THE HARRINGTON-SHELAH MODEL WITH LARGE CONTINUUM

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Abstract. We prove from the existence of a Mahlo cardinal the consistency of the statement that $2^\omega = \omega_3$ holds and every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects to an ordinal less than ω_2 with cofinality ω_1.

Let us say that *stationary set reflection holds at* ω_2 if for any stationary set $S \subseteq \omega_2 \cap \mathrm{cof}(\omega)$ there is an ordinal $\alpha \in \omega_2 \cap \mathrm{cof}(\omega_1)$ such that $S \cap \alpha$ is stationary in α (that is, S *reflects* to α). In a classic forcing construction, Harrington and Shelah [3] proved the equiconsistency of stationary set reflection at ω_2 with the existence of a Mahlo cardinal. Specifically, if stationary set reflection holds at ω_2, then $\Box_{\omega_1}$ fails, and hence ω_2 is a Mahlo cardinal in L. Conversely, if κ is a Mahlo cardinal, then the generic extension obtained by Lévy collapsing κ to become ω_2 and then iterating to kill the stationarity of nonreflecting sets satisfies stationary set reflection at ω_2. The Harrington-Shelah argument is notable because the majority of stationary set reflection principles are derived by extending large cardinal elementary embeddings, and thus use large cardinal principles much stronger than the existence of a Mahlo cardinal.

The original Harrington-Shelah model satisfies the generalized continuum hypothesis, and in particular, that $2^\omega = \omega_1$. Suppose we would like to obtain a model of stationary set reflection at ω_2 together with $2^\omega = \omega_2$. A natural construction would be to iterate forcing with countable support of length a weakly compact cardinal κ, alternating between adding reals and collapsing ω_2 to have size ω_1. Such an iteration $\mathbb{P}$ would be proper, κ-c.c., collapse κ to become ω_2, and satisfy that $2^\omega = \omega_2$. The fact that stationary set reflection holds in any generic extension $V[G]$ by $\mathbb{P}$ follows from the ability to extend an elementary embedding j with critical point κ after forcing with the proper forcing $j(\mathbb{P})/G$ over $V[G]$.

Consider the problem of obtaining a model satisfying stationary set reflection at ω_2 together with $2^\omega > \omega_2$. Since in that case not all reals would be added by the iteration collapsing κ to become ω_2, extending the elementary embedding becomes more difficult. Indeed, in the model referred to in the previous paragraph, a stronger stationary set reflection principle holds, namely $\mathsf{WRP}(\omega_2)$, which asserts that any stationary subset of $[\omega_2]^\omega$ reflects to $[\beta]^\omega$ for some uncountable $\beta < \omega_2$, and by a result of Todorčević, $\mathsf{WRP}(\omega_2)$ implies $2^\omega \leq \omega_2$ (see [5, Lemma 2.9]).

In this paper we demonstrate that the cardinality of the continuum provides a natural separation between ordinary stationary set reflection and higher order

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reflection principles such as $\mathsf{WRP}(\omega_2)$. We prove that, in contrast to $\mathsf{WRP}(\omega_2)$, stationary set reflection at ω_2 is consistent with $2^\omega = \omega_3$. This result provides a natural variation of the Harrington-Shelah model with a large value of the continuum. Our argument adapts the method of mixed support forcing iterations into the context of iterating distributive forcings. We expect that the technicalities worked out in this paper will be applicable to a broad range of similar problems.

We assume that the reader is familiar with the basics of forcing and has had some exposure to iterated forcing and proper forcing. Other than assuming some general knowledge of these areas, the paper is self-contained.

In Section 1 we provide an abstract definition and development of the kind of mixed support forcing iteration we will use in the consistency result. This iteration combines adding Cohen reals together with adding club subsets of ω_2, with finite support on the Cohen forcing and supports of size ω_1 on the club adding forcing. This kind of mixed support forcing iteration is reminiscent of Mitchell's classic forcing for constructing a model in which there is no Aronszajn tree on ω_2 [4], as well as the term forcing analysis provided in Abraham's extension of Mitchell's result to two successive cardinals [1].

The main challenge in proving our consistency result will be to verify that the forcing iteration preserves ω_1 and ω_2. In Section 2 we analyze the features of this kind of forcing iteration relevant to the issue of cardinal preservation. In Section 3 we put the pieces worked out in Sections 1 and 2 together to prove the consistency of stationary set reflection at ω_2 together with $2^\omega = \omega_3$.

1. Suitable Mixed Support Forcing Iterations

In this section we introduce and develop the basic properties of the type of mixed support forcing iteration which we will use in the consistency result. This kind of iteration will alternate between adding Cohen subsets of ω and adding clubs disjoint from certain subsets of ω_2. The support of a condition in such an iteration will be finite on the Cohen part and of size less than ω_2 on the club adding part.

We let *even* denote the class of even ordinals, and *odd* the class of odd ordinals.

Definition 1.1. *Let $\alpha \leq \omega_3$. Let $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ be a sequence of forcing posets and $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ a sequence such that for all odd $\gamma < \alpha$, $\dot{S}_\gamma$ is a nice $\mathbb{P}_\gamma$-name for a subset of $\omega_2 \cap \mathrm{cof}(\omega)$. Assume that for all $\beta \leq \alpha$, every member of $\mathbb{P}_\beta$ is a function whose domain is a subset of β, and define*

$$\mathbb{P}^c_\beta := \{p \in \mathbb{P}_\beta : \mathrm{dom}(p) \subseteq \mathrm{even}\}.$$

We say that the sequence of forcing posets is a suitable mixed support forcing iteration of length α based on the sequence of names *if the following statements are satisfied:*

1. *$\mathbb{P}_0 = \{\emptyset\}$ is the trivial forcing;*
2. *if $\gamma < \alpha$ is even, then $p \in \mathbb{P}_{\gamma+1}$ iff p is a function whose domain is a subset of $\gamma + 1$ such that $p \upharpoonright \gamma \in \mathbb{P}_\gamma$ and, if $\gamma \in \mathrm{dom}(p)$, then $p(\gamma) \in Add(\omega)$;*
3. *if $\gamma < \alpha$ is odd, then $p \in \mathbb{P}_{\gamma+1}$ iff p is a function whose domain is a subset of $\gamma + 1$ such that $p \upharpoonright \gamma \in \mathbb{P}_\gamma$ and, if $\gamma \in \mathrm{dom}(p)$, then $p(\gamma)$ is a nice $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2 such that*

$$p \upharpoonright \gamma \Vdash_{\mathbb{P}_\gamma} p(\gamma) \cap \dot{S}_\gamma = \emptyset;$$

(4) *if* $\delta \leq \alpha$ *is a limit ordinal, then* $p \in \mathbb{P}_\delta$ *iff* p *is a function whose domain is a subset of* δ *such that* $|\mathrm{dom}(p) \cap \mathrm{even}| < \omega$, $|\mathrm{dom}(p) \cap \mathrm{odd}| < \omega_2$, *and for all* $\beta < \delta$, $p \restriction \beta \in \mathbb{P}_\beta$*;*

(5) *for all* $\beta \leq \alpha$, $q \leq p$ *in* $\mathbb{P}_\beta$ *iff* $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, *and for all* $\gamma \in \mathrm{dom}(p)$, *if* γ *is even then* $p(\gamma) \subseteq q(\gamma)$, *and if* γ *is odd then*

$$q \restriction (\gamma \cap \mathrm{even}) \Vdash_{\mathbb{P}^c_\gamma} q(\gamma) \text{ is an end-extension of } p(\gamma).$$

The definition makes sense without assuming that the forcing iterations preserve cardinals, if we interpret ω_2 in the definition as meaning ω_2 of the ground model. In any case, the only such forcing iterations we will consider in this paper will preserve ω_1 and ω_2, although cardinal preservation will not be verified until the end of the paper.

The requirement in (3) that $p(\gamma)$ is a nice $\mathbb{P}^c_\gamma$-name, rather than a $\mathbb{P}_\gamma$-name, is made in order to prove the following absoluteness result.

Lemma 1.2. *Let* M *be a transitive model of* $\mathsf{ZFC} - \mathsf{Powerset}$ *with* $\omega_2 \in M$ *and* $M^{\omega_1} \subseteq M$. *Suppose that* $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ *is a sequence of forcing posets in* M *and* $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ *is a sequence in* M *so that for each odd* $\gamma \in \alpha$, $\dot{S}_\gamma$ *is a nice* $\mathbb{P}_\gamma$*-name for a subset of* $\omega_2 \cap \mathrm{cof}(\omega)$. *Then* $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ *is a suitable mixed support forcing iteration based on the sequence of names* $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$ *iff* M *models that it is.*

The proof, which we omit, is a straightforward verification that each property of Definition 1.1 is absolute between M and V. The closure of M is used to see that M contains all names described in Definition 1.1(3) (see Lemma 1.3 below).

For the remainder of the section we fix a particular suitable mixed support forcing iteration $\langle \mathbb{P}_\beta : \beta \leq \alpha \rangle$ based on a sequence of names $\langle \dot{S}_\gamma : \gamma \in \alpha \cap \mathrm{odd} \rangle$. For $\beta \leq \alpha$, we will write $q \leq_\beta p$ to mean that $q \leq p$ in $\mathbb{P}_\beta$, and we will abbreviate $\Vdash_{\mathbb{P}_\beta}$ as $\Vdash_\beta$.

When p is a condition in $\mathbb{P}_\beta$ and $\gamma < \beta$, for simplicity we will sometimes write $p(\gamma)$ without knowing whether or not $\gamma \in \mathrm{dom}(p)$; in the case that it is not, then $p(\gamma)$ means the empty set.

The proof of the next lemma is straightforward.

Lemma 1.3. *Let* $\beta \leq \alpha$. *The forcing poset* $\mathbb{P}^c_\beta$ *is a regular suborder of* $\mathbb{P}_\beta$, *and* $\mathbb{P}^c_\beta$ *is isomorphic to* $Add(\omega, \mathrm{ot}(\beta \cap \mathrm{even}))$.

It follows that if G is a generic filter on $\mathbb{P}_\beta$, then $G^c := G \cap \mathbb{P}^c_\beta$ is a generic filter on $\mathbb{P}^c_\beta$. Also, for any condition $q \in G$, $q \leq_\beta (q \restriction \mathrm{even})$ implies that $q \restriction \mathrm{even} \in G^c$. If $\dot{x}$ is a $\mathbb{P}^c_\beta$-name, then it is also a $\mathbb{P}_\beta$-name and $\dot{x}^G = \dot{x}^{G^c}$.

The next two lemmas state some basic facts about the forcing iteration. The proofs, which we omit, are straightforward.

Lemma 1.4. *Let* $\gamma < \beta \leq \alpha$.

(1) $\mathbb{P}_\gamma \subseteq \mathbb{P}_\beta$, *and for all* $p \in \mathbb{P}_\beta$, $p \restriction \gamma \in \mathbb{P}_\gamma$*;*

(2) *if* p *and* q *are in* $\mathbb{P}_\gamma$, *then* $q \leq_\gamma p$ *iff* $q \leq_\beta p$*;*

(3) *if* $p \in \mathbb{P}_\gamma$, $r \in \mathbb{P}_\beta$, *and* $r \leq_\beta p$, *then* $r \restriction \gamma \leq_\gamma p$*;*

(4) *if* $q \in \mathbb{P}_\beta$ *and* $r \leq_\gamma q \restriction \gamma$, *then* $r \cup q \restriction [\gamma, \beta)$ *is in* $\mathbb{P}_\beta$ *and is* $\leq_\beta$*-below* r *and* q*;*

(5) $\mathbb{P}_\gamma$ *is a regular suborder of* $\mathbb{P}_\beta$.

Lemma 1.5. *Let* $\beta \leq \alpha$ *and* p *and* q *be in* $\mathbb{P}_\beta$.

(1) *If β is a limit ordinal, then $q \leq_\beta p$ iff for all $\gamma < \beta$, $q \restriction \gamma \leq_\gamma p \restriction \gamma$;*
(2) *if $\beta = \gamma+1$, where γ is even, then $q \leq_\beta p$ iff $q \restriction \gamma \leq_\gamma p \restriction \gamma$ and $p(\gamma) \subseteq q(\gamma)$;*
(3) *if $\beta = \gamma+1$, where γ is odd, then $q \leq_\beta p$ iff $q \restriction \gamma \leq_\gamma p \restriction \gamma$ and $q \restriction (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}^c_\gamma$ that $q(\gamma)$ is an end-extension of $p(\gamma)$.*

Notation 1.6. *Let $\beta \leq \alpha$. For p and q in $\mathbb{P}_\beta$, let $q \leq^*_\beta p$ mean that $q \leq_\beta p$ and $q \restriction \mathrm{even} = p \restriction \mathrm{even}$. For p and q in $\mathbb{P}^c_\beta$, let $q \leq^c_\beta p$ mean that $q \leq_\beta p$. We will abbreviate the forcing poset $(\mathbb{P}_\beta, \leq^*_\beta)$ as $\mathbb{P}^*_\beta$ and $(\mathbb{P}^c_\beta, \leq^c_\beta)$ as $\mathbb{P}^c_\beta$.*

Consider $p \in \mathbb{P}_\beta$ and $a \in \mathbb{P}^c_\beta$. Then a and p are compatible in $\mathbb{P}_\beta$ iff a and $p \restriction \mathrm{even}$ are compatible in $\mathbb{P}^c_\beta$ iff for all even $\gamma \in \mathrm{dom}(p) \cap \mathrm{dom}(a)$, $p(\gamma)$ and $a(\gamma)$ are compatible in $\mathrm{Add}(\omega)$, that is, $p(\gamma) \cup a(\gamma)$ is a function.

Notation 1.7. *Let $\beta \leq \alpha$. If $a \in \mathbb{P}^c_\beta$ and $p \in \mathbb{P}_\beta$, and a and p are compatible in $\mathbb{P}_\beta$, let $p + a$ denote the function s such that $\mathrm{dom}(s) := \mathrm{dom}(a) \cup \mathrm{dom}(p)$, for all even $\gamma \in \mathrm{dom}(s)$, $s(\gamma) := a(\gamma) \cup p(\gamma)$, and for all odd $\gamma \in \mathrm{dom}(s)$, $s(\gamma) := p(\gamma)$.*

The proofs of the next four lemmas are straightforward.

Lemma 1.8. *Let $\beta \leq \alpha$. If $a \in \mathbb{P}^c_\beta$ and $p \in \mathbb{P}_\beta$, and a and p are compatible in $\mathbb{P}_\beta$, then $p + a$ is in $\mathbb{P}_\beta$ and $p + a \leq_\beta p, a$. Moreover, $p + a$ is the greatest lower bound of p and a.*

Lemma 1.9. *Let $\beta \leq \alpha$. Let $p \in \mathbb{P}_\beta$ and $a \in \mathbb{P}^c_\beta$. Let G be a generic filter on $\mathbb{P}_\beta$. If p and a are both in G, then so is $p + a$.*

Lemma 1.10. *Let $\beta \leq \alpha$.*
(1) *For all $p \in \mathbb{P}_\beta$, $p \leq_\beta p \restriction \mathrm{even}$;*
(2) *if $q \leq_\beta p$ then $q \restriction \mathrm{even} \leq^c_\beta p \restriction \mathrm{even}$;*
(3) *if $q \leq^*_\beta p$, $a \in \mathbb{P}^c_\beta$, and a and p are compatible in $\mathbb{P}_\beta$, then a and q are compatible in $\mathbb{P}_\beta$ and $q + a \leq_\beta p + a$.*

Lemma 1.11. *Let $\beta \leq \alpha$. Suppose that $b \leq^c_\beta a$ and $q \leq_\beta p$, where a and p are compatible in $\mathbb{P}_\beta$ and b and q are compatible in $\mathbb{P}_\beta$. Then $q + b \leq_\beta p + a$.*

Lemma 1.12. *Let $\beta \leq \alpha$, $q \in \mathbb{P}_\beta$, $\dot{x}$ a $\mathbb{P}^c_\beta$-name, and $\varphi(x)$ a Δ_0-formula. Then*

$$q \Vdash_\beta \varphi(\dot{x}) \quad \textit{iff} \quad (q \restriction \mathrm{even}) \Vdash_{\mathbb{P}^c_\beta} \varphi(\dot{x}).$$

Proof. For the backwards implication, assume that $q \restriction \mathrm{even}$ forces in $\mathbb{P}^c_\beta$ that $\varphi(\dot{x})$ holds. If G is a generic filter on $\mathbb{P}_\beta$ which contains q, then $q \restriction \mathrm{even} \in G^c$ implies that $\dot{x}^{G^c} = \dot{x}^G$ satisfies φ in $V[G^c]$ and hence in $V[G]$. For the forward implication, suppose that q forces in $\mathbb{P}_\beta$ that $\varphi(\dot{x})$ holds. Consider any $b \leq^c_\beta q \restriction \mathrm{even}$. Fix a generic filter G on $\mathbb{P}_\beta$ which contains $q+b$, and let $x := \dot{x}^G = \dot{x}^{G^c}$. Since $q+b \leq_\beta q$, $q \in G$, and therefore $\varphi(x)$ holds in $V[G]$ and hence in $V[G^c]$. But $q+b \leq_\beta b$ implies that $b \in G \cap \mathbb{P}^c_\beta = G^c$. Thus, b does not force the negation of $\varphi(\dot{x})$. Since b was arbitrary, $q \restriction \mathrm{even}$ forces in $\mathbb{P}^c_\beta$ that $\varphi(\dot{x})$ holds. $\square$

In particular, in Definition 1.1(5) the property

$$q \restriction (\gamma \cap \mathrm{even}) \Vdash_{\mathbb{P}^c_\gamma} q(\gamma) \text{ is an end-extension of } p(\gamma)$$

is equivalent to

$$q \restriction \gamma \Vdash_\gamma q(\gamma) \text{ is an end-extension of } p(\gamma).$$

The next technical proposition will be crucial to the arguments in Section 2.

Proposition 1.13. *Let $\beta \le \alpha$. Suppose that $q \le_\beta p$. Let $b := q \restriction \mathrm{even}$. Then there exists $q' \in \mathbb{P}_\beta$ such that*

$$q \le_\beta q' \le_\beta^* p$$

and

$$q \le_\beta q' + b \le_\beta q.$$

Proof. Let $q' \restriction \mathrm{even} := p \restriction \mathrm{even}$. Let $\mathrm{dom}(q') \cap \mathrm{odd} := \mathrm{dom}(q) \cap \mathrm{odd}$. Consider $\gamma \in \mathrm{dom}(q') \cap \mathrm{odd}$. By the maximality principle for names, we can find a nice $\mathbb{P}^c_\gamma$-name $q'(\gamma)$ for a nonempty closed and bounded subset of ω_2 which end-extends $p(\gamma)$ such that, if $b \restriction \gamma$ is in the generic filter on $\mathbb{P}^c_\gamma$, then $q'(\gamma) = q(\gamma)$, and otherwise $q'(\gamma)$ is $p(\gamma)$ together with the least ordinal of cofinality ω_1 strictly above all members of $p(\gamma)$.

Assume for a moment that q' is a condition. Note that for all odd $\gamma \in \mathrm{dom}(q')$, $q \restriction (\gamma \cap \mathrm{even}) = b \restriction \gamma$ forces that $q'(\gamma) = q(\gamma)$. Based on this fact, it is easy to check that $q \le_\beta q'$. Also, $q' \restriction \mathrm{even} = p \restriction \mathrm{even}$, and for all odd $\gamma \in \mathrm{dom}(q')$, $\mathbb{P}^c_\gamma$ forces that $q'(\gamma)$ is an end-extension of $p(\gamma)$. It easily follows that $q' \le^*_\beta p$, which verifies the first pair of inequalities.

For the second pair, since $q \le_\beta p$, $b = q \restriction \mathrm{even} \le^c_\beta p \restriction \mathrm{even} = q' \restriction \mathrm{even}$. So b and q' are compatible in $\mathbb{P}_\beta$. Also, $q \le_\beta q'$ from the previous paragraph. By Lemma 1.11, $q = q + b \le_\beta q' + b$. Now if $\gamma \in \mathrm{dom}(q')$ is odd, and assuming $(q' + b) \restriction \gamma \le_\gamma q \restriction \gamma$, it follows that $(q' + b) \restriction (\gamma \cap \mathrm{even}) = b \restriction \gamma$ forces that $q'(\gamma) = q(\gamma)$, and hence $(q' + b) \restriction (\gamma + 1) \le_{\gamma+1} q \restriction (\gamma + 1)$. It easily follows by an inductive argument that $q' + b \le_\beta q$.

Thus, we have shown that if $q' \in \mathbb{P}_\beta$, then all of the inequalities stated in the proposition hold. Moreover, the above argument also shows that if, for a fixed $\xi \le \beta$, $q' \restriction \xi \in \mathbb{P}_\xi$, then all of the inequalities stated in the proposition hold for the conditions restricted to ξ.

It remains to show that q' is a condition. By Definition 1.1, it suffices to show that whenever $\gamma \in \mathrm{dom}(q')$ is odd, if we assume that $q' \restriction \gamma$ is in $\mathbb{P}_\gamma$ and is $\le^*_\gamma$-below $p \restriction \gamma$, then

$$q' \restriction \gamma \Vdash_\gamma q'(\gamma) \cap \dot{S}_\gamma = \emptyset.$$

Let G be a generic filter on $\mathbb{P}_\gamma$ which contains $q' \restriction \gamma$. Let $S_\gamma := \dot{S}^G_\gamma$, $G^c := G \cap \mathbb{P}^c_\gamma$, and $x := q'(\gamma)^{G^c}$. We will show that $x \cap S_\gamma = \emptyset$.

By the choice of $q'(\gamma)$, x is equal to $q(\gamma)^{G^c}$ provided that $b \restriction \gamma \in G^c$, and otherwise is equal to $p(\gamma)^{G^c}$ together with an ordinal of cofinality ω_1. In the latter case, since $q' \restriction \gamma \le_\gamma p \restriction \gamma$ and $p \restriction \gamma \Vdash_\gamma p(\gamma) \cap \dot{S}_\gamma = \emptyset$, we have that $p \restriction \gamma \in G$ and $p(\gamma)^{G^c}$ is disjoint from S_γ. Since x is equal to $p(\gamma)^{G^c}$ together with an ordinal of cofinality ω_1, whereas S_γ consists of ordinals of cofinality ω, x is disjoint from S_γ. So assume that $b \restriction \gamma \in G^c$. Then by Lemma 1.9, $(q' \restriction \gamma) + (b \restriction \gamma) \in G$. But this condition is $\le_\gamma$-below $q \restriction \gamma$. So $q \restriction \gamma \in G$. As $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $q(\gamma) \cap \dot{S}_\gamma = \emptyset$, it follows that $q(\gamma)^{G^c} = q'(\gamma)^{G^c} = x$ is disjoint from S_γ. □

Definition 1.14. *Let $\beta \le \alpha$. Define $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ as the forcing poset consisting of pairs (a, p), where $a \in \mathbb{P}^c_\beta$ and $p \in \mathbb{P}_\beta$, such that a and p are compatible in $\mathbb{P}_\beta$, with the ordering $(a_1, p_1) \le (a_0, p_0)$ if $a_1 \le^c_\beta a_0$ and $p_1 \le^*_\beta p_0$.*

Observe that if $p \in \mathbb{P}_\beta$, then $(p \restriction \mathrm{even}, p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$.

For any forcing poset $\mathbb{Q}$ and $q \in \mathbb{Q}$, we will use the notation $\mathbb{Q}/q$ for the suborder $\{r \in \mathbb{Q} : r \le_{\mathbb{Q}} q\}$.

The next lemma reveals that $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ is essentially a product forcing.

Lemma 1.15. *Let $\beta \leq \alpha$. Let $(a,p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$, and assume that $a \leq^c_\beta p \restriction \text{even}$. Then $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/(a,p)$ is equal to the product forcing*

$$(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p).$$

Proof. Let $(b,q) \leq (a,p)$ in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Then $b \leq^c_\beta a$ and $q \leq^*_\beta p$. Thus, $(b,q) \in (\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$.

Now consider $(b,q) \in (\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$. Then $b \leq^c_\beta a$ and $q \leq^*_\beta p$. By the choice of (a,p), $b \leq^c_\beta a \leq^c_\beta p \restriction \text{even} = q \restriction \text{even}$, and in particular, b and q are compatible in $\mathbb{P}_\beta$. Therefore, (b,q) is in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. And $b \leq^c_\beta a$ and $q \leq^*_\beta p$ means that $(b,q) \leq (a,p)$ in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Finally, it is immediate by definition that these two forcings have the same ordering. $\square$

Note that there are densely many conditions (a,p) in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ such that $a \leq^c_\beta p \restriction$ even. This observation together with Lemma 1.15 easily implies the next result.

Lemma 1.16. *Let $\beta \leq \alpha$. Suppose that H is a generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Then there is a condition $(a,p) \in H$ such that $a \leq^c_\beta p \restriction \text{even}$. Moreover, if (a,p) is any such condition in H, then letting $K := H \cap ((\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/(a,p))$, we have that K is a generic filter on $(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$ and $V[H] = V[K]$.*

To provide some additional clarification, let us describe the forcing poset $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ as a disjoint sum of product forcings. Namely, for each $b \in \mathbb{P}^c_\beta$, observe that $\mathbb{P}^*_\beta/b = \{p \in \mathbb{P}_\beta : p \restriction \text{even} = b\}$. In particular, if $b \neq c$ then $\mathbb{P}^*_\beta/b$ and $\mathbb{P}^*_\beta/c$ are disjoint, and moreover, any condition in $\mathbb{P}^*_\beta/b$ and any condition in $\mathbb{P}^*_\beta/c$ are $\leq^*_\beta$-incomparable.

Let D be the dense set of conditions (a,p) in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ such that $a \leq^c_\beta p \restriction$ even. It is easy to check that

$$D = \bigcup\{(\mathbb{P}^c_\beta/b) \times (\mathbb{P}^*_\beta/b) : b \in \mathbb{P}^c_\beta\}.$$

Thus, $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ contains a dense subset which is a disjoint sum of product forcings.

Definition 1.17. *Let $\beta \leq \alpha$. Define $\tau_\beta : \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta \to \mathbb{P}_\beta$ by $\tau_\beta(a,p) := p + a$.*

Note that this definition makes sense by Lemma 1.8.

Lemma 1.18. *Let $\beta \leq \alpha$. The function $\tau_\beta : \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta \to \mathbb{P}_\beta$ is a surjective projection mapping.*

Proof. Suppose that $(b,q) \leq (a,p)$ in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Then by definition, $b \leq^c_\beta a$ and $q \leq^*_\beta p$. Hence, $q \leq_\beta p$. By Lemma 1.11, $\tau_\beta(b,q) = q + b \leq_\beta p + a = \tau_\beta(a,p)$.

Consider a condition $p \in \mathbb{P}_\beta$. Then $(p \restriction \text{even}, p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$, and $\tau_\beta(p \restriction \text{even}, p) = p$. So τ_β is surjective.

Now assume that $q \leq_\beta \tau_\beta(a,p) = p + a$. We will find $(b,q') \leq (a,p)$ in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ such that $\tau_\beta(b,q') \leq_\beta q$. Now $q \leq_\beta p + a \leq_\beta p$, so $q \leq_\beta p$. Let $b := q \restriction \text{even}$. Then by Lemma 1.10(2),

$$b = q \restriction \text{even} \leq^c_\beta (p+a) \restriction \text{even} \leq^c_\beta a, p \restriction \text{even}.$$

So $b \leq^c_\beta a$ and $b \leq^c_\beta p \restriction$ even. Apply Proposition 1.13 to find $q' \in \mathbb{P}_\beta$ such that $q \leq_\beta q' \leq^*_\beta p$ and $q \leq_\beta q' + b \leq_\beta q$.

Since $b \leq^c_\beta p \restriction \text{even} = q' \restriction \text{even}$, b and $q' \restriction \text{even}$ are compatible in $\mathbb{P}^c_\beta$. Hence, b and q' are compatible in $\mathbb{P}_\beta$. Therefore, $(b, q') \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Also, as noted above, $b \leq^c_\beta a$ and $q' \leq^*_\beta p$, and therefore $(b, q') \leq (a, p)$ in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Finally, $\tau_\beta(b, q') = q' + b \leq_\beta q$. $\square$

The final result from this section will be used in the cardinal preservation arguments needed for the consistency result.

Lemma 1.19. *Assume that $2^{\omega_1} = \omega_2$. Then:*

(1) *for all $\beta \leq \alpha$ with $|\beta| \leq \omega_2$, $|\mathbb{P}_\beta| \leq \omega_2$;*
(2) *if $\alpha = \omega_3$, then $\mathbb{P}_\alpha = \bigcup\{\mathbb{P}_\beta : \beta < \omega_3\}$ has size ω_3 and $\mathbb{P}_\alpha$ is ω_3-c.c.;*
(3) *if $\alpha = \omega_3$, then for all $a \in \mathbb{P}^c_\alpha$, $\mathbb{P}^*_\alpha/a = \bigcup\{\mathbb{P}^*_\beta/a : \beta < \omega_3\}$ has size ω_3 and is ω_3-c.c.*

Proof. (1) Since $\alpha \leq \omega_3$, for all $\gamma \in \alpha$, $\mathbb{P}^c_\gamma$ is ω_1-c.c. and has size at most ω_2. Hence, there are at most $2^{\omega_1} = \omega_2$ many nice $\mathbb{P}^c_\gamma$-names for bounded subsets of ω_2. With this observation, (1) easily follows by induction on β.

(2) The first part of (2) easily follows from Definition 1.1. If $\{p_i : i < \omega_3\} \subseteq \mathbb{P}_\alpha$, then a Δ-system argument implies that there is a set $X \subseteq \omega_3$ of size ω_3 and a function r such that for all $i < j$ in X, $\text{dom}(p_i) \cap \text{dom}(p_j) = \text{dom}(r)$ and for all $\gamma \in \text{dom}(r)$, $p_i(\gamma) = p_j(\gamma)$. It easily follows that $p_i \cup p_j$ is a condition in $\mathbb{P}_\alpha$ below p_i and p_j, proving that $\mathbb{P}_\alpha$ is ω_3-c.c.

(3) The proof of (3) is similar to the proof of (2). $\square$

Note that if $\alpha = \omega_3$, then $\mathbb{P}^*_\alpha$ is not ω_3-c.c., since any two conditions in $\mathbb{P}^*_\alpha$ with different even parts are incompatible in $\mathbb{P}^*_\alpha$.

2. Distributivity and cardinal preservation

The most challenging part of our main consistency result will be in the verification that a particular suitable mixed support forcing iteration $\langle \mathbb{P}_\beta : \beta \leq \omega_3 \rangle$, which destroys the stationarity of nonreflecting subsets of $\omega_2 \cap \text{cof}(\omega)$, preserves ω_1 and ω_2. By Propositions 2.1 and 2.2 below, it will suffice to prove that $\mathbb{P}^*_\beta$ is ω_2-distributive for all $\beta < \omega_3$.

For some perspective, let us review in rough outline the original Harrington-Shelah argument [3]. Start with a model of GCH in which κ is a Mahlo cardinal, and let G be a generic filter on the Lévy collapse $\text{Col}(\omega_1, {<}\kappa)$. In $V[G]$, define a forcing iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \leq \omega_3,\ \beta < \omega_3 \rangle$ so that for all $\alpha < \omega_3$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a forcing which kills the stationarity of a nonreflecting subset of $\omega_2 \cap \text{cof}(\omega)$, bookkeeping so that all nonreflecting stationary sets are handled. To prove that this forcing iteration is ω_2-distributive, fix $\alpha < \omega_3$, and consider an appropriate elementary substructure M containing $\mathbb{P}_\alpha$ with transitive collapsing map π. Then show that any condition in $M \cap \mathbb{P}_\alpha$ has an extension which lies in every dense open subset of $\mathbb{P}_\alpha$ in M.

The fact that $\mathbb{P}_\alpha$ is an iteration of natural posets adding clubs disjoint from nonreflecting subsets of ω_2 implies that in $V[G \restriction (M \cap \kappa)]$, $\pi(\mathbb{P}_\alpha)$ is an iteration of natural posets adding clubs disjoint from *nonstationary* subsets of $M \cap \kappa$. As such, $\pi(\mathbb{P}_\alpha)$ contains an $(M \cap \kappa)$-closed dense subset. It follows that the tail of the Lévy collapse provides a $V[G \restriction (M \cap \kappa)]$-generic filter on $\pi(\mathbb{P}_\alpha)$ in $V[G]$, and the image of this filter under π^{-1} is an M-generic filter on $\mathbb{P}_\alpha$. Hence, a lower bound of this filter, which does exist, is a member of every dense open subset of $\mathbb{P}_\alpha$ in M.

Let us compare these arguments with our situation. Instead of forcing with a Lévy collapse, our preparation forcing will be a countable support iteration of proper forcings which is designed to collapse κ to become ω_2 and ensure the existence of sufficiently generic filters for certain forcings. Let G be a generic filter for the preparation forcing. In $V[G]$, we define a suitable mixed support forcing iteration $\mathbb{P}$ which adds reals and clubs disjoint from nonreflecting sets.

Consider an elementary substructure M with transitive collapsing map π. In order to prove that $\mathbb{P}^*$ is ω_2-distributive, one might try to argue similarly as above that in $V[G \restriction (M \cap \kappa)]$, $\pi(\mathbb{P})$ is a suitable mixed support forcing iteration for adding reals and adding clubs disjoint from nonstationary sets. It turns out, however, that we can only show that the product $\pi(\mathbb{P}^c \otimes \mathbb{P}^*)$ forces that the collapse of a nonreflecting set is nonstationary, rather than $\pi(\mathbb{P})$. Nonetheless, by some technical arguments this will suffice to prove that $\mathbb{P}^*$ is ω_2-distributive, and hence that $\mathbb{P}$ preserves cardinals.

Proposition 2.1. *Let $\langle \mathbb{P}_\beta : \beta \le \alpha \rangle$ be a suitable mixed support forcing iteration. Let $\beta \le \alpha$. If $\mathbb{P}^*_\beta$ is ω_2-distributive, then $\mathbb{P}_\beta$ preserves ω_1 and ω_2.*

Proof. Suppose for a contradiction that $p \in \mathbb{P}_\beta$ forces that either ω_1^V or ω_2^V is no longer a cardinal in $V^{\mathbb{P}_\beta}$. Let $a := p \restriction \text{even}$. Let H be a generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ which contains the condition (a, p). Let $G := \tau_\beta[H]$. Then G is a generic filter on $\mathbb{P}_\beta$ by Lemma 1.18 and $p = p + a = \tau_\beta(a, p)$ is in G. Therefore, either ω_1^V or ω_2^V is no longer a cardinal in $V[G]$, and hence in $V[H]$.

By Lemma 1.16, $V[H] = V[K]$, where $K = K_1 \times K_2$ is a generic filter on $(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$. Now $\mathbb{P}^*_\beta$ is ω_2-distributive by assumption, so ω_1^V and ω_2^V remain cardinals in $V[K_2]$. By absoluteness, in $V[K_2]$, $\mathbb{P}^c_\beta$ is still isomorphic to Cohen forcing, and hence is ω_1-c.c. Therefore, ω_1^V and ω_2^V remain cardinals in $V[K_2][K_1] = V[K_1][K_2] = V[K] = V[H]$, which is a contradiction. □

Proposition 2.2. *Assume that $2^{\omega_1} = \omega_2$. Let $\langle \mathbb{P}_\beta : \beta \le \omega_3 \rangle$ be a suitable mixed support forcing iteration. Suppose that for all $\beta < \omega_3$, $\mathbb{P}^*_\beta$ is ω_2-distributive. Then $\mathbb{P}^*_{\omega_3}$ is ω_2-distributive, and hence preserves ω_1 and ω_2.*

Proof. Let $\mathbb{P} := \mathbb{P}^*_{\omega_3}$. Consider $p \in \mathbb{P}$. Let $a := p \restriction \text{even}$. Then easily $p \in \mathbb{P}/a$.

Suppose that p forces in $\mathbb{P}$ that $\{\dot\alpha_i : i < \omega_1\}$ is a set of ordinals. We will find q below p in $\mathbb{P}$ which decides the value of $\dot\alpha_i$, for all $i < \omega_1$. Without loss of generality, we can assume that each $\dot\alpha_i$ is a nice $(\mathbb{P}/a)$-name for an ordinal. It easily follows by Lemma 1.19(3) that each $\dot\alpha_i$ is a nice $(\mathbb{P}^*_\beta/a)$-name for an ordinal for some $\beta < \omega_3$. Thus, we can find an ordinal $\xi < \omega_3$ such that $p \in \mathbb{P}^*_\xi/a$ and each $\dot\alpha_i$ is a $(\mathbb{P}^*_\xi/a)$-name for an ordinal.

Since $\mathbb{P}^*_\xi$ is ω_2-distributive by assumption, fix $q \le^*_\xi p$ which decides in $\mathbb{P}^*_\xi$ the value of $\dot\alpha_i$ for all $i < \omega_1$. Then $q \le_{\mathbb{P}} p$ and q decides in $\mathbb{P}$ the value of $\dot\alpha_i$ for all $i < \omega_1$. □

For the remainder of the section, fix a suitable mixed support forcing iteration $\langle \mathbb{P}_\beta : \beta \le \alpha \rangle$, where $\alpha < \omega_3$, based on a sequence of names $\langle \dot S_\gamma : \gamma \in \alpha \cap \text{odd} \rangle$.

Before stating the next result, we make some clarifying remarks about names. Consider $\beta \le \alpha$. Then we have four forcing posets associated with β: $\mathbb{P}^c_\beta$, $\mathbb{P}_\beta$, $\mathbb{P}^*_\beta$, and $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. If H is a generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$, then $G := \tau_\beta[H]$ is a generic filter on $\mathbb{P}_\beta$, and in turn $G^c := G \cap \mathbb{P}^c_\beta$ is a generic filter on $\mathbb{P}^c_\beta$. Accordingly, if $\dot x$ is either a $\mathbb{P}_\beta$-name or a $\mathbb{P}^c_\beta$-name, when we talk about $\dot{x}$ in the context of statements in the forcing language of $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$, we really mean the $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$-name for the interpretation of $\dot{x}$ under $\tau_\beta[\dot{H}]$ or $\tau_\beta[\dot{H}] \cap \mathbb{P}^c_\beta$ respectively. Similar comments apply to $\mathbb{P}^c_\beta$-names in the context of the forcing language for $\mathbb{P}_\beta$.

The next two technical results will be crucial for the rest of the paper.

Proposition 2.3. *Let $\beta \le \alpha$, and assume that $\mathbb{P}^*_\beta$ is ω_2-distributive. Suppose that $\dot{x}$ is a $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$-name for a set of ordinals of size less than ω_2. Then for all $p \in \mathbb{P}_\beta$, there is $q \le^*_\beta p$ and a nice $\mathbb{P}^c_\beta$-name $\dot{b}$ of size ω_1 such that $(q \restriction \mathrm{even}, q)$ forces in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ that $\dot{x} = \dot{b}$.*

Proof. Let $\dot{x}$ and p be as above. Let $a := p \restriction \mathrm{even}$. For the purpose of finding the condition q and the name $\dot{b}$, let us consider a generic filter H on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ which contains the condition (a, p). By Lemma 1.16, $V[H] = V[K]$, where $K = K_1 \times K_2$ is a generic filter on $(\mathbb{P}^c_\beta/a) \times (\mathbb{P}^*_\beta/p)$.

Let $x := \dot{x}^H$. Then $x \in V[K_2][K_1]$. Since $\mathbb{P}^c_\beta$ is still isomorphic to Cohen forcing in $V[K_2]$, we can cover x by some set of ordinals $y \in V[K_2]$ of size ω_1. Now fix in $V[K_2]$ a nice $(\mathbb{P}^c_\beta/a)$-name $\dot{b}$ for a subset of y such that $\dot{b}^{K_1} = x$. Moreover, by the maximality principle applied in $V[K_2]$, we can find such a nice name so that $\mathbb{P}^c_\beta/a$ forces over $V[K_2]$ that $\dot{b}$ is equal to $\dot{x}$ (interpreted by the appropriate generic filters).

Since $\dot{b}$ is a nice name for a subset of y and $\mathbb{P}^c_\beta/a$ is ω_1-c.c. in $V[K_2]$, $\dot{b}$ has size ω_1 in $V[K_2]$. Easily $\dot{b} \subseteq V$. Therefore, since $\mathbb{P}^*_\beta$ is ω_2-distributive, the name $\dot{b}$ is in V. As K_2 is a V-generic filter on $\mathbb{P}^*_\beta/p$, we can find $q \le^*_\beta p$ in K_2 which forces in $\mathbb{P}^*_\beta$ that $\mathbb{P}^c_\beta/a$ forces that $\dot{b}$ equals $\dot{x}$. It is now straightforward to check that q and $\dot{b}$ are as required. □

Proposition 2.4. *Let $\beta \le \alpha$, and assume that $\mathbb{P}^*_\beta$ is ω_2-distributive. Suppose that $\dot{x}$ is a $\mathbb{P}_\beta$-name for a set of ordinals of size less than ω_2. Then for all $p \in \mathbb{P}_\beta$, there is $q \le^*_\beta p$ and a nice $\mathbb{P}^c_\beta$-name $\dot{b}$ of size ω_1 such that q forces in $\mathbb{P}_\beta$ that $\dot{x} = \dot{b}$.*

Proof. Let $\dot{x}'$ be a $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$-name for the interpretation of $\dot{x}$ by $\tau_\beta[\dot{H}]$, where $\dot{H}$ is the canonical name for the generic filter on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Then obviously $\dot{x}'$ is a $(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$-name for a set of ordinals of size less than ω_2. By Proposition 2.3, there is $q \le^*_\beta p$ and a nice $\mathbb{P}^c_\beta$-name $\dot{b}$ of size ω_1 such that $(q \restriction \mathrm{even}, q)$ forces in $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ that $\dot{x}' = \dot{b}$.

It remains to show that q forces in $\mathbb{P}_\beta$ that $\dot{x} = \dot{b}$. Suppose for a contradiction that $r \le_\beta q$ and r forces in $\mathbb{P}_\beta$ that $\dot{x} \ne \dot{b}$. Let $a := r \restriction \mathrm{even}$. By Proposition 1.13, fix $r' \in \mathbb{P}_\beta$ such that $r \le_\beta r' \le^*_\beta q$ and $r \le_\beta r' + a \le_\beta r$. Then r' and a are compatible in $\mathbb{P}_\beta$, so $(a, r') \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$.

Fix a generic filter H on $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ which contains (a, r'). Note that $(a, r') \le (q \restriction \mathrm{even}, q)$, so $(q \restriction \mathrm{even}, q) \in H$. Let $G := \tau_\beta[H]$ and $G^c := G \cap \mathbb{P}^c_\beta$. Then $\tau_\beta(a, r') = r' + a \in G$. Since $r' + a \le_\beta r$, $r \in G$. By the choice of r, $\dot{x}^G \ne \dot{b}^{G^c}$. By the choice of q, $(\dot{x}')^H = \dot{b}^{G^c}$. Finally, by the choice of $\dot{x}'$, $(\dot{x}')^H = \dot{x}^G$. Thus, $\dot{x}^G \ne \dot{b}^{G^c}$ and yet $\dot{x}^G = (\dot{x}')^H = \dot{b}^{G^c}$, which is a contradiction. □

For a set $A \subseteq \omega_2$, let $\mathrm{CU}(A)$ denote the forcing poset consisting of closed and bounded subsets of A, ordered by end-extension. Assuming that A is unbounded in ω_2, it is easy to check that $\mathrm{CU}(A)$ adds a closed and cofinal subset of ω_2 which is contained in A.

One of the main consequences of Proposition 2.4 is that our suitable mixed support forcing iteration will in fact add the desired generic filters for the club adding forcings.

Proposition 2.5. *Let $\gamma < \alpha$ be odd, and assume that $\mathbb{P}^*_\gamma$ is ω_2-distributive. Then $\mathbb{P}_{\gamma+1}$ is forcing equivalent to $\mathbb{P}_\gamma * CU(\omega_2 \setminus \dot{S}_\gamma)$.*

Proof. Let $\mathbb{Q} := \mathbb{P}_\gamma * \mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)$. Define $f : \mathbb{P}_{\gamma+1} \to \mathbb{Q}$ by $f(p) := (p \restriction \gamma) * p(\gamma)$. Let us check that f actually maps into $\mathbb{Q}$. For a condition $p \in \mathbb{P}_{\gamma+1}$, Definition 1.1(3) implies that

(1) $p \restriction \gamma \in \mathbb{P}_\gamma$;
(2) $p(\gamma)$ is a $\mathbb{P}^c_\gamma$-name for a closed and bounded subset of ω_2;
(3) $p \restriction \gamma \Vdash_\gamma p(\gamma) \cap \dot{S}_\gamma = \emptyset$.

By Lemma 1.12, (2) implies that $p(\gamma)$ is a $\mathbb{P}_\gamma$-name for a closed and bounded subset of ω_2. So by (3), $p \restriction \gamma \Vdash_\gamma p(\gamma) \in \mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)$. Hence, $f(p) = (p \restriction \gamma) * p(\gamma)$ is in $\mathbb{Q}$.

We claim that f is a dense embedding. It suffices to show that for all p and q in $\mathbb{P}_{\gamma+1}$, $q \leq_{\gamma+1} p$ iff $f(q) \leq_{\mathbb{Q}} f(p)$, and the range of f is dense in $\mathbb{Q}$.

Consider p and q in $\mathbb{P}_{\gamma+1}$. Then by Lemma 1.5(3), $q \leq_{\gamma+1} p$ iff

(a) $q \restriction \gamma \leq_\gamma p \restriction \gamma$;
(b) $q \restriction (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}^c_\gamma$ that $q(\gamma)$ end-extends $p(\gamma)$.

Assume that $q \leq_{\gamma+1} p$. Then $q \restriction \gamma \leq_\gamma p \restriction \gamma$. To see that $f(q) = (q \restriction \gamma) * q(\gamma) \leq_{\mathbb{Q}} (p \restriction \gamma) * p(\gamma)$, it remains to show that $q \restriction \gamma \Vdash_\gamma q(\gamma) \leq_{\mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)} p(\gamma)$, or in other words, that $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $q(\gamma)$ end-extends $p(\gamma)$. By Lemma 1.12, this follows from (b) above.

Assume conversely that $f(q) \leq_{\mathbb{Q}} f(p)$. Then $q \restriction \gamma \leq_\gamma p \restriction \gamma$, and $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $q(\gamma) \leq_{\mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)} p(\gamma)$. Hence, $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $q(\gamma)$ end-extends $p(\gamma)$. By Lemma 1.12, $q \restriction (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}^c_\gamma$ that $q(\gamma)$ end-extends $p(\gamma)$. By Lemma 1.5(3), $q \leq_{\gamma+1} p$.

To show that f is dense, consider $r \in \mathbb{Q}$. Then $r = r_0 * \dot{r}_1$, where $r_0 \in \mathbb{P}_\gamma$ and r_0 forces in $\mathbb{P}_\gamma$ that $\dot{r}_1 \in \mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)$. We will find $w \in \mathbb{P}_{\gamma+1}$ such that $f(w) \leq_{\mathbb{Q}} r$. By extending r if necessary, we may assume without loss of generality that r_0 forces that $\dot{r}_1$ is nonempty.

By Proposition 2.4, fix $t \leq^*_\gamma r_0$ and a nice $\mathbb{P}^c_\gamma$-name $\dot{b}$ such that $t \Vdash_\gamma \dot{r}_1 = \dot{b}$. By the maximality principle for names, we may assume that $\dot{b}$ is a nice $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2. We claim that $w := t \cup \{(\gamma, \dot{b})\}$ is a condition in $\mathbb{P}_{\gamma+1}$ and $f(w) \leq_{\mathbb{Q}} r$. We know that $w \restriction \gamma = t$ is in $\mathbb{P}_\gamma$, $w(\gamma) = \dot{b}$ is a nice $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2, and $w \restriction \gamma = t$ forces in $\mathbb{P}_\gamma$ that $w(\gamma) = \dot{b}$ is equal to $\dot{r}_1$, which is in $\mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)$ and hence is disjoint from $\dot{S}_\gamma$. By Definition 1.1, $w \in \mathbb{P}_{\gamma+1}$. Since $t \leq^*_\gamma r_0$, we have that $t \leq_\gamma r_0$. Also, t forces in $\mathbb{P}_\gamma$ that $\dot{r}_1 = \dot{b}$, and hence obviously that $\dot{b} \leq \dot{r}_1$ in $\mathrm{CU}(\omega_2 \setminus \dot{S}_\gamma)$. Therefore, $f(w) = t * \dot{b}$ extends $r = r_0 * \dot{r}_1$ in $\mathbb{Q}$. $\square$

We now turn to studying conditions under which $\mathbb{P}^*_\alpha$ is ω_2-distributive. The main result is Proposition 2.9 below.

Lemma 2.6. *Let $\gamma < \alpha$ be odd. Assume that $\dot{C}$ is a $(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$-name for a club subset of ω_2 which is disjoint from $\dot{S}_\gamma$. Let $p \in \mathbb{P}_\gamma$ and $\dot{\zeta}$ be a $\mathbb{P}_\gamma$-name for an ordinal. If $(p \upharpoonright \mathrm{even}, p)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\dot{\zeta}$ is in $\dot{C}$, then p forces in $\mathbb{P}_\gamma$ that $\dot{\zeta}$ is not in $\dot{S}_\gamma$.*

Proof. Suppose for a contradiction that there is $q \leq_\gamma p$ which forces in $\mathbb{P}_\gamma$ that $\dot{\zeta}$ is in $\dot{S}_\gamma$. Let $b := q \upharpoonright \mathrm{even}$. Apply Proposition 1.13 to fix $q' \in \mathbb{P}_\gamma$ such that $q \leq_\gamma q' \leq^*_\gamma p$ and $q \leq_\gamma q' + b \leq_\gamma q$.

Let H be a generic filter on $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ which contains the condition (b, q'). Let $G := \tau_\gamma[H]$, which is a generic filter on $\mathbb{P}_\gamma$. Let $\zeta := \dot{\zeta}^G$, $S_\gamma := \dot{S}^G_\gamma$, and $C := \dot{C}^H$. Then $C \cap S_\gamma = \emptyset$.

Since $q' \leq^*_\gamma p$ and $b \leq^c_\gamma p \upharpoonright \mathrm{even}$, it follows that $(b, q') \leq (p \upharpoonright \mathrm{even}, p)$, and hence $(p \upharpoonright \mathrm{even}, p) \in H$. Therefore, $\zeta \in C$. Since C is disjoint from S_γ, $\zeta \notin S_\gamma$. On the other hand, $\tau_\gamma(b, q') = q' + b \in G$ and $q' + b \leq_\gamma q$, so $q \in G$. By the choice of q, $\zeta \in S_\gamma$, and we have a contradiction. $\square$

Notation 2.7. *Let $\beta \leq \alpha$. Define the relation $\leq^{*,s}_\beta$ on $\mathbb{P}_\beta$ by letting $q \leq^{*,s}_\beta p$ if for all $r \leq^*_\beta q$, r and p are compatible in $\mathbb{P}^*_\beta$. We will abbreviate the forcing poset $(\mathbb{P}_\beta, \leq^{*,s}_\beta)$ as $\mathbb{P}^{*,s}_\beta$.*

Note that $q \leq^*_\beta p$ implies that $q \leq^{*,s}_\beta p$. It is easy to verify that the forcing poset $\mathbb{P}^{*,s}_\beta$ is separative, and the identity function is a dense embedding of $\mathbb{P}^*_\beta$ into $\mathbb{P}^{*,s}_\beta$.

Lemma 2.8. *Let $\beta \leq \alpha$. Assume that $q \leq^{*,s}_\beta p$. Then:*

1. $p \upharpoonright \mathrm{even} = q \upharpoonright \mathrm{even}$*;*
2. $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$*;*
3. *for all odd $\gamma \in \mathrm{dom}(p)$, $p \upharpoonright (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}^c_\gamma$ that one of $p(\gamma)$ and $q(\gamma)$ is an end-extension of the other.*

Proof. (1) By the definition of $\leq^{*,s}_\beta$, clearly p and q are compatible in $\mathbb{P}^*_\beta$. Fix $r \leq^*_\beta p, q$. Then $p \upharpoonright \mathrm{even} = r \upharpoonright \mathrm{even} = q \upharpoonright \mathrm{even}$.

(2) If not, then by (1) we can fix an odd ordinal $\gamma \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$. Fix a $\mathbb{P}^c_\gamma$-name $\dot{a}$ for the singleton consisting of the least member of $\omega_2 \cap \mathrm{cof}(\omega_1)$ which is strictly larger than $\max(p(\gamma))$ (we are using the fact that $p(\gamma)$ is forced to be nonempty by Definition 1.1(3)). Clearly, $\mathbb{P}^c_\gamma$ forces that $\dot{a}$ and $p(\gamma)$ have no common end-extension, and since $\mathbb{P}_\gamma$ forces that $\dot{S}_\gamma$ consists of ordinals of cofinality ω, $\mathbb{P}_\gamma$ forces that $\dot{a}$ is disjoint from $\dot{S}_\gamma$. Define $s := q \cup \{(\gamma, \dot{a})\}$. Then $s \in \mathbb{P}_\beta$, $s \leq^*_\beta q$, and s and p are incompatible in $\mathbb{P}^*_\beta$. This contradicts the assumption that $q \leq^{*,s}_\beta p$.

(3) Let $\gamma \in \mathrm{dom}(p) \cap \mathrm{odd}$. Then by (2), $\gamma \in \mathrm{dom}(q)$. Since p and q are compatible in $\mathbb{P}^*_\beta$, fix $r \leq^*_\beta p, q$. As $\gamma \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$, $r \upharpoonright (\gamma \cap \mathrm{even})$ forces in $\mathbb{P}^c_\gamma$ that $r(\gamma)$ is an end-extension of both $p(\gamma)$ and $q(\gamma)$. In particular, it forces that $p(\gamma)$ and $q(\gamma)$ have a common end-extension, and hence that one of them is an end-extension of the other. But $r \leq^*_\beta p$ implies that $r \upharpoonright \mathrm{even} = p \upharpoonright \mathrm{even}$, so $p \upharpoonright (\gamma \cap \mathrm{even})$ forces the same. $\square$

Proposition 2.9. *Assume that for all odd $\gamma < \alpha$, $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ forces that $\dot{S}_\gamma$ is a nonstationary subset of ω_2. Then both $\mathbb{P}^*_\alpha$ and $\mathbb{P}^{*,s}_\alpha$ contain an ω_2-closed dense subset.*

Proof. For each odd $\gamma < \alpha$, fix a $(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$-name $\dot{C}_\gamma$ for a club subset of ω_2 which is disjoint from $\dot{S}_\gamma$. For each $\beta \le \alpha$, define D_β as the set of conditions $p \in \mathbb{P}_\beta$ such that for all odd $\gamma \in \mathrm{dom}(p)$, $(p \restriction (\gamma \cap \mathrm{even}), p \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(p(\gamma)) \in \dot{C}_\gamma$. Observe that for all $\xi \le \beta \le \alpha$, $D_\xi \subseteq D_\beta$, and if $p \in D_\beta$, then $p \restriction \xi \in D_\xi$.

We claim that for all $\beta \le \alpha$, D_β is an ω_2-closed dense subset of both $\mathbb{P}^*_\beta$ and $\mathbb{P}^{*,s}_\beta$. The proof will be by induction on β, with the case $\beta = \alpha$ concluding the proof of the proposition. So fix $\beta \le \alpha$, and assume that for all $\xi < \beta$, D_ξ is an ω_2-closed dense subset of both $\mathbb{P}^*_\xi$ and $\mathbb{P}^{*,s}_\xi$. It follows that for all $\xi < \beta$, the forcing poset $\mathbb{P}^*_\xi$ is ω_2-distributive, since it is forcing equivalent to an ω_2-closed forcing poset.

We begin by proving closure. We will show that any $\le^{*,s}_\beta$-descending sequence of conditions in D_β of length a limit ordinal less than ω_2 has a $\le^*_\beta$-lower bound in D_β. Note that this implies that D_β is ω_2-closed in both $\mathbb{P}^*_\beta$ and $\mathbb{P}^{*,s}_\beta$. So consider a $\le^{*,s}_\beta$-descending sequence $\langle p_i : i < \delta \rangle$ of conditions in D_β, where $\delta < \omega_2$ is a limit ordinal. We will find $q \in D_\beta$ such that $q \le^*_\beta p_i$ for all $i < \delta$. Let $a := p_0 \restriction \mathrm{even}$. Then by Lemma 2.8(1), for all $i < \delta$, $p_i \restriction \mathrm{even} = a$.

Define q as follows. Let $q \restriction \mathrm{even} := a$. Let $\mathrm{dom}(q) \cap \mathrm{odd} := \bigcup\{\mathrm{dom}(p_i) \cap \mathrm{odd} : i < \delta\}$. Consider an odd ordinal γ in $\mathrm{dom}(q)$. By Lemma 2.8(3), $a \restriction \gamma$ forces in $\mathbb{P}^c_\gamma$ that $\{p_i(\gamma) : i < \delta\}$ is a family of closed and bounded subsets of ω_2 which are pairwise comparable under end-extension. It easily follows that $a \restriction \gamma$ forces that the union of this family is bounded in ω_2 and is closed below its supremum. Let $q(\gamma)$ be a nice $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2 which, if $a \restriction \gamma$ is in the generic filter on $\mathbb{P}^c_\gamma$, is equal to the union of $\{p_i(\gamma) : i < \delta\}$ together with the ordinal $\sup\{\max(p_i(\gamma)) : i < \delta\}$.

We prove by induction on $\xi \le \beta$ that $q \restriction \xi \in D_\xi$ and $q \restriction \xi \le^*_\xi p_i \restriction \xi$ for all $i < \delta$. It then follows that $q \in D_\beta$ and $q \le^*_\beta p_i$ for all $i < \delta$. Referring to Definition 1.1, the only nontrivial case to consider is when $\xi = \gamma + 1$ for an odd ordinal γ.

So assume that $\gamma < \beta$ is odd and $q \restriction \gamma$ is as required. Then $q \restriction \gamma \le^*_\gamma p_i \restriction \gamma$ for all $i < \delta$. By the definition of D_β, each p_i with $\gamma \in \mathrm{dom}(p_i)$ satisfies that $(p_i \restriction (\gamma \cap \mathrm{even}), p_i \restriction \gamma) = (a \restriction \gamma, p_i \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(p_i(\gamma)) \in \dot{C}_\gamma$. The fact that $q \restriction \gamma \le^*_\gamma p_i \restriction \gamma$ implies that $(q \restriction (\gamma \cap \mathrm{even}), q \restriction \gamma) = (a \restriction \gamma, q \restriction \gamma)$ is below $(a \restriction \gamma, p_i \restriction \gamma)$ in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$. Therefore, $(q \restriction (\gamma \cap \mathrm{even}), q \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\max(p_i(\gamma)) \in \dot{C}_\gamma$.

Since the above is true for all $i < \delta$ and $\dot{C}_\gamma$ is a name for a club, it follows that $(q \restriction (\gamma \cap \mathrm{even}), q \restriction \gamma)$ forces in $\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma$ that $\sup\{\max(p_i(\gamma)) : i < \delta\} = \max(q(\gamma)) \in \dot{C}_\gamma$. By Lemma 2.6, $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $\max(q(\gamma)) \notin \dot{S}_\gamma$. Since $q \restriction \gamma$ forces that any other member of $q(\gamma)$ is in $p_i(\gamma)$ for some $i < \delta$, and $q \restriction \gamma \le_\gamma p_i \restriction \gamma$ for all $i < \delta$, it follows that $q \restriction \gamma$ forces that $q(\gamma)$ is disjoint from $\dot{S}_\gamma$. Thus, $q \restriction (\gamma + 1)$ is in $\mathbb{P}_{\gamma+1}$. Now the inductive hypothesis and the above arguments imply that $q \restriction (\gamma + 1) \in D_{\gamma+1}$ and $q \restriction (\gamma + 1) \le^*_{\gamma+1} p_i \restriction (\gamma + 1)$ for all $i < \delta$. This completes the proof of closure.

It remains to show that D_β is a dense subset of $\mathbb{P}^*_\beta$ and $\mathbb{P}^{*,s}_\beta$. Note that it suffices to prove that D_β is dense in $\mathbb{P}^*_\beta$. Consider $p \in \mathbb{P}^*_\beta$, and we will find $q \le^*_\beta p$ in D_β. First, assume that $\beta = \xi + 1$ is a successor ordinal. If ξ is even, then fix $q_0 \le^*_\xi p \restriction \xi$ in D_ξ by the inductive hypothesis. Then $q_0 \cup \{(\xi, p(\xi))\} \le^*_\beta p$ is in D_β. Now suppose that ξ is odd. If $\xi \notin \mathrm{dom}(p)$, then fix $q_0 \le^*_\xi p \restriction \xi$ in D_ξ by the inductive hypothesis. Then $q_0 \le^*_\beta p$ and $q_0 \in D_\beta$.

Suppose that $\xi \in \mathrm{dom}(p)$. Let $\dot{x}$ be a $(\mathbb{P}^c_\xi \otimes \mathbb{P}^*_\xi)$-name for $p(\xi)$ together with the least member of $\dot{C}_\xi$ strictly above $\max(p(\xi))$. Since $\mathbb{P}^*_\xi$ is ω_2-distributive by the inductive hypothesis, by Proposition 2.3 we can fix $q_0 \leq^*_\xi p \restriction \xi$ and a nice $\mathbb{P}^c_\xi$-name $\dot{b}$ such that $(q_0 \restriction \mathrm{even}, q_0)$ forces in $\mathbb{P}^c_\xi \otimes \mathbb{P}^*_\xi$ that $\dot{b} = \dot{x}$. By the maximality principle for names, we may assume without loss of generality that $\dot{b}$ is a nice $\mathbb{P}^c_\xi$-name for a nonempty closed and bounded subset of ω_2. Note that $(q_0 \restriction \mathrm{even}, q_0)$ forces in $\mathbb{P}^c_\xi \otimes \mathbb{P}^*_\xi$ that $\max(\dot{b}) = \max(\dot{x}) \in \dot{C}_\xi$. By Lemma 2.6, q_0 forces in $\mathbb{P}_\xi$ that $\max(\dot{b}) \notin \dot{S}_\xi$. Now fix $r_0 \leq^*_\xi q_0$ in D_ξ by the inductive hypothesis. Let $r := r_0 \cup \{(\xi, \dot{b})\}$. Since $r_0 \leq_\xi q_0$, r_0 forces in $\mathbb{P}_\xi$ that $\max(\dot{b}) \notin \dot{S}_\xi$. As $r_0 \leq_\xi p \restriction \xi$, r_0 forces in $\mathbb{P}_\xi$ that $\dot{b}$ is disjoint from $\dot{S}_\xi$. Thus, $r \in \mathbb{P}_\beta$. Also, clearly r is in D_β and $r \leq^*_\beta p$.

Secondly, assume that β is a limit ordinal. If $\mathrm{cf}(\beta) \geq \omega_2$, then for some $\xi < \beta$, $\mathrm{dom}(p) \subseteq \xi$, and hence $p \in \mathbb{P}_\xi$. By the inductive hypothesis, we can fix $q \leq^*_\xi p$ in D_ξ. Then $q \leq^*_\beta p$ is in D_β.

Suppose that $\mathrm{cf}(\beta) < \omega_2$. Fix a strictly increasing and continuous sequence $\langle \beta_i : i < \mathrm{cf}(\beta) \rangle$ which is cofinal in β, and let $\beta_{\mathrm{cf}(\beta)} = \beta$. Since $\mathrm{dom}(p) \cap \mathrm{even}$ is finite, we may assume that $\mathrm{dom}(p) \cap \mathrm{even} \subseteq \beta_0$. We define by induction a $\leq^*_\beta$-descending sequence of conditions $\langle p_i : i \leq \mathrm{cf}(\beta) \rangle$ below p such that for each $i \leq \mathrm{cf}(\beta)$, $p_i \restriction \beta_i \in D_{\beta_i}$ if $i > 0$, and $p_i \restriction [\beta_i, \beta) = p \restriction [\beta_i, \beta)$.

Let $p_0 := p$. Let $i < \mathrm{cf}(\beta)$, and assume that p_j is defined as required for all $j \leq i$. By the inductive hypothesis, fix $p^-_{i+1} \leq^*_{\beta_{i+1}} p_i \restriction \beta_{i+1}$ in $D_{\beta_{i+1}}$. Now let $p_{i+1} := p^-_{i+1} \cup p \restriction [\beta_{i+1}, \beta)$. Then easily p_{i+1} is as required.

Let $\delta \leq \mathrm{cf}(\beta)$ be a limit ordinal, and assume that p_i is defined as required for all $i < \delta$. Then for all $i < j < \delta$, $p_j \leq^*_\beta p_i$. Since $\mathrm{dom}(p) \cap \mathrm{even} \subseteq \beta_0$, it easily follows that for all $i < j < \delta$, $p_j \restriction \beta_j \leq^*_{\beta_\delta} p_i \restriction \beta_i$. Therefore, $\langle p_i \restriction \beta_i : i < \delta \rangle$ is a $\leq^*_{\beta_\delta}$-descending sequence in D_{β_δ}. Since we have already proven the ω_2-closure of D_{β_δ}, we can find $p^-_\delta \in D_{\beta_\delta}$ such that $p^-_\delta \leq^*_{\beta_\delta} p_i \restriction \beta_i$ for all $i < \delta$. As $\sup_{i<\delta} \beta_i = \beta_\delta$, it easily follows that $p^-_\delta \leq^*_{\beta_\delta} p_i \restriction \beta_\delta$ for all $i < \delta$. Let $p_\delta := p^-_\delta \cup p \restriction [\beta_\delta, \beta)$. Then $p_\delta \leq^*_\beta p_i$ for all $i < \delta$ and $p_\delta \restriction \beta_\delta = p^-_\delta \in D_{\beta_\delta}$. $\square$

The next result describes how we will use the preparation forcing in the proof of the main consistency result.

Lemma 2.10. *Assume that $2^{\omega_1} = \omega_2$. Suppose that the forcing poset $\mathbb{P}^{*,s}_\alpha$ contains an ω_2-closed dense subset. Let $G \times H$ be a generic filter on $Add(\omega, \omega_2) \times Add(\omega_2)$. Then in $V[G \times H]$, for any condition $(a, p) \in \mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ such that $a \leq^c_\alpha p \restriction \mathrm{even}$, there exists a generic filter K on $\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ which contains (a, p), and moreover, $V[G \times H]$ is a generic extension of $V[K]$ by an ω_1-c.c. forcing poset.*

Proof. Fix an ω_2-closed dense subset D of $\mathbb{P}^{*,s}_\alpha$. Consider a condition $(a, p) \in \mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ such that $a \leq^c_\alpha p \restriction \mathrm{even}$. Let $D_p := \{q \in D : q \leq^{*,s}_\alpha p\}$. Then clearly D_p is an ω_2-closed dense subset of $\mathbb{P}^{*,s}_\alpha / p$. Since $\mathbb{P}^{*,s}_\alpha$ is separative, obviously $(D_p, \leq^{*,s}_\alpha)$ is also separative, and since $\mathbb{P}_\alpha$ has size ω_2, so does D_p. By standard forcing facts, it follows that $(D_p, \leq^{*,s}_\alpha)$ is forcing equivalent to $\mathrm{Add}(\omega_2)$.

We also know by Lemma 1.3 that $\mathbb{P}^c_\alpha$ is isomorphic to $\mathrm{Add}(\omega, \mathrm{ot}(\alpha \cap \mathrm{even}))$. Since $\alpha < \omega_3$, $\mathbb{P}^c_\alpha$ is isomorphic to a regular suborder of $\mathrm{Add}(\omega, \omega_2)$ of the form $\mathrm{Add}(\omega, \delta)$ for some $\delta \leq \omega_2$. By standard facts, for any $s \in \mathrm{Add}(\omega, \delta)$, $\mathrm{Add}(\omega, \delta)/s$

is isomorphic to $\mathrm{Add}(\omega, \delta)$. Hence, $\mathbb{P}^c_\alpha/a$ is isomorphic to $\mathrm{Add}(\omega, \delta)$. So $\mathrm{Add}(\omega, \omega_2)$ is isomorphic to $(\mathbb{P}^c_\alpha/a) \times \mathrm{Add}(\omega, \omega_2 \setminus \delta)$.

From these facts, we can obtain in $V[H]$ a V-generic filter H_1 on $(D_p, \leq^{*,s}_\alpha)$ such that $V[H] = V[H_1]$, and in $V[H][G]$ we can obtain a $V[H]$-generic filter H_2 on $\mathbb{P}^c_\alpha/a$ such that $V[G \times H] = V[H][G]$ is a generic extension of $V[H][H_2]$ by the ω_1-c.c. forcing $\mathrm{Add}(\omega, \omega_2 \setminus \delta)$.

Now the upwards closure H_1' of H_1 in $\mathbb{P}^{*,s}_\alpha$ is a V-generic filter on $\mathbb{P}^{*,s}_\alpha$ which contains p, and $V[H] = V[H_1] = V[H_1']$. Since the identity function is a dense embedding of $\mathbb{P}^*_\alpha$ into $\mathbb{P}^{*,s}_\alpha$, H_1' is also a V-generic filter on $\mathbb{P}^*_\alpha$ which contains p. So H_1'/p is a V-generic filter on $\mathbb{P}^*_\alpha/p$ and $V[H] = V[H_1'] = V[H_1'/p]$. Thus, $H_2 \times (H_1'/p)$ is a V-generic filter on $(\mathbb{P}^c_\alpha/a) \times (\mathbb{P}^*_\alpha/p) = (\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha)/(a, p)$. Letting K be the upwards closure of this filter in $\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$, K is a generic filter on $\mathbb{P}^c_\alpha \otimes \mathbb{P}^*_\alpha$ which contains (a, p), and $V[K] = V[H_2 \times (H_1'/p)] = V[H][H_2]$. And from the above, $V[G \times H]$ is a generic extension of $V[H][H_2] = V[K]$ by an ω_1-c.c. forcing poset. □

We need one more lemma before proceeding to the main result of the paper.

Lemma 2.11. *Assume that for all $\beta < \alpha$, $\mathbb{P}_\beta$ preserves ω_1. Suppose that $\langle p_i : i < \delta \rangle$ is a $\leq^*_\alpha$-descending sequence of conditions, where $\delta \in \omega_2 \cap \mathrm{cof}(\omega_1)$. Then there is q such that $q \leq^*_\alpha p_i$ for all $i < \delta$.*

Proof. Let $a := p_0 \restriction \text{even}$. Then $a = p_i \restriction \text{even}$ for all $i < \delta$. Define q as follows. Let $q \restriction \text{even} = a$ and $\mathrm{dom}(q) \cap \text{odd} := \bigcup \{\mathrm{dom}(p_i) \cap \text{odd} : i < \delta\}$. For each odd $\gamma \in \mathrm{dom}(q)$, let $q(\gamma)$ be a $\mathbb{P}^c_\gamma$-name for a nonempty closed and bounded subset of ω_2 such that, assuming $a \restriction \gamma$ is in the generic filter, then $q(\gamma)$ is the union of $\{p_i(\gamma) : i < \delta\}$ together with the supremum of $\{\max(p_i(\gamma)) : i < \delta\}$.

To see that q is a condition, it suffices to show that for all odd $\gamma < \alpha$, assuming that $q \restriction \gamma$ is in $\mathbb{P}_\gamma$ and is $\leq^*_\gamma$-below $p_i \restriction \gamma$ for all $i < \delta$, then $q \restriction \gamma$ forces in $\mathbb{P}_\gamma$ that $\max(q(\gamma)) \notin \dot{S}_\gamma$. But since δ has cofinality ω_1, $a \restriction \gamma$ forces that $\max(q(\gamma))$ has cofinality ω_1, or for some $i < \delta$, $\max(q(\gamma)) = \max(p_j(\gamma))$ for all $i \leq j < \delta$. As $\dot{S}_\gamma$ is a $\mathbb{P}_\gamma$-name for a subset of $\omega_2 \cap \mathrm{cof}(\omega)$ and $\mathbb{P}_\gamma$ preserves ω_1, in either case $q \restriction \gamma$ forces that $\max(q(\gamma))$ is not in $\dot{S}_\gamma$. □

3. The Consistency Result

Let κ be a Mahlo cardinal and assume that GCH holds. For example, if κ is Mahlo, then κ is Mahlo in L, so we can take our ground model to be L. We will prove that there exists a forcing poset which collapses κ to become ω_2, forces that $2^\omega = \omega_3$, and forces that every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects to an ordinal in ω_2 with cofinality ω_1. The forcing poset will be of the form $\mathbb{R}_\kappa * \mathbb{P}_{\kappa^+}$, where $\mathbb{R}_\kappa$ is a preparation forcing which collapses κ to become ω_2 and $\mathbb{P}_{\kappa^+}$ is a suitable mixed support forcing iteration in $V^{\mathbb{R}_\kappa}$ for killing nonreflecting sets.

To begin, let us define in the ground model V a countable support forcing iteration

$$\langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\beta : \alpha \leq \kappa,\ \beta < \kappa \rangle$$

of proper forcings as follows. Let $\alpha < \kappa$, and assume that $\mathbb{R}_\beta$ and $\dot{\mathbb{S}}_\gamma$ are defined for all $\beta \leq \alpha$ and $\gamma < \alpha$. If α is not inaccessible, then let $\dot{\mathbb{S}}_\alpha$ be an $\mathbb{R}_\alpha$-name for the collapse $\mathrm{Col}(\omega_1, \omega_2)$. Then $\mathbb{R}_\alpha$ forces that $\dot{\mathbb{S}}_\alpha$ is ω_1-closed, and hence proper. Let $\mathbb{R}_{\alpha+1} := \mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha$.

Now assume that α is inaccessible. Also, assume as a recursion hypothesis that $\mathbb{R}_\alpha$ is α-c.c., has size α, and collapses α to become ω_2. Let $\dot{\mathbb{S}}_\alpha$ be an $\mathbb{R}_\alpha$-name for $\mathrm{Add}(\omega, \omega_2) \times \mathrm{Add}(\omega_2)$ (in other words, $\mathrm{Add}(\omega, \alpha) \times \mathrm{Add}(\alpha)$). Note that this product is forcing equivalent to the two-step forcing iteration $\mathrm{Add}(\omega_2) * \mathrm{Add}(\omega, \omega_2)$, which is an ω_1-closed forcing followed by an ω_1-c.c. forcing, and hence is proper. Let $\mathbb{R}_{\alpha+1} := \mathbb{R}_\alpha * \dot{\mathbb{S}}_\alpha$.

Now let $\delta \leq \kappa$ be a limit ordinal, and assume that $\mathbb{R}_\beta$ and $\dot{\mathbb{S}}_\beta$ are defined for all $\beta < \delta$. Let $\mathbb{R}_\delta$ be the countable support limit of $\langle \mathbb{R}_\alpha : \alpha < \delta \rangle$. By standard arguments, it is easy to check that if δ is inaccessible, then the recursion hypothesis stated in the inaccessible case above holds for $\mathbb{R}_\delta$.

This completes the definition. The iteration $\mathbb{R}_\kappa$ is proper, κ-c.c., and has size κ. So $\mathbb{R}_\kappa$ preserves ω_1 and collapses κ to become ω_2. Standard nice name arguments show that $\mathbb{R}_\kappa$ forces that $2^\omega = 2^{\omega_1} = \omega_2$ and $2^\mu = \mu^+$ for all cardinals $\mu \geq \kappa$.

Let G be a generic filter on $\mathbb{R}_\kappa$. In $V[G]$, we define a sequence of forcing posets $\langle \mathbb{P}_\beta : \beta \leq \kappa^+ \rangle$. This sequence will be a suitable mixed support forcing iteration based on a sequence of names $\langle \dot{S}_\gamma : \gamma \in \kappa^+ \cap \mathrm{odd} \rangle$. Definition 1.1 provides a recursive description which will determine the iteration, provided that we specify the names $\dot{S}_\gamma$ for all $\gamma \in \kappa^+ \cap \mathrm{odd}$. Each name $\dot{S}_\gamma$ will be a nice $\mathbb{P}_\gamma$-name for a subset of $\omega_2 \cap \mathrm{cof}(\omega)$ such that $\mathbb{P}_\gamma$ forces that $\dot{S}_\gamma$ does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$.

We will assume two recursion hypotheses in $V[G]$. Let $\beta < \kappa^+$, and suppose that $\langle \mathbb{P}_\delta : \delta \leq \beta \rangle$ and $\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle$ are defined. The first recursion hypothesis is:

Recursion Hypothesis 3.1. *For all $\xi \leq \beta$, the forcing poset $\mathbb{P}^*_\xi$ is ω_2-distributive, and therefore $\mathbb{P}_\xi$ preserves ω_1 and ω_2.*

Let us see how we can prove the consistency result assuming that this first recursion hypothesis holds for all $\beta < \kappa^+$. By Lemma 1.19(2) and Proposition 2.2, $\mathbb{P}_{\kappa^+}$ is κ^+-c.c. and preserves ω_1 and ω_2. It easily follows that any nice $\mathbb{P}_{\kappa^+}$-name for a subset of $\kappa \cap \mathrm{cof}(\omega)$ which does not reflect to any ordinal of uncountable cofinality in κ is also a nice $\mathbb{P}_\beta$-name for a set of the same kind for some $\beta < \kappa^+$. Since $\mathbb{P}_\beta$ has size κ and $2^\kappa = \kappa^+$, after we define $\mathbb{P}_\beta$ we can enumerate all such $\mathbb{P}_\beta$-names in order type κ^+. When we select the names $\dot{S}_\gamma$, we use a standard bookkeeping function argument to arrange that any such name is equal to $\dot{S}_\gamma$ for some $\gamma < \kappa^+$. Since $\mathbb{P}_{\gamma+1}$ is a regular suborder of $\mathbb{P}_{\kappa^+}$ and is forcing equivalent to $\mathbb{P}_\gamma * \mathrm{CU}(\kappa \setminus \dot{S}_\gamma)$ by Proposition 2.5, this nonreflecting set will become nonstationary after forcing with $\mathbb{P}_{\kappa^+}$. Thus, in the model $V^{\mathbb{R}_\kappa * \mathbb{P}_{\kappa^+}}$, every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects to an ordinal in ω_2 with cofinality ω_1. Since $\mathbb{P}_{\kappa^+}$ adds κ^+ many reals, standard arguments show that in this final model, $2^\omega = \omega_3$.

In order to maintain the first recursion hypothesis, we will need a second more technical recursion hypothesis. Before stating it, we introduce some terminology.

Notation 3.2. *A set N in the ground model V is said to be* suitable *if N is an elementary substructure of $H(\kappa^+)$ of size less than κ, $\kappa_N := N \cap \kappa$ is inaccessible, $|N| = \kappa_N$, $N^{<\kappa_N} \subseteq N$, and the forcing iteration $\vec{\mathbb{R}} := \langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\delta : \alpha \leq \kappa,\ \delta < \kappa \rangle$ is a member of N.*

The fact that κ is Mahlo implies by standard arguments that there are stationarily many suitable sets in $P_\kappa(H(\kappa^+))$. The same comment applies regarding Notation 3.4 below.

Lemma 3.3. *Suppose that N is suitable. Let $\pi_N : N \to N_0$ be the transitive collapse of N. Let $\pi_{N[G]} : N[G] \to M_0$ be the transitive collapse of $N[G]$ in $V[G]$. Then:*

(1) $\pi_N(\vec{\mathbb{R}}) = \langle \mathbb{R}_\alpha, \dot{\mathbb{S}}_\delta : \alpha \leq \kappa_N,\ \delta < \kappa_N \rangle$;
(2) $M_0 = N_0[G \restriction \kappa_N]$, *and therefore* $M_0 \in V[G \restriction \kappa_N]$;
(3) $\pi_{N[G]} \restriction N = \pi_N$.

The proof is straightforward.

Notation 3.4. *A set N is said to be β-*suitable *if N is suitable and N contains $\mathbb{R}_\kappa$-names for the objects $\langle \mathbb{P}_i : i \leq \beta \rangle$ and $\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle$.*

Observe that if N is β-suitable, then for all $\beta' \in N \cap \beta$, N is β'-suitable.

Lemma 3.5. *Let N be β-suitable, $\pi_N : N \to N_0$ the transitive collapse of N, and $\pi : N[G] \to N_0[G \restriction \kappa_N]$ the transitive collapse of $N[G]$. Then in $V[G \restriction \kappa_N]$, $\langle \mathbb{P}^\pi_i : i \leq \pi(\beta) \rangle := \pi(\langle \mathbb{P}_i : i \leq \beta \rangle)$ is a suitable mixed support forcing iteration based on the sequence of names $\langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \mathrm{odd} \rangle := \pi(\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle)$. Moreover, $\pi(\mathbb{P}^c_\beta) = (\mathbb{P}^\pi_{\pi(\beta)})^c$, $\pi(\mathbb{P}^*_\beta) = (\mathbb{P}^\pi_{\pi(\beta)})^*$, $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta) = (\mathbb{P}^\pi_{\pi(\beta)})^c \otimes (\mathbb{P}^\pi_{\pi(\beta)})^*$, and $\pi(\mathbb{P}^{*,s}_\beta) = (\mathbb{P}^\pi_{\pi(\beta)})^{*,s}$.*

Proof. Let $M := N_0[G \restriction \kappa_N]$. Then $\kappa_N = \pi(\kappa) \in M$ and κ_N equals ω_2 in $V[G \restriction \kappa_N]$. Since $N^{<\kappa_N} \subseteq N$ and N and N_0 are isomorphic, $N_0^{<\kappa_N} \subseteq N_0$. As $\mathbb{R}_{\kappa_N}$ is κ_N-c.c., it follows by standard facts that $M = N_0[G \restriction \kappa_N]$ is closed under sequences of length less than κ_N in $V[G \restriction \kappa_N]$. In particular, $M^{\omega_1} \subseteq M$ in $V[G \restriction \kappa_N]$. Since M is isomorphic to $N[G]$, which is a model of $\mathsf{ZFC} - \mathsf{Powerset}$, M is a model of $\mathsf{ZFC} - \mathsf{Powerset}$.

Using absoluteness, $\pi(\langle \mathbb{P}_i : i \leq \beta \rangle)$ is a sequence of forcing posets $\langle \mathbb{P}^\pi_i : i \leq \pi(\beta) \rangle$, and $\pi(\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle)$ is a sequence $\langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \mathrm{odd} \rangle$ such that for each $\gamma \in \pi(\beta) \cap \mathrm{odd}$, $\dot{S}^\pi_\gamma$ is a nice $\mathbb{P}^\pi_\gamma$-name for a subset of $\kappa_N \cap \mathrm{cof}(\omega)$.

Since π is an isomorphism, M models that $\langle \mathbb{P}^\pi_i : i \leq \pi(\beta) \rangle$ is a suitable mixed support forcing iteration based on the sequence of names $\langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \mathrm{odd} \rangle$. By Lemma 1.2, it follows that in $V[G \restriction \kappa_N]$, $\langle \mathbb{P}^\pi_i : i \leq \pi(\beta) \rangle$ is a suitable mixed support forcing iteration based on the sequence of names $\langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \mathrm{odd} \rangle$. The remaining statements are easy to verify. $\square$

We are now ready to state the second recursion hypothesis.

Recursion Hypothesis 3.6. *Let N be β-suitable and π be the transitive collapsing map of $N[G]$. Then for all odd $\gamma \in N \cap \beta$, in the model $V[G \restriction \kappa_N]$, $\pi(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$ forces that $\pi(\dot{S}_\gamma)$ is a nonstationary subset of κ_N.*

It remains to prove that the two recursion hypotheses hold for all $\beta < \kappa^+$. The proof will proceed as follows. For a fixed $\beta < \kappa^+$, we will assume that the recursion hypotheses hold for all $\gamma \leq \beta$, and then prove that they hold for $\beta + 1$ by first verifying the second recursion hypothesis for $\beta + 1$, and then using that hypothesis to prove the first recursion hypothesis for $\beta + 1$. Then, for a fixed limit ordinal $\alpha < \kappa^+$, we will assume that both recursion hypotheses hold for all $\beta < \alpha$. Observe that the second recursion hypothesis then holds immediately for α. So in the limit case it will suffice to prove the first recursion hypothesis for α.

The proof of the first recursion hypothesis is the same for both successor and limit stages. Observe that if the second recursion hypothesis holds for β, where β is even, then it immediately holds for $\beta + 1$. Putting it all together, it will suffice to prove the second recursion hypothesis only in the successor case $\beta + 1$ where β is odd, and then prove the first recursion hypothesis in an independent way.

The proofs of both recursion hypotheses will use the following lemma.

Lemma 3.7. *Assume that both recursion hypotheses hold for all $\gamma < \beta$ and the second recursion hypothesis holds for β. Let N be β-suitable and $(a,p) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. Let π be the transitive collapsing map of $N[G]$. Then in $V[G]$ there exists a $V[G \restriction \kappa_N]$-generic filter K on $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ which contains $\pi(a,p)$ such that $V[G]$ is a generic extension of $V[G \restriction \kappa_N][K]$ by a proper forcing poset.*

Furthermore, letting $J := \pi(\tau_\beta)[K]$, $K^+ := \pi^{-1}(K)$, and $J^+ := \pi^{-1}(J)$, then K^+ is a filter on $N[G] \cap (\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ containing (a,p) which is $N[G]$-generic, J^+ is a filter on $N[G] \cap \mathbb{P}_\beta$ which is $N[G]$-generic, and $J^+ = \tau_\beta[K^+]$. Moreover, there exists $s \in \mathbb{P}_\beta$ such that for all (b,q) in K^+, $s \leq^*_\beta q$.

Proof. By extending further if necessary, we may assume without loss of generality that $a \leq^c_\beta p \restriction \mathrm{even}$. Let $\pi(\langle \mathbb{P}_i : i \leq \beta \rangle) = \langle \mathbb{P}^\pi_i : i \leq \pi(\beta) \rangle$ and $\pi(\langle \dot{S}_\gamma : \gamma \in \beta \cap \mathrm{odd} \rangle) = \langle \dot{S}^\pi_\gamma : \gamma \in \pi(\beta) \cap \mathrm{odd} \rangle$. Then the second recursion hypothesis means that in $V[G \restriction \kappa_N]$, for all $\gamma \in \pi(\beta) \cap \mathrm{odd}$, $(\mathbb{P}^\pi_\gamma)^c \otimes (\mathbb{P}^\pi_\gamma)^*$ forces that $\dot{S}^\pi_\gamma$ is nonstationary in κ_N. By Proposition 2.9, in $V[G \restriction \kappa_N]$ the forcing poset $\pi(\mathbb{P}^{*,s}_\beta)$ contains a κ_N-closed dense subset.

At stage κ_N in the preparation forcing iteration $\mathbb{R}_\kappa$ we forced with $\mathrm{Add}(\omega, \kappa_N) \times \mathrm{Add}(\kappa_N)$. Therefore, $V[G \restriction (\kappa_N + 1)] = V[G \restriction \kappa_N][L]$, where L is a $V[G \restriction \kappa_N]$-generic filter on $\mathrm{Add}(\omega, \kappa_N) \times \mathrm{Add}(\kappa_N)$. By Lemma 2.10, there exists in $V[G \restriction \kappa_N][L]$ a $V[G \restriction \kappa_N]$-generic filter K on $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ which contains $\pi(a,p)$ such that $V[G \restriction \kappa_N][L]$ is a generic extension of $V[G \restriction \kappa_N][K]$ by an ω_1-c.c. forcing poset. Since $V[G]$ is a generic extension of $V[G \restriction \kappa_N][L]$ by a proper forcing, namely, the tail of the iteration $\mathbb{R}_\kappa$ after forcing with $\mathbb{R}_{\kappa_N+1}$, it follows that $V[G]$ is a generic extension of $V[G \restriction \kappa_N][K]$ by a proper forcing.

Recall that the map $\tau_\beta : \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta \to \mathbb{P}_\beta$ defined by $\tau_\beta(b,q) = q + b$ is a surjective projection mapping by Lemma 1.18. Since π is an isomorphism and by absoluteness, in $V[G \restriction \kappa_N]$ we have that $\pi(\tau_\beta)$ is a surjective projection mapping from $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ onto $\pi(\mathbb{P}_\beta)$. Let $J := \pi(\tau_\beta)[K]$. Then J is a $V[G \restriction \kappa_N]$-generic filter on $\pi(\mathbb{P}_\beta)$.

Let $K^+ := \pi^{-1}(K)$ and $J^+ := \pi^{-1}(J)$. Since $\pi(a,p) \in K$, $(a,p) \in K^+$. It is easy to check that K^+ and J^+ are filters on $N[G] \cap (\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ and $N[G] \cap \mathbb{P}_\beta$ respectively, and $J^+ = \tau_\beta[K^+]$. If D is a dense open subset of $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ in $N[G]$, then since π is an isomorphism and by absoluteness, $\pi(D)$ is a dense open subset of $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ in $V[G \restriction \kappa_N]$. Since K is $V[G \restriction \kappa_N]$-generic, we can fix $w \in \pi(D) \cap K$. Then $\pi^{-1}(w) \in D \cap K^+$. This shows that K^+ is $N[G]$-generic for $\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$. A similar argument shows that J^+ is $N[G]$-generic for $\mathbb{P}_\beta$.

By Lemma 1.16, we can write $V[G \restriction \kappa_N][K] = V[G \restriction \kappa_N][K_1 \times K_2]$, where $K_1 \times K_2 := K \cap (\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/\pi(a,p))$ is a $V[G \restriction \kappa_N]$-generic filter on $(\pi(\mathbb{P}_\beta)^c/\pi(a)) \times (\pi(\mathbb{P}_\beta)^*/\pi(p))$. By Proposition 2.9, $\pi(\mathbb{P}_\beta)^*$ contains a κ_N-closed dense subset.

By standard arguments, it follows that there exists in $V[G \upharpoonright \kappa_N][K]$ a $\pi(\leq^*_\beta)$-descending sequence $\langle q_i : i < \kappa_N \rangle$ below $\pi(p)$ which is dense in K_2. Let $r_i := \pi^{-1}(q_i)$ for all $i < \kappa_N$. Then $\langle r_i : i < \kappa_N \rangle$ is a $\leq^*_\beta$-descending sequence of conditions in $N[G] \cap \mathbb{P}^*_\beta$ below p which is dense in $\pi^{-1}(K_2)$.

Now κ_N has cofinality ω_1 in $V[G]$, and since both recursion hypotheses hold for all $\gamma < \beta$, we also have that for all $\gamma < \beta$, $\mathbb{P}_\gamma$ preserves ω_1. By Lemma 2.11, there is $s \in \mathbb{P}_\beta$ such that $s \leq^*_\beta r_i$ for all $i < \kappa_N$. Then $s \leq^*_\beta r$ for all $r \in \pi^{-1}(K_2)$. Consider (b, q) in K^+. Since $(a, p) \in K^+$, without loss of generality $(b, q) \leq (a, p)$. Then $\pi(b, q) \in K$, so $\pi(q) \in K_2$. Hence, $q \in \pi^{-1}(K_2)$. Therefore, $s \leq^*_\beta q$, which completes the proof. □

The next proposition verifies the second recursion hypothesis. We will use the standard result that proper forcings preserve the stationarity of stationary subsets of $\alpha \cap \mathrm{cof}(\omega)$, for any ordinal α with uncountable cofinality. This result is true because any set $S \subseteq \alpha \cap \mathrm{cof}(\omega)$ is stationary in α iff the set $\{a \in [\alpha]^\omega : \sup(a) \in S\}$ is stationary in $[\alpha]^\omega$, and proper forcings preserve the stationarity of subsets of $[\alpha]^\omega$.

Proposition 3.8. *Let $\beta < \omega_3$ be odd, and assume that the two recursion hypotheses hold for all $\gamma \leq \beta$. Let N be $(\beta+1)$-suitable and π be the transitive collapsing map of $N[G]$. Then for all odd $\gamma \in N \cap (\beta + 1)$, in the model $V[G \upharpoonright \kappa_N]$, $\pi(\mathbb{P}^c_\gamma \otimes \mathbb{P}^*_\gamma)$ forces that $\pi(\dot{S}_\gamma)$ is a nonstationary subset of κ_N.*

Proof. Since N is $(\beta+1)$-suitable, $\beta \in N$ by elementarity, so N is also β-suitable. By the second recursion hypothesis holding at β, the conclusion of the proposition is true for all odd $\gamma \in N \cap \beta$. So it suffices to show that in $V[G \upharpoonright \kappa_N]$, $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ forces that $\pi(\dot{S}_\beta)$ is a nonstationary subset of κ_N.

Let $(a_0, p_0) \in \pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$, and we will find $(a, p) \leq (a_0, p_0)$ which forces that $\pi(\dot{S}_\beta)$ is nonstationary in κ_N. By extending further if necessary, we may assume without loss of generality that $a_0 \leq p_0 \upharpoonright \mathrm{even}$ in $\pi(\mathbb{P}_\beta)^c$. Then by Lemma 1.15, $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)/(a_0, p_0)$ is equal to the product forcing $(\pi(\mathbb{P}_\beta)^c/a_0) \times (\pi(\mathbb{P}_\beta)^*/p_0)$.

Let K, J, K^+, J^+, and s be as described in Lemma 3.7, where $(a_0, p_0) \in K$. Use J^+ to interpret the name $\dot{S}_\beta$ by letting S be the set of $\alpha < \kappa_N$ such that for some $u \in J^+$, $u \Vdash_\beta \check{\alpha} \in \dot{S}_\beta$. We claim that $S = \pi(\dot{S}_\beta)^J$. Clearly $\pi(\dot{S}_\beta)^J$ is a subset of κ_N, since $\pi(\dot{S}_\beta)$ is a $\pi(\mathbb{P}_\beta)$-name for a subset of κ_N.

Consider $\alpha < \kappa_N$. In $V[G]$, let D be the dense open set of conditions in $\mathbb{P}_\beta$ which decide whether or not α is in $\dot{S}_\beta$. By the elementarity of $N[G]$, $D \in N[G]$. Since J^+ is $N[G]$-generic, fix $w \in J^+ \cap D$. Let $w' := \pi(w)$, which is in $\pi(D)$. Since π is an isomorphism and by absoluteness, w' decides in $\pi(\mathbb{P}_\beta)$ whether or not $\pi(\alpha) = \alpha$ is in $\pi(\dot{S}_\beta)$ the same way that w decides whether α is in $\dot{S}_\beta$. As J and J^+ are filters, it easily follows that $\alpha \in S$ iff $w \Vdash^{V[G]}_\beta \alpha \in \dot{S}_\beta$ iff $w' \Vdash^{V[G \upharpoonright \kappa_N]}_{\pi(\mathbb{P}_\beta)} \alpha \in \pi(\dot{S}_\beta)$ iff $\alpha \in \pi(\dot{S}_\beta)^J$. Thus, $S = \pi(\dot{S}_\beta)^J$.

By the choice of $\dot{S}_\beta$, we know that in $V[G]$ the forcing poset $\mathbb{P}_\beta$ forces that $\dot{S}_\beta$ does not reflect to any ordinal in κ with cofinality ω_1. Now κ_N has cofinality ω_1 in $V[G]$, and by the recursion hypotheses ω_1 is preserved by $\mathbb{P}_\beta$. Thus, $\mathbb{P}_\beta$ forces that there exists a club subset of κ_N with order type ω_1 which is disjoint from $\dot{S}_\beta \cap \kappa_N$. Let $\dot{c}$ be a $\mathbb{P}_\beta$-name for such a club.

By the first recursion hypothesis holding for β, $\mathbb{P}^*_\beta$ is ω_2-distributive in $V[G]$. By Proposition 2.4, we can find $t \leq^*_\beta s$ and a $\mathbb{P}^c_\beta$-name $\dot{c}_0$ such that $t \Vdash_\beta \dot{c} = \dot{c}_0$. By the maximality principle for names, we may assume without loss of generality that $\dot{c}_0$ is a $\mathbb{P}^c_\beta$-name for a club subset of κ_N with order type ω_1. As $\mathbb{P}^c_\beta$ is ω_1-c.c., we can find a set d in $V[G]$ which is a club subset of κ_N such that $\mathbb{P}^c_\beta$ forces that $d \subseteq \dot{c}_0$. Then $t \Vdash_\beta d \cap \dot{S}_\beta = \emptyset$.

We claim that $d \cap S = \emptyset$. If not, then fix $\alpha \in d \cap S$. By the definition of S, there exists $u \in J^+$ which forces in $\mathbb{P}_\beta$ that α is in $\dot{S}_\beta$. Since $J^+ = \tau_\beta[K^+]$ by Lemma 3.7, there is $(b, z) \in K^+$ such that $u = z + b$.

By Lemma 3.7, $s \leq^*_\beta z$. So $t \leq^*_\beta z$. By Lemma 1.10(3), t and b are compatible in $\mathbb{P}_\beta$ and $t + b \leq_\beta z + b = u$. It follows that $t + b$ forces in $\mathbb{P}_\beta$ that $\alpha \in \dot{S}_\beta$. This is impossible since $\alpha \in d$ and t forces in $\mathbb{P}_\beta$ that $d \cap \dot{S}_\beta = \emptyset$.

So indeed $d \cap S = \emptyset$, and hence S is a nonstationary subset of κ_N in the model $V[G]$. Since $S = \pi(\dot{S}_\beta)^J$, $S \in V[G \restriction \kappa_N][J]$. As $V[G \restriction \kappa_N][J] \subseteq V[G \restriction \kappa_N][K]$, $S \in V[G \restriction \kappa_N][K]$. But $V[G]$ is a generic extension of $V[G \restriction \kappa_N][K]$ by a proper forcing poset by Lemma 3.7. Since S is a set of ordinals of cofinality ω, S must be nonstationary in $V[G \restriction \kappa_N][K]$. As $(a_0, p_0) \in K$, we can find $(a, p) \leq (a_0, p_0)$ in K which forces in $\pi(\mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta)$ that $\pi(\dot{S}_\beta)$ is nonstationary in κ_N, which completes the proof. □

We now verify the first recursion hypothesis for β, which will finish the proof of the consistency result.

Proposition 3.9. *Let $\beta < \kappa^+$, and assume that the first and second recursion hypotheses hold for all $\gamma < \beta$ and the second recursion hypothesis holds for β. Then $\mathbb{P}^*_\beta$ is ω_2-distributive.*

Proof. Assume that $p \in \mathbb{P}_\beta$ forces in $\mathbb{P}^*_\beta$ that $\langle \dot{\alpha}_i : i < \omega_1 \rangle$ is a sequence of ordinals. We will find $q \leq^*_\beta p$ which decides in $\mathbb{P}^*_\beta$ the value of $\dot{\alpha}_i$ for all $i < \omega_1$, and hence forces that this sequence is in the ground model.

Fix a β-suitable model N such that $N[G]$ contains p and $\langle \dot{\alpha}_i : i < \omega_1 \rangle$, and let π be the transitive collapsing map of $N[G]$. Fix K, J, K^+, J^+, and s as in Lemma 3.7, where $\pi(p \restriction \text{even}, p) \in K$. Then $(p \restriction \text{even}, p) \in K^+$.

Let $i < \omega_1$, and we will show that s decides the value of $\dot{\alpha}_i$. Let D be the set of $(b, q) \in \mathbb{P}^c_\beta \otimes \mathbb{P}^*_\beta$ below $(p \restriction \text{even}, p)$ such that q decides in $\mathbb{P}^*_\beta$ the value of $\dot{\alpha}_i$. Then $D \in N[G]$ by elementarity, and easily D is dense below $(p \restriction \text{even}, p)$. Since K^+ is $N[G]$-generic and contains $(p \restriction \text{even}, p)$, fix $(b, q) \in D \cap K^+$. Then by Lemma 3.7, $s \leq^*_\beta q$. Since $q \in D$, q decides the value of $\dot{\alpha}_i$. So s decides the value of $\dot{\alpha}_i$. □

Postscript

After this article was completed, I. Neeman discovered a shorter proof of the consistency of stationary set reflection at ω_2 together with an arbitrarily large continuum. Specifically, starting with a model in which stationary set reflection holds, adding any number of Cohen reals preserves stationary set reflection (see [2, Theorem 3.1]). This new proof is, however, somewhat limited in its applications. For example, in the model of Harrington and Shelah [3], there exists a special ω_2-Aronszajn tree, and that fact cannot be changed by Cohen forcing. In contrast, the methods developed in this paper can be used to construct models of stationary set reflection together with a variety of combinatorial properties, such as the non-existence of special ω_2-Aronszajn trees and the failure of the weak Kurepa hypothesis. The details will be worked out in the first author's upcoming Ph.D. dissertation.

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