

A NOTE ON THE EIGHTFOLD WAY

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ABSTRACT. Assuming the existence of a Mahlo cardinal, we construct a model in which there exists an ω_2 -Aronszajn tree, the ω_1 -approachability property fails, and every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects. This solves an open problem of [1].

Cummings, Friedman, Magidor, Rinot, and Sinapova [1] proved the consistency of any logical Boolean combination of the statements which assert the ω_1 -approachability property, the tree property on ω_2 , and stationary reflection at ω_2 . For most of these combinations, they assumed the existence of a weakly compact cardinal in order to construct the desired model. This is a natural assumption to make, since the ω_2 -tree property implies that ω_2 is weakly compact in L . On the other hand, Harrington and Shelah [4] proved that stationary reflection at ω_2 is equiconsistent with the existence of a Mahlo cardinal. Cummings et al. [1] asked whether a Mahlo cardinal is sufficient to prove the consistency of the existence of an ω_2 -Aronszajn tree, the failure of the ω_1 -approachability property, and stationary reflection at ω_2 . In this article we answer this question in the affirmative.

We begin by reviewing the relevant definitions and facts. We refer the reader to [1] for a more detailed discussion of these ideas and their history. A stationary set $S \subseteq \omega_2 \cap \text{cof}(\omega)$ is said to *reflect* at an ordinal $\beta \in \omega_2 \cap \text{cof}(\omega_1)$ if $S \cap \beta$ is a stationary subset of β . If S does not reflect at any such ordinal, S is *non-reflecting*. We say that *stationary reflection* holds at ω_2 if every stationary subset of $\omega_2 \cap \text{cof}(\omega)$ reflects to some ordinal in $\omega_2 \cap \text{cof}(\omega_1)$.

An ω_2 -Aronszajn tree is a tree of height ω_2 , whose levels have size less than ω_2 , and which has no cofinal branches. The ω_2 -tree property is the statement that there does not exist an ω_2 -Aronszajn tree. A well-known fact is that if the ω_2 -tree property holds, then ω_2 is a weakly compact cardinal in L . Therefore, if one starts with a Mahlo cardinal κ which is not weakly compact in L (for example, if κ is the least Mahlo cardinal in L), then in any subsequent forcing extension in which κ equals ω_2 , there exists an ω_2 -Aronszajn tree.

The ω_1 -approachability property is the statement that there exists a sequence $\vec{a} = \langle a_i : i < \omega_2 \rangle$ of countable subsets of ω_2 and a club $C \subseteq \omega_2$ such that for all limit ordinals $\alpha \in C$, α is *approachable* by \vec{a} in the following sense: there exists a cofinal set $c \subseteq \alpha$ with order type equal to $\text{cf}(\alpha)$ such that for all $\beta < \alpha$, $c \cap \beta$ is a member of $\{a_i : i < \alpha\}$. Essentially, this property is a very weak form of the square principle \square_{ω_1} . The failure of the ω_1 -approachability property is known to

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hold in Mitchell's model [6] in which there does not exist a special ω_2 -Aronszajn tree, which he constructed using a Mahlo cardinal.

A solution to the problem of [1] addressed in this article was originally discovered by the first author, using a mixed support forcing iteration similar to the forcings appearing in [1] and [2]. Later, the second author found a different proof using the idea of a disjoint stationary sequence. The latter proof is somewhat easier, since it avoids the technicalities of mixed support iterations, and also can be easily adapted to arbitrarily large continuum. In this article we present the second proof.

In Section 1, we discuss the idea of a disjoint stationary sequence, which was originally introduced by the second author in [5]. In Section 2, we prove the main result of the paper. In Section 3, we adapt our model to arbitrarily large continuum using an argument of I. Neeman, which we include with his kind permission.

1. DISJOINT STATIONARY SEQUENCES

Recall that for an uncountable ordinal $\alpha \in \omega_2$, $P_{\omega_1}(\alpha)$ denotes the set of all countable subsets of α . A set $c \subseteq P_{\omega_1}(\alpha)$ is *club* if it is cofinal in $P_{\omega_1}(\alpha)$ and closed under unions of countable increasing sequences. A set $s \subseteq P_{\omega_1}(\alpha)$ is *stationary* if it has non-empty intersection with every club in $P_{\omega_1}(\alpha)$. For an infinite cardinal κ , a forcing \mathbb{P} is said to be κ -*distributive* if it adds no new subsets of V of size less than κ .

Let α be an uncountable ordinal in ω_2 . Fix an increasing and continuous sequence $\langle b_i : i < \omega_1 \rangle$ of countable sets with union equal to α (for example, fix a bijection $f : \omega_1 \rightarrow \alpha$ and let $b_i := f[i]$). Note that the set $\{b_i : i < \omega_1\}$ is club in $P_{\omega_1}(\alpha)$. A set $s \subseteq P_{\omega_1}(\alpha)$ is stationary in $P_{\omega_1}(\alpha)$ iff the set $x := \{i < \omega_1 : b_i \in s\}$ is a stationary subset of ω_1 . Indeed, if $C \subseteq \omega_1$ is a club which is disjoint from x , then the set $\{b_i : i \in C\}$ is a club subset of $P_{\omega_1}(\alpha)$ which is obviously disjoint from s . On the other hand, if $c \subseteq P_{\omega_1}(\alpha)$ is a club which is disjoint from s , then the set $\{i < \omega_1 : b_i \in c\}$ is a club in ω_1 , and this club is clearly disjoint from x .

Definition 1.1. A disjoint stationary sequence on ω_2 is a sequence $\langle s_\alpha : \alpha \in S \rangle$, where S is a stationary subset of $\omega_2 \cap \text{cof}(\omega_1)$, satisfying:

- (1) for all $\alpha \in S$, s_α is a stationary subset of $P_{\omega_1}(\alpha)$;
- (2) for all $\alpha < \beta$ in S , $s_\alpha \cap s_\beta = \emptyset$.

As we will show below, the existence of a disjoint stationary sequence $\langle s_\alpha : \alpha \in S \rangle$ on ω_2 implies the failure of the ω_1 -approachability property (more specifically, that the set S is not in the approachability ideal $I[\omega_2]$). In our main result, the failure of the ω_1 -approachability property will follow from the existence of a disjoint stationary sequence.

One of the advantages of disjoint stationary sequences over other methods for obtaining the failure of approachability, such as using the ω_1 -approximation property, is their upward absoluteness.

Lemma 1.2. Suppose that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Let \mathbb{P} be a forcing poset which preserves ω_1 and ω_2 , preserves the stationarity of S , and preserves stationary subsets of ω_1 . Then \mathbb{P} forces that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence.

The proof is straightforward.

Corollary 1.3. *Assume that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Let \mathbb{P} be a forcing poset which is either c.c.c., or ω_2 -distributive and preserves the stationarity of S . Then \mathbb{P} forces that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence.*

The next result describes a well-known consequence of approachability; we include a proof for completeness.

Proposition 1.4. *Assume that the ω_1 -approachability property holds. Then for any stationary set $S \subseteq \omega_2 \cap \text{cof}(\omega_1)$, there exists an ω_2 -distributive forcing which adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$.*

Proof. Fix a sequence $\vec{a} = \langle a_i : i < \omega_2 \rangle$ of countable subsets of ω_2 and a club $C \subseteq \omega_2$ such that for all limit ordinals $\alpha \in C$, there exists a set $e \subseteq \alpha$ which is cofinal in α , has order type $\text{cf}(\alpha)$, and for all $\beta < \alpha$, $e \cap \beta \in \{a_i : i < \alpha\}$.

Define \mathbb{P} as the forcing poset consisting of all closed and bounded subsets of $S \cup (\omega_2 \cap \text{cof}(\omega))$, ordered by end-extension. We will show that \mathbb{P} is ω_2 -distributive. Observe that if $c \in \mathbb{P}$ and $\gamma < \omega_2$, then there is $d \leq c$ with $\text{sup}(d) \geq \gamma$ (for example, $d := c \cup \min(S \setminus \max\{\text{sup}(c), \gamma\})$). Using this, a straightforward argument shows that, if \mathbb{P} is ω_2 -distributive, then \mathbb{P} adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$.

To show that \mathbb{P} is ω_2 -distributive, fix $c \in \mathbb{P}$ and a family $\{D_i : i < \omega_1\}$ of dense open subsets of \mathbb{P} . We will find $d \leq c$ in $\bigcap \{D_i : i < \omega_1\}$.

Fix a regular cardinal θ large enough so that all of the objects mentioned so far are members of $H(\theta)$. Fix a well-ordering \trianglelefteq of $H(\theta)$. Since S is stationary, we can find an elementary substructure N of $(H(\theta), \in, \trianglelefteq)$ such that \vec{a} , C , S , \mathbb{P} , c , and $\langle D_i : i < \omega_1 \rangle$ are members of N and $\alpha := N \cap \omega_2 \in S$. In particular, $\alpha \in C \cap \text{cof}(\omega_1)$. Fix a cofinal set $e \subseteq \alpha$ with order type ω_1 such that for all $\beta < \alpha$, $e \cap \beta \in \{a_i : i < \alpha\}$. Enumerate e in increasing order as $\langle \gamma_i : i < \omega_1 \rangle$. Note that since $\{a_i : i < \alpha\}$ is a subset of N by elementarity, for all $\beta < \alpha$, $e \cap \beta \in N$. Consequently, for each $\delta < \omega_1$, the sequence $\langle \gamma_i : i < \delta \rangle$ is a member of N .

We define by induction a strictly descending sequence of conditions $\langle c_i : i < \omega_1 \rangle$, starting with $c_0 := c$, together with some auxiliary objects. We will maintain that for each $\delta < \omega_1$, the sequence $\langle c_i : i < \delta \rangle$ is definable in $H(\theta)$ from parameters in N , and hence is a member of N .

Given a limit ordinal $\delta < \omega_1$, assuming that c_i is defined for all $i < \delta$, we define $c_{\delta,0}$ to be equal to $\bigcup \{c_i : i < \delta\}$. Then clearly $\text{sup}(c_{\delta,0})$ is an ordinal of cofinality ω . Hence, $c_\delta := c_{\delta,0} \cup \{\text{sup}(c_{\delta,0})\}$ is a condition and is a strict end-extension of c_i for all $i < \delta$. Now assume that $\xi < \omega_1$ and c_i is defined for all $i \leq \xi$. Let $c_{\xi,0}$ be the \trianglelefteq -least strict end-extension of c_ξ such that $\max(c_{\xi,0}) \geq \gamma_\xi$. Now let $c_{\xi+1}$ be the \trianglelefteq -least condition in D_ξ which is below $c_{\xi,0}$. This completes the construction. Define $d_0 := \bigcup \{c_i : i < \omega_1\}$.

Reviewing the inductive definition of the sequence $\langle c_i : i < \omega_1 \rangle$, we see that for all $\delta < \omega_1$, $\langle c_i : i < \delta \rangle$ is definable in $H(\theta)$ from parameters in N , including specifically the sequence $\langle \gamma_i : i < \delta \rangle$. Therefore, each c_i is in N . In addition, for each $i < \omega_1$, $\max(c_{i+1}) \geq \gamma_i$. Since $\{\gamma_i : i < \omega_1\} = e$ is cofinal in α , $\text{sup}(d_0) = \alpha$. Let $d := d_0 \cup \{\alpha\}$. Then d is a condition since $\alpha \in S$, and $d \leq c_i$ for all $i < \omega_1$, and in particular, $d \leq c$. For each $i < \omega_1$, $c_{i+1} \in D_i$, so $d \in D_i$. \square

Proposition 1.5. *Suppose that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Then $(\omega_2 \cap \text{cof}(\omega_1)) \setminus S$ is stationary.*

Proof. Let C be club in ω_2 . By induction, it is easy to define an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ satisfying:

- (1) each N_i is a countable elementary substructure of $H(\omega_3)$ containing the objects $\langle s_\alpha : \alpha \in S \rangle$ and C ;
- (2) for each $i < \omega_1$, $N_i \in N_{i+1}$.

Let $N := \bigcup\{N_i : i < \omega_1\}$. Then by elementarity, $\omega_1 \subseteq N$ and $\beta := N \cap \omega_2$ has cofinality ω_1 and is in C .

We claim that $\beta \notin S$, which completes the proof. Suppose for a contradiction that $\beta \in S$. Then s_β is defined and is a stationary subset of $P_{\omega_1}(\beta)$. On the other hand, $\langle N_i \cap \omega_2 : i < \omega_1 \rangle$ is a club subset of $P_{\omega_1}(\beta)$. So we can fix $i < \omega_1$ such that $N_i \cap \omega_2 \in s_\beta$.

Now the sequence $\langle s_\alpha : \alpha \in S \rangle$ is a member of N , and also $N_i \cap \omega_2 \in N \cap s_\beta$. So by elementarity, there exists $\alpha \in N \cap S$ such that $N_i \cap \omega_2 \in s_\alpha$. Then $\alpha \in N \cap \omega_2 = \beta$, so $\alpha < \beta$. Thus, we have that $N_i \cap \omega_2$ is a member of both s_α and s_β , which contradicts that $s_\alpha \cap s_\beta = \emptyset$. \square

Corollary 1.6. *Assume that there exists a disjoint stationary sequence on ω_2 . Then the ω_1 -approachability property fails.*

Proof. Suppose for a contradiction that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence and the ω_1 -approachability property holds. By Proposition 1.4, fix an ω_2 -distributive forcing \mathbb{P} which adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$. In particular, \mathbb{P} forces that $(\omega_2 \cap \text{cof}(\omega_1)) \setminus S$ is non-stationary in ω_2 . By Proposition 1.5, the sequence $\langle s_\alpha : \alpha \in S \rangle$ is not a disjoint stationary sequence in $V^{\mathbb{P}}$.

Now \mathbb{P} is ω_2 -distributive, and it preserves the stationarity of S because it adds a club subset of $S \cup (\omega_2 \cap \text{cof}(\omega))$. By Corollary 1.3, $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence in $V^{\mathbb{P}}$, which is a contradiction. \square

2. THE MAIN RESULT

Assume for the rest of the section that κ is a Mahlo cardinal. Without loss of generality, we may also assume that $2^\kappa = \kappa^+$, since this can be forced while preserving Mahloness. Define S as the set of inaccessible cardinals below κ .

We will define a two-step forcing iteration $\mathbb{P} * \dot{\mathbb{A}}$ with the following properties. The forcing \mathbb{P} collapses κ to become ω_2 and adds a disjoint stationary sequence on S . In $V^{\mathbb{P}}$, $\dot{\mathbb{A}}$ is an iteration for destroying the stationarity of non-reflecting subsets of $\kappa \cap \text{cof}(\omega)$. The forcing $\dot{\mathbb{A}}$ will be κ -distributive and preserve the stationarity of S , which implies by Corollary 1.3 that there exists a disjoint stationary sequence in $V^{\mathbb{P} * \dot{\mathbb{A}}}$. Thus, in $V^{\mathbb{P} * \dot{\mathbb{A}}}$ we have that stationary reflection holds at ω_2 and the ω_1 -approachability property fails. If, in addition, we assume that the Mahlo cardinal κ is not weakly compact in L , then there exists an ω_2 -Aronszajn tree in $V^{\mathbb{P} * \dot{\mathbb{A}}}$ as discussed above.

The remainder of this section is divided into two parts. In the first part we will develop the forcing \mathbb{P} , and in the second we will handle the forcing $\dot{\mathbb{A}}$ in $V^{\mathbb{P}}$. We will use the following theorem of Gitik [3]. Suppose that $V \subseteq W$ are transitive models of ZFC with the same ordinals and the same ω_1 and ω_2 . If $(P(\omega) \cap W) \setminus V$ is non-empty, then in W the set $P_{\omega_1}(\omega_2) \setminus V$ is stationary in $P_{\omega_1}(\omega_2)$. For a regular cardinal κ , we let $\text{Add}(\kappa)$ denote the usual Cohen forcing consisting of all functions from some $\gamma < \kappa$ into 2, ordered by reverse inclusion.

We define by induction a forcing iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle.$$

This iteration will be a countable support forcing iteration of proper forcings. We will then let $\mathbb{P} := \mathbb{P}_\kappa$.

Fix $\alpha < \kappa$ and assume that \mathbb{P}_α has been defined. We split the definition of \dot{Q}_α into three cases. If α is an inaccessible cardinal, then let \dot{Q}_α be a \mathbb{P}_α -name for the forcing $\text{Add}(\alpha)$. If $\alpha = \beta + 1$ where β is inaccessible, then let \dot{Q}_α be a \mathbb{P}_α -name for $\text{Add}(\omega)$. For all other cases, let \dot{Q}_α be a \mathbb{P}_α -name for $\text{Col}(\omega_1, \omega_2)$. Note that in any case, \dot{Q}_α is forced to be proper. Now let $\mathbb{P}_{\alpha+1}$ be $\mathbb{P}_\alpha * \dot{Q}_\alpha$. At limit stages $\delta \leq \kappa$, assuming that \mathbb{P}_α is defined for all $\alpha < \delta$, we let \mathbb{P}_δ denote the countable support limit of these forcings.

This completes the construction. For each $\alpha \leq \kappa$, \mathbb{P}_α is a countable support iteration of proper forcings, and hence is proper. Also, by standard facts, if $\beta < \alpha$, then \mathbb{P}_β is a regular suborder of \mathbb{P}_α , and in $V^{\mathbb{P}_\beta}$, the quotient forcing $\mathbb{P}_\alpha / \dot{G}_{\mathbb{P}_\beta}$ is forcing equivalent to a countable support iteration of proper forcings, and hence is itself proper. We let $\dot{\mathbb{P}}_{\beta, \alpha}$ be a \mathbb{P}_β -name for this proper forcing iteration which is equivalent to $\mathbb{P}_\alpha / \dot{G}_{\mathbb{P}_\beta}$ in $V^{\mathbb{P}_\beta}$.

One can show by well-known arguments that for all inaccessible cardinals $\alpha \leq \kappa$, \mathbb{P}_α has size α , is α -c.c., and forces that $\alpha = \omega_2$. Namely, since α is inaccessible, for all $\beta < \alpha$, $|\mathbb{P}_\beta| < \alpha$. Hence \mathbb{P}_α has size α by definition. A standard Δ -system argument shows that \mathbb{P}_α is α -c.c., and since collapses are used at cofinally many stages below α , \mathbb{P}_α turns α into ω_2 .

Let $\mathbb{P} := \mathbb{P}_\kappa$. In $V^{\mathbb{P}}$, let us define a disjoint stationary sequence. Recall that S is the set of inaccessible cardinals in κ in the ground model V . Since κ is Mahlo, S is a stationary subset of κ in V . As \mathbb{P} is κ -c.c., S remains stationary in $V^{\mathbb{P}}$. And since \mathbb{P} is proper and forces that $\kappa = \omega_2$, each member of S has cofinality ω_1 in $V^{\mathbb{P}}$.

The set S will be the domain of the disjoint stationary sequence in $V^{\mathbb{P}}$. Consider $\alpha \in S$. Then \mathbb{P}_α forces that $\alpha = \omega_2$. We have that $\mathbb{P}_{\alpha+1}$ is forcing equivalent to $\mathbb{P}_\alpha * \text{Add}(\alpha)$ and $\mathbb{P}_{\alpha+2}$ is forcing equivalent to

$$\mathbb{P}_\alpha * \text{Add}(\alpha) * \text{Add}(\omega).$$

Clearly, α is still equal to ω_2 after forcing with $\mathbb{P}_{\alpha+1}$ or $\mathbb{P}_{\alpha+2}$.

Since there exists a subset of ω in $V^{\mathbb{P}_{\alpha+2}} \setminus V^{\mathbb{P}_{\alpha+1}}$, in $V^{\mathbb{P}_{\alpha+2}}$ the set

$$s_\alpha := P_{\omega_1}(\alpha) \setminus V^{\mathbb{P}_{\alpha+1}}$$

is a stationary subset of $P_{\omega_1}(\alpha)$ by Gitik's theorem. Now the tail of the iteration $\mathbb{P}_{\alpha+2, \kappa}$ is proper in $V^{\mathbb{P}_{\alpha+2}}$. Therefore, s_α remains stationary in $P_{\omega_1}(\alpha)$ in $V^{\mathbb{P}}$.

Observe that if $\alpha < \beta$ are both in S , then by definition $s_\alpha \subseteq V^{\mathbb{P}_{\alpha+2}} \subseteq V^{\mathbb{P}_\beta}$, whereas $s_\beta \cap V^{\mathbb{P}_\beta} = \emptyset$. Thus, $s_\alpha \cap s_\beta = \emptyset$. It follows that in $V^{\mathbb{P}}$, $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence on ω_2 .

For the second part of our proof, we work in $V^{\mathbb{P}}$ to define a forcing iteration \mathbb{A} of length κ^+ which is designed to destroy the stationarity of any subset of $\omega_2 \cap \text{cof}(\omega)$ which does not reflect to an ordinal in $\omega_2 \cap \text{cof}(\omega_1)$. This forcing will be shown to be κ -distributive and preserve the stationarity of S . It follows from Corollary 1.3 that \mathbb{A} preserves the fact that $\langle s_\alpha : \alpha \in S \rangle$ is a disjoint stationary sequence. Note that since \mathbb{P} is κ -c.c. and has size κ , easily $2^\kappa = \kappa^+$ in $V^{\mathbb{P}}$.

The definition of and arguments involving \mathbb{A} are essentially the same as in the original construction of Harrington and Shelah [4]. The main differences are that we are using \mathbb{P} to collapse κ to become ω_2 instead of $\text{Col}(\omega_1, < \kappa)$, and that we are now required to show that \mathbb{A} preserves the stationarity of S . We will sketch the

main points of the construction, but leave some of the routine technical details to be checked by the reader in consultation with [4].

Many of the facts which we will need to know about \mathbb{A} can be abstracted out more generally to a kind of forcing iteration which we will call a suitable iteration. So before defining \mathbb{A} , let us describe this kind of iteration in detail. We will assume in what follows that $2^{\omega_1} = \omega_2$.

Let us define abstractly the idea of a *suitable iteration*

$$\langle \mathbb{A}_i, \dot{T}_j : i \leq \alpha, j < \alpha \rangle,$$

where $\alpha \leq \omega_3$. Such an iteration is determined by the following recursion. A condition in \mathbb{A}_i is any function p whose domain is a subset of i of size less than ω_2 such that for all $j \in \text{dom}(p)$, $p(j)$ is a non-empty closed and bounded subset of ω_2 such that $p \upharpoonright j$ forces in \mathbb{A}_j that $p(j) \cap \dot{T}_j = \emptyset$. We let $q \leq p$ if $\text{dom}(q) \subseteq \text{dom}(p)$ and for all $i \in \text{dom}(p)$, $q(i)$ is an end-extension of $p(i)$. And \dot{T}_i is a nice \mathbb{A}_i -name for a subset of $\omega_2 \cap \text{cof}(\omega)$.¹

Suppose that M is a transitive model of ZFC^- which is closed under ω_1 -sequences. Then if M models that $\langle \mathbb{A}_i, \dot{T}_j : i \leq \alpha, j < \alpha \rangle$ is a suitable iteration, then in fact it is. Specifically, all the notions used in the recursion above are upwards absolute for such a model, since M contains all ω_1 -sized sets. For example, M contains all closed and bounded subsets of ω_2 and being a nice name is absolute.

Observe that if $\alpha < \omega_3$, then $2^{\omega_1} = \omega_2$ immediately implies that \mathbb{A}_α has size ω_2 . On the other hand, if $\alpha = \omega_3$, then a straightforward application of the Δ -system lemma shows that \mathbb{A}_{ω_3} is ω_3 -c.c. Using a covering and nice name argument, it then follows that if \mathbb{A}_β is ω_2 -distributive for all $\beta < \omega_3$, then so is \mathbb{A}_{ω_3} .

Lemma 2.1. *Suppose that for all $i < \alpha$, \mathbb{A}_i forces that \dot{T}_i is non-stationary. Then for any $q \in \mathbb{A}_\alpha$, \mathbb{A}_α/q is forcing equivalent to $\text{Add}(\omega_2)$.*

Proof. First we claim that \mathbb{A}_α contains an ω_2 -closed dense subset. For each i let \dot{E}_i be an \mathbb{A}_i -name for a club disjoint from \dot{T}_i . Define D as the set of conditions p such that for all $i \in \text{dom}(p)$, $p \upharpoonright i$ forces that $\max(p(i)) \in \dot{E}_i$. It is easy to prove that D is dense and ω_2 -closed.

Reviewing the definition of \mathbb{A}_α , clearly \mathbb{A}_α is separative and every condition in it has ω_2 -many incompatible extensions. By a well-known fact, any ω_2 -closed separative forcing of size ω_2 for which any condition has ω_2 -many incompatible extensions is forcing equivalent to $\text{Add}(\omega_2)$. \square

Having described the main facts which we will use about a suitable iteration, let us show how this kind of iteration can be used to obtain a model satisfying that stationary reflection holds at ω_2 . Suppose that we have a ground model in which $2^{\omega_2} = \omega_3$. Using a standard bookkeeping argument, we can define a suitable iteration

$$\langle \mathbb{A}_i, \dot{T}_j : i \leq \omega_3, j < \omega_3 \rangle,$$

so that every nice name for a non-reflecting subset of $\omega_2 \cap \text{cof}(\omega)$ is equal to some \dot{T}_j . Specifically, assuming that \mathbb{A}_i is defined for some $i < \omega_3$, then using $2^{\omega_2} = \omega_3$ and the fact that \mathbb{A}_i has size ω_2 , we can list out all nice \mathbb{A}_i -names for subsets of $\omega_2 \cap \text{cof}(\omega)$ in order type ω_3 . Now choose \dot{T}_i to be the first name (according to the

¹In our construction below, our specific suitable iteration will be shown to be ω_2 -distributive. However, being ω_2 -distributive is not a part of the abstract definition of a suitable iteration.

bookkeeping function) which was listed at some stage less than or equal to i which is forced by \mathbb{A}_i to be non-reflecting. In this manner, we can arrange that after ω_3 -many stages, all names which arise during the iteration are handled, and thus that the iteration destroys the stationarity of all non-reflecting sets. Of course this construction breaks down if we reach some i such that \mathbb{A}_i is not ω_2 -distributive. So proving the ω_2 -distributivity of such a suitable iteration will be the main remaining goal.

This completes the abstract description of a suitable iteration and how it will be used to obtain stationary reflection at ω_2 . Let us now return to our construction. Fix a generic filter G on \mathbb{P} . Then in $V[G]$ we have that $\kappa = \omega_2$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3 = \kappa^+$. Working in $V[G]$, we define a suitable iteration $\langle \mathbb{A}_i, \dot{T}_j : i \leq \kappa^+, j < \kappa^+ \rangle$. We will prove that each \mathbb{A}_i is ω_2 -distributive and preserves the stationarity of S . By the discussion above, this will complete the proof of our main result.

Fix $\alpha < \kappa^+$. In V , fix \mathbb{P} -names $\dot{\mathbb{A}}_i$ for all $i \leq \alpha$ and \dot{T}_j for all $j < \alpha$ which are forced to satisfy the definitions of these objects given above (we will abuse notation by writing \dot{T}_j for the \mathbb{P} -name for the \mathbb{A}_j -name \dot{T}_j).

We would like to prove that \mathbb{A}_α is κ -distributive and preserves the stationarity of S . In order to prove this, we will make two inductive hypotheses. The first inductive hypothesis is that for all $\beta < \alpha$, \mathbb{A}_β is κ -distributive and preserves the stationarity of S .

Before describing the second inductive hypothesis, we need to develop some ideas and notation. For each $\beta \leq \alpha$, define in V the set \mathcal{X}_β to consist of all sets N satisfying:

- (1) $N \prec H(\kappa^+)$;
- (2) N contains as members \mathbb{P} and $\langle \dot{\mathbb{A}}_i, \dot{T}_j : i \leq \beta, j < \beta \rangle$;
- (3) $\kappa_N := |N| = N \cap \kappa$ and $N^{<\kappa_N} \subseteq N$;
- (4) $\kappa_N \in S$.

An easy application of the stationarity of S and the inaccessibility of κ shows that each \mathcal{X}_β is a stationary subset of $P_\kappa(H(\kappa^+))$. Also note that if $N \in \mathcal{X}_\beta$ and $\gamma \in N \cap \beta$, then $N \in \mathcal{X}_\gamma$.

Consider N in \mathcal{X}_α . Since \mathbb{P} is κ -c.c., the maximal condition in \mathbb{P} is (N, \mathbb{P}) -generic. So if G is a V -generic filter on \mathbb{P} , then $N[G] \cap V = N$. In particular, $N[G] \cap \kappa = N \cap \kappa = \kappa_N \in S$. Let $\pi : N[G] \rightarrow \overline{N[G]}$ be the transitive collapsing map of $N[G]$ in $V[G]$. Let $G^* := G \cap \mathbb{P}_{\kappa_N}$, which is a V -generic filter on \mathbb{P}_{κ_N} .

Lemma 2.2. *The following statements hold.*

- (1) $\pi \upharpoonright N : N \rightarrow \overline{N}$ is the transitive collapsing map of N in V ;
- (2) $\pi(\mathbb{P}) = \mathbb{P}_{\kappa_N}$, $\pi(G) = G^*$, and $\overline{N[G]} = \overline{N[G^]}$; in particular, $\overline{N[G]}$ is a member of $V[G^*]$;
- (3) $\overline{N[G]} = \overline{N[G^]}$ is closed under $< \kappa_N$ -sequences in $V[G^*]$.

Proof. (1) and (2) are straightforward. Since $\overline{N}^{<\kappa_N} \subseteq \overline{N}$ in V by the closure of N and \mathbb{P}_{κ_N} is κ_N -c.c., (3) follows immediately by a standard fact. \square

Now we are ready to state our second inductive hypothesis: for all $\beta < \alpha$ and for all $N \in \mathcal{X}_\beta$, letting $\pi : N[G] \rightarrow \overline{N[G]}$ be the transitive collapsing map of $N[G]$ and $G^* := \pi(G)$, for all $q \in \pi(\mathbb{A}_\beta)$, the forcing poset $\pi(\mathbb{A}_\beta)/q$ is forcing equivalent to $\text{Add}(\omega_2)$ in $V[G^*]$.

We begin the proof of the two inductive hypotheses for α , assuming that they hold for all $\beta < \alpha$. Let $N \in \mathcal{X}_\alpha$. Let $\pi : N[G] \rightarrow \overline{N[G]}$ be the transitive collapsing map of $N[G]$ and $G^* := \pi(G)$. Since π is an isomorphism, by the absoluteness of suitable iterations we have that in $V[G^*]$,

$$\langle \mathbb{A}_i^*, \dot{T}_j^* : i \leq \pi(\alpha), j < \pi(\alpha) \rangle := \pi(\langle \mathbb{A}_i, \dot{T}_j : i \leq \alpha, j < \alpha \rangle)$$

is a suitable iteration of length $\pi(\alpha) < \omega_3$. Applying Lemma 2.1 to this suitable iteration in the model $V[G^*]$, the second inductive hypothesis for α will follow from the next lemma.

Lemma 2.3. *For all $\gamma \in N \cap \alpha$, $\pi(\mathbb{A}_\gamma) = \mathbb{A}_{\pi(\gamma)}^*$ forces over $V[G^*]$ that $\pi(\dot{T}_\gamma) = \dot{T}_{\pi(\gamma)}^*$ is non-stationary in κ_N .*

Proof. Consider $\gamma \in N \cap \alpha$. Then by the choice of the names used in the iteration, \mathbb{A}_γ forces that \dot{T}_γ is a subset of $\kappa \cap \text{cof}(\omega)$ which does not reflect to any ordinal in $\kappa \cap \text{cof}(\omega_1)$. In particular, \mathbb{A}_γ forces that $\dot{T}_\gamma \cap \kappa_N$ is non-stationary in κ_N .

Consider $q \in \pi(\mathbb{A}_\gamma)$. We will find a $V[G^*]$ -generic filter H on $\pi(\mathbb{A}_\gamma)$ which contains q such that in $V[G^*][H]$, $\pi(\dot{T}_\gamma)^H$ is non-stationary in κ_N . Because q is arbitrary, this proves that $\pi(\mathbb{A}_\gamma)$ forces that $\pi(\dot{T}_\gamma)$ is non-stationary. Since N is in \mathcal{X}_α and $\gamma \in N \cap \alpha$, N is in \mathcal{X}_γ . By the second inductive hypothesis, $\pi(\mathbb{A}_\gamma)/q$ is forcing equivalent to $\text{Add}(\kappa_N)$ in $V[G^*]$. By definition, the forcing iteration \mathbb{P} forces with $\text{Add}(\kappa_N)$ at stage κ_N . Hence, we can write $V[G \cap \mathbb{P}_{\kappa_N+1}]$ as $V[G^*][H]$, where H is some $V[G^*]$ -generic filter on $\pi(\mathbb{A}_\gamma)/q$.

Now $\pi \upharpoonright \mathbb{A}_\gamma$ is an isomorphism between the posets $N[G] \cap \mathbb{A}_\gamma$ and $\pi(\mathbb{A}_\gamma)$. Therefore, $I := \pi^{-1}(H)$ is a filter on $N[G] \cap \mathbb{A}_\gamma$. The fact that H is a $V[G^*]$ -generic filter on $\pi(\mathbb{A}_\gamma)$ easily implies that I meets every dense subset of \mathbb{A}_γ which is a member of $N[G]$. Now a lower bound t of I can be easily constructed by taking the coordinate-wise closure of the union of the clubs appearing in the conditions of I . Namely, the fact that I meets every dense set in $N[G]$ implies that the maximum member of any such club is equal to κ_N , which has cofinality ω_1 in $V[G]$ and hence is not in any of the sets \dot{T}_j .

Fix a $V[G]$ -generic filter h on \mathbb{A}_γ which contains t . Now $\pi^{-1} : \overline{N[G^*]} \rightarrow N[G]$ is an elementary embedding of $\overline{N[G^*]}$ into $H(\kappa^+)^{V[G]}$ which satisfies that $\pi^{-1}(H) = I \subseteq h$. So by a standard fact about extending elementary embeddings, we can extend π^{-1} to an elementary embedding $\tau : \overline{N[G^*]}[H] \rightarrow N[G][h]$ which maps H to h . Let $T^* := \pi(\dot{T}_\gamma)^H$ and $T_\gamma := (\dot{T}_\gamma)^h$. Then clearly, $\tau(T^*) = T_\gamma$.

Since κ_N is the critical point of τ , $T_\gamma \cap \kappa_N = T^*$. As \mathbb{A}_γ forces that \dot{T}_γ does not reflect to κ_N , T^* is a non-stationary subset of κ_N in the model $V[G][h]$. By the first inductive hypothesis, \mathbb{A}_γ is κ -distributive. Therefore, any club of κ_N in $V[G][h]$ is actually in $V[G]$. Thus, T^* is non-stationary in $V[G]$. But $V[G]$ is a generic extension of $V[G^*][H]$ by the proper forcing $\mathbb{P}_{\kappa_N+1, \kappa}$. So T^* is non-stationary in $V[G^*][H]$. \square

This completes the proof of the second inductive hypothesis. It remains to prove the first inductive hypothesis that \mathbb{A}_α is κ -distributive and preserves the stationarity of S .

Lemma 2.4. *For all $N \in \mathcal{X}_\alpha$, for all $a \in N[G] \cap \mathbb{A}_\alpha$, there exists a filter I on $N[G] \cap \mathbb{A}_\alpha$ in $V[G]$ containing a which meets every dense subset of \mathbb{A}_α in $N[G]$.*

Proof. This is similar to a part of the proof of the Lemma 2.3. Let $\pi : N[G] \rightarrow \overline{N[G]}$ be the transitive collapsing map of $N[G]$ and $G^* := \pi(G)$. Let $a \in N[G] \cap \mathbb{A}_\alpha$. Then $\pi(a) \in \pi(\mathbb{A}_\alpha)$. By the second inductive hypothesis which we have now verified for α , $\pi(\mathbb{A}_\alpha)/\pi(a)$ is forcing equivalent to $\text{Add}(\kappa_N)$ in $V[G^*]$. By definition, the forcing iteration \mathbb{P} forces with $\text{Add}(\kappa_N)$ at stage κ_N . Hence, we can write $V[G \cap \mathbb{P}_{\kappa_N+1}]$ as $V[G^*][H]$, where H is some $V[G^*]$ -generic filter on $\pi(\mathbb{A}_\alpha)/\pi(a)$.

Now $\pi \upharpoonright \mathbb{A}_\alpha$ is an isomorphism between the posets $N[G] \cap \mathbb{A}_\alpha$ and $\pi(\mathbb{A}_\alpha)$. Therefore, $I := \pi^{-1}(H)$ is a filter on $N[G] \cap \mathbb{A}_\alpha$. The fact that H is a $V[G^*]$ -generic filter on $\pi(\mathbb{A}_\alpha)$ easily implies that I meets every dense subset of \mathbb{A}_α which is a member of $N[G]$. \square

We can now complete the proof that \mathbb{A}_α is κ -distributive and preserves the stationarity of S . Given a family \mathcal{D} of fewer than κ many dense open subsets of \mathbb{A}_α and a condition $a \in \mathbb{A}_\alpha$, we may pick $N \in \mathcal{X}_\alpha$ so that \mathcal{D} and a are members of $N[G]$. Then $\mathcal{D} \subseteq N[G]$. By Lemma 2.4, fix a filter I on $N[G] \cap \mathbb{A}_\alpha$ in $V[G]$ which contains a and meets every dense subset of \mathbb{A}_α in $N[G]$ (and in particular, meets every dense set in \mathcal{D}). It is easy to define a lower bound t of I in \mathbb{A}_α by taking the coordinate-wise closure of the union of the clubs appearing in the conditions in I . Then $t \leq a$ and t is in every dense open set in \mathcal{D} .

Similarly, given an \mathbb{A}_α -name \dot{C} for a club subset of κ and $a \in \mathbb{A}_\alpha$, we may choose $N \in \mathcal{X}_\alpha$ such that \dot{C} and a are in N . Fix a filter I on $N[G] \cap \mathbb{A}_\alpha$ in $V[G]$ which contains a and meets every dense subset of \mathbb{A}_α in $N[G]$. As usual, let t be a lower bound of I . Then t is an $(N[G], \mathbb{A}_\alpha)$ -generic condition, which implies that t forces that $N[G] \cap \kappa = \kappa_N$ is in $S \cap \dot{C}$.

3. ARBITRARILY LARGE CONTINUUM

In the model of the previous section, $2^\omega = \omega_2$ holds. A violation of CH is necessary, since CH implies the ω_1 -approximation property, as witnessed by any enumeration of all countable subsets of ω_2 in order type ω_2 . In this section, we will show how to modify this model to obtain arbitrarily large continuum. This modification will use an unpublished result of I. Neeman.

Theorem 3.1 (Neeman). *Assume that stationary reflection holds at ω_2 . Then for any ordinal μ , $\text{Add}(\omega, \mu)$ forces that stationary reflection still holds at ω_2 .*

Proof. We first prove the result in the special case that $\mu = \omega_2$. Let $p \in \text{Add}(\omega, \omega_2)$, and suppose that p forces that \dot{S} is a stationary subset of $\omega_2 \cap \text{cof}(\omega)$. We will find $q \leq p$ and an ordinal $\beta \in \omega_2 \cap \text{cof}(\omega_1)$ such that q forces that $\dot{S} \cap \beta$ is stationary in β .

Let T be the set of ordinals $\alpha < \omega_2$ such that for some $s \leq p$, s forces that $\alpha \in \dot{S}$. Then $T \subseteq \omega_2 \cap \text{cof}(\omega)$. An easy observation is that p forces that $\dot{S} \subseteq T$, and consequently T is a stationary subset of ω_2 . For each $\alpha \in T$, fix a witness $s_\alpha \leq p$ which forces that $\alpha \in \dot{S}$, and define

$$a_\alpha := s_\alpha \upharpoonright (\alpha \times \omega) \text{ and } b_\alpha := s_\alpha \upharpoonright ([\alpha, \omega_2) \times \omega).$$

Using Fodor's lemma, we can find a stationary set $U \subseteq T$ and a set x satisfying that for all $\alpha \in U$, $a_\alpha = x$. Observe that $q := x \cup p$ is a condition which extends p . Applying the fact that stationary reflection holds in the ground model together with an easy closure argument, we can fix $\beta \in \omega_2 \cap \text{cof}(\omega_1)$ such that $U \cap \beta$ is stationary in β and for all $\alpha < \beta$, $\text{dom}(s_\alpha) \subseteq \beta \times \omega$.

We claim that q forces that $\dot{S} \cap \beta$ is stationary in β , which finishes the proof. Suppose for a contradiction that there is $r \leq q$ which forces that $\dot{S} \cap \beta$ is non-stationary in β . Using the fact that $\text{Add}(\omega, \omega_2)$ is c.c.c. and $\text{cf}(\beta) = \omega_1$, there exists a club $D \subseteq \beta$ in the ground model such that r forces that $D \cap \dot{S} = \emptyset$. As r is finite, we can fix $\delta < \beta$ such that $\text{dom}(r) \cap (\beta \times \omega) \subseteq \delta \times \omega$.

Since $U \cap \beta$ is stationary in β , fix $\alpha \in U \cap D$ larger than δ . We claim that s_α and r are compatible. By the choice of U , $s_\alpha \upharpoonright (\alpha \times \omega) = x$, and by the choice of β , $\text{dom}(s_\alpha) \subseteq \beta \times \omega$. Suppose that $(\xi, n) \in \text{dom}(s_\alpha) \cap \text{dom}(r)$. Then $\xi < \beta$, so $(\xi, n) \in \text{dom}(r) \cap (\beta \times \omega) \subseteq \delta \times \omega$. Thus, $\xi < \delta < \alpha$. So $(\xi, n) \in \alpha \times \omega$, and hence $s_\alpha(\xi, n) = a_\alpha(\xi, n) = x(\xi, n)$. On the other hand, $r \leq q \leq x$, and so $r(\xi, n) = x(\xi, n) = s_\alpha(\xi, n)$.

This proves that r and s_α are compatible. Fix $t \leq r, s_\alpha$. Since $t \leq s_\alpha$, t forces that $\alpha \in \dot{S}$. On the other hand, $\alpha \in D$, and r forces that $\dot{S} \cap D = \emptyset$. So r , and hence t , forces that $\alpha \notin \dot{S}$, which is a contradiction.

Now we prove the result for arbitrary ordinals μ . If $\mu < \omega_2$, then $\text{Add}(\omega, \omega_2)$ is isomorphic to $\text{Add}(\omega, \mu) \times \text{Add}(\omega, \omega_2 \setminus \mu)$. Since stationary reflection holds in $V^{\text{Add}(\omega, \omega_2)}$, it also holds in the submodel $V^{\text{Add}(\omega, \mu)}$, since a non-reflecting stationary set in the latter model would remain a non-reflecting stationary set in the former model.

Suppose that $\mu > \omega_2$. Let p be a condition in $\text{Add}(\omega, \mu)$ which forces that \dot{S} is a stationary subset of $\omega_2 \cap \text{cof}(\omega)$, for some nice name \dot{S} . Then by the c.c.c. property of $\text{Add}(\omega, \mu)$ and the fact that conditions are finite, it is easy to show there exists a set $X \subseteq \mu$ of size ω_2 such that \dot{S} is a nice $\text{Add}(\omega, X)$ -name and $p \in \text{Add}(\omega, X)$. Since X has size ω_2 , $\text{Add}(\omega, X)$ is isomorphic to $\text{Add}(\omega, \omega_2)$. By the first result above, we can find $q \leq p$ in $\text{Add}(\omega, X)$ and $\beta \in \omega_2 \cap \text{cof}(\omega_1)$ such that q forces in $\text{Add}(\omega, X)$ that $\dot{S} \cap \beta$ is stationary in β . Since $\text{Add}(\omega, \mu)$ is isomorphic to $\text{Add}(\omega, X) \times \text{Add}(\omega, \mu \setminus X)$ and $\text{Add}(\omega, \mu \setminus X)$ is c.c.c. in $V^{\text{Add}(\omega, X)}$, an easy argument shows that q forces in $\text{Add}(\omega, \mu)$ that $\dot{S} \cap \beta$ is stationary in β . \square

Now start with the model $W := V^{\mathbb{P} * \dot{A}}$ from the previous section. Then ω_2 is not weakly compact in L , there exists a disjoint stationary sequence in W , and stationary reflection holds at ω_2 in W . Let μ be any ordinal and let H be a W -generic filter on $\text{Add}(\omega, \mu)$. Since $\text{Add}(\omega, \mu)$ is c.c.c., Corollary 1.3 implies that there exists a disjoint stationary sequence in $W[H]$. As ω_2 is not weakly compact in L , there exists an ω_2 -Aronszajn tree in $W[H]$. And stationary reflection holds in $W[H]$ by Theorem 3.1.

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