MEAN VALUE PROPERTIES: OLD AND NEW

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A classical result states that harmonic functions, solutions to the Laplace equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

are characterized by the mean value property

$$u(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} u(y)\, dy.$$

In this lecture we explore analogues of this result for p-harmonic functions, solution of the p-Laplace equation,

$$\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) = 0,$$

where $1 \leq p \leq \infty$ but $p \neq 2$.

We show how characterize p-harmonic functions, including $p = 1$ and $p = \infty$, by using asymptotic mean value properties, extending several classical results from the linear case $p = 2$ to all $p's$. It turns out that a continuous function is p-harmonic in $\mathbb{R}^n$ for $1 < p \leq \infty$ if and only the following expansion

$$u(x) = \frac{p-2}{2(p+n)} \left\{ \max_{B(x,\varepsilon)} u_\varepsilon + \min_{B(x,\varepsilon)} u_\varepsilon \right\} + \frac{n+2}{n+p} \fint_{B(x,\varepsilon)} u(y)\, dy + o(\varepsilon^2)$$

holds is a certain weak sense, which turns out to be equivalent to the viscosity sense. We will show how to modify this formula to include the case $p = 1$.

This type of non-linear averages appears naturally as the dynamic programming principle equation that are satisfied by the value functions of stochastic tug-of-war games.

This is joint work with Mikko Parviainen (Helsinki), Julio Rossi (Alicante), and Bernd Kawohl (Cologne).