EXPONENTIALS OF DIFFERENCE MATRICES : HEAT EQUATION AND WAVE EQUATION

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In discretizing $u_t = u_{xx}$ and $u_{tt} = u_{xx}$, we replace the right hand sides by a second difference matrix acting on a vector U. With $u = 0$ on the boundaries, the natural choice is the $1, -2, 1$ tridiagonal Toeplitz matrix $-K$. Then the solutions involve the matrix exponential $\exp(-Kt)$, and also $S = \sqrt{K}$ and $\exp(iSt)$ for the wave equation. We look for simple and close approximations to these matrix functions.

In the process a remarkable property appears: all matrix functions of K are Toeplitz plus Hankel shift-invariant and anti-shift-invariant. For the doubly infinite case the Hankel part disappears (no boundaries) and the entries of $\sqrt{K} = s$ quare root (Laplacian) are nice fractions. The entries of $\exp(-Kt)$ and $\exp(iSt)$ and $\sinh(St)$ come from Bessel functions.