]^2$.
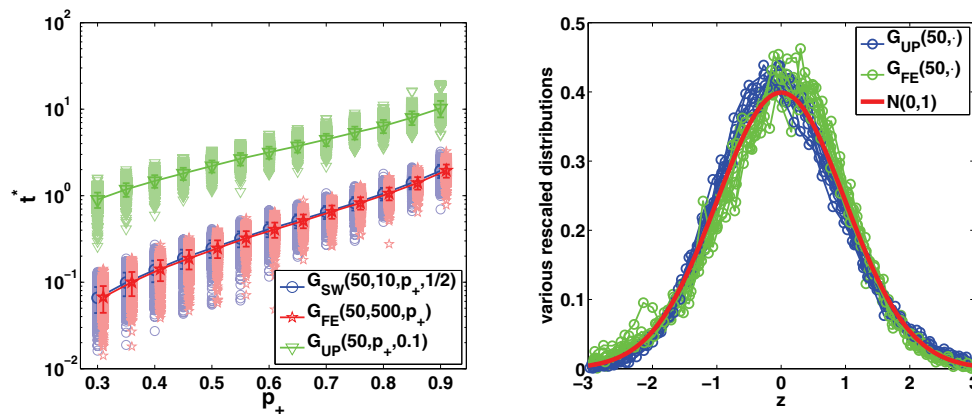

Fig. 4.1. *The left panel is a plot of the mean, standard deviation, and individual realizations of $t^\star$ for different random matrix ensembles for various parameters; the right panel contains rescaled distributions for $t^\star$ (see text for details).*

Some properties of these models are apparent. In $\mathcal{G}_{\mathsf{UP}}(N, p_+, p_-)$, the number of edges is random, whereas in $\mathcal{G}_{\mathsf{FE}}(N, M, p_+)$ it is fixed at M. Moreover, the small-world ensemble is chosen in such a way that $\mathcal{G}_{\mathsf{SW}}(N, K, p_+, 1)$ is the same ensemble as $\mathcal{G}_{\mathsf{FE}}(N, NK, p_+)$. We performed a series of numerical experiments on these random ensembles and plot the results in Figure 4.1. In the left frame, we plot the mean, standard deviation, and the realizations for the three ensembles $\mathcal{G}_{\mathsf{UP}}(50, p_+, 0.1)$ (i.e., we fixed $p_- = 0.1$, line with triangles), $\mathcal{G}_{\mathsf{SW}}(50, 10, p_+, 1/2)$ (i.e., fixed $p_{\mathsf{rewire}} = 1/2$, line with circles), and $\mathcal{G}_{\mathsf{FE}}(50, 500, p_+)$ (line with stars), all as a function of p_+. We make several observations: first, the mean of $t^\star$ has very close to exponential dependence on p_+ (note the linear-log scale); second, p_{rewire} does not seem crucial, since the star and circle curves match quite well, in this parameter regime. In the right frame, we plot the (mean zero, variance one) rescaled distributions for $\mathcal{G}_{\mathsf{UP}}(50, p_+)$ and $\mathcal{G}_{\mathsf{FE}}(50, p_+)$ for all of the values plotted in the left frame with $p_+ \geq 0.4$ (i.e., we threw out the first two ensembles), and we see that these ensembles look quite close to normal (the star curve is the actual normal distribution for comparison). There is an interesting deviation, in that both ensembles seem to be slightly more concentrated near zero than is to be expected from normality, and, moreover, $\mathcal{G}_{\mathsf{UP}}$ has a small leftward skew, while $\mathcal{G}_{\mathsf{FE}}$ has a small rightward skew. The code used to perform all of these computations was written in MATLAB and is available at http://www.math.uiuc.edu/~rdeville/code/paper-SpectralTheorySIAMJAM/.

4.2. Feuds in social networks. We consider two datasets from social networks and compute the flexibility of the graphs and, in one case, the bifurcations. Our computations were facilitated by the `matlab_bgl` library [58].

The first dataset we consider is from Read [59] and represents sympathetic and antagonistic relationships among sixteen subtribes of the Gahuku–Gama people in the highlands of New Guinea. This has become a somewhat popular dataset to analyze; see the pioneering work of Hage and Harary [18] and the recent work of Kunegis et al. [60]. Warfare was an important social interaction within this society, and the edges in this graph represent traditional relationships between subtribes. The positive (hina) edges represent closely allied subtribes. Warfare between these subtribes occurs but is limited, and is often resolved by payment of blood money or other concessions. The negative (rova) edges represent relations between subtribes that are traditionally more antagonistic. The network of these interactions is plotted in the top frame of Figure 4.2. The graph has $c(\Gamma_+) = 2$ and $c(\Gamma_-) = 3$, giving $\tau = 12$. A symbolic computation using *Mathematica* gives the crossing polynomial as

$$
\begin{aligned}
\mathcal{M}(t) = {} & -45432223t^{13} + 657635624t^{12} - 4187415940t^{11} + 15505043366t^{10} \\
& - 37159886129t^9 + 60647687776t^8 - 68960526571t^7 + 54844706645t^6 \\
& - 30103762121t^5 + 11015925656t^4 - 2508107376t^3 + 308319872t^2 - 14192640t.
\end{aligned}
$$

As $t \to 0+$, we have one positive eigenvalue, and as t increases we move up to 13. We can compute numerically that the first two eigenvalue crossings occur at $t \approx 0.11$ and $t \approx 0.42$. In the bottom frame of Figure 4.2, we plot three vectors: the solid line plots the eigenvector corresponding to the positive eigenvalue at $t = 0.01$, and the dotted (resp., dashed) line plots the eigenvector corresponding to the eigenvalue that crosses at $t \approx 0.11$ (resp., $t \approx 0.42$). The eigenvectors are normalized to have unit ℓ^2 norm.

The first positive eigenvector (solid line) picks out the two connected components of Γ_+, namely, (A) a collection of four tribes (Gaveve, Kotuni, Gama, Nagamidzhuha), all of whom have friendly relations, and (B) the remaining twelve tribes (Seu've, Kohika, Notohana, Uheto, Nagamiza, Masilakidzuha, Asarodzuha, Gahuku, Gehamo, Ove, Ukudzuha, and Alikadzuha), all of whom are connected by at least one chain of sympathetic relationships. This is not surprising, and can be read off from the original topology of the graph. The second unstable eigenvector (dotted line) is roughly zero on (A), but separates (B) into two components. By reexamining the network, we see that component (B) can be naturally separated by the removal of the Masilakidzuha tribe, and this is exactly the fracture that the eigenvector predicts. Finally, the third eigenvector (dashed line) shows significant fractures inside each of these three main groups. The interpretation here is that t could serve as a global "uneasiness" index, and we see that at all times we will have two main subgroups that are feuding, but if there is a significant amount of uneasiness there is a natural splitting into three mutually antagonistic subgroups. Moreover, this knowledge of these three groups allow us to plot the interaction graph in such a way that one can read off these relationships almost by eye, which is, in fact, what we have done in the left frame of Figure 4.2.

The second dataset is from the Slashdot Zoo. Slashdot is a user-run website where links are submitted and commented on by the userbase. Users can vote up or down on the links and the comments by other users. A significant amount of discussion occurs on this website, both friendly and fractious, with many relationships developing between users. Users are able to tag other users in the database as "fan" or "foe." In the Slashdot Zoo [61] database, the edge (i, j) is given weight $+1$ (resp., -1) if
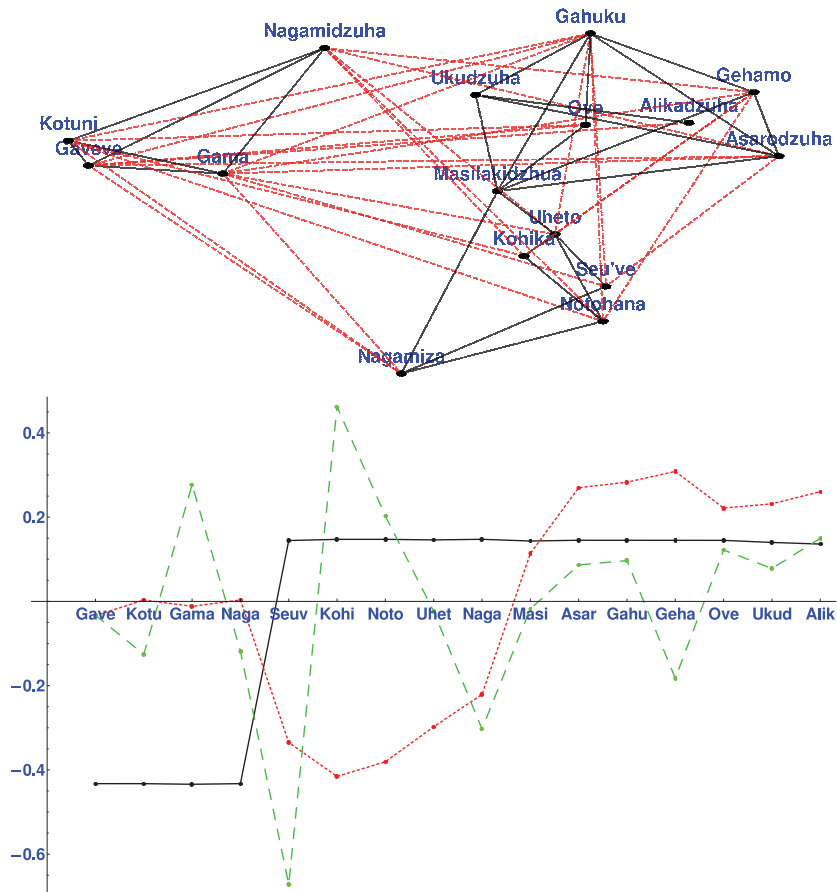
FIG. 4.2. *The PNG dataset from* [59]. *In the top frame we plot the interaction network among the subtribes. In the bottom frame, we plot the first three unstable vectors as described in the text.*

user i is a fan (resp., foe) of user j. This dataset contained connection data on 82,144 users, with 549,202 edges in the graph. This network is not a priori symmetric, since the fan/foe operations have directionality, so we imposed symmetry. Given users i and j, if i and j are both fans of each other, or i is a fan to j and j is neutral to i, then we placed a $+1$ in edge (i, j)—and similarly for foes. In short, we simply extended unidirectional relationships to be bidirectional as long as the other direction was neutral. The only nonobvious choice is when the two directions are of opposite sign, i.e., if i was a fan of j and j a foe of i; in this case we decided to assume that the relationship canceled and placed a 0 on edge (i, j). As expected, this is relatively rare, and this happened only 1,949 times in this dataset.

We consider the component of the Slashdot database that is friendly to CmdrTaco, the founder of the site and user number 1. More specifically, we considered only those users for which there existed a "friendly path" from that user to CmdrTaco. This subset is the largest connected component of the full network and contains 23,514 users. This graph contains 415,118 positive edges and 117,024 negative edges, and we computed components. By definition, $c(\Gamma_+) = 1$, and we compute that $c(\Gamma_-) = 10,327$, giving $\tau(\Gamma) = 13,187$. We conjecture that data from social networks will show

this pattern, that even amongst a group of "friends" or common "fans" of a particular user, there will be a large number of instabilities in exactly this manner.

5. Conclusions. The signed Laplacian on a graph or network occurs in many problems, including the evolution of a system of coupled oscillators, the analysis of social networks, data mining, and many others. We have shown that the topology of the network puts many constraints on the index of the associated Laplacian, and in some cases determines it uniquely. The topological picture also defines certain subspaces that give a nice splitting of the underlying vector space into the modes which undergo an eigenvalue crossing and the modes which do not. We have applied these ideas to analyze two social networks: the Slashdot zoo and relations among the subtribes of the Gahuku–Gama people of New Guinea. We expect that these ideas might be much more fruitful in the future in studying the potential for feud formation in real and synthetic social network datasets.

We have also considered a family of random graph problems and presented some statistical data from certain ensembles. There has been a large degree of interest in the spectral distributions of various random graph ensembles (for just a sampling of this literature, see [62, 63, 64, 65, 66, 67, 68]). The numerical experiments considered above seem to be a nontrivial generalization of these ensemble models, and their analysis could be as rich. One could imagine considering random graph models as above where the weights are normally distributed as well as being signed, for example. We also point out that we have only considered random graph models that have small diameters (both Erdős–Rényi and small world have these properties, the latter by construction), and this might have something to do with the normal statistics that arise; a numerical study of random graph models with large diameters may very well give nonnormal statistics, and this should be investigated.

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