MITCHELL'S THEOREM REVISITED

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Abstract. Mitchell's theorem on the approachability ideal states that it is consistent relative to a greatly Mahlo cardinal that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$. In this paper we give a new proof of Mitchell's theorem, deriving it from an abstract framework of side condition methods.

Contents

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Introduction

The approachability ideal $I[\lambda^+]$, for an uncountable cardinal λ, is defined as follows. For a given sequence $\vec{a} = \langle a_i : i < \lambda^+ \rangle$ of bounded subsets of λ^+, let $S_{\vec{a}}$ denote the set of limit ordinals $\alpha < \lambda^+$ for which there exists a set $c \subseteq \alpha$, which is club in α with order type $\mathrm{cf}(\alpha)$, such that for all $\beta < \alpha$, there is $i < \alpha$ with $c \cap \beta = a_i$. Intuitively speaking, the set $S_{\vec{a}}$ carries a kind of weak square sequence, namely a sequence of clubs such that for each α in $S_{\vec{a}}$, the club attached to α has its initial segments enumerated at stages prior to α. Define $I[\lambda^+]$ as the collection of sets $S \subseteq \lambda^+$ for which there exists a sequence $\vec{a}$ as above and a club $C \subseteq \lambda^+$ such that $S \cap C \subseteq S_{\vec{a}}$. In other words, $I[\lambda^+]$ is the ideal of subsets of λ^+ which is generated modulo the club filter by sets of the form $S_{\vec{a}}$.

Let λ be a regular uncountable cardinal. Shelah [14] proved that the set $\lambda^+ \cap \mathrm{cof}(< \lambda)$ is in $I[\lambda^+]$. Therefore the structure of $I[\lambda^+]$ is determined by which subsets of $\lambda^+ \cap \mathrm{cof}(\lambda)$ belong to it. At one extreme, the weak square principle $\square^*_\lambda$ implies that $\lambda^+ \cap \mathrm{cof}(\lambda)$ is in $I[\lambda^+]$; therefore $I[\lambda^+]$ is just the power set of λ^+. The opposite extreme would be that no stationary subset of $\lambda^+ \cap \mathrm{cof}(\lambda)$ belongs to $I[\lambda^+]$, in other words, that $I[\lambda^+]$ is the nonstationary ideal when restricted to cofinality λ. Whether the second extreme is consistent was open for several decades, and was eventually solved by Mitchell [12]. Mitchell proved that it is consistent, relative to the consistency of a greatly Mahlo cardinal, that there does not exist a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in $I[\omega_2]$. We will refer to this result as *Mitchell's theorem.*

Mitchell's theorem is important not only for solving a deep and long-standing open problem in combinatorial set theory, but also for introducing powerful new techniques in forcing. A basic tool in the proof is a forcing poset for adding a club subset of ω_2 with finite conditions, using finite sets of countable models as side conditions. A similar forcing poset was introduced by Friedman [3] around the same time. The use of countable models in Friedman's and Mitchell's forcing posets for adding a club expanded the original side condition method of Todorčević [15], which was designed to add a generic object of size ω_1, to adding a generic object of size ω_2. In addition, Mitchell's proof introduced the new concepts of strongly generic conditions and strongly proper forcing posets, which are closely related to the approximation property.

Several years later, Neeman [13] developed a general framework of side conditions, which he called *sequences of models of two types.* An important distinction between Neeman's side conditions and those of Friedman and Mitchell is that the two-type side conditions include both countable and uncountable models. A couple of years later, Krueger [6] developed an alternative framework of side conditions called *adequate sets.* This approach bases the analysis of side conditions on the ideas of the *comparison point* and *remainder points* of two countable models. Notably, this approach has led to the solution of an open problem of Friedman [3], by showing how to add a club subset of ω_2 with finite conditions while preserving the continuum hypothesis ([10]). Other applications are given in [8], [7], [9], and [2].

Notwithstanding the merits of the frameworks of Neeman [13] and Krueger [6], these frameworks are limited in the sense that they are intended to add a single subset of ω_2 (or of a cardinal κ which is collapsed to become ω_2). The proof of Mitchell's theorem, on the other hand, involves adding κ^+ many club subsets of a cardinal κ. Many consistency proofs in set theory about a cardinal κ involve adding

κ^+ many subsets of κ by forcing, so that each of the potential counterexamples to the statement being forced is captured in some intermediate generic extension and dealt with by the rest of the forcing extension.

The goal of this paper is to extend the framework of adequate sets to allow for adding many subsets of ω_2, or of a cardinal κ which is collapsed to become ω_2. The purpose of this extension is to provide general tools which will be useful for proving new consistency results on ω_2. In Part III we give an example by deriving Mitchell's theorem from the abstract framework developed in Parts I and II. The paper includes a very detailed treatment of adequate sets and remainder points in Sections 1 and 2, and of Mitchell's application of the square principle to side conditions in Sections 7 and 8. We also develop some new ideas, including canonical models in Sections 9 and 10, and the main proxy lemma in Section 11.

We will analyze finite sets of countable elementary substructures of $H(\kappa^+)$. The method of adequate sets handles the interaction of the models below κ. Following Mitchell, we employ the square principle $\square_\kappa$ to describe and control the interaction of countable models between κ and κ^+. We introduce a new kind of side condition, which we call an *$\vec{S}$-obedient side condition*. We show that the forcing poset consisting of $\vec{S}$-obedient side conditions on $H(\kappa^+)$, where κ is a greatly Mahlo cardinal, ordered by component-wise inclusion, forces that $\kappa = \omega_2$ and there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$.

This project began with the M.S. thesis of Gilton at the University of North Texas, in which he reconstructed the original proof of Mitchell's theorem in the context of adequate sets. Krueger is indebted to Gilton for explaining to him many of the details of Mitchell's proof, especially the use of $\square_\kappa$. Gilton isolated a workable requirement on remainder points which later evolved into the idea of $\vec{S}$-obedient side conditions.

After Gilton's thesis was complete, Krueger returned to the problem and made a number of advances. Krueger developed the new idea of canonical models, which is dealt with in Sections 9 and 10. Canonical models are models which appear in a given model N, reflect information about models lying outside of N, and are determined by canonical parameters which arise in the comparison of models. He isolated the main proxy lemma, Lemma 11.5, which significantly simplifies the method of proxies used by Mitchell. And he introduced the idea of $\vec{S}$-obedient side conditions, and showed that forcing with pure side conditions on a greatly Mahlo cardinal produces a generic extension in which the approachability ideal on ω_2 restricted to cofinality ω_1 is the nonstationary ideal.

This paper was written for an audience with a minimum background of one year of graduate studies in set theory, with a working knowledge of forcing and proper forcing, and with some familiarity with generalized stationarity.

For a regular uncountable cardinal μ and a set X with $\mu \subseteq X$, we let $P_\mu(X)$ denote the set $\{a \subseteq X : |a| < \mu\}$. A set $S \subseteq P_\mu(X)$ being stationary is equivalent to the statement that for any function $F : X^{<\omega} \to X$, there exists $a \in S$ such that $a \cap \mu \in \mu$ and a is closed under F.

If a is a set of ordinals, then $\lim(a)$ denotes the set of ordinals β such that for all $\gamma < \beta$, $a \cap (\gamma, \beta) \neq \emptyset$. We let $\mathrm{cl}(a) = a \cup \lim(a)$. If M is a set, we write $\sup(M)$ to denote $\sup(M \cap On)$.

If $\mathcal{A}$ is a structure in a first order language, and $X_1, \ldots, X_k$ are subsets of the underlying set of $\mathcal{A}$, then we write $(\mathcal{A}, X_1, \ldots, X_k)$ to denote the expansion of the structure $\mathcal{A}$ obtained by adding $X_1, \ldots, X_k$ as predicates.

Part 1. **Basic side condition methods**

§1. Adequate sets

We begin the paper by working out the basic framework of adequate sets. Roughly speaking, this framework provides methods for describing and handling the interaction of countable elementary substructures below ω_2, or below κ for some regular uncountable cardinal κ which is intended to become ω_2 in a forcing extension.

Adequate sets were introduced by Krueger [6]; many of the results of this section appear in [6], although in a slightly different form.

We fix objects κ, λ, T^*, π^*, C^*, Λ, $\mathcal{X}_0$, and $\mathcal{Y}_0$ as follows.

Notation 1.1. *For the remainder of the paper, κ is a regular cardinal with $\omega_2 \leq \kappa$.*

In [6] we only considered the case when $\kappa = \omega_2$. In the proof of Mitchell's theorem given in Part III, κ is a greatly Mahlo cardinal.

Notation 1.2. *Fix a cardinal λ such that $\kappa \leq \lambda$. In Parts II and III we will let $\lambda = \kappa^+$.*

Definition 1.3. *A set $T \subseteq P_{\omega_1}(\kappa)$ is* thin *if for all $\beta < \kappa$,*

$$|\{a \cap \beta : a \in T\}| < \kappa.$$

The idea of a thin stationary set was introduced by Friedman [3], who used a thin stationary set to develop a forcing poset for adding a club subset of a fat stationary subset of ω_2 with finite conditions.

Observe that if $|\beta^\omega| < \kappa$ for all $\beta < \kappa$, then $P_{\omega_1}(\kappa)$ itself is thin. Krueger proved that the existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ is independent of ZFC; see [4].

Notation 1.4. *Fix a thin stationary set $T^* \subseteq P_{\omega_1}(\kappa)$ which satisfies the property that for all $\beta < \kappa$ and $a \in T^*$, $a \cap \beta \in T^*$. In Part III, we will let $T^* = P_{\omega_1}(\kappa)$.*

Note that if T is a thin stationary set, then the set $\{a \cap \beta : a \in T,\ \beta < \kappa\}$ is a thin stationary set which satisfies the property of being closed under initial segments which is described in Notation 1.4.

Observe that if T is a thin stationary set, then $|T| = \kappa$.

Notation 1.5. *Fix a bijection $\pi^* : T^* \to \kappa$.*

Notation 1.6. *Let C^* denote the set of $\beta < \kappa$ such that whenever a is a bounded subset of β in T^*, then $\pi^*(a) < \beta$.*

The fact that T^* is thin easily implies that C^* is a club subset of κ.

Notation 1.7. *Let Λ denote the set $C^* \cap \operatorname{cof}(>\omega)$.*

Notation 1.8. *For the remainder of the paper, let $\trianglelefteq$ denote a well-ordering of $H(\lambda)$.*

Notation 1.9. *Let $\mathcal{X}_0$ denote the set of M in $P_{\omega_1}(H(\lambda))$ such that $M \cap \kappa \in T^*$ and M is an elementary substructure of $(H(\lambda), \in, \trianglelefteq, \kappa, T^*, \pi^*, C^*, \Lambda)$.*

Notation 1.10. *Let $\mathcal{Y}_0$ denote the set of P in $P_\kappa(H(\lambda))$ such that $P \cap \kappa \in \kappa$ and P is an elementary substructure of $(H(\lambda), \in, \trianglelefteq, \kappa, T^*, \pi^*, C^*, \Lambda)$.*

Note that if P and Q are in $\mathcal{Y}_0$, then $P \cap Q$ is in $\mathcal{Y}_0$. And if $M \in \mathcal{X}_0$ and $P \in \mathcal{Y}_0$, then $M \cap P$ is in $\mathcal{X}_0$. For the presence of the well-ordering $\trianglelefteq$ implies that $P \cap Q$ and $M \cap P$ are elementary substructures, and $M \cap P \cap \kappa$ is an initial segment of $M \cap \kappa$ and hence is in T^*. For the intersection of models in $\mathcal{X}_0$, see Lemma 1.23.

This completes the introduction of the basic objects.

Next we will define comparison points and a way to compare two models in $\mathcal{X}_0$.

Definition 1.11. *For $M \in \mathcal{X}_0$, let Λ_M denote the set of $\beta \in \Lambda$ such that*

$$\beta = \min(\Lambda \setminus \sup(M \cap \beta)).$$

Observe that since any member of Λ_M is determined by an ordinal in $\mathrm{cl}(M)$, and $\mathrm{cl}(M)$ is countable, it follows that Λ_M is countable.

Lemma 1.12. *Let $M \in \mathcal{X}_0$. If $\beta \in \Lambda_M$ and $\beta_0 \in \Lambda \cap \beta$, then $M \cap [\beta_0, \beta) \neq \emptyset$.*

Proof. If $M \cap [\beta_0, \beta) = \emptyset$, then $\sup(M \cap \beta) \leq \beta_0$. So

$$\beta = \min(\Lambda \setminus \sup(M \cap \beta)) \leq \beta_0 < \beta,$$

which is a contradiction. □

Lemma 1.13. *Let M and N be in $\mathcal{X}_0$. Then $\Lambda_M \cap \Lambda_N$ has a maximum element.*

Proof. Note that the first member of Λ is in both Λ_M and Λ_N, and therefore $\Lambda_M \cap \Lambda_N$ is nonempty. Suppose for a contradiction that $\gamma := \sup(\Lambda_M \cap \Lambda_N)$ is not in $\Lambda_M \cap \Lambda_N$. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ in $\Lambda_M \cap \Lambda_N$ which is cofinal in γ. Then for each $n < \omega$, $M \cap [\gamma_n, \gamma_{n+1})$ is nonempty by Lemma 1.12. So γ is a limit point of M. Similarly, γ is a limit point of N. Let $\beta = \min(\Lambda \setminus \gamma)$. Since γ has cofinality ω, $\gamma < \beta$, and since γ is a limit point of M and a limit point of N, easily $\beta \in \Lambda_M \cap \Lambda_N$. This contradicts that $\gamma = \sup(\Lambda_M \cap \Lambda_N)$ and $\gamma < \beta$. □

Definition 1.14. *For M and N in $\mathcal{X}_0$, let $\beta_{M,N}$ be the maximum element of $\Lambda_M \cap \Lambda_N$. The ordinal $\beta_{M,N}$ is called the* comparison point of M and N.

The most important property of $\beta_{M,N}$ is described in the next lemma.

Lemma 1.15. *Let M and N be in $\mathcal{X}_0$. Then*

$$\mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa) \subseteq \beta_{M,N}.$$

Proof. Suppose for a contradiction that ξ is in $\mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa)$ but $\beta_{M,N} \leq \xi$. Let $\beta = \min(\Lambda \setminus (\xi + 1))$. Since β is a limit ordinal, $\beta_{M,N} \leq \xi < \xi + 1 < \beta$. We claim that $\beta \in \Lambda_M \cap \Lambda_N$. Then by the maximality of $\beta_{M,N}$, $\beta \leq \beta_{M,N}$, which is a contradiction.

First, assume that $\xi \in \Lambda$. Then ξ has uncountable cofinality. So ξ cannot be a limit point of M or of N. Hence $\xi \in M \cap N$. By elementarity, $\xi + 1 \in M \cap N$. Since $\xi + 1 \in M \cap \beta$, $\xi + 1 \leq \sup(M \cap \beta) < \beta$. As $\beta = \min(\Lambda \setminus (\xi + 1))$, clearly

$\beta = \min(\Lambda \setminus \sup(M \cap \beta))$. So $\beta \in \Lambda_M$. The same argument shows that $\beta \in \Lambda_N$, and we are done.

Secondly, assume that $\xi \notin \Lambda$. Then $\min(\Lambda \setminus \xi) = \min(\Lambda \setminus (\xi+1)) = \beta$. Since $\xi < \beta$ and ξ is either in $M \cap \kappa$ or is a limit point of $M \cap \kappa$, clearly $\xi \leq \sup(M \cap \beta) < \beta$. Hence $\beta = \min(\Lambda \setminus \sup(M \cap \beta))$, and therefore $\beta \in \Lambda_M$. The same argument shows that $\beta \in \Lambda_N$, finishing the proof. □

The next lemma provides some useful technical facts about comparison points. Statement (4) is not very intuitive; however it turns out that this observation simplifies some of the material in the original development of adequate sets in [6].

Lemma 1.16. *Let L, M, and N be in $\mathcal{X}_0$.*

1. *If $L \cap \kappa \subseteq M \cap \kappa$ then $\Lambda_L \subseteq \Lambda_M$. Hence $\beta_{L,N} \leq \beta_{M,N}$.*
2. *If $L \cap \kappa \subseteq \beta$ where $\beta \in \Lambda$, then $\Lambda_L \subseteq \beta + 1$. Hence $\beta_{L,M} \leq \beta$.*
3. *If $\beta < \beta_{M,N}$ and $\beta \in \Lambda$, then $M \cap [\beta, \beta_{M,N}) \neq \emptyset$.*
4. *Suppose that $M \cap \beta_{L,M} \subseteq N$. Then $\beta_{L,M} \leq \beta_{L,N}$.*

Proof. Statements (1) and (2) can be proven in a straightforward way from the definitions, and (3) follows immediately from Lemma 1.12. (4) By definition, $\beta_{L,M} \in \Lambda_L$. Since $M \cap \beta_{L,M} \subseteq N$, $\sup(M \cap \beta_{L,M}) \leq \sup(N \cap \beta_{L,M})$. As $\beta_{L,M} \in \Lambda_M$, by definition $\beta_{L,M} = \min(\Lambda \setminus \sup(M \cap \beta_{L,M}))$. So clearly $\beta_{L,M} = \min(\Lambda \setminus \sup(N \cap \beta_{L,M}))$. Hence $\beta_{L,M} \in \Lambda_N$. So $\beta_{L,M} \in \Lambda_L \cap \Lambda_N$. Therefore $\beta_{L,M} \leq \max(\Lambda_L \cap \Lambda_N) = \beta_{L,N}$. □

Now we introduce our way of comparing models.

Definition 1.17. *Let M and N be in $\mathcal{X}_0$.*

1. *Let $M < N$ if $M \cap \beta_{M,N} \in N$.*
2. *Let $M \sim N$ if $M \cap \beta_{M,N} = N \cap \beta_{M,N}$.*
3. *Let $M \leq N$ if either $M < N$ or $M \sim N$.*

Definition 1.18. *A finite set $A \subseteq \mathcal{X}_0$ is said to be* adequate *if for all M and N in A, either $M < N$, $M \sim N$, or $N < M$.*

If $M < N$, then by elementarity $\mathrm{cl}(M \cap \beta_{M,N})$ is a member of N. Since $\mathrm{cl}(M \cap \beta_{M,N})$ is countable, $\mathrm{cl}(M \cap \beta_{M,N}) \subseteq N$. Also every initial segment of $M \cap \beta_{M,N}$ is in N. For any proper initial segment has the form $M \cap \gamma = M \cap \beta_{M,N} \cap \gamma$ for some $\gamma \in M \cap \beta_{M,N}$, and since $M \cap \beta_{M,N}$ and γ are in N, so is $M \cap \gamma$.

The next lemma provides some useful technical facts about the relation on models just introduced.

Lemma 1.19. *Let $\{M, N\}$ be adequate.*

1. *If $(N \cap \beta_{M,N}) \setminus M$ is nonempty, then $M < N$.*
2. *If $M \leq N$ then $M \cap \beta_{M,N} = M \cap N \cap \kappa = M \cap N \cap \beta_{M,N}$.*
3. *$\beta_{M,N} = \min(\Lambda \setminus \sup(M \cap N \cap \kappa))$.*
4. *If $M < N$ then $\beta_{M,N} \in N$.*
5. *If $\beta < \beta_{M,N}$ and $\beta \in \Lambda$, then $(M \cap N) \cap [\beta, \beta_{M,N}) \neq \emptyset$.*

Proof. The assumption of (1) implies that $M \sim N$ and $N < M$ are impossible. (2) Both $M \cap \beta_{M,N} \in N$ and $M \cap \beta_{M,N} = N \cap \beta_{M,N}$ imply that $M \cap \beta_{M,N} \subseteq N$. So $M \cap \beta_{M,N} \subseteq M \cap N \cap \kappa$. Conversely by Lemma 1.15, $M \cap N \cap \kappa \subseteq \beta_{M,N}$, so $M \cap N \cap \kappa \subseteq M \cap \beta_{M,N}$. This proves that $M \cap N \cap \kappa = M \cap \beta_{M,N}$. Since $M \cap N \cap \kappa \subseteq \beta_{M,N}$ by Lemma 1.15, $M \cap N \cap \kappa = M \cap N \cap \beta_{M,N}$.

(3) Without loss of generality assume that $M \leq N$. Then $M \cap N \cap \kappa = M \cap \beta_{M,N}$ by (2). Since $\beta_{M,N} \in \Lambda_M$, by definition

$$\beta_{M,N} = \min(\Lambda \setminus (\sup(M \cap \beta_{M,N}))) = \min(\Lambda \setminus (\sup(M \cap N \cap \kappa))).$$

(4) If $M < N$ then $M \cap \beta_{M,N} \in N$. By (2), $M \cap \beta_{M,N} = M \cap N \cap \kappa$. So $M \cap N \cap \kappa \in N$. By (3), $\beta_{M,N} = \min(\Lambda \setminus \sup(M \cap N \cap \kappa))$. So $\beta_{M,N} \in N$ by elementarity.

(5) Without loss of generality assume that $M \leq N$. Then by (2), $M \cap N \cap \beta_{M,N} = M \cap \beta_{M,N}$. Since $\beta_{M,N} \in \Lambda_M$, Lemma 1.12 implies that $M \cap [\beta, \beta_{M,N})$ is nonempty. Fix $\xi \in M \cap [\beta, \beta_{M,N})$. Then $\xi \in M \cap \beta_{M,N} = M \cap N \cap \beta_{M,N}$. So $(M \cap N) \cap [\beta, \beta_{M,N})$ is nonempty. □

Lemma 1.20. *Let M and N be in $\mathcal{X}_0$, and assume that $\{M, N\}$ is adequate. Then*

$$\mathrm{cl}(M \cap N \cap \kappa) = \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa).$$

Proof. The forward inclusion is immediate. Suppose that α is in $\mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa)$. Then by Lemma 1.15, $\alpha < \beta_{M,N}$. Without loss of generality, assume that $M \leq N$. Then

$$\alpha \in \mathrm{cl}(M \cap \kappa) \cap \beta_{M,N} = \mathrm{cl}(M \cap \beta_{M,N}) = \mathrm{cl}(M \cap N \cap \kappa)$$

by Lemma 1.19(2). □

If $\{M, N\}$ is adequate, then the relation which holds between M and N is determined by the intersection of M and N with ω_1.

Lemma 1.21. *Let $\{M, N\}$ be adequate. Then:*

1. *$M < N$ iff $M \cap \omega_1 < N \cap \omega_1$;*
2. *$M \sim N$ iff $M \cap \omega_1 = N \cap \omega_1$.*

Proof. Suppose that $M < N$. Then $M \cap \beta_{M,N} \in N$. Since $\beta_{M,N}$ has uncountable cofinality, $\omega_1 \leq \beta_{M,N}$. So $M \cap \omega_1$ is an initial segment of $M \cap \beta_{M,N}$, and hence $M \cap \omega_1 \in N$. So $M \cap \omega_1 < N \cap \omega_1$.

Suppose that $M \sim N$. Then $M \cap \beta_{M,N} = N \cap \beta_{M,N}$. Since $\omega_1 \leq \beta_{M,N}$, $M \cap \omega_1 = N \cap \omega_1$.

Conversely if $M \cap \omega_1 < N \cap \omega_1$, then the facts just proved imply that $M < N$ is the only possibility of how M and N relate. Similarly $M \cap \omega_1 = N \cap \omega_1$ implies that $M \sim N$. □

Lemma 1.22. *Let A be an adequate set. Then the relation $<$ is irreflexive and transitive on A, $\sim$ is an equivalence relation on A, and the relations $<$ and $\leq$ respect $\sim$.*

Proof. Immediate from Lemma 1.21. □

In proving amalgamation results over countable models, we will need to be able to enlarge an adequate set A by adding $M \cap N$ to A, where $M < N$ are in A. Let us show that we can do this while preserving adequacy.

First we note that $M \cap N$ is in $\mathcal{X}_0$.

Lemma 1.23. *Let $\{M, N\}$ be adequate. Then $M \cap N$ is in $\mathcal{X}_0$.*

Proof. Without loss of generality, assume that $M \leq N$. Then by Lemma 1.19(2), $M \cap N \cap \kappa = M \cap \beta_{M,N}$. Since T^* is closed under initial segments and $M \cap \kappa \in T^*$, it follows that $M \cap \beta_{M,N} \in T^*$. Hence $M \cap N \cap \kappa \in T^*$. Also clearly $M \cap N$ is an elementary substructure. □

Lemma 1.24. *Let K, M, and N be in $\mathcal{X}_0$. Suppose that $M < N$ and $\{K, M\}$ is adequate. Then:*

1. $\beta_{K,M\cap N} \leq \beta_{K,M}$ *and* $\beta_{K,M\cap N} \leq \beta_{M,N}$*;*
2. $M < K$ *iff* $M \cap N < K$*;*
3. $K \sim M$ *iff* $K \sim M \cap N$*;*
4. $K < M$ *iff* $K < M \cap N$.

In particular, $\{K, M \cap N\}$ is adequate.

Proof. (1) Since $M \cap N \subseteq M$, $\beta_{K,M\cap N} \leq \beta_{K,M}$ by Lemma 1.16(1). Also $M \cap N \cap \kappa \subseteq \beta_{M,N}$ by Lemma 1.15, which implies that $\beta_{K,M\cap N} \leq \beta_{M,N}$ by Lemma 1.16(2). This proves (1).

Since $M \cap N \cap \beta_{M,N} = M \cap \beta_{M,N}$ by Lemma 1.19(2) and $\beta_{K,M\cap N} \leq \beta_{M,N}$, it follows that

$$M \cap N \cap \beta_{K,M\cap N} = M \cap \beta_{K,M\cap N}.$$

(2,3,4) First we will prove the forward implications of (2), (3), and (4). If $M < K$ then $M \cap \beta_{K,M}$ is in K. But since $\beta_{K,M\cap N} \leq \beta_{K,M}$, $M \cap \beta_{K,M\cap N}$ is an initial segment of $M \cap \beta_{K,M}$, and hence is in K. So $M \cap N \cap \beta_{K,M\cap N} = M \cap \beta_{K,M\cap N}$ is in K, and therefore $M \cap N < K$.

If $K \sim M$, then $K \cap \beta_{K,M} = M \cap \beta_{K,M}$. Since $\beta_{K,M\cap N} \leq \beta_{K,M}$,

$$K \cap \beta_{K,M\cap N} = M \cap \beta_{K,M\cap N} = M \cap N \cap \beta_{K,M\cap N}.$$

Therefore $K \sim M \cap N$.

Suppose that $K < M$. Then $K \cap \beta_{K,M} \in M$. Since $\beta_{K,M\cap N} \leq \beta_{K,M}$, $K \cap \beta_{K,M\cap N} \in M$. So to show that $K < M \cap N$, it suffices to show that $K \cap \beta_{K,M\cap N} \in N$.

Since $K \cap \kappa \in T^*$ by the definition of $\mathcal{X}_0$, $K \cap \beta_{K,M\cap N} \in T^*$ as T^* is closed under initial segments. Recall from Notation 1.5 that $\pi^* : T^* \to \kappa$ is a bijection. As M is closed under π^* by elementarity, $\pi^*(K \cap \beta_{K,M\cap N}) \in M \cap \kappa$. Since $K \cap \beta_{K,M\cap N}$ is a bounded subset of $\beta_{K,M\cap N}$ and $\beta_{K,M\cap N} \leq \beta_{M,N}$, we have that $K \cap \beta_{K,M\cap N}$ is a bounded subset of $\beta_{M,N}$. Since $\beta_{M,N} \in \Lambda$, it follows that $\pi^*(K \cap \beta_{K,M\cap N}) < \beta_{M,N}$ by the definitions of C^* and Λ from Notations 1.6 and 1.7. Hence $\pi^*(K \cap \beta_{K,M\cap N}) \in M \cap \beta_{M,N} \subseteq N$. Since N is closed under the inverse of π^* by elementarity, $K \cap \beta_{K,M\cap N} \in N$.

Now we consider the reverse implications of (2), (3), and (4). Suppose that $M \cap N < K$. Since $\{K, M\}$ is adequate, either $K < M$, $K \sim M$, or $M < K$. But $K \sim M$ and $K < M$ are ruled out by the forward implications of (3) and (4). So $M < K$. The other converses are proved similarly. □

Proposition 1.25. *Let A be an adequate set and $N \in \mathcal{X}_0$. Let M be in A, and suppose that $M < N$. Then $A \cup \{M \cap N\}$ is adequate.*

Proof. Immediate from Lemma 1.24. □

Our next goal is to prove the first amalgamation result over countable models, which is stated in Proposition 1.29 below. See Proposition 13.1 for a much deeper result.

Lemma 1.26. *Let L, M, and N be in $\mathcal{X}_0$. Suppose that $N \leq M$ and $L \in N$. Then $L < M$.*

Proof. Since $L \in N$, $\beta_{L,M} \leq \beta_{M,N}$ by Lemma 1.16(1). Also $L \cap \beta_{L,M}$ is in $N \cap T^*$, since it is an initial segment of $L \cap \kappa$. As N is closed under π^* by elementarity, the ordinal $\pi^*(L \cap \beta_{L,M})$ is in $N \cap \kappa$. And as $\beta_{M,N} \in \Lambda$ and $L \cap \beta_{L,M}$ is a bounded subset of $\beta_{M,N}$ in T^*, it follows that $\pi^*(L \cap \beta_{L,M}) < \beta_{M,N}$ by the definition of Λ. Hence $\pi^*(L \cap \beta_{L,M}) \in N \cap \beta_{M,N} \subseteq M$. By elementarity, M is closed under the inverse of π^*, so $L \cap \beta_{L,M} \in M$. □

Lemma 1.27. *Let L, M, and N be in $\mathcal{X}_0$. Suppose that $M < N$ and $L \in N$. Then:*

(1) $\beta_{L,M} = \beta_{L,M\cap N}$;
(2) $L \sim M \cap N$ *iff* $L \sim M$;
(3) $L < M \cap N$ *iff* $L < M$;
(4) $M \cap N < L$ *iff* $M < L$.

Proof. (1) Since $M \cap N \cap \kappa \subseteq M \cap \kappa$, $\beta_{L,M\cap N} \leq \beta_{L,M}$ by Lemma 1.16(1), which proves one direction of the equality. Since $L \cap \kappa \subseteq N \cap \kappa$, $\beta_{L,M} \leq \beta_{M,N}$ by Lemma 1.16(1). So

$$M \cap \beta_{L,M} \subseteq M \cap \beta_{M,N} \subseteq M \cap N.$$

By Lemma 1.16(4), $\beta_{L,M} \leq \beta_{L,M\cap N}$.

(2,3,4) First we will prove the forward implications of (2), (3), and (4). As $\beta_{L,M} \leq \beta_{M,N}$ and $M \cap \beta_{M,N} = M \cap N \cap \beta_{M,N}$, it follows that

$$M \cap \beta_{L,M} = M \cap N \cap \beta_{L,M}.$$

If $L \sim M \cap N$, then

$$L \cap \beta_{L,M} = L \cap \beta_{L,M\cap N} = M \cap N \cap \beta_{L,M\cap N} = M \cap N \cap \beta_{L,M} = M \cap \beta_{L,M}.$$

So $L \cap \beta_{L,M} = M \cap \beta_{L,M}$, and hence $L \sim M$. And if $L < M \cap N$, then

$$L \cap \beta_{L,M} = L \cap \beta_{L,M\cap N} \in M \cap N \subseteq M.$$

So $L \cap \beta_{L,M} \in M$, and hence $L < M$. If $M \cap N < L$, then

$$M \cap \beta_{L,M} = M \cap N \cap \beta_{L,M} = M \cap N \cap \beta_{L,M\cap N} \in L.$$

So $M \cap \beta_{L,M} \in L$, and therefore $M < L$.

For the reverse implications, each of the assumptions $L \sim M$, $L < M$, and $M < L$ implies that $\{L, M\}$ is adequate. Hence these assumptions imply that $L \sim M \cap N$, $L < M \cap N$, and $M \cap N < L$ respectively by Lemma 1.24. □

Lemma 1.28. *Let L, M, and N be in $\mathcal{X}_0$. Suppose that $M < N$ and $L \in N$. If $\{L, M \cap N\}$ is adequate, then $\{L, M\}$ is adequate.*

Proof. Immediate from Lemma 1.27. □

Proposition 1.29. *Let A be adequate, $N \in A$, and suppose that for all $M \in A$, if $M < N$ then $M \cap N \in A \cap N$. Suppose that B is adequate and $A \cap N \subseteq B \subseteq N$. Then $A \cup B$ is adequate.*

Proof. Let $L \in B$ and $M \in A$, and we will show that $\{L, M\}$ is adequate. Since $L \in B$ and $B \subseteq N$, $L \in N$. If $N \leq M$, then $L < M$ by Lemma 1.26. Suppose that $M < N$. Then $M \cap N \in A \cap N$ by assumption. Since $A \cap N \subseteq B$, $M \cap N \in B$. As B is adequate, $\{L, M \cap N\}$ is adequate. Since $L \in N$ and $\{L, M \cap N\}$ is adequate, $\{L, M\}$ is adequate by Lemma 1.28. $\square$

In the last proposition, we assumed that $M < N$ implies that $M \cap N \in N$, for $M \in A$. At this point we do not have any reason to believe this implication is true in general. In Section 7, we will define a subclass of $\mathcal{X}_0$ on which this implication holds. See Notation 7.7 and Lemma 8.2.

So far we have discussed the interaction of countable models in $\mathcal{X}_0$. We now turn our attention to how models in $\mathcal{X}_0$ relate to models in $\mathcal{Y}_0$.

Lemma 1.30. *Let M and N be in $\mathcal{X}_0 \cup \mathcal{Y}_0$. Suppose that:*
1. *M and N are in $\mathcal{X}_0$ and $M < N$, or*
2. *M and N are in $\mathcal{Y}_0$ and $M \cap \kappa < N \cap \kappa$, or*
3. *$M \in \mathcal{X}_0$, $N \in \mathcal{Y}_0$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$.*

Then $M \cap N \cap \kappa \in N$.

Proof. (1) If M and N are in $\mathcal{X}_0$, then $M < N$ implies that $M \cap \beta_{M,N} \in N$. By Lemma 1.19(2), $M \cap \beta_{M,N} = M \cap N \cap \kappa$, so $M \cap N \cap \kappa \in N$. (2) If M and N are in $\mathcal{Y}_0$, then since $M \cap \kappa < N \cap \kappa$, $M \cap N \cap \kappa = M \cap \kappa \in N$.

(3) Suppose that $M \in \mathcal{X}_0$, $N \in \mathcal{Y}_0$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$. Let $\beta := N \cap \kappa$. By the elementarity of N, β is a limit point of Λ. So fix $\gamma \in N \cap \Lambda$ such that $\sup(M \cap \beta) < \gamma$. Then $M \cap N \cap \kappa = M \cap \gamma$. Since γ has uncountable cofinality, $M \cap \gamma$ is a bounded subset of γ, and as $M \in \mathcal{X}_0$, $M \cap \gamma \in T^*$. By the definition of C^* and Λ, $\pi^*(M \cap \gamma) < \gamma < N \cap \kappa$. Since N is closed under the inverse of π^* by elementarity, $M \cap \gamma = M \cap N \cap \kappa \in N$. $\square$

Note that (3) holds if $\mathrm{cf}(N \cap \kappa) > \omega$, which is the typical situation that we will consider.

Lemma 1.31. *Let $M \in \mathcal{X}_0$ and $N \in \mathcal{Y}_0$, and assume that $\sup(M \cap N \cap \kappa) < N \cap \kappa$. Then*
$$\mathrm{cl}(M \cap N \cap \kappa) = \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa) \cap (N \cap \kappa).$$

Proof. The forward inclusion is immediate. Let
$$\alpha \in \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa) \cap (N \cap \kappa).$$
Since $\mathrm{cl}(N \cap \kappa) = (N \cap \kappa) \cup \{N \cap \kappa\}$,
$$\alpha \in \mathrm{cl}(M \cap \kappa) \cap (N \cap \kappa) = \mathrm{cl}(M \cap N \cap \kappa).$$
$\square$

Recall that if $M \in \mathcal{X}_0$ and $P \in \mathcal{Y}_0$, then $M \cap P \in \mathcal{X}_0$. We show next that we can add $M \cap P$ to an adequate set and preserve adequacy.

Lemma 1.32. *Let K and M be in $\mathcal{X}_0$ and P in $\mathcal{Y}_0$. Assume that $\{K, M\}$ is adequate and $\sup(M \cap P \cap \kappa) < P \cap \kappa$. Then:*
1. *$\beta_{K,M \cap P} \leq \beta_{K,M}$ and $\beta_{K,M \cap P} < P \cap \kappa$;*
2. *$M < K$ iff $M \cap P < K$;*

(3) *$K \sim M$ iff $K \sim M \cap P$;*
(4) *$K < M$ iff $K < M \cap P$.*

In particular, $\{K, M \cap P\}$ is adequate.

Proof. (1) Since $M \cap P \subseteq M$, $\beta_{K,M\cap P} \leq \beta_{K,M}$ by Lemma 1.16(1). As $\sup(M \cap P \cap \kappa) < P \cap \kappa$ and Λ is unbounded in $P \cap \kappa$ by elementarity, we can fix $\beta \in \Lambda$ with $\sup(M \cap P \cap \kappa) < \beta < P \cap \kappa$. By Lemma 1.16(2),

$$\beta_{K,M\cap P} \leq \beta < P \cap \kappa.$$

This proves (1). It follows that

$$M \cap \beta_{K,M\cap P} = M \cap P \cap \beta_{K,M\cap P}.$$

(2,3,4) First we will prove the forward implications of (2), (3), and (4). Assume that $M < K$. Then $M \cap \beta_{K,M} \in K$. Since $\beta_{K,M\cap P} \leq \beta_{K,M}$, $M \cap \beta_{K,M\cap P} \in K$. So

$$M \cap P \cap \beta_{K,M\cap P} = M \cap \beta_{K,M\cap P} \in K.$$

Hence $M \cap P < K$.

Suppose that $K \sim M$. Then $K \cap \beta_{K,M} = M \cap \beta_{K,M}$. Since $\beta_{K,M\cap P} \leq \beta_{K,M}$, it follows that

$$K \cap \beta_{K,M\cap P} = M \cap \beta_{K,M\cap P} = M \cap P \cap \beta_{K,M\cap P}.$$

Therefore $K \sim M \cap P$.

Finally, assume that $K < M$. Then $K \cap \beta_{K,M} \in M$. Since $\beta_{K,M\cap P} \leq \beta_{K,M}$, $K \cap \beta_{K,M\cap P} \in M$. As $\beta_{K,M\cap P} < P \cap \kappa$, by elementarity there is $\gamma \in P \cap \Lambda$ with $\beta_{K,M\cap P} < \gamma$. Then $K \cap \beta_{K,M\cap P}$ is a bounded subset of γ in T^*. Hence $\pi^*(K \cap \beta_{K,M\cap P}) < \gamma$. In particular, $\pi^*(K \cap \beta_{K,M\cap P}) \in P \cap \kappa$. By the elementarity of P, P is closed under the inverse of π^*. So $K \cap \beta_{K,M\cap P} \in P$. Thus $K \cap \beta_{K,M\cap P} \in M \cap P$, and therefore $K < M \cap P$.

Conversely, assume that $M \cap P < K$. Since $\{K, M\}$ is adequate, either $M < K$, $M \sim K$, or $K < M$. But the forward implications of (3) and (4) rule out $M \sim K$ and $K < M$. Hence $M < K$. The other converses are proved similarly. □

Proposition 1.33. *Let A be an adequate set. Let M be in A and P in $\mathcal{Y}_0$, and assume that $\sup(M \cap P \cap \kappa) < P \cap \kappa$. Then $A \cup \{M \cap P\}$ is adequate.*

Proof. Immediate from Lemma 1.32. □

Next we will prove an amalgamation result for uncountable models. See Proposition 13.2 for a deeper result.

Lemma 1.34. *Let L and M be in $\mathcal{X}_0$ and $P \in \mathcal{Y}_0$. Assume that $L \in P$ and $\sup(M \cap P \cap \kappa) < P \cap \kappa$. Then:*

(1) *$\beta_{L,M} = \beta_{L,M\cap P}$ and $\beta_{L,M} < P \cap \kappa$;*
(2) *$L \sim M \cap P$ iff $L \sim M$;*
(3) *$M \cap P < L$ iff $M < L$;*
(4) *$L < M \cap P$ iff $L < M$.*

In particular, $\{L, M \cap P\}$ is adequate iff $\{L, M\}$ is adequate.

Proof. (1) Since $M \cap P \subseteq M$, $\beta_{L,M\cap P} \leq \beta_{L,M}$ by Lemma 1.16(1), which proves one direction of the equality. As $L \in P$, by elementarity, $\Lambda_L \in P$. Since Λ_L is countable, $\Lambda_L \subseteq P$. As $\beta_{L,M} \in \Lambda_L$, $\beta_{L,M} \in P \cap \kappa$. So $M \cap \beta_{L,M} \subseteq M \cap P$. By Lemma 1.16(4), it follows that $\beta_{L,M} \leq \beta_{L,M\cap P}$.

(2,3,4) First we will prove the forward implications of (2), (3), and (4). Since $\beta_{L,M} \in P$ as noted above,

$$M \cap \beta_{L,M} = M \cap P \cap \beta_{L,M}.$$

If $L \sim M \cap P$, then

$$L \cap \beta_{L,M} = L \cap \beta_{L,M\cap P} = M \cap P \cap \beta_{L,M\cap P} = M \cap P \cap \beta_{L,M} = M \cap \beta_{L,M}.$$

So $L \cap \beta_{L,M} = M \cap \beta_{L,M}$, and hence $L \sim M$.

If $M \cap P < L$, then

$$M \cap \beta_{L,M} = M \cap P \cap \beta_{L,M} = M \cap P \cap \beta_{L,M\cap P} \in L.$$

So $M \cap \beta_{L,M} \in L$, and therefore $M < L$. And if $L < M \cap P$, then

$$L \cap \beta_{L,M} = L \cap \beta_{L,M\cap P} \in M \cap P \subseteq M.$$

So $L \cap \beta_{L,M} \in M$, and therefore $L < M$.

Conversely, the assumptions $M < L$, $L \sim M$, and $L < M$ imply that $\{L, M\}$ is adequate. Hence each of these assumptions imply that $M \cap P < L$, $L \sim M \cap P$, $L < M \cap P$ respectively by Lemma 1.32. □

Proposition 1.35. *Let A be adequate, $P \in \mathcal{Y}_0$, and assume that for all $M \in A$, $M \cap P \in A \cap P$. Suppose that B is adequate and $A \cap P \subseteq B \subseteq P$. Then $A \cup B$ is adequate.*

Proof. Let $L \in B$ and $M \in A$. Then $M \cap P \in A \cap P \subseteq B$. Since B is adequate, $\{L, M \cap P\}$ is adequate. As $M \cap P \in P$, $\sup(M \cap P \cap \kappa) < P \cap \kappa$. By Lemma 1.34, $\{L, M\}$ is adequate. □

We conclude the discussion about models in $\mathcal{X}_0$ and $\mathcal{Y}_0$ with the following useful lemma.

Lemma 1.36. *Let M and N be in $\mathcal{X}_0$, and assume that $\{M, N\}$ is adequate. Let $P \in \mathcal{Y}_0$. Then either $\beta_{M,N} = \beta_{M\cap P,N}$, or $P \cap \kappa < \beta_{M,N}$.*

Proof. Since $M \cap P \subseteq M$, $\beta_{M\cap P,N} \leq \beta_{M,N}$. If $\beta_{M,N} = \beta_{M\cap P,N}$, then we are done. So assume that $\beta_{M\cap P,N} < \beta_{M,N}$. We claim that $P \cap \kappa < \beta_{M,N}$. Suppose for a contradiction that $\beta_{M,N} \leq P \cap \kappa$. Since $\beta_{M\cap P,N} < \beta_{M,N}$, by Lemma 1.19(5), we can fix

$$\xi \in (M \cap N) \cap [\beta_{M\cap P,N}, \beta_{M,N}).$$

As $\beta_{M,N} \leq P \cap \kappa$,

$$\xi \in (M \cap N) \cap P \cap \kappa = (M \cap P) \cap N \cap \kappa.$$

Therefore $\xi < \beta_{M\cap P,N}$ by Lemma 1.15, which contradicts the choice of ξ. □

Finally, we prove an amalgamation result over transitive models.

Lemma 1.37. *Let M, M', N, and N' be in $\mathcal{X}_0$. Assume that $M \cap \kappa = M' \cap \kappa$ and $N \cap \kappa = N' \cap \kappa$. Then:*

(1) $\beta_{M,N} = \beta_{M',N'}$;

(2) *$M \sim N$ iff $M' \sim N'$;*
(3) *$M < N$ iff $M' < N'$;*
(4) *$N < M$ iff $N' < M'$.*

In particular, $\{M, N\}$ is adequate iff $\{M', N'\}$ is adequate.

Proof. (1) Since $M \cap \kappa \subseteq M' \cap \kappa$ and $N \cap \kappa \subseteq N' \cap \kappa$, it follows that $\beta_{M,N} \leq \beta_{M',N} \leq \beta_{M',N'}$ by Lemma 1.16(1). Similarly, the reverse inclusions imply that $\beta_{M',N'} \leq \beta_{M,N}$. So $\beta_{M,N} = \beta_{M',N'}$.

(2,3,4) It suffices to prove the forward direction of the iff's of (2), (3), and (4), since the converses hold by symmetry. If $M \sim N$, then

$$M' \cap \beta_{M',N'} = M \cap \beta_{M,N} = N \cap \beta_{M,N} = N' \cap \beta_{M',N'},$$

which proves (2). Suppose that $M < N$. Then

$$M' \cap \beta_{M',N'} = M \cap \beta_{M,N} \in N.$$

By elementarity,

$$\pi^*(M' \cap \beta_{M',N'}) \in N \cap \kappa = N' \cap \kappa.$$

Since N' is closed under the inverse of π^* by elementarity, $M' \cap \beta_{M',N'} \in N'$. (4) is similar. □

Proposition 1.38. *Let A be an adequate set. Assume that $X \prec (H(\kappa^+), \in)$, $|X| = \kappa$, and $X \cap \kappa^+ \in \kappa^+$. Let B be an adequate set such that $A \cap X \subseteq B \subseteq X$. Suppose that for all $M \in A$, there is $M' \in B$ such that $M \cap \kappa = M' \cap \kappa$. Then $A \cup B$ is adequate.*

Proof. Let $M \in A$ and $K \in B$ be given. Fix $M' \in B$ such that $M \cap \kappa = M' \cap \kappa$. As $\{M', K\} \subseteq B$, $\{M', K\}$ is adequate. Therefore $\{M, K\}$ is adequate by Lemma 1.37. □

§2. Analysis of remainder points

In this section we will provide a detailed analysis of remainder points; some of these arguments appeared previously in [8] and [9], although in a less complete form. This analysis will be the foundation from which we derive the amalgamation results of Section 13.

Definition 2.1. *Let $\{M, N\}$ be adequate. Let $R_M(N)$, the* set of remainder points of N over M, *be defined as the set of ζ satisfying either:*

(1) *$\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$, provided that $M \sim N$, or*
(2) *there is $\gamma \in (M \cap \kappa) \setminus \beta_{M,N}$ such that $\zeta = \min((N \cap \kappa) \setminus \gamma)$.*

Note that if $N < M$, then $\beta_{M,N} \in M$ by Lemma 1.19(4). It follows that $\min((N \cap \kappa) \setminus \beta_{M,N}) \in R_M(N)$ by Definition 2.1(2).

The next lemma describes some basic properties of remainder points.

Lemma 2.2. *Let $\{M, N\}$ be adequate. Then:*

(1) *$R_M(N) \cap \mathrm{cl}(M \cap \kappa) = \emptyset$;*
(2) *$R_M(N)$ is finite;*
(3) *suppose that $\zeta \in R_M(N)$ and $\zeta > \min(R_M(N) \cup R_N(M))$; then $\sigma := \min((M \cap \kappa) \setminus \sup(N \cap \zeta)) \in R_N(M)$ and $\zeta = \min((N \cap \kappa) \setminus \sigma)$.*

Proof. (1) If $\zeta \in R_M(N)$, then by definition, $\zeta \in N$ and $\beta_{M,N} \leq \zeta$. Hence $\zeta \notin \mathrm{cl}(M \cap \kappa)$ by Lemma 1.15.

(2) Suppose for a contradiction that $\langle \zeta_n : n < \omega \rangle$ is a strictly increasing sequence from $R_M(N)$. Then by definition, for each $n > 0$ there is $\gamma_n \in M$ such that $\zeta_n = \min((N \cap \kappa) \setminus \gamma_n)$. Let $\zeta := \sup\{\zeta_n : n < \omega\}$. Then $\zeta = \sup\{\gamma_n : n < \omega\}$. Therefore

$$\zeta \in \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa).$$

Hence $\zeta < \beta_{M,N}$ by Lemma 1.15. But

$$\beta_{M,N} \leq \zeta_0 < \zeta,$$

which is a contradiction.

(3) Since $\zeta > \min(R_M(N) \cup R_N(M))$ and $R_M(N)$ and $R_N(M)$ are finite, let σ_0 be the largest member of $R_M(N) \cup R_N(M)$ less than ζ. We claim that $\sigma_0 \in R_N(M)$. If not, then $\sigma_0 \in R_M(N)$, and in particular, $\sigma_0 \in (N \cap \zeta) \setminus \beta_{M,N}$. Since $\zeta \in R_M(N)$, by the definition of $R_M(N)$ we have that $M \cap (\sigma_0, \zeta) \neq \emptyset$. But then $\min((M \cap \kappa) \setminus \sigma_0)$ is in $R_N(M)$ and is between σ_0 and ζ, which contradicts the maximality of σ_0.

We claim that $\zeta = \min((N \cap \kappa) \setminus \sigma_0)$. Otherwise $\min((N \cap \kappa) \setminus \sigma_0)$ is in $R_M(N)$ and is between σ_0 and ζ, which contradicts the maximality of σ_0. It follows that $\sup(N \cap \zeta) \leq \sigma_0$. Finally, we show that $\sigma_0 = \min((M \cap \kappa) \setminus \sup(N \cap \zeta))$. Therefore $\sigma = \sigma_0$, and we are done. Suppose for a contradiction that $\sigma < \sigma_0$. As $\sup(N \cap \zeta) \leq \sigma$, we have that $N \cap (\sigma, \sigma_0) = \emptyset$.

Observe that $\beta_{M,N} \leq \sigma$. For if $\sigma < \beta_{M,N}$, then $\sigma \in (M \cap \beta_{M,N}) \setminus N$, which implies that $N < M$. And since $\sup(N \cap \zeta) \leq \sigma$, it follows that $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$. So $\zeta = \min(R_M(N) \cup R_N(M))$, which is a contradiction. Hence $\beta_{M,N} \leq \sigma < \sigma_0$. Since $\sigma_0 \in R_N(M)$, there is $\gamma \in N$ such that $\sigma_0 = \min((M \cap \kappa) \setminus \gamma)$. But then $\sigma < \gamma < \sigma_0$, which contradicts that $N \cap (\sigma, \sigma_0) = \emptyset$. □

The rest of the section follows roughly the same sequence of topics covered in the previous section. Lemma 2.3 describes the remainder points which appear when adding $M \cap N$ to an adequate set, where $M < N$, as in Lemma 1.24 and Proposition 1.25. Then Lemmas 2.4–2.6 analyze remainder points which appear in the process of amalgamating over countable models, as in Proposition 1.29.

Lemma 2.3. *Let K, M, and N be in $\mathcal{X}_0$. Suppose that $M < N$ and $\{K, M, N\}$ is adequate. Then:*

1. *$R_K(M \cap N) \subseteq R_K(M)$;*
2. *$R_{M \cap N}(K) \subseteq R_M(K) \cup R_N(K)$.*

Proof. Note that by Lemma 1.24, $\{K, M \cap N\}$ is adequate, $\beta_{K,M \cap N} \leq \beta_{K,M}$, and $\beta_{K,M \cap N} \leq \beta_{M,N}$.

(1) Let $\zeta \in R_K(M \cap N)$, and we will show that $\zeta \in R_K(M)$. Then either (a) $K \sim M \cap N$ and $\zeta = \min((M \cap N \cap \kappa) \setminus \beta_{K,M \cap N})$, or (b) there is $\gamma \in (K \cap \kappa) \setminus \beta_{K,M \cap N}$ such that $\zeta = \min((M \cap N \cap \kappa) \setminus \gamma)$.

Case a: $K \sim M \cap N$ and $\zeta = \min((M \cap N \cap \kappa) \setminus \beta_{K,M \cap N})$. Then by Lemma 1.24, $K \sim M$. We claim that $\beta_{K,M} \leq \zeta$. Suppose for a contradiction that $\zeta < \beta_{K,M}$. Then since $K \sim M$ and $\zeta \in M \cap \beta_{K,M}$, it follows that $\zeta \in K$. But this contradicts that $\zeta \in R_K(M \cap N)$.

Since $\beta_{K,M\cap N} \leq \beta_{K,M} \leq \zeta$, it follows that $\zeta = \min((M \cap N \cap \kappa) \setminus \beta_{K,M})$. As $M < N$, $M \cap N \cap \kappa = M \cap \beta_{M,N}$, which is an initial segment of $M \cap \kappa$. So $\zeta = \min((M \cap \kappa) \setminus \beta_{K,M})$, and hence $\zeta \in R_K(M)$.

Case b: There is $\gamma \in (K \cap \kappa) \setminus \beta_{K,M\cap N}$ such that $\zeta = \min((M \cap N \cap \kappa) \setminus \gamma)$. Since $M \cap N \cap \kappa = M \cap \beta_{M,N}$ is an initial segment of $M \cap \kappa$, it follows that $\zeta = \min((M\cap\kappa)\setminus\gamma)$. If $\beta_{K,M} \leq \gamma$, then since $\gamma \in K$, $\zeta \in R_K(M)$. So assume that $\gamma < \beta_{K,M}$.

Now $\zeta \in M \cap N \cap \kappa$ implies that $\zeta < \beta_{M,N}$. So $\gamma < \beta_{M,N}$. Since $\gamma \in (K \cap \kappa) \setminus \beta_{K,M\cap N}$, $\gamma \notin M \cap N$. But $M \cap N \cap \kappa = M \cap \beta_{M,N}$, so $\gamma \notin M \cap \kappa$. Since $\gamma < \beta_{K,M}$ and $\gamma \in K \setminus M$, we have that $M < K$. So $M \cap \beta_{K,M} \subseteq K$. As $\zeta \in R_K(M \cap N)$, $\zeta \notin K$. Since $M \cap \beta_{K,M} \subseteq K$ and $\zeta \in M \setminus K$, it follows that $\beta_{K,M} \leq \zeta$. In conclusion, $\gamma < \beta_{K,M} \leq \zeta$. Hence $\zeta = \min((M \cap \kappa) \setminus \beta_{K,M})$. Since $M < K$, this implies that $\zeta \in R_K(M)$.

(2) Let $\zeta \in R_{M\cap N}(K)$. Then either (a) $K \sim M \cap N$ and $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M\cap N})$, or (b) there is $\gamma \in (M \cap N) \setminus \beta_{K,M\cap N}$ such that $\zeta = \min((K \cap \kappa) \setminus \gamma)$. We will show that either $\zeta \in R_M(K)$ or $\zeta \in R_N(K)$.

Case a: $K \sim M \cap N$ and $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M\cap N})$. Then $K \sim M$ by Lemma 1.24. Assume first that $\beta_{K,M} \leq \zeta$. Then $\beta_{K,M\cap N} \leq \beta_{K,M} \leq \zeta$. So $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M})$, which implies that $\zeta \in R_M(K)$.

Now assume that $\zeta < \beta_{K,M}$. Since $K \sim M$ and $\zeta \in K \cap \beta_{K,M}$, $\zeta \in M$. As $\zeta \in R_{M\cap N}(K)$, $\zeta \notin M \cap N$, so $\zeta \notin N$. Since $K \sim M < N$, $K < N$. As $\zeta \in K \setminus N$ and $K < N$, $\beta_{K,N} \leq \zeta$. Since $M \cap N \subseteq N$, $\beta_{K,M\cap N} \leq \beta_{K,N}$. Hence

$$\beta_{K,M\cap N} \leq \beta_{K,N} \leq \zeta.$$

So $\zeta = \min((K \cap \kappa) \setminus \beta_{K,N})$, and therefore $\zeta \in R_N(K)$.

Case b: $\zeta = \min((K \cap \kappa) \setminus \gamma)$, for some $\gamma \in (M \cap N) \setminus \beta_{K,M\cap N}$. If $\beta_{K,M} \leq \gamma$, then $\gamma \in (M \cap \kappa) \setminus \beta_{K,M}$, and hence $\zeta \in R_M(K)$. Suppose that $\gamma < \beta_{K,M} \leq \zeta$. Then $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M})$. Since $\gamma \in (M \cap \beta_{K,M}) \setminus K$, $K < M$. Therefore $\zeta \in R_M(K)$.

The remaining case is that $\gamma < \zeta < \beta_{K,M}$. Since $\beta_{K,M\cap N} \leq \gamma$ and $\gamma \in M \cap N$, $\gamma \notin K$. So $\gamma \in (M \cap \beta_{K,M}) \setminus K$. It follows that $K < M$. But $\zeta \in K \cap \beta_{K,M}$, so $\zeta \in M$. As $\zeta \in R_{M\cap N}(K)$ and $\zeta \in M$, $\zeta \notin N$. But $K < M < N$, so $K < N$. As $\zeta \in K \setminus N$, $\beta_{K,N} \leq \zeta$.

If $\beta_{K,N} \leq \gamma$, then $\gamma \in (N \cap \kappa) \setminus \beta_{K,N}$, and therefore $\zeta \in R_N(K)$. Suppose that $\gamma < \beta_{K,N} \leq \zeta$. Then $\zeta = \min((K \cap \kappa) \setminus \beta_{K,N})$. Since $K < N$, $\zeta \in R_N(K)$. □

Lemmas 2.4 and 2.5 describe the same situation we considered in Lemmas 1.26 and 1.27.

Lemma 2.4. *Let $N \leq M$ and $L \in N$, where L, M, and N are in $\mathcal{X}_0$. Then:*

1. *for all $\zeta \in R_L(M)$, $\beta_{M,N} \leq \zeta$ and $\zeta \in R_N(M)$;*
2. *for all $\zeta \in R_M(L)$, there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$.*

Proof. Note that by Lemma 1.26, $L < M$.

(1) Let $\zeta \in R_L(M)$. Since $L < M$, there is $\gamma \in (L \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((M \cap \kappa) \setminus \gamma)$. Since $\gamma \in L$ and $L \in N$, $\gamma \in N$. So $\gamma \in N \setminus M$. Since $N \leq M$, $\beta_{M,N} \leq \gamma$. Hence $\beta_{M,N} \leq \zeta$. As $\zeta = \min((M \cap \kappa) \setminus \gamma)$, $\zeta \in R_N(M)$.

(2) Let $\zeta \in R_M(L)$. Since $L < M$, there is $\gamma \in (M \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((L \cap \kappa) \setminus \gamma)$. Now $\zeta \in L \setminus M$ and $L \in N$. So $\zeta \in N \setminus M$. Since $N \leq M$, this implies that $\beta_{M,N} \leq \zeta$.

If $\gamma < \beta_{M,N}$, then let $\xi := \min((N \cap \kappa) \setminus \beta_{M,N})$. Since $N \leq M$, $\xi \in R_M(N)$. As $L \subseteq N$, clearly $\zeta = \min((L \cap \kappa) \setminus \xi)$. If $\beta_{M,N} \leq \gamma$, then let $\xi := \min((N \cap \kappa) \setminus \gamma)$, which exists since $\zeta \in N$. Then $\xi \in R_M(N)$, and since $L \subseteq N$, $\zeta = \min((L \cap \kappa) \setminus \xi)$. □

Lemma 2.5. *Let $M < N$ and $L \in N$, where L, M, and N are in $\mathcal{X}_0$. Then:*

1. *for all $\zeta \in R_L(M)$, either $\zeta < \beta_{M,N}$ and $\zeta \in R_L(M \cap N)$, or $\beta_{M,N} \leq \zeta$ and $\zeta \in R_N(M)$;*
2. *for all $\zeta \in R_M(L)$, either $\zeta \in R_{M \cap N}(L)$ or there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$.*

Proof. Note that by Lemma 1.27, $\beta_{L,M} = \beta_{L,M \cap N}$. And since $M < N$, $M \cap \beta_{M,N} = M \cap N \cap \kappa$.

(1) Let $\zeta \in R_L(M)$. Then either (a) $L \sim M$ and $\zeta = \min((M \cap \kappa) \setminus \beta_{L,M})$, or (b) there is $\gamma \in (L \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((M \cap \kappa) \setminus \gamma)$. Assume first that $\zeta < \beta_{M,N}$. In case (a), $L \sim M \cap N$ by Lemma 1.27. Since $\zeta < \beta_{M,N}$, $\zeta = \min((M \cap N \cap \kappa) \setminus \beta_{L,M \cap N})$. In case (b), $\gamma \in (L \cap \kappa) \setminus \beta_{L,M \cap N}$ and $\zeta = \min((M \cap N \cap \kappa) \setminus \gamma)$. In either case, $\zeta \in R_L(M \cap N)$.

Now assume that $\beta_{M,N} \leq \zeta$. In case (a), since

$$\beta_{L,M} \leq \beta_{M,N} \leq \zeta,$$

$\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$. Since $M < N$, this implies that $\zeta \in R_N(M)$. In case (b), if $\gamma < \beta_{M,N}$, then again $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$, and so $\zeta \in R_N(M)$. Otherwise $\gamma \in (N \cap \kappa) \setminus \beta_{M,N}$ and $\zeta = \min((M \cap \kappa) \setminus \gamma)$, so $\zeta \in R_N(M)$.

(2) Let $\zeta \in R_M(L)$. Then either (a) $L \sim M$ and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M})$, or (b) there is $\gamma \in (M \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((L \cap \kappa) \setminus \gamma)$. In case (a), $L \sim M \cap N$ by Lemma 1.27 and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M \cap N})$. Hence $\zeta \in R_{M \cap N}(L)$.

Assume (b). First consider the case that $\gamma < \beta_{M,N}$. Then

$$\gamma \in M \cap \beta_{M,N} \subseteq M \cap N.$$

So

$$\gamma \in (M \cap N \cap \kappa) \setminus \beta_{L,M \cap N}$$

and $\zeta = \min((L \cap \kappa) \setminus \gamma)$. Hence $\zeta \in R_{M \cap N}(L)$. Now consider the case that $\beta_{M,N} \leq \gamma$. Then $\gamma \in (M \cap \kappa) \setminus \beta_{M,N}$. Let $\xi := \min((N \cap \kappa) \setminus \gamma)$, which exists since $\zeta \in N$. Then $\xi \in R_M(N)$ and $\zeta = \min((L \cap \kappa) \setminus \xi)$. □

When amalgamating over a countable model N, the presence of $M \cap N$ prevents certain incompatibilities between M and the object we build in N. But oftentimes $M \cap N$ does not have enough information about M. In that case, we will use a model M' in N which is more representative of M than $M \cap N$.

Lemma 2.6. *Let L, M, M', and N be in $\mathcal{X}_0$. Assume that $M < N$ and $L \in N$. Also suppose that $M' \in N$, $\{L, M \cap N, M'\}$ is adequate, and $M \cap \beta_{M,N} = M' \cap \beta_{M,N}$. Then:*

1. *either $\beta_{L,M} = \beta_{L,M'}$ or $\beta_{M,N} < \beta_{L,M'}$;*
2. *if $\beta_{L,M} = \beta_{L,M'}$ and $\zeta \in R_{M \cap N}(L)$, then $\zeta \in R_{M'}(L)$.*

Proof. Note that $\{L, M\}$ is adequate by Lemma 1.28. We claim that $\beta_{L,M} \le \beta_{L,M'}$. Otherwise $\beta_{L,M'} < \beta_{L,M}$. Since $\{L, M\}$ is adequate, we can fix $\xi \in (L \cap M) \cap [\beta_{L,M'}, \beta_{L,M})$ by Lemma 1.19(5). Since $L \in N$, $\xi \in N$. So

$$\xi \in M \cap N \cap \kappa = M \cap \beta_{M,N} \subseteq M'.$$

Hence $\xi \in (L \cap M' \cap \kappa) \setminus \beta_{L,M'}$, which is impossible.

(1) If $\beta_{L,M} = \beta_{L,M'}$, then we are done. So assume that $\beta_{L,M} < \beta_{L,M'}$. We claim that $\beta_{M,N} < \beta_{L,M'}$. Otherwise

$$\beta_{L,M} < \beta_{L,M'} \le \beta_{M,N}.$$

Since $\{L, M'\}$ is adequate, we can fix $\xi \in (L \cap M') \cap [\beta_{L,M}, \beta_{L,M'})$ by Lemma 1.19(5). Then

$$\xi \in M' \cap \beta_{M,N} \subseteq M.$$

So $\xi \in (L \cap M \cap \kappa) \setminus \beta_{L,M}$, which is a contradiction.

(2) Assume that $\beta_{L,M} = \beta_{L,M'}$ and $\zeta \in R_{M \cap N}(L)$. By Lemma 1.27,

$$\beta_{L,M'} = \beta_{L,M} = \beta_{L,M \cap N}.$$

First, assume that $L \sim M \cap N$ and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M \cap N})$. Then $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M'})$. Also

$$L \cap \omega_1 = (M \cap N) \cap \omega_1 = M \cap \beta_{M,N} \cap \omega_1 = M' \cap \beta_{M,N} \cap \omega_1 = M' \cap \omega_1.$$

Since $\{L, M'\}$ is adequate and $L \cap \omega_1 = M' \cap \omega_1$, $L \sim M'$ by Lemma 1.21. Since $L \sim M'$ and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M'})$, $\zeta \in R_{M'}(L)$.

Secondly, suppose that $\gamma \in (M \cap N \cap \kappa) \setminus \beta_{L,M \cap N}$ and $\zeta = \min((L \cap \kappa) \setminus \gamma)$. Then

$$\gamma \in (M \cap N \cap \kappa) \setminus \beta_{L,M'}.$$

Since

$$M \cap N \cap \kappa = M \cap \beta_{M,N} \subseteq M',$$

$\gamma \in (M' \cap \kappa) \setminus \beta_{L,M'}$. So $\zeta \in R_{M'}(L)$. □

The statement of the next technical lemma is not very intuitive. But its discovery led to substantial simplifications of some of the arguments from [8].

Lemma 2.7. *Let K, M, and N be in $\mathcal{X}_0$ such that $\{K, M, N\}$ is adequate. Suppose that*

$$\zeta \in R_M(N),\ \zeta \notin K,\ \theta = \min((K \cap \kappa) \setminus \zeta),\ \textit{and } \theta < \beta_{K,N}.$$

Then $\theta \in R_M(K)$.

Proof. Since $\zeta < \theta < \beta_{K,N}$ and $\zeta \in N \setminus K$, it follows that $K < N$. In particular, $K \cap (\theta + 1) \subseteq N$.

Case 1: $N \le M$. Then $K < N \le M$, so $K < M$. We claim that $\beta_{K,M} \le \beta_{M,N}$. Otherwise $\beta_{M,N} < \beta_{K,M}$, which implies that

$$(K \cap M) \cap [\beta_{M,N}, \beta_{K,M}) \ne \emptyset$$

by Lemma 1.19(5). Let $\gamma = \min((K \cap \kappa) \setminus \beta_{M,N})$. Then since the intersection above is nonempty, $\gamma < \beta_{K,M}$, and hence $\gamma \in K \cap M$. But $\beta_{M,N} \le \zeta < \theta$ and $\theta \in K$ implies that $\gamma \le \theta$. Since $K \cap (\theta + 1) \subseteq N$, $\gamma \in N$. So $\gamma \in (M \cap N) \setminus \beta_{M,N}$, which is impossible. This proves that $\beta_{K,M} \le \beta_{M,N}$.

Suppose that $\zeta = \min((N \cap \kappa) \setminus \gamma)$ for some $\gamma \in (M \cap \kappa) \setminus \beta_{M,N}$. Since $\beta_{K,M} \leq \beta_{M,N}$, it follows that $\gamma \in (M \cap \kappa) \setminus \beta_{K,M}$. As $K \cap (\theta+1) \subseteq N$, $\theta = \min((K \cap \kappa) \setminus \gamma)$. Hence $\theta \in R_M(K)$.

Suppose that $M \sim N$ and $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$. Since $K \cap (\theta + 1) \subseteq N$, it follows that $\theta = \min((K \cap \kappa) \setminus \beta_{M,N})$. We claim that $\theta = \min((K \cap \kappa) \setminus \beta_{K,M})$, which implies that $\theta \in R_M(K)$ as desired. If not, then there is $\pi \in K \cap [\beta_{K,M}, \beta_{M,N})$. But $\beta_{M,N} \leq \zeta < \theta < \beta_{K,N}$, so $\pi \in K \cap \beta_{K,N} \subseteq N$. Hence $\pi \in N \cap \beta_{M,N} \subseteq M$. So $\pi \in M$. Therefore $\pi \in (K \cap M) \setminus \beta_{K,M}$, which is impossible.

Case 2: $M < N$. Since $\zeta \in R_M(N)$, there is $\gamma \in (M \cap \kappa) \setminus \beta_{M,N}$ such that $\zeta = \min((N \cap \kappa) \setminus \gamma)$. If $\beta_{K,M} \leq \gamma$, then $\gamma \in (M \cap \kappa) \setminus \beta_{K,M}$, and since $K \cap (\theta+1) \subseteq N$, $\theta = \min((K \cap \kappa) \setminus \gamma)$. Hence $\theta \in R_M(K)$.

Otherwise $\gamma < \beta_{K,M}$. Since $\gamma \notin N$, $\gamma < \theta$, and $K \cap (\theta + 1) \subseteq N$, it follows that $\gamma \notin K$. So $\gamma \in (M \cap \beta_{K,M}) \setminus K$, which implies that $K < M$. Since $K \cap (\theta+1) \subseteq N$, it follows that $\theta = \min((K \cap \kappa) \setminus \gamma)$. As $\theta \in (N \cap \kappa) \setminus \beta_{M,N}$, $\theta \notin M$. As $K < M$ and $\theta \in K \cap \kappa$, $\beta_{K,M} \leq \theta$. So $\gamma < \beta_{K,M} \leq \theta$. Hence $\theta = \min((K \cap \kappa) \setminus \beta_{K,M})$, which implies that $\theta \in R_M(K)$. $\square$

The next three lemmas are analogues of Lemmas 2.3, 2.5, and 2.6, where the countable model N in $\mathcal{X}_0$ is replaced by an uncountable model P in $\mathcal{Y}_0$.

Lemma 2.8. *Let K and M be in $\mathcal{X}_0$ and $P \in \mathcal{Y}_0$. Assume that $\{K, M\}$ is adequate and $\sup(M \cap P \cap \kappa) < P \cap \kappa$. Then:*

1. *$R_K(M \cap P) \subseteq R_K(M)$;*
2. *if $\zeta \in R_{M \cap P}(K)$, then either $\zeta \in R_M(K)$ or $\zeta = \min((K \cap \kappa) \setminus (P \cap \kappa))$.*

Proof. Note that by Lemma 1.32, $\beta_{K,M \cap P} \leq \beta_{K,M}$, $\beta_{K,M \cap P} < P \cap \kappa$, and $\{K, M \cap P\}$ is adequate.

(1) Let $\zeta \in R_K(M \cap P)$. Then either (a) $K \sim M \cap P$ and $\zeta = \min((M \cap P \cap \kappa) \setminus \beta_{K,M \cap P})$, or (b) there is $\gamma \in (K \cap \kappa) \setminus \beta_{K,M \cap P}$ such that $\zeta = \min((M \cap P \cap \kappa) \setminus \gamma)$.

Case a: $K \sim M \cap P$ and $\zeta = \min((M \cap P \cap \kappa) \setminus \beta_{K,M \cap P})$. Then $K \sim M$ by Lemma 1.32. By Lemma 1.36, either $\beta_{K,M} = \beta_{K,M \cap P}$, or $P \cap \kappa < \beta_{K,M}$.

We claim that $\beta_{K,M} = \beta_{K,M \cap P}$. Suppose for a contradiction that $P \cap \kappa < \beta_{K,M}$. Since $\zeta \in M \cap P \cap \kappa \subseteq P \cap \kappa$, $\zeta < \beta_{K,M}$. But since $K \sim M$ and $\zeta \in M \cap \beta_{K,M}$, $\zeta \in K$. So $\zeta \in K \cap (M \cap P) \cap \kappa$, which contradicts that $\zeta \in R_K(M \cap P)$.

So $\beta_{K,M} = \beta_{K,M \cap P}$. Since $M \cap P \cap \kappa$ is an initial segment of $M \cap \kappa$, $\zeta = \min((M \cap \kappa) \setminus \beta_{K,M})$. Hence $\zeta \in R_K(M)$.

Case b: $\zeta = \min((M \cap P \cap \kappa) \setminus \gamma)$, for some $\gamma \in (K \cap \kappa) \setminus \beta_{K,M \cap P}$. Since $M \cap P \cap \kappa$ is an initial segment of $M \cap \kappa$, $\zeta = \min((M \cap \kappa) \setminus \gamma)$. By Lemma 1.36, either $\beta_{K,M} = \beta_{K,M \cap P}$ or $P \cap \kappa < \beta_{K,M}$. In the first case, $\gamma \in (K \cap \kappa) \setminus \beta_{K,M}$, so $\zeta \in R_K(M)$.

We prove that the other case is impossible. Suppose for a contradiction that $P \cap \kappa < \beta_{K,M}$. Since $\gamma < \zeta < P \cap \kappa$, $\gamma \in P$. But $\gamma \in (K \cap \kappa) \setminus \beta_{K,M \cap P}$ implies that $\gamma \notin M \cap P$. So $\gamma \notin M$. As $\gamma < P \cap \kappa < \beta_{K,M}$, we have that $\gamma \in (K \cap \beta_{K,M}) \setminus M$. Hence $M < K$. Since $\zeta \in M \cap P \cap \kappa$, $\zeta \in M \cap \beta_{K,M}$. As $M < K$, $\zeta \in K$. But this is impossible since $\zeta \in R_K(M \cap P)$.

(2) Let $\zeta \in R_{M \cap P}(K)$. We will prove that either $\zeta \in R_M(K)$, or $\zeta = \min((K \cap \kappa) \setminus (P \cap \kappa))$. Either (a) $K \sim M \cap P$ and $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M \cap P})$, or (b) there is $\gamma \in (M \cap P \cap \kappa) \setminus \beta_{K,M \cap P}$ such that $\zeta = \min((K \cap \kappa) \setminus \gamma)$.

Case a: $K \sim M \cap P$ and $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M \cap P})$. Then $K \sim M$ by Lemma 1.32. Also by Lemma 1.36, either $\beta_{K,M} = \beta_{K,M \cap P}$ or $P \cap \kappa < \beta_{K,M}$.

First, assume that $\beta_{K,M} = \beta_{K,M \cap P}$. Then $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M})$, so $\zeta \in R_M(K)$.

Secondly, assume that $P \cap \kappa < \beta_{K,M}$. Suppose that $\beta_{K,M} \leq \zeta$. Since $\beta_{K,M \cap P} \leq \beta_{K,M}$, it follows that $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M})$. Therefore $\zeta \in R_M(K)$.

Otherwise $\zeta < \beta_{K,M}$. But then $K \sim M$ and $\zeta \in K \cap \beta_{K,M}$ imply that $\zeta \in M$. Since $\zeta \in R_{M \cap P}(K)$ and $\zeta \in M$, $\zeta \notin P \cap \kappa$. Therefore $\beta_{K,M \cap P} < P \cap \kappa \leq \zeta$. So $\zeta = \min((K \cap \kappa) \setminus (P \cap \kappa))$.

Case b: $\zeta = \min((K \cap \kappa) \setminus \gamma)$ for some $\gamma \in (M \cap P \cap \kappa) \setminus \beta_{K,M \cap P}$. If $P \cap \kappa \leq \zeta$, then $\gamma < P \cap \kappa \leq \zeta$ implies that $\zeta = \min((K \cap \kappa) \setminus (P \cap \kappa))$.

Suppose that $\zeta < P \cap \kappa$. If $\beta_{K,M} \leq \gamma$, then $\gamma \in (M \cap \kappa) \setminus \beta_{K,M}$, and therefore $\zeta \in R_M(K)$. So assume that $\gamma < \beta_{K,M}$. First consider the case that $\beta_{K,M} \leq \zeta$. Then $\zeta = \min((K \cap \kappa) \setminus \beta_{K,M})$. Since $\gamma \in (M \cap \beta_{K,M}) \setminus K$, it follows that $K < M$. So $\zeta \in R_M(K)$.

In the final case, assume that $\gamma < \zeta < \beta_{K,M}$. We will show that this case does not occur. Then

$$\beta_{K,M \cap P} \leq \gamma < \zeta < \beta_{K,M}.$$

Since $\gamma \in (M \cap \beta_{K,M}) \setminus K$, it follows that $K < M$. So as $\zeta \in K \cap \beta_{K,M}$, $\zeta \in M$. But also $\zeta \in P \cap \kappa$. So $\zeta \in M \cap P$, which contradicts that $\zeta \in R_{M \cap P}(K)$. $\square$

Lemma 2.9. *Let L and M be in $\mathcal{X}_0$ and P in $\mathcal{Y}_0$. Assume that $L \in P$, $\{L, M \cap P\}$ is adequate, and $\sup(M \cap P \cap \kappa) < P \cap \kappa$. Then:*

1. *if $\zeta \in R_L(M)$, then either $\zeta \in R_L(M \cap P)$ or $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$;*
2. *$R_M(L) \subseteq R_{M \cap P}(L)$.*

Proof. Note that by Lemma 1.34, $\beta_{L,M} = \beta_{L,M \cap P}$, $\beta_{L,M} < P \cap \kappa$, and $\{L, M\}$ is adequate.

(1) Let $\zeta \in R_L(M)$. Then either (a) $L \sim M$ and $\zeta = \min((M \cap \kappa) \setminus \beta_{L,M})$, or (b) there is $\gamma \in (L \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((M \cap \kappa) \setminus \gamma)$.

Case a: $L \sim M$ and $\zeta = \min((M \cap \kappa) \setminus \beta_{L,M})$. Then $L \sim M \cap P$ by Lemma 1.34. If $P \cap \kappa \leq \zeta$, then since $\beta_{L,M} < P \cap \kappa$, it follows that $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$. Suppose that $\zeta < P \cap \kappa$. Then

$$\zeta = \min((M \cap P \cap \kappa) \setminus \beta_{L,M}) = \min((M \cap P \cap \kappa) \setminus \beta_{L,M \cap P}).$$

So $\zeta \in R_L(M \cap P)$.

Case b: There is $\gamma \in (L \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((M \cap \kappa) \setminus \gamma)$. Then $\gamma \in (L \cap \kappa) \setminus \beta_{L,M \cap P}$. If $\zeta < P \cap \kappa$, then $\zeta = \min((M \cap P \cap \kappa) \setminus \gamma)$, so $\zeta \in R_L(M \cap P)$. Otherwise $P \cap \kappa \leq \zeta$, and since $\gamma \in L$, $\gamma < P \cap \kappa$. So $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$.

(2) Let $\zeta \in R_M(L)$, and we will show that $\zeta \in R_{M \cap P}(L)$. Either (a) $L \sim M$ and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M})$, or (b) there is $\gamma \in (M \cap \kappa) \setminus \beta_{L,M}$ such that $\zeta = \min((L \cap \kappa) \setminus \gamma)$.

Assume (a). Then $L \sim M \cap P$ by Lemma 1.34. Also $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M\cap P})$, so $\zeta \in R_{M\cap P}(L)$.

Assume (b). Since $\zeta \in L$ and $L \in P$, $\zeta \in P$. As $\gamma < \zeta$ and $\zeta \in P \cap \kappa$, $\gamma < P \cap \kappa$. So $\gamma \in M \cap P$. Thus $\gamma \in (M \cap P \cap \kappa) \setminus \beta_{L,M\cap P}$ and $\zeta = \min((L \cap \kappa) \setminus \gamma)$. So $\zeta \in R_{M\cap P}(L)$. $\square$

Lemma 2.10. *Let L, M, and M' be in $\mathcal{X}_0$, and let P and P' be in $\mathcal{Y}_0$. Assume that $\{L, M, M'\}$ is adequate, and L, M', and P' are in P. Let $\beta := P \cap \kappa$ and $\beta' := P' \cap \kappa$. Suppose that $\sup(M \cap \beta) < \beta'$ and $M \cap \beta = M' \cap \beta'$. Then:*

1. *$\beta_{L,M} < \beta'$;*
2. *either $\beta_{L,M} = \beta_{L,M'}$ or $\beta' < \beta_{L,M'}$;*
3. *if $\beta_{L,M} = \beta_{L,M'}$ and $\zeta \in R_{M\cap P}(L)$, then $\zeta \in R_{M'}(L)$.*

Proof. (1) Since $M \cap \beta \subseteq \beta'$ and $L \cap \kappa \subseteq \beta$, $L \cap M \cap \kappa \subseteq \beta'$. As $\sup(M \cap \beta) < \beta'$, $L \cap M \cap \kappa$ is a bounded subset of β'. By the elementarity of P', fix $\gamma \in \Lambda$ such that $\sup(L \cap M \cap \kappa) < \gamma < \beta'$. By Lemma 1.19(3),

$$\beta_{L,M} = \min(\Lambda \setminus \sup(L \cap M \cap \kappa)) \leq \gamma < \beta'.$$

(2) If $\beta_{L,M} = \beta_{L,M'}$, then we are done. So suppose not. We claim that $\beta_{L,M} < \beta_{L,M'}$. Suppose for a contradiction that $\beta_{L,M'} < \beta_{L,M}$. By Lemma 1.19(3), $\beta_{L,M'} = \min(\Lambda \setminus \sup(L \cap M' \cap \kappa))$. But

$$\beta_{L,M'} < \beta_{L,M} < \beta'$$

by (1) and the assumption just made. So $\beta_{L,M'} < \beta'$. Hence $L \cap M' \cap \kappa = L \cap M' \cap \beta'$. Since $M \cap \beta = M' \cap \beta'$, it follows that

$$\sup(L \cap M' \cap \kappa) = \sup(L \cap M' \cap \beta') = \sup(L \cap M \cap \beta) = \sup(L \cap M \cap \kappa).$$

So

$$\beta_{L,M'} = \min(\Lambda \setminus \sup(L \cap M \cap \kappa)) = \beta_{L,M}.$$

But this contradicts the assumption that $\beta_{L,M'} < \beta_{L,M}$.

This proves that $\beta_{L,M} < \beta_{L,M'}$. By Lemma 1.19(5), we can fix $\xi \in (L \cap M') \cap [\beta_{L,M}, \beta_{L,M'})$. Since $\beta_{L,M} \leq \xi$ and $\xi \in L$, it follows that $\xi \notin M$. But $M \cap \beta = M' \cap \beta'$. Since $\xi \in (M' \cap \kappa) \setminus M$, $\beta' \leq \xi$. As $\xi < \beta_{L,M'}$, it follows that $\beta' < \beta_{L,M'}$.

(3) Suppose that $\beta_{L,M} = \beta_{L,M'}$ and $\zeta \in R_{M\cap P}(L)$. We will prove that $\zeta \in R_{M'}(L)$. Since $\beta_{L,M} = \beta_{L,M\cap P}$ by Lemma 1.34, $\beta_{L,M'} = \beta_{L,M\cap P}$. First assume that $L \sim M \cap P$ and $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M\cap P})$. Hence $\zeta = \min((L \cap \kappa) \setminus \beta_{L,M'})$. As $L \sim M \cap P$,

$$L \cap \omega_1 = M \cap P \cap \omega_1 = M' \cap \omega_1.$$

So $L \sim M'$ by Lemma 1.21. Hence $\zeta \in R_{M'}(L)$.

Now assume that $\zeta = \min((L \cap \kappa) \setminus \gamma)$, where $\gamma \in (M \cap P \cap \kappa) \setminus \beta_{L,M\cap P}$. Since $M \cap P \cap \kappa = M \cap \beta \subseteq M'$, $\gamma \in M'$. And

$$\beta_{L,M\cap P} = \beta_{L,M} = \beta_{L,M'} \leq \gamma.$$

So $\gamma \in (M' \cap \kappa) \setminus \beta_{L,M'}$. Therefore $\zeta \in R_{M'}(L)$. $\square$

The final lemma concerning remainder points will be used when amalgamating over transitive models.

Lemma 2.11. *Let M, M', N, and N' be in $\mathcal{X}_0$. Assume that $M \cap \kappa = M' \cap \kappa$ and $N \cap \kappa = N' \cap \kappa$. Then $R_M(N) = R_{M'}(N')$.*

Proof. We will show that $R_M(N) \subseteq R_{M'}(N')$. The reverse inclusion follows by symmetry. So let $\zeta \in R_M(N)$.

First, assume that $M \sim N$ and $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$. Then by Lemma 1.37, $\beta_{M,N} = \beta_{M',N'}$ and $M' \sim N'$. Since $N' \cap \kappa = N \cap \kappa$, clearly $\zeta = \min((N' \cap \kappa) \setminus \beta_{M',N'})$. So $\zeta \in R_{M'}(N')$.

Secondly, assume that $\zeta = \min((N \cap \kappa) \setminus \gamma)$, for some $\gamma \in (M \cap \kappa) \setminus \beta_{M,N}$. By Lemma 1.37, $\beta_{M,N} = \beta_{M',N'}$. Since $M \cap \kappa = M' \cap \kappa$, $\gamma \in (M' \cap \kappa) \setminus \beta_{M',N'}$. As $N \cap \kappa = N' \cap \kappa$, $\zeta = \min((N' \cap \kappa) \setminus \gamma)$. So $\zeta \in R_{M'}(N')$. $\square$

§3. Strong genericity and cardinal preservation

In this section we will discuss the idea of a strongly generic condition, which is due to Mitchell [12]. Then we will use the existence of strongly generic conditions to prove cardinal preservation results. All of the results in this section are either due to Mitchell, or are based on standard proper forcing arguments.

Definition 3.1. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and N a set. We say that q is a* strongly N-generic condition *if for any set D which is a dense subset of $N \cap \mathbb{Q}$, D is predense in $\mathbb{Q}$ below q.*

Note that if q is strongly N-generic and $r \leq q$, then r is strongly N-generic.

Notation 3.2. *For a forcing poset $\mathbb{Q}$, let $\lambda_{\mathbb{Q}}$ denote the least cardinal such that $\mathbb{Q} \subseteq H(\lambda_{\mathbb{Q}})$.*

Note that q is strongly N-generic iff q is strongly $(N \cap H(\lambda_{\mathbb{Q}}))$-generic.

The following proposition gives a more intuitive description of strong genericity.

Lemma 3.3. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and $N \prec (H(\chi), \in, \mathbb{Q})$, where $\lambda_{\mathbb{Q}} \leq \chi$ is a cardinal. Then q is a strongly N-generic condition iff q forces that $N \cap \dot{G}$ is a V-generic filter on $N \cap \mathbb{Q}$.*

Proof. Suppose that q is a strongly N-generic condition, and let G be a V-generic filter on $\mathbb{Q}$ containing q. We will show that $N \cap G$ is a V-generic filter on $N \cap \mathbb{Q}$.

First, we show that $N \cap G$ is a filter on $N \cap \mathbb{Q}$. If $p \in N \cap G$ and $t \in N \cap \mathbb{Q}$ with $p \leq t$, then $t \in G$ since G is a filter, and hence $t \in N \cap G$. Suppose that s and t are in $N \cap G$, and we will find $p \in N \cap G$ such that $p \leq s, t$. The set D of p in $N \cap \mathbb{Q}$ which are either incompatible with one of s and t, or below both s and t, is a dense subset of $N \cap \mathbb{Q}$ by the elementarity of N. Since q is strongly N-generic, D is predense below q. As $q \in G$ and G is a V-generic filter, we can fix $p \in G \cap D$. Since s, t, and p are in G, p is compatible with s and t, and therefore $p \leq s, t$ by the definition of D. As $D \subseteq N$, $p \in N \cap G$.

Secondly, we prove that $N \cap G$ is V-generic on $N \cap \mathbb{Q}$. So let D be a dense subset of $N \cap \mathbb{Q}$. Since q is a strongly N-generic condition, D is predense below q. As $q \in G$, it follows that $D \cap G \neq \emptyset$. But $D \subseteq N$, so $D \cap N \cap G \neq \emptyset$.

Conversely, suppose that q forces that $N \cap \dot{G}$ is a V-generic filter on $N \cap \mathbb{Q}$, and we will show that q is strongly N-generic. Let D be a dense subset of $N \cap \mathbb{Q}$. If D is not predense below q, then we can fix $r \leq q$ which is incompatible with every condition in D. Let G be a V-generic filter on $\mathbb{Q}$ containing r. Since $r \leq q$, $q \in G$.

Hence by assumption, $N \cap G$ is a V-generic filter on $N \cap \mathbb{Q}$. Since D is dense in $N \cap \mathbb{Q}$, we can fix $s \in G \cap D$. Then r is incompatible with s by the choice of r, and yet r and s are compatible since they are both in the filter G. $\square$

The following combinatorial characterization of strong genericity is very useful in practice.

Lemma 3.4. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and N a set. Then the following are equivalent:*

(1) *q is strongly N-generic;*
(2) *for all $r \leq q$, there exists $v \in N \cap \mathbb{Q}$ such that for all $w \leq v$ in $N \cap \mathbb{Q}$, r and w are compatible.*

Proof. For the forward direction, suppose that there is $r \leq q$ for which there does not exist a condition $v \in N \cap \mathbb{Q}$ all of whose extensions in $N \cap \mathbb{Q}$ are compatible with r. Let D be the set of $w \in N \cap \mathbb{Q}$ which are incompatible with r. The assumption on r implies that D is dense in $N \cap \mathbb{Q}$. But D is not predense below q since every condition in D is incompatible with r. So q is not strongly N-generic.

Conversely, assume that there is a function $r \mapsto v_r$ as described in (2). Let D be dense in $N \cap \mathbb{Q}$, and let $r \leq q$. Since D is dense in $N \cap \mathbb{Q}$, we can fix $w \leq v_r$ in D. Then r and w are compatible by the choice of v_r. So D is predense below q. $\square$

The next idea was introduced by Cox-Krueger [2].

Definition 3.5. *Let $\mathbb{Q}$ be a forcing poset, $q \in \mathbb{Q}$, and N a set. We say that q is a* universal strongly N-generic condition *if q is a strongly N-generic condition and for all $p \in N \cap \mathbb{Q}$, p and q are compatible.*

The strongly generic conditions used in this paper are universal. This fact allows us to factor forcing posets over elementary substructures in such a way that the quotient forcing has nice properties. See Section 6 for more details on this topic.

Definition 3.6. *Let $\mathbb{Q}$ be a forcing poset and $\mu \leq \lambda_{\mathbb{Q}}$ a regular uncountable cardinal. We say that $\mathbb{Q}$ is* μ-strongly proper on a stationary set *if there are stationarily many N in $P_\mu(H(\lambda_{\mathbb{Q}}))$ such that for all $p \in N \cap \mathbb{Q}$, there is $q \leq p$ such that q is strongly N-generic.*

When we say that $\mathbb{Q}$ is strongly proper on a stationary set, we will mean that it is ω_1-strongly proper on a stationary set.

By standard arguments, $\mathbb{Q}$ is μ-strongly proper on a stationary set iff for any cardinal $\lambda_{\mathbb{Q}} \leq \chi$, there are stationarily many N in $P_\mu(H(\chi))$ such that for all $p \in N \cap \mathbb{Q}$, there is $q \leq p$ such that q is strongly N-generic.

Lemma 3.7. *Let $\mathbb{Q}$ be a forcing poset and $\mu \leq \lambda_{\mathbb{Q}}$ a regular uncountable cardinal. If there are stationarily many N in $P_\mu(H(\lambda_{\mathbb{Q}}))$ such that there exists a universal strongly N-generic condition, then $\mathbb{Q}$ is μ-strongly proper on a stationary set.*

Proof. Let $N \in P_\mu(H(\lambda_{\mathbb{Q}}))$ be such that there exists a universal strongly N-generic condition q_N. Let $p \in N \cap \mathbb{Q}$, and we will find $r \leq p$ which is strongly N-generic. Since q_N is universal, p and q_N are compatible. So fix $r \leq p, q_N$. Then $r \leq p$ and r is strongly N-generic. $\square$

Definition 3.8. *Let μ be a regular uncountable cardinal. A forcing poset $\mathbb{Q}$ is said to satisfy the* μ-covering property *if $\mathbb{Q}$ forces that for any set $a \subseteq On$ in the generic extension, if a has size less than μ in the generic extension, then there is b in the ground model with size less than μ in the ground model such that $a \subseteq b$.*

Note that if $\mathbb{Q}$ has the μ-covering property, then $\mathbb{Q}$ forces that μ is regular.

Proposition 3.9. *Let $\mathbb{Q}$ be a forcing poset, and let $\mu \leq \lambda_{\mathbb{Q}}$ be a regular uncountable cardinal. Suppose that $\mathbb{Q}$ is μ-strongly proper on a stationary set. Then $\mathbb{Q}$ satisfies the μ-covering property.*[1]

Proof. Let p be a condition, and suppose that p forces that $\dot{a}$ is a set of ordinals of size less than μ. We will find $q \leq p$ and a set x of size less than μ such that q forces that $\dot{a} \subseteq x$. Extending p if necessary, we can assume that p forces that $\dot{a}$ has size μ_0, for some cardinal $\mu_0 < \mu$. Fix a sequence $\langle \dot{\alpha}_i : i < \mu_0 \rangle$ of $\mathbb{Q}$-names such that p forces that $\dot{a} = \{\dot{\alpha}_i : i < \mu_0\}$.

Fix a regular cardinal $\lambda_{\mathbb{Q}} \leq \chi$ such that $\mathbb{Q}$, $\dot{a}$, and $\langle \dot{\alpha}_i : i < \mu_0 \rangle$ are members of $H(\chi)$. Fix $N \in P_\mu(H(\chi))$ such that $N \prec (H(\chi), \in, \mathbb{Q}, p, \dot{a}, \langle \dot{\alpha}_i : i < \mu_0 \rangle)$, $\mu_0 \subseteq N$, and for all $p_0 \in N \cap \mathbb{Q}$, there is $q \leq p_0$ which is a strongly N-generic condition. In particular, since $p \in N \cap \mathbb{Q}$, we can fix $q \leq p$ such that q is strongly N-generic.

We claim that q forces that for all $i < \mu_0$, $\dot{\alpha}_i \in N$. Let $i < \mu_0$. Let D be the set of $s \in N \cap \mathbb{Q}$ such that s decides the value of $\dot{\alpha}_i$. By the elementarity of N, it is easy to see that D is dense in $N \cap \mathbb{Q}$. Since q is strongly N-generic, D is predense below q. Therefore q forces that $\dot{\alpha}_i$ is decided by a condition in N. By elementarity, the value of the name $\dot{\alpha}_i$ decided by a condition in N lies in N. Hence q forces that $\dot{\alpha}_i \in N$. It follows that q forces that $\dot{a} \subseteq N \cap On$. Since N has size less than μ, we are done. □

Corollary 3.10. *Let $\mathbb{Q}$ be a forcing poset, and let $\mu \leq \lambda_{\mathbb{Q}}$ be a regular uncountable cardinal. Suppose that there are stationarily many N in $P_\mu(H(\lambda_{\mathbb{Q}}))$ for which there exists a universal strongly N-generic condition. Then $\mathbb{Q}$ satisfies the μ-covering property. In particular, $\mathbb{Q}$ forces that μ is a regular cardinal.*

Proof. Immediate from Lemma 3.7 and Proposition 3.9. □

Proposition 3.11. *Let $\mathbb{Q}$ be a forcing poset, and let $\mu \leq \lambda_{\mathbb{Q}}$ be a regular uncountable cardinal. Suppose that there are stationarily many $N \in P_\mu(H(\lambda_{\mathbb{Q}}))$ such that every condition in $\mathbb{Q}$ is a strongly N-generic condition. Then $\mathbb{Q}$ is μ-c.c.*

Note that if $\mathbb{Q}$ has a maximum condition, then every condition in $\mathbb{Q}$ being strongly N-generic is equivalent to the maximum condition being strongly N-generic.

Proof. Let A be a maximal antichain in $\mathbb{Q}$, and we will show that $|A| < \mu$. Let $N \in P_\mu(H(\lambda_{\mathbb{Q}}))$ be such that $N \prec (H(\lambda_{\mathbb{Q}}), \in, \mathbb{Q}, A)$ and every condition in $\mathbb{Q}$ is strongly N-generic.

Note that by the elementarity of N and since A is a maximal antichain, $N \cap A$ is predense in $N \cap \mathbb{Q}$. Namely, if $u \in N \cap \mathbb{Q}$, then u is compatible with some member of A. By elementarity, u is compatible with some member of $N \cap A$. Let D be the

[1]The proof of this proposition is basically the same as a standard proof that proper forcing posets preserve ω_1.

set of $t \in N \cap \mathbb{Q}$ such that for some $s \in N \cap A$, $t \leq s$. Then easily D is dense in $N \cap \mathbb{Q}$. Since every condition in $\mathbb{Q}$ is strongly N-generic, D is predense in $\mathbb{Q}$.

We claim that $A \subseteq N$, which implies that $|A| \leq |N| < \mu$. So let $p \in A$ be given, and we will show that $p \in N$. Since D is predense in $\mathbb{Q}$, fix $t \in D$ which is compatible with p. By the definition of D, we can fix $s \in N \cap A$ such that $t \leq s$. Then p and s are compatible. But p and s are both in A and A is an antichain, so $p = s$. Since $s \in N$, $p \in N$. $\square$

In general, forcings which include adequate sets as side conditions will collapse κ to become ω_2. In other words, all the cardinals μ with $\omega_1 < \mu < \kappa$ will be collapsed to have size ω_1. The next result describes some general properties of a forcing poset which imply that such collapsing takes place.

Proposition 3.12. *Suppose that $\mathbb{Q}$ is a forcing poset which preserves ω_1 and satisfies:*

1. *there exists an integer $k < \omega$ such that the conditions of $\mathbb{Q}$ are of the form $(x_1, \ldots, x_k, A)$, where $x_1, \ldots, x_k$ are finite subsets of $H(\lambda)$, and A is an adequate set;*
2. *if $(y_1, \ldots, y_k, B) \leq (x_1, \ldots, x_k, A)$, then $A \subseteq B$;*
3. *there are stationarily many $N \in \mathcal{X}_0$ such that whenever $(x_1, \ldots, x_k, A) \in N \cap \mathbb{Q}$, then $(x_1, \ldots, x_k, A \cup \{N\})$ is a condition below $(x_1, \ldots, x_k, A)$.*

Then for any cardinal $\omega_1 < \mu < \kappa$, $\mathbb{Q}$ collapses μ to have size ω_1.

Proof. It suffices to show that $\mathbb{Q}$ singularizes all regular cardinals μ with $\omega_1 < \mu < \kappa$. For suppose that this is true, but there is a cardinal μ in the interval (ω_1, κ) in some generic extension. Assume moreover that μ is the least such cardinal. Then $\mu = \omega_2$ in the generic extension. By downwards absoluteness, μ is regular in the ground model. This contradicts our assumption that all regular cardinals in the interval (ω_1, κ) are singularized.

Let G be a V-generic filter on $\mathbb{Q}$. Define

$$X := \{N : \exists (x_1, \ldots, x_k, A) \in G,\ N \in A\}.$$

Let

$$X_\mu := \{N \in X : \mu \in N\}.$$

Then by (2) and the fact that G is a filter, for any M and N in X_μ, there is a condition $(x_1, \ldots, x_k, A) \in G$ such that M and N are in A. Since A is adequate, $\{M, N\}$ is adequate. As $\mu \in M \cap N \cap \kappa$, $\mu < \beta_{M,N}$. Therefore either $M \cap \mu = N \cap \mu$, $M \cap \mu \in N$, or $N \cap \mu \in M$. Moreover, which of these three relations holds is determined by how $M \cap \omega_1$ and $N \cap \omega_1$ are ordered, by Lemma 1.21. It follows that $\{\sup(N \cap \mu) : N \in X_\mu\}$ is a strictly increasing sequence of ordinals with order type at most ω_1.

We claim that the set $\{\sup(N \cap \mu) : N \in X_\mu\}$ is cofinal in μ. The claim implies that μ has cofinality less than or equal to ω_1 in $V[G]$, finishing the proof. Fix a name $\dot{X}_\mu$ which is forced to be equal to the set X_μ defined above.

Let $\gamma < \mu$ and $(x_1, \ldots, x_k, A)$ be a condition. By (3), there is $N \in \mathcal{X}_0$ such that $(x_1, \ldots, x_k, A)$, γ, and μ are in N, and $(x_1, \ldots, x_k, A \cup \{N\})$ is a condition below $(x_1, \ldots, x_k, A)$. Since $\mu \in N$, $(x_1, \ldots, x_k, A \cup \{N\})$ forces that $N \in \dot{X}_\mu$. As $\gamma \in N$, $\gamma < \sup(N \cap \mu)$. $\square$

§4. Adding a club

In this section we give an example to illustrate the methods developed so far, by showing how to add a club subset of a stationary subset of ω_2 using adequate sets of models. Adding a club with finite conditions was the original application of the side conditions of Friedman [3] and Mitchell [12]. Later Neeman [13] defined a forcing for adding a club using his method of two-type side conditions. The forcing poset we develop in this section is the first example of a forcing which adds a club subset of ω_2 using conditions which are just finite sets of models ordered by reverse inclusion.

The following general lemma will be used frequently in this section.

Lemma 4.1. *Let A be an adequate set. Let K, M, and N be in A, and $\zeta \in R_M(N)$. Suppose that $\theta = \min((K \cap \kappa) \setminus \zeta)$. Then*

$$\theta \in R_M(N) \cup R_M(K) \cup R_N(K).$$

Proof. If $\theta = \zeta$, then $\theta \in R_M(N)$ and we are done. Assume that $\zeta < \theta$, which means that $\zeta \notin K$. If $\theta < \beta_{K,N}$, then $\theta \in R_M(K)$ by Lemma 2.7.

Suppose that $\beta_{K,N} \leq \theta$. If $\beta_{K,N} \leq \zeta$, then since $\zeta \in N$, we have that

$$\theta = \min((K \cap \kappa) \setminus \zeta) \in R_N(K).$$

Otherwise $\zeta < \beta_{K,N} \leq \theta$. Then $\theta = \min((K \cap \kappa) \setminus \beta_{K,N})$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, it follows that $K < N$. So $\theta \in R_N(K)$. $\square$

For the remainder of this section, let $\kappa = \lambda = \omega_2$. Recall that T^* is a thin stationary subset of $P_{\omega_1}(\omega_2)$. We will also assume that $2^{\omega_1} = \omega_2$, and hence that $H(\omega_2)$ has size ω_2. Fix a bijection $g^* : \omega_2 \to H(\omega_2)$.

Let $\mathcal{B}$ denote the structure

$$(H(\omega_2), \in, \trianglelefteq, T^*, \pi^*, C^*, \Lambda, g^*).$$

Note that if N is a countable elementary substructure of $\mathcal{B}$ and $N \cap \omega_2 \in T^*$, then $N \in \mathcal{X}_0$. Also note that if M and N are countable elementary substructures of $\mathcal{B}$ and $M \cap \omega_2 \in N$, then by the elementarity of M, $M = g^*[M \cap \omega_2]$, and hence by the elementarity of N, $M \in N$.

Fix a stationary set $S \subseteq \omega_2 \cap \mathrm{cof}(\omega_1)$. We will define a forcing poset which adds a club subset of $S \cup \mathrm{cof}(\omega)$.[2]

Definition 4.2. *A finite set A of elementary substructures of $\mathcal{B}$ in $\mathcal{X}_0$ is* S-adequate *if it is adequate, and for all M and N in A, $R_M(N) \subseteq S$.*

Recall that $(\mathcal{B}, S)$ is the structure $\mathcal{B}$ augmented with the additional predicate S. Note that the property of being S-adequate is definable in the structure $(\mathcal{B}, S)$.

Definition 4.3. *Let $\mathbb{P}$ be the forcing poset consisting of S-adequate sets, ordered by reverse inclusion.*

We will show that $\mathbb{P}$ preserves all cardinals, and adds a club subset of $S \cup \mathrm{cof}(\omega)$.

Note that since $H(\omega_2)$ has size ω_2 and $\mathbb{P} \subseteq H(\omega_2)$, $\mathbb{P}$ has size ω_2 and thus preserves all cardinals greater than ω_2.

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[2]More generally, it is possible to add a club subset to a fat stationary subset of ω_2 using adequate sets as side conditions, but the argument is more complicated than the one which we give here. See [9].

Proposition 4.4. *The forcing poset $\mathbb{P}$ is strongly proper on a stationary set. Therefore $\mathbb{P}$ satisfies the ω_1-covering property and preserves ω_1.*

Proof. Let N be a countable elementary substructure of $(\mathcal{B}, S)$ such that $N \cap \omega_2 \in T^*$. Note that $N \in \mathcal{X}_0$. Let A_0 be in $N \cap \mathbb{P}$. Define $A_1 := A_0 \cup \{N\}$. Observe that A_1 is adequate, since for all $M \in A_0$, $M \cap \beta_{M,N} = M \cap \omega_2$, which is in N. Also A_1 is S-adequate, because for all $M \in A_0$, $R_M(N)$ and $R_N(M)$ are empty. Thus A_1 is in $\mathbb{P}$ and $A_1 \leq A_0$.

We claim that A_1 is a strongly N-generic condition. By Lemma 3.4, it suffices to show that for all $A_2 \leq A_1$, there exists $B \in N \cap \mathbb{P}$ such that for all $C \leq B$ in $N \cap \mathbb{P}$, $A_2 \cup C$ is S-adequate. Let $A_2 \leq A_1$.

We claim that whenever $A_3 \leq A_2$, K and M are in A_3, and $M < N$, then

$$R_{M \cap N}(K) \cup R_K(M \cap N) \subseteq S.$$

But this follows immediately from Lemma 2.3 and the fact that A_3 is S-adequate.

By applying Proposition 1.25 and the last claim finitely many times, we get that the set

$$A := A_2 \cup \{M \cap N : M \in A_2,\ M < N\}$$

is S-adequate. Hence $A \in \mathbb{P}$ and $A \leq A_2$.

Let

$$x := \bigcup\{R_M(N) : M \in A\}.$$

Since $x \subseteq N$ and x is finite, $x \in N$.

The sets A and N witness that the following statement holds in $(\mathcal{B}, S)$:

There are B and N' such that B is S-adequate, $A \cap N \subseteq B$, $N' \in B$, and $x = \bigcup\{R_M(N') : M \in B\}$.

The parameters which appear in the above statement, namely $A \cap N$ and x, are members of N. By the elementarity of N, we can find B and N' in N which satisfy the same statement.

Suppose that $C \in N \cap \mathbb{P}$ and $C \leq B$. We claim that $A \cup C$ is S-adequate, which finishes the proof. Note that if $M \in A$ and $M < N$, then $M \cap N \in A$. By Lemma 1.19(2), $M \cap \beta_{M,N} = M \cap N \cap \omega_2$. Since $M < N$, it follows that $M \cap N \cap \omega_2 \in N$. But $M \cap N = g^*[M \cap N \cap \omega_2]$ by the elementarity of $M \cap N$, so $M \cap N \in N$ by the elementarity of N. Hence the assumptions of Proposition 1.29 hold. Therefore $A \cup C$ is adequate.

To show that $A \cup C$ is S-adequate, let $M \in A$ and $L \in C$. Let $\zeta \in R_L(M)$, and we will show that $\zeta \in S$. By Lemmas 2.4(1) and 2.5(1), we have that

$$\zeta \in R_N(M) \cup R_L(M \cap N).$$

Since A and C are S-adequate, M and N are in A, and L and $M \cap N$ are in C, it follows that $\zeta \in S$.

Let $\theta \in R_M(L)$. By Lemmas 2.4(2) and 2.5(2), either $M < N$ and $\theta \in R_{M \cap N}(L)$, or there is $\xi \in R_M(N)$ such that $\theta = \min((L \cap \omega_2) \setminus \xi)$. In the first case, $\theta \in S$ since C is S-adequate and $M \cap N$ and L are in C. In the second case, $\xi \in x$, and hence for some $K \in B$, $\xi \in R_K(N')$. By Lemma 4.1,

$$\theta \in R_K(N') \cup R_K(L) \cup R_{N'}(L).$$

Since K, L, and N' are in C and C is S-adequate, it follows that $\theta \in S$. $\square$

Lemma 4.5. *Suppose that $M \in \mathcal{X}_0$ and Q is in $\mathcal{Y}_0$. Let $\beta := Q \cap \omega_2$, and assume that* $\mathrm{cf}(\beta) = \omega_1$ *and $\beta \in M$. Then $M \sim M \cap Q$, $R_M(M \cap Q) = \emptyset$, and $R_{M \cap Q}(M) = \{\beta\}$.*

Proof. By the comments after Notation 1.10, $M \cap Q \in \mathcal{X}_0$. By the elementarity of Q, β is a limit point of Λ. Hence the ordinal

$$\beta_0 := \min(\Lambda \setminus \sup(M \cap \beta))$$

is less than β. By the definition of β_0, clearly

$$\beta_0 \in \Lambda_M \cap \Lambda_{M \cap Q}.$$

And since $M \cap Q \cap \omega_2 \subseteq \beta_0$, β_0 is the maximal element of $\Lambda_M \cap \Lambda_{M \cap Q}$. Therefore $\beta_0 = \beta_{M,M \cap Q}$. As $M \cap \beta_0 = M \cap Q \cap \beta_0$, we have that $M \sim M \cap Q$.

Since $M \cap Q \cap \omega_2 \subseteq \beta_0 = \beta_{M,M \cap Q}$, $R_M(M \cap Q) = \emptyset$. As $M \sim M \cap Q$ and

$$\beta = \min((M \cap \omega_2) \setminus \beta_0) = \min((M \cap \omega_2) \setminus \beta_{M,M \cap Q}),$$

we have that $\beta \in R_{M \cap Q}(M)$. And the fact that $M \cap Q \cap \omega_2 \subseteq \beta_0$ implies that $R_{M \cap Q}(M) = \{\beta\}$. □

Lemma 4.6. *Let Q be an elementary substructure of $(\mathcal{B}, S)$ such that Q has size ω_1 and $Q \cap \omega_2 \in S$. Let $\beta := Q \cap \omega_2$. Let $A_0 \in Q \cap \mathbb{P}$. Suppose that $M \in \mathcal{X}_0$, and A_0 and β are in M. Then $\beta \in R_{M \cap Q}(M)$, and*

$$A_0 \cup \{M\} \cup \{M \cap Q\}$$

is a strongly Q-generic condition.

Proof. Define $A_1 := A_0 \cup \{M\}$. Then A_1 is S-adequate and $A_1 \leq A_0$. Namely, for all $K \in A_0$, $K \in M$ implies that $K \cap \beta_{K,M} = K \cap \omega_2 \in M$. So $K < M$ for all $K \in A_0$. Also $R_K(M)$ and $R_M(K)$ are both empty.

Define $A := A_1 \cup \{M \cap Q\}$. By Proposition 1.33, A is adequate. We claim that A is S-adequate. So let K be in A_1. If $K \in A_0$, then $K \in M \cap Q$. So $R_K(M \cap Q)$ and $R_{M \cap Q}(K)$ are empty. Suppose that $K = M$. Then by Lemma 4.5, $M \sim M \cap Q$, $R_M(M \cap Q) = \emptyset$, and $R_{M \cap Q}(M) = \{\beta\}$. Since $\beta \in S$, we are done.

Thus we have established that A is S-adequate. We will show that A is strongly Q-generic. So let $A_2 \leq A$ be given.

We claim that for all $A_3 \leq A_2$, for all $K \in A_3$, $A_3 \cup \{K \cap Q\}$ is S-adequate. By Proposition 1.33, $A_3 \cup \{K \cap Q\}$ is adequate. To show that it is S-adequate, let $N \in A_3$, and we will show that $R_N(K \cap Q)$ and $R_{K \cap Q}(N)$ are subsets of S.

By Lemma 2.8(1), $R_N(K \cap Q) \subseteq R_N(K)$. Since K and N are in A_3 and A_3 is S-adequate, $R_N(K) \subseteq S$. Thus $R_N(K \cap Q) \subseteq S$. Now suppose that $\zeta \in R_{K \cap Q}(N)$. Then by Lemma 2.8(2), either $\zeta \in R_K(N)$, or $\zeta = \min((N \cap \omega_2) \setminus \beta)$. In the first case, since K and N are in A_3, we have that $\zeta \in R_K(N) \subseteq S$. Assume the second case. Since $\beta \in R_{M \cap Q}(M)$ by Lemma 4.5, and $M \cap Q$, M, and N are in A_3, Lemma 4.1 implies that ζ is in $R_{M \cap Q}(M)$, $R_{M \cap Q}(N)$, or $R_M(N)$. Since A_3 is S-adequate, $\zeta \in S$, which proves the claim.

By applying the claim finitely many times, we get that the set

$$A^* := A_2 \cup \{K \cap Q : K \in A_2\}$$

is S-adequate.

Next we claim that for all $K \in A_2$, $K \cap Q$ is in Q. By Lemma 1.30, $K \cap Q \cap \omega_2$ is in Q. Since K and Q are elementary in $\mathcal{B}$, $K \cap Q = g^*[K \cap Q \cap \omega_2]$. As Q is elementary in $\mathcal{B}$, $K \cap Q = g^*[K \cap Q \cap \omega_2]$ is in Q. It follows that for all $K \in A^*$, $K \cap Q \in Q$.

Let $B := A^* \cap Q$. We will show that for any $C \leq B$ in $\mathbb{P} \cap Q$, $A^* \cup C$ is S-adequate, which finishes the proof. So let $C \leq B$ be in $\mathbb{P} \cap Q$. By the previous claim, for all $K \in A^*$, $K \cap Q \in A^* \cap Q$. And $A^* \cap Q \subseteq C \subseteq Q$. By Proposition 1.35, $A^* \cup C$ is adequate.

To prove that $A^* \cup C$ is S-adequate, let $L \in C$ and $N \in A^*$, and we will show that

$$R_L(N) \cup R_N(L) \subseteq S.$$

By Lemma 2.9(2), $R_N(L) \subseteq R_{N \cap Q}(L)$. Since L and $N \cap Q$ are in C, $R_{N \cap Q}(L) \subseteq S$. Hence $R_N(L) \subseteq S$.

Let $\zeta \in R_L(N)$. Then by Lemma 2.9(1), either $\zeta \in R_L(N \cap Q)$, or $\zeta = \min((N \cap \omega_2) \setminus \beta)$. In the first case, since L and $N \cap Q$ are in C and C is S-adequate, it follows that $\zeta \in S$. Assume the second case. Then since $\beta \in R_{M \cap Q}(M)$, and M, $M \cap Q$, and N are in A^*, by Lemma 4.1 we have that ζ is in $R_{M \cap Q}(M)$, $R_{M \cap Q}(N)$, or $R_M(N)$. Since A^* is S-adequate, it follows that $\zeta \in S$. □

Corollary 4.7. *The forcing poset $\mathbb{P}$ is ω_2-strongly proper on a stationary set. Therefore $\mathbb{P}$ preserves ω_2 and has the ω_2-covering property.*

Proof. Immediate from Lemma 4.6. □

Proposition 4.8. *The forcing poset $\mathbb{P}$ adds a club subset of $S \cup \mathrm{cof}(\omega)$.*

Proof. Let $\dot{C}$ be a $\mathbb{P}$-name for the set

$$\bigcup \{R_M(N) : \exists A \in \dot{G},\ M, N \in A\}.$$

It follows easily from Lemma 4.6 that $\mathbb{P}$ forces that $\dot{C}$ is cofinal in ω_2.

We claim that $\mathbb{P}$ forces that

$$\lim(\dot{C}) \subseteq S \cup \mathrm{cof}(\omega),$$

which completes the proof. Let $\beta < \omega_2$, and assume that A is a condition which forces that β is a limit point of $\dot{C}$. We will prove that β is in $S \cup \mathrm{cof}(\omega)$. If β has cofinality ω, then we are done. So assume that $\mathrm{cf}(\beta) = \omega_1$. We will show that $\beta \in S$.

Fix $N \in \mathcal{X}_0$ such that A and β are in N. Then $A \cup \{N\}$ is an S-adequate set, and thus is in $\mathbb{P}$. Since $A \cup \{N\} \leq A$, $A \cup \{N\}$ forces that β is a limit point of $\dot{C}$ with uncountable cofinality. Hence we can fix $B \leq A \cup \{N\}$, K and M in B, and $\gamma \in R_K(M)$ such that

$$\sup(N \cap \beta) < \gamma < \beta.$$

Since $\beta \in N$, we have that $\beta = \min((N \cap \omega_2) \setminus \gamma)$. By Lemma 4.1,

$$\beta \in R_K(M) \cup R_K(N) \cup R_M(N).$$

As B is S-adequate, it follows that $\beta \in S$. □

We remark that it is not necessary to assume that κ is ω_2. If $\kappa > \omega_2$, we can fix any stationary set $S \subseteq \kappa \cap \mathrm{cof}(>\omega)$, and then the forcing poset $\mathbb{P}$ defined above will add a club subset of $S \cup \mathrm{cof}(\omega)$, and collapse κ to become ω_2 by Proposition 3.12.

§5. $\vec{S}$-obedient side conditions

We now generalize the idea of an S-adequate set to the case when we have a sequence $\vec{S}$ of sets, instead of a single set S. For the remainder of this section fix a sequence $\vec{S} = \langle S_\eta : \eta < \lambda \rangle$, where each S_η is a subset of $\kappa \cap \mathrm{cof}(>\omega)$.

Definition 5.1. *A set $P \in \mathcal{Y}_0$ is* $\vec{S}$-strong *if for all $\tau \in P \cap \kappa^+$, $P \cap \kappa \in S_\tau$.*

Note that if P is $\vec{S}$-strong, then $\mathrm{cf}(P \cap \kappa) > \omega$, since $P \cap \kappa \in S_0 \subseteq \kappa \cap \mathrm{cof}(>\omega)$.

For the next two definitions, we fix a class $\mathcal{Y} \subseteq \mathcal{Y}_0$. The definitions of $\vec{S}$-adequate and $\vec{S}$-obedient are made relative to the class $\mathcal{Y}$.

Definition 5.2. *Let A be an adequate set. We say that A is* $\vec{S}$-adequate *if for all M and N in A and $\zeta \in R_M(N)$:*

(1) *for all $\tau \in M \cap N \cap \kappa^+$, $\zeta \in S_\tau$;*
(2) *if $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, then for all $\tau \in N \cap P \cap \kappa^+$, $\zeta \in S_\tau$.*

Definition 5.3. *A pair (A, B) is an* $\vec{S}$-obedient side condition *if:*

(1) *A is an $\vec{S}$-adequate set;*
(2) *B is a finite set of $\vec{S}$-strong models in $\mathcal{Y}_0$;*
(3) *for all $M \in A$ and $P \in B$, if $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$, then for all $\tau \in P \cap M \cap \kappa^+$, $\zeta \in S_\tau$.*

The next two lemmas show that we can add certain models to an $\vec{S}$-obedient side condition and preserve $\vec{S}$-obedience.

Lemma 5.4. *Let (A, B) be an $\vec{S}$-obedient side condition.*

(1) *If $N \in \mathcal{X}_0$ and $(A, B) \in N$, then $(\{N\}, \emptyset)$ and $(A \cup \{N\}, B)$ are $\vec{S}$-obedient side conditions.*
(2) *If $P \in \mathcal{Y}_0$ is $\vec{S}$-strong and $(A, B) \in P$, then $(\emptyset, \{P\})$ and $(A, B \cup \{P\})$ are $\vec{S}$-obedient side conditions.*

Proof. (2) is trivial. (1) The fact that $(\{N\}, \emptyset)$ is an $\vec{S}$-obedient side condition is easy. The set $A \cup \{N\}$ is $\vec{S}$-adequate because for all $M \in A$, $M \cap \beta_{M,N} = M \cap \kappa$ is in N, and $R_M(N)$ and $R_N(M)$ are empty. If $P \in B$, then $\min((N \cap \kappa) \setminus (P \cap \kappa)) = P \cap \kappa$. And if $\tau \in P \cap N \cap \kappa^+$, then $P \cap \kappa \in S_\tau$ since P is $\vec{S}$-strong. □

Lemma 5.5. *Let (A, B) be an $\vec{S}$-obedient side condition.*

(1) *Let M and N be in A, and suppose that $M < N$. Then $(A \cup \{M \cap N\}, B)$ is an $\vec{S}$-obedient side condition.*
(2) *Let $M \in A$ and $P \in B$. Then $(A \cup \{M \cap P\}, B)$ is an $\vec{S}$-obedient side condition.*
(3) *Suppose that P and Q are in B and $P \cap \kappa < Q \cap \kappa$. Then $(A, B \cup \{P \cap Q\})$ is an $\vec{S}$-obedient side condition.*

Proof. (1) The set $A \cup \{M \cap N\}$ is adequate by Proposition 1.25. To show that $A \cup \{M \cap N\}$ is $\vec{S}$-adequate, it suffices to show that for all $K \in A$, $\{K, M \cap N\}$ is $\vec{S}$-adequate. So let $K \in A$ be given.

Let $\zeta \in R_K(M \cap N)$. Then $\zeta \in R_K(M)$ by Lemma 2.3. Let $\tau \in K \cap (M \cap N) \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Then $\tau \in K \cap M$, which implies that $\zeta \in S_\tau$ since $\zeta \in R_K(M)$.

Suppose that $P \in K \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(M \cap N \cap \zeta) < P \cap \kappa < \zeta$. Let $\tau \in (M \cap N) \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\zeta \in M \cap N \cap \kappa$, $\zeta < \beta_{M,N}$. And since $M \cap N \cap \kappa = M \cap \beta_{M,N}$ is an initial segment of $M \cap \kappa$, $\sup(M \cap N \cap \zeta) = \sup(M \cap \zeta)$. So $\sup(M \cap \zeta) < P \cap \kappa < \zeta$. Since $\zeta \in R_K(M)$, it follows that $\zeta \in S_\tau$.

Let $\zeta \in R_{M \cap N}(K)$. Then by Lemma 2.3, either (i) $\zeta \in R_M(K)$ or (ii) $\zeta \in R_N(K)$. Consider $\tau \in K \cap (M \cap N) \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Then $\tau \in K \cap M$, so in case (i), $\zeta \in S_\tau$. Also $\tau \in K \cap N$, so in case (ii), $\zeta \in S_\tau$. Suppose that $P \in (M \cap N) \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(K \cap \zeta) < P \cap \kappa < \zeta$. Let $\tau \in K \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Then $P \in M \cap \mathcal{Y}$, so in case (i), $\zeta \in S_\tau$. And $P \in N \cap \mathcal{Y}$, so in case (ii), $\zeta \in S_\tau$. This completes the proof that $A \cup \{M \cap N\}$ is $\vec{S}$-adequate.

Let $Q \in B$, and suppose that $\xi = \min((M \cap N \cap \kappa) \setminus (Q \cap \kappa))$. Since $M < N$, $M \cap N \cap \kappa = M \cap \beta_{M,N}$, which is an initial segment of $M \cap \kappa$. Hence $\xi = \min((M \cap \kappa) \setminus (Q \cap \kappa))$. Let $\tau \in (M \cap N) \cap Q \cap \kappa^+$, and we will show that $\xi \in S_\tau$. Then $\tau \in M \cap Q$, so $\xi \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

(2) Since P is $\vec{S}$-strong, $\mathrm{cf}(P \cap \kappa) > \omega$. So clearly $\sup(M \cap P \cap \kappa) < P \cap \kappa$. It follows that $A \cup \{M \cap P\}$ is adequate by Proposition 1.33.

To show that $A \cup \{M \cap P\}$ is $\vec{S}$-adequate, let $K \in A$ be given, and we will show that $\{K, M \cap P\}$ is $\vec{S}$-adequate.

Let $\zeta \in R_K(M \cap P)$. Then $\zeta \in R_K(M)$ by Lemma 2.8. Let $\tau \in K \cap (M \cap P) \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Then $\tau \in K \cap M$ implies that $\zeta \in S_\tau$.

Suppose that $Q \in K \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(M \cap P \cap \zeta) < Q \cap \kappa < \zeta$. Let $\tau \in Q \cap (M \cap P) \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\zeta \in P \cap \kappa$ and $P \cap \kappa$ is an ordinal, $\sup(M \cap P \cap \zeta) = \sup(M \cap \zeta)$. So $\sup(M \cap \zeta) < Q \cap \kappa < \zeta$. Since $\zeta \in R_K(M)$, $Q \in K \cap \mathcal{Y}$, and $\tau \in M \cap Q$, it follows that $\zeta \in S_\tau$.

Let $\zeta \in R_{M \cap P}(K)$. Then by Lemma 2.8, either (i) $\zeta \in R_M(K)$ or (ii) $\zeta = \min((K \cap \kappa) \setminus (P \cap \kappa))$.

Let $\tau \in (M \cap P) \cap K \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. In case (i), $\tau \in M \cap K$ implies that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate. In case (ii), $\tau \in K \cap P$ implies that $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

Suppose that $Q \in (M \cap P) \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(K \cap \zeta) < Q \cap \kappa < \zeta$. Let $\tau \in K \cap Q \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. In case (i), $Q \in M \cap \mathcal{Y}$ implies that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate. In case (ii), since $\tau \in Q$ and $Q \in P$, $\tau \in P$. So $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient. This completes the proof that $A \cup \{M \cap P\}$ is $\vec{S}$-adequate.

Let $Q \in B$, and suppose that $\zeta = \min((M \cap P \cap \kappa) \setminus (Q \cap \kappa))$. Let $\tau \in (M \cap P) \cap Q \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $P \cap \kappa \in \kappa$, $M \cap P \cap \kappa$ is an initial segment of $M \cap \kappa$. Hence $\zeta = \min((M \cap \kappa) \setminus (Q \cap \kappa))$. Since $\tau \in M \cap Q$, it follows that $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

(3) Note that since $P \cap \kappa < Q \cap \kappa$, $P \cap Q \cap \kappa = P \cap \kappa$.

To show that $P \cap Q$ is $\vec{S}$-strong, let $\tau \in P \cap Q \cap \kappa^+$. Then $\tau \in P$. Since P is $\vec{S}$-strong, $P \cap Q \cap \kappa = P \cap \kappa \in S_\tau$.

Let $M \in A$, and suppose that $\zeta = \min((M \cap \kappa) \setminus (P \cap Q \cap \kappa))$. Then $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$. Let $\tau \in M \cap (P \cap Q) \cap \kappa^+$. Then $\tau \in M \cap P$, so $\zeta \in S_\tau$. $\square$

We conclude the section with an easy lemma which will be used frequently for checking that certain models are $\vec{S}$-strong.

Lemma 5.6. *Suppose that $N \in \mathcal{X}_0 \cup \mathcal{Y}_0$, $Q \in N \cap \mathcal{Y}_0$, and $P \in \mathcal{Y}_0$. Suppose that P is $\vec{S}$-strong and $N \prec (H(\lambda), \in, \mathcal{Y}_0, \vec{S})$. Assume that $Q \cap N \cap \kappa^+ \subseteq P$ and $P \cap \kappa = Q \cap \kappa$. Then Q is $\vec{S}$-strong.*

Proof. Since $Q \in N$, it suffices to show that N models that Q is $\vec{S}$-strong. So let $\tau \in Q \cap N \cap \kappa^+$. Since $Q \cap N \cap \kappa^+ \subseteq P$, $\tau \in P$. As P is $\vec{S}$-strong, $Q \cap \kappa = P \cap \kappa \in S_\tau$. □

§6. The approximation property and factorization

We briefly discuss the approximation property, and state the theorem on factoring a generic extension which we will use in the proof of Mitchell's theorem in Part III.

Let (W_1, W_2) be a pair of transitive class models of ZFC such that $W_1 \subseteq W_2$. We say that the pair (W_1, W_2) satisfies the ω_1-*approximation property* if, whenever $X \in W_2$ is a set of ordinals such that $a \cap X \in W_1$ whenever $a \in W_1$ is countable in W_1, then the set X itself is in W_1.

The approximation property is due to Hamkins [5], and is similar to properties studied in Mitchell's construction of a model with no Aronszajn trees on ω_2 [11]. It plays a crucial role in the original proof of Mitchell's theorem on the approachability ideal, as well as in the proof presented in Part III.

We will use the following easy consequence of the approximation property.

Lemma 6.1. *Suppose that (W_1, W_2) satisfies the ω_1-approximation property. Assume that c is a set of ordinals of order type ω_1 in W_2 such that for all $\beta < \sup(c)$, $c \cap \beta \in W_1$. Then $c \in W_1$.*

Proof. To show that $c \in W_1$, it suffices to show that for any set $a \in W_1$ which is countable in W_1, $a \cap c \in W_1$. So let $a \in W_1$ be countable in W_1. Then a is countable in W_2. Since c has order type ω_1, there is $\beta < \sup(c)$ such that $a \cap c \subseteq c \cap \beta$. By the assumption on c, $c \cap \beta \in W_1$. Since $c \cap \beta$ and a are in W_1, $a \cap c = a \cap (c \cap \beta)$ is in W_1. □

In the original proof of Mitchell's theorem, being able to factor a generic extension in a way which satisfies the approximation property relied on what was called *tidy* strongly generic conditions (see Lemma 2.22 of [12]). However, the strongly generic conditions used in the present paper are not tidy. Therefore we need a different factorization theorem which is applicable in the present context; such a theorem was provided by Cox-Krueger [2].

Let us recall the property $*$ introduced in [2].

Definition 6.2. *Let $\mathbb{P}_1$ be a suborder of a forcing poset $\mathbb{P}_2$, where $\mathbb{P}_2$ has greatest lower bounds. We say that $\mathbb{P}_1$ satisfies property $*(\mathbb{P}_1, \mathbb{P}_2)$ if for all $p \in \mathbb{P}_1$ and $q, r \in \mathbb{P}_2$, if p, q, and r are pairwise compatible in $\mathbb{P}_2$, then p is compatible in $\mathbb{P}_2$ with $q \wedge r$.*

Note that if a forcing poset $\mathbb{Q}$ satisfies property $*(\mathbb{Q},\mathbb{Q})$, then for any suborder $\mathbb{P}$ of $\mathbb{Q}$, $*(\mathbb{P},\mathbb{Q})$.

Notation 6.3. *Let $\mathbb{Q}$ be a forcing poset. If $q \in \mathbb{Q}$ and K is a subset of $\mathbb{Q}$, let $(\mathbb{Q}/q)/K$ denote the forcing poset consisting of conditions $s \in \mathbb{Q}$ such that $s \le q$, and s is compatible in $\mathbb{Q}$ with all conditions in K.*

The following result appears as Theorem 4.3 in [2].

Theorem 6.4 (Factorization theorem)**.** *Let $\mathbb{Q}$ be a forcing poset with greatest lower bounds satisfying $*(\mathbb{Q},\mathbb{Q})$, χ a regular cardinal with $\lambda_{\mathbb{Q}} \le \chi$, and $N \prec (H(\chi),\in,\mathbb{Q})$. Suppose that there are stationarily many models in $P_{\omega_1}(H(\chi))$ which have universal strongly generic conditions. Assume that q is a universal strongly N-generic condition.*

Then for any V-generic filter G on $\mathbb{Q}$ which contains q, $V[G] = V[G\cap N][H]$, where $G \cap N$ is a V-generic filter on $\mathbb{Q}\cap N$, H is a $V[G\cap N]$-generic filter on $(\mathbb{Q}/q)/(G\cap N)$, and the pair $(V[G\cap N], V[G])$ satisfies the ω_1-approximation property.

This theorem will be used in the final argument of the proof of Mitchell's theorem in Section 16. It is interesting to note that not all intermediate extensions of a strongly proper forcing extension satisfy the ω_1-approximation property; see Section 5 of [2] for a counterexample.

Part 2. Advanced side condition methods

§7. Mitchell's use of $\square_\kappa$[3]

For the remainder of the paper we will assume $\square_\kappa$ and $2^\kappa = \kappa^+$. Also we let the cardinal λ from Part I equal κ^+. Since $2^\kappa = \kappa^+$, $H(\kappa^+)$ has size κ^+.

Notation 7.1. *Let f^* denote a bijection from κ^+ to $H(\kappa^+)$.*

Notation 7.2. *Fix a sequence $\vec{C} = \langle C_\alpha : \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ satisfying that for all limit $\alpha < \kappa^+$:*

1. *C_α is a club subset of α with $\mathrm{ot}(C_\alpha) \le \kappa$; in particular, if $\mathrm{cf}(\alpha) < \kappa$ then $\mathrm{ot}(C_\alpha) < \kappa$;*
2. *if $\beta \in \lim(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$;*
3. *if α is a limit of limit ordinals, then every ordinal in C_α is a limit ordinal;*
4. *if $\alpha = \alpha_0 + \omega$ for a limit ordinal α_0, then $\alpha_0 \in \lim(C_\alpha)$, and hence $C_{\alpha_0} = C_\alpha \cap \alpha_0$.*

Properties (1) and (2) embody the standard definition of a square sequence. It is easy to modify a square sequence to also satisfy properties (3) and (4). For example, start by replacing each ordinal in C_α with the greatest limit ordinal less than or equal to it. The details are left to the reader.

Notation 7.3. *For each limit ordinal $\alpha < \kappa^+$ and $\beta < \mathrm{ot}(C_\alpha)$, let $c_{\alpha,\beta}$ denote the β-th member of C_α, that is, the unique γ in C_α such that $\mathrm{ot}(C_\alpha \cap \gamma) = \beta$.*

[3]Almost all of the arguments in this section and the next are due to Mitchell, but adopted to the present context.

Notation 7.4. *Fix a sequence $\vec{A} = \langle A_{\eta,\beta} : \eta < \kappa^+,\ \beta < \kappa\rangle$ satisfying the following properties:*

(1) *for each $\eta < \kappa^+$, $\{A_{\eta,\beta} : \beta < \kappa\}$ is an increasing and continuous sequence of sets with union equal to η;*
(2) *$A_{\eta+1,\beta} = A_{\eta,\beta} \cup \{\eta\}$;*
(3) *for all $\eta < \kappa^+$ and $\beta < \kappa$, $|A_{\eta,\beta}| \leq |\beta| \cdot \omega$;*
(4) *if $\xi \in \lim(C_\eta) \cup A_{\eta,\beta} \cup \lim(A_{\eta,\beta})$, then $A_{\xi,\beta} = A_{\eta,\beta} \cap \xi$;*
(5) *there exists a function $c^* : \kappa \to \kappa$ such that for all $\eta < \kappa^+$ and $\beta < \kappa$, $\mathrm{ot}(A_{\eta,\beta}) < c^*(\beta)$;*
(6) *if $\beta < \mathrm{ot}(C_\eta)$, then $A_{\eta,\beta} \subseteq c_{\eta,\beta}$ and $\lim(C_\eta) \cap c_{\eta,\beta} \subseteq A_{\eta,\beta}$;*
(7) *if $\gamma \in \eta \setminus C_\eta$, then $\gamma \in A_{\eta,\beta}$ iff*
$$\gamma \in A_{\min(C_\eta \setminus \gamma),\beta} \textit{ and } \min(C_\eta \setminus \gamma) \in A_{\eta,\beta};$$
(8) *if $\xi \in \lim(A_{\eta,\beta}) \cap \eta$ and $\mathrm{ot}(C_\xi) < \beta$, then $\xi \in A_{\eta,\beta}$.*

Properties (1), (2), and (3) describe a typical kind of filtration of each ordinal $\eta < \kappa^+$. The coherence property (4) is one of the most often used facts in the paper. It gives sufficient conditions for coherence to hold between $A_{\xi,\beta}$ and $A_{\eta,\beta}$, where $\xi < \eta$. If ξ is a limit point of C_η, then
$$\forall \beta < \kappa\ A_{\xi,\beta} = A_{\eta,\beta} \cap \xi.$$
And if $\beta < \kappa$ and ξ is either in $A_{\eta,\beta}$, or a limit point of $A_{\eta,\beta}$, then
$$A_{\xi,\beta} = A_{\eta,\beta} \cap \xi.$$
We recommend that the reader memorize this important fact before proceeding.

Property (5) follows immediately from property (3) in the case when κ is weakly inaccessible, by letting $c^*(\beta) = \beta^+$. This property is only used in one lemma in the paper, namely Lemma 8.6. Likewise, properties (6), (7), and (8) are technical facts about $\vec{A}$ which are only used in Lemmas 8.10 and 8.11, and in several places in Section 12. There is no harm in the reader forgetting about properties (5)–(8) for now, and just looking back at them later in the rare places that they are used.

Theorem 7.5 (Mitchell [12])**.** *Assume that κ is weakly inaccessible and $\vec{C}$ is a sequence as in Notation 7.2. Then there exists a sequence $\vec{A}$ as described in Notation 7.4.*

Mitchell constructs the sequence $\vec{A}$ using the square sequence $\vec{C}$ in a careful way. The only place where the weak inaccessibility of κ is used is to derive property (5) from property (3), as mentioned above. If κ is weakly inaccessible in an inner model W which satisfies $\Box_\kappa$, and $(\kappa^+)^W = (\kappa^+)^V$, then the sequence $\vec{A}$ constructed in W still satisfies properties (1)–(8) in V by upwards absoluteness. For example, if V is obtained from W by collapsing κ to become ω_2 while preserving κ^+, then there is a sequence $\vec{A}$ as above in V.

The construction of $\vec{A}$ appears in [12, Section 3.1]. We do not repeat it here because it is technical and not helpful for understanding the other material in our paper.

Notation 7.6. *Let $\mathcal{A}$ denote some expansion of the structure*
$$(H(\kappa^+), \in, \trianglelefteq, \kappa, T^*, \pi^*, C^*, \Lambda, \mathcal{Y}_0, f^*, \vec{C}, \vec{A}, c^*).$$

Note that $\mathcal{A}$ is an expansion of the structure described in Notations 1.9 and 1.10. Thus any elementary substructure of $\mathcal{A}$ is also an elementary substructure of that structure.

Notation 7.7. *Let $\mathcal{X}$ denote the set of M in $P_{\omega_1}(H(\kappa^+))$ such that $M \cap \kappa \in T^*$, $M \prec \mathcal{A}$, and $\lim(C_{\sup(M)}) \cap M$ is cofinal in $\sup(M)$.*

Notation 7.8. *Let $\mathcal{Y}$ denote the set of P in $P_\kappa(H(\kappa^+))$ such that $P \cap \kappa \in \kappa$, $P \prec \mathcal{A}$, and $\lim(C_{\sup(P)}) \cap P$ is cofinal in $\sup(P)$.*

Note that $\mathcal{X} \subseteq \mathcal{X}_0$ and $\mathcal{Y} \subseteq \mathcal{Y}_0$, where $\mathcal{X}_0$ and $\mathcal{Y}_0$ were defined in Notations 1.9 and 1.10.[4]

Observe that by elementarity, for any $M \in \mathcal{X}$ and $P \in \mathcal{Y}$, $M = f^*[M \cap \kappa^+]$ and $P = f^*[P \cap \kappa^+]$. In particular, if M and N are in $\mathcal{X} \cup \mathcal{Y}$ and $M \cap \kappa^+ \in N$, then by elementarity, $M \in N$.

As a result of the presence of the well-ordering $\trianglelefteq$, the structure $\mathcal{A}$ described in Notation 7.6 has definable Skolem functions. Let $\langle \tau_n : n < \omega \rangle$ be a complete list of definable Skolem terms for $\mathcal{A}$. For any set $a \subseteq H(\kappa^+)$, let $Sk(a)$ denote the closure of a under the Skolem terms.

For $n < \omega$ and m equal to the arity of τ_n, we define a partial function $\tau_n' : (\kappa^+)^m \to \kappa^+$ by letting $\tau_n'(\alpha_0, \ldots, \alpha_{m-1}) = \tau_n(\alpha_0, \ldots, \alpha_{m-1})$, provided that this is an ordinal, and otherwise is undefined. Note that τ_n' is also definable in $\mathcal{A}$.

Notation 7.9. *Let $H^* : (\kappa^+)^{<\omega} \to \kappa^+$ be a function such that any elementary substructure of $\mathcal{A}$ is closed under H^*, and whenever $a \subseteq \kappa^+$ is closed under H^*, then $Sk(a) \cap \kappa^+ = a$. In addition, a is closed under H^* iff a is closed under τ_n' for all $n < \omega$.*

The existence of such a function H^* is proved by standard arguments. Note that if a is a set of ordinals closed under H^*, then $Sk(a) = f^*[a]$. In particular, if $M \in \mathcal{X} \cup \mathcal{Y}$ and $a \in M$ is a set of ordinals which is closed under H^*, then by elementarity, $Sk(a) \in M$.

The next simple lemma will prove very useful throughout the paper.

Lemma 7.10. *Suppose that $N \in \mathcal{X} \cup \mathcal{Y}$, a is a set of ordinals in N, and for some set b which is closed under H^*, $N \cap a = N \cap b$. Then a is closed under H^*.*

Proof. It suffices to show that a is closed under τ_n' for all $n < \omega$. Fix $n < \omega$, and let k be the arity of τ_n'. Since $a \in N$ and τ_n' is definable in $\mathcal{A}$, it suffices to show that N models that a is closed under τ_n'. Let $\alpha_0, \ldots, \alpha_{k-1} \in N \cap a$. Then $\alpha_0, \ldots, \alpha_{k-1} \in N \cap b$. Since N and b are both closed under H^*, they are closed under τ_n'. So $\tau_n'(\alpha_0, \ldots, \alpha_{k-1}) \in N \cap b$. Since $N \cap b \subseteq a$, $\tau_n'(\alpha_0, \ldots, \alpha_{k-1}) \in a$. $\square$

In the remainder of this section, we will provide a thorough analysis of the models in $\mathcal{X}$ and $\mathcal{Y}$.

The following notation will be useful.

Notation 7.11. *Let $N \subseteq H(\kappa^+)$ be a set and $\gamma \in \kappa^+ \cap \sup(N)$. Let γ_N denote the ordinal $\min((N \cap \kappa^+) \setminus \gamma)$.*

Recall that if a is a set of ordinals, then $\mathrm{cl}(a)$ denotes the set $a \cup \lim(a)$.

---

[4]The requirement that $\lim(C_{\sup(X)}) \cap X$ is cofinal in $\sup(X)$ appears in Mitchell's definition of a *model* ([12, Definition 3.14]). We do not, however, assume that $\mathrm{ot}(C_{\sup(X)}) \notin X$, as in his definition.

Lemma 7.12. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ and $\eta \in \mathrm{cl}(N)$, and suppose that $\eta < \sup(N)$. Then either $\eta \in N$ or $\eta \in \lim(C_{\eta_N})$. Hence:*

(1) $C_\eta = C_{\eta_N} \cap \eta$;
(2) $A_{\eta,\xi} = A_{\eta_N,\xi} \cap \eta$ *for all* $\xi < \kappa$;
(3) $N \cap A_{\eta_N,\xi} = N \cap A_{\eta,\xi}$ *for all* $\xi < \kappa$.

Proof. If $\eta \in N$, then $\eta_N = \eta$, and (1), (2), and (3) are trivial.

Suppose that $\eta < \eta_N$, and we will show that $\eta \in \lim(C_{\eta_N})$. Let $\gamma < \eta$. Since $\eta \in \mathrm{cl}(N) \setminus N$, $\eta \in \lim(N)$. So we can fix $\sigma \in (N \cap \eta) \setminus \gamma$. Now $\eta_N \in N$ and $\sigma \in N \cap \eta_N$, so by elementarity there is $\delta \in C_{\eta_N} \cap N$ larger than σ. Then $\delta \in N \cap \eta_N \subseteq \eta$. So $\gamma < \delta < \eta$ and $\delta \in C_{\eta_N}$. This proves that $\eta \in \lim(C_{\eta_N})$.

(1) follows from the definition of a square sequence, and (2) follows from Notation 7.4(4). For (3), since $N \cap \eta_N = N \cap \eta$, it follows that for all $\xi < \kappa$,

$$N \cap A_{\eta_N,\xi} = N \cap A_{\eta_N,\xi} \cap \eta = N \cap A_{\eta,\xi}.$$

□

Lemma 7.13. *Let $N \in \mathcal{X} \cup \mathcal{Y}$, and suppose that $\eta \in \mathrm{cl}(N) \setminus N$. Then $\lim(C_\eta) \cap N$ is cofinal in η.*

Proof. Note that $\eta \in \lim(N)$. If $\eta = \sup(N)$, then the statement of the lemma follows from the definitions of $\mathcal{X}$ and $\mathcal{Y}$. Otherwise by Lemma 7.12, $C_\eta = C_{\eta_N} \cap \eta$. Let $\gamma < \eta$. Since $\eta \in \lim(N)$, we can fix $\sigma \in N \cap \eta$ larger than γ. As $\eta < \eta_N$, η_N has uncountable cofinality. So certainly $\lim(C_{\eta_N})$ is cofinal in η_N. By elementarity, we can find $\delta \in \lim(C_{\eta_N}) \cap \eta_N \cap N$ which is larger than σ. Then $\delta < \eta$. Since $C_\eta = C_{\eta_N} \cap \eta$, it follows that $\delta \in \lim(C_\eta)$. Thus $\gamma \leq \delta$ and $\delta \in \lim(C_\eta) \cap N$. □

The next lemma is standard.

Lemma 7.14. *Suppose that $P \in P_\kappa(H(\kappa^+))$, $P \prec (H(\kappa^+), \in)$, $P \cap \kappa \in \kappa$, and $\mathrm{cf}(P \cap \kappa) > \omega$. Assume that γ is a limit point of $P \cap \kappa^+$ below $\sup(P)$, and $\mathrm{cf}(\gamma) < \mathrm{cf}(P \cap \kappa)$. Then $\gamma \in P$.*

Proof. Suppose for a contradiction that $\gamma \notin P$. Then γ_P is in P and $\gamma < \gamma_P$. By elementarity, we can fix an increasing and cofinal function $f : \mathrm{cf}(\gamma_P) \to \gamma_P$ which is in P. Since $\gamma_P < \kappa^+$, either $\mathrm{cf}(\gamma_P) < \kappa$ or $\mathrm{cf}(\gamma_P) = \kappa$. In the first case, $\mathrm{cf}(\gamma_P) \in P \cap \kappa \in \kappa$ implies that $\mathrm{cf}(\gamma_P) \subseteq P$. By elementarity, $f[\mathrm{cf}(\gamma_P)] \subseteq P \cap \gamma_P \subseteq \gamma$, which is impossible since $f[\mathrm{cf}(\gamma_P)]$ is cofinal in γ_P and $\gamma < \gamma_P$. Therefore $\mathrm{cf}(\gamma_P) = \kappa$. By elementarity, $f \restriction P \cap \kappa$ is cofinal in $P \cap \gamma_P$, and hence is cofinal in γ. But then γ has cofinality equal to $\mathrm{cf}(P \cap \kappa)$, which contradicts our assumption on γ. □

Lemma 7.15. *Let $P \in P_\kappa(H(\kappa^+))$ with $P \cap \kappa \in \kappa$ and $P \prec \mathcal{A}$. If $\mathrm{cf}(P \cap \kappa) > \omega$ and $\mathrm{cf}(\sup(P)) > \omega$, then $P \in \mathcal{Y}$.*

Proof. Let $\sigma := \sup(P)$. By the definition of $\mathcal{Y}$, it suffices to show that $\lim(C_\sigma) \cap P$ is cofinal in σ. So let $\xi < \sigma$. Since $\sup(P) = \sigma$ has uncountable cofinality, there exists a sequence $\langle \gamma_n : n < \omega \rangle$ increasing and bounded below σ such that $\xi < \gamma_0$, $\gamma_n \in P$ if n is even, and $\gamma_n \in C_\sigma$ if n is odd. Let γ be the supremum of this sequence. Then γ is a limit point of P which is strictly below $\sup(P)$ with cofinality ω. Since $\mathrm{cf}(P \cap \kappa) > \omega$, $\mathrm{cf}(\gamma) < \mathrm{cf}(P \cap \kappa)$. So $\gamma \in P$ by Lemma 7.14. On the other hand, γ is a limit point of C_σ. So $\xi < \gamma$ and $\gamma \in \lim(C_\sigma) \cap P$. □

Lemma 7.16. *Let M and N be in $\mathcal{X} \cup \mathcal{Y}$, and assume that $\{M, N\}$ is adequate in the case that M and N are in $\mathcal{X}$. Then $M \cap N \in \mathcal{X} \cup \mathcal{Y}$. Specifically:*

(1) *if $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, then $M \cap N \in \mathcal{X}$;*
(2) *if $M \in \mathcal{Y}$ and $N \in \mathcal{Y}$, then $M \cap N \in \mathcal{Y}$.*

Proof. Obviously $M \cap N$ is an elementary substructure of $\mathcal{A}$. If $M \in \mathcal{X}$, then $M \cap N \in \mathcal{X}_0$ by Lemma 1.23 and the comment after Notation 1.10. Hence $M \cap N \cap \kappa \in T^*$. And if M and N are in $\mathcal{Y}$, then $M \cap N \cap \kappa = \min\{M \cap \kappa, N \cap \kappa\} \in \kappa$. Let $\alpha := \sup(M \cap N)$. It remains to show that $\lim(C_\alpha) \cap (M \cap N)$ is cofinal in α.

First we claim that $\lim(C_\alpha) \cap M$ and $\lim(C_\alpha) \cap N$ are cofinal in α. Since $M \cap N \cap \kappa^+$ is closed under successors, it does not have a maximum element, and therefore α is a limit point of $M \cap N \cap \kappa^+$. As $\alpha \in \lim(M)$, if $\alpha \notin M$ then $\lim(C_\alpha) \cap M$ is cofinal in α by Lemma 7.13, and similarly with N. So if α is neither in M nor N, then the claim is proved. Assume that α is in one of them. Since $\alpha = \sup(M \cap N)$, α cannot be in both in M and N. Without loss of generality, assume that $\alpha \in N \setminus M$. Then $\lim(C_\alpha) \cap M$ is cofinal in α as just observed, and so in particular, $\lim(C_\alpha)$ is cofinal in α. By the elementarity of N, and since $\alpha \in N$ and also α is a limit point of N, easily $\lim(C_\alpha) \cap N$ is cofinal in α. This completes the proof of the claim.

To show that $\lim(C_\alpha) \cap (M \cap N)$ is cofinal in α, let $\gamma < \alpha$. Fix $\gamma' \in M \cap N \cap \kappa^+$ with $\gamma < \gamma'$. Let $\sigma = \min(\lim(C_\alpha) \setminus \gamma')$. We claim that $\sigma \in M \cap N$, which completes the proof. Since $\lim(C_\alpha) \cap M$ is cofinal in α, we can fix $\eta \in \lim(C_\alpha) \cap M$ with $\sigma < \eta$. As $\eta \in \lim(C_\alpha)$, $C_\eta = C_\alpha \cap \eta$. Therefore $\sigma = \min(\lim(C_\eta) \setminus \gamma')$. Since η and γ' are in M, $\sigma \in M$ by elementarity. The same argument shows that $\sigma \in N$. $\square$

We now introduce the idea of a simple model.[5] These are the models for which there exist strongly generic conditions. To motivate the definition, we prove a bound on $\mathrm{ot}(C_{\sup(N)})$.

Lemma 7.17. *Let $N \in \mathcal{X} \cup \mathcal{Y}$. If $\eta \in \lim(N)$, then $\mathrm{ot}(C_\eta) \in \mathrm{cl}(N \cap \kappa)$. In particular, $\mathrm{ot}(C_{\sup(N)}) \le \sup(N \cap \kappa)$.*

Proof. Since $\eta \in \lim(N)$ and $|N| < \kappa$, it follows that $\mathrm{cf}(\eta) < \kappa$. If $\eta \in N$, then $\mathrm{ot}(C_\eta) \in N \cap \kappa$ by elementarity. Assume that η is not in N. Then $\eta \in \mathrm{cl}(N) \setminus N$. By Lemma 7.13, $\lim(C_\eta) \cap N$ is cofinal in η. We claim that $\mathrm{ot}(C_\eta)$ is a limit point of $N \cap \kappa$. Let $\gamma < \mathrm{ot}(C_\eta)$. Then we can find $\delta \in \lim(C_\eta) \cap N$ such that

$$\gamma < \mathrm{ot}(C_\eta \cap \delta) = \mathrm{ot}(C_\delta) < \mathrm{ot}(C_\eta).$$

Since $\delta \in N$, $\mathrm{ot}(C_\delta) \in N \cap \mathrm{ot}(C_\eta)$. So $\gamma < \mathrm{ot}(C_\delta) < \mathrm{ot}(C_\eta)$ and $\mathrm{ot}(C_\delta) \in N \cap \kappa$. $\square$

Definition 7.18. *Let $N \in \mathcal{X} \cup \mathcal{Y}$. We say that N is* simple *if $\mathrm{ot}(C_{\sup(N)}) = \sup(N \cap \kappa)$.*

We prove next that there exist stationarily many simple models in $\mathcal{X}$.

Lemma 7.19. *Let $N \in \mathcal{X} \cup \mathcal{Y}$ and $\delta := \sup(N)$. Then for all $\xi \in N \cap \mathrm{ot}(C_\delta)$, $c_{\delta,\xi} \in N$.*

Proof. Let $\xi \in N \cap \mathrm{ot}(C_\delta)$. Then $\xi < \mathrm{ot}(C_\delta)$. As $\lim(C_\delta) \cap N$ is cofinal δ, we can fix $\eta \in \lim(C_\delta) \cap N$ such that $\xi < \mathrm{ot}(C_\delta \cap \eta) = \mathrm{ot}(C_\eta)$. Hence $c_{\delta,\xi} = c_{\eta,\xi}$. Since η and ξ are in N, by elementarity, $c_{\eta,\xi}$ is in N. $\square$

[5] This is the same idea as described in [12, Section 3.3].

Proposition 7.20. *The collection of models in $\mathcal{X}$ which are simple is stationary in $P_{\omega_1}(H(\kappa^+))$.*

Proof. Let $F : H(\kappa^+)^{<\omega} \to H(\kappa^+)$, and we will find a simple model in $\mathcal{X}$ which is closed under F. Fix X of size κ such that $X \prec \mathcal{A}$, X is closed under F, and $\theta := X \cap \kappa^+$ has cofinality κ. Let $c : \kappa \to \theta$ be the function $c(\xi) = c_{\theta,\xi}$ for all $\xi < \kappa$. Since T^* is stationary in $P_{\omega_1}(\kappa)$ and $X \prec \mathcal{A}$, we can find $M \in P_{\omega_1}(X)$ which is closed under F such that

$$M \cap \kappa \in T^*,\ M \prec \mathcal{A},\ \text{and}\ M \prec (X, \in, C_\theta, c).$$

Let $\delta := \sup(M)$.

We claim that M is in $\mathcal{X}$ and is simple. To show that $M \in \mathcal{X}$, it suffices to show that $\lim(C_\delta) \cap M$ is cofinal in δ. By elementarity, clearly $\delta \in \lim(C_\theta)$. Hence $C_\delta = C_\theta \cap \delta$. As M is closed under c, for all $\xi \in M \cap \kappa$,

$$c(\xi) = c_{\theta,\xi} \in M \cap \kappa^+ \subseteq \delta.$$

Thus $c(\xi) \in C_\theta \cap \delta = C_\delta$. So $c(\xi) = c_{\theta,\xi} = c_{\delta,\xi}$. It follows by elementarity that $\{c(\xi) : \xi \in M \cap \kappa\}$ is increasing and cofinal in $M \cap \delta$. In particular, the set $\{c(\xi) : \xi \in M \cap \kappa,\ \xi \text{ limit}\}$ witnesses that $\lim(C_\delta) \cap M$ is cofinal δ.

It remains to show that M is simple, which means that $\mathrm{ot}(C_\delta) = \sup(M \cap \kappa)$. We know that $\mathrm{ot}(C_\delta) \leq \sup(M \cap \kappa)$ by Lemma 7.17. Suppose for a contradiction that $\mathrm{ot}(C_\delta) < \sup(M \cap \kappa)$. Fix $\beta \in (M \cap \kappa) \setminus \mathrm{ot}(C_\delta)$. Then $c(\beta) \in M$ by elementarity. But $c(\beta) = c_{\theta,\beta} = c_{\delta,\beta}$, as previously observed, which is absurd since $\mathrm{ot}(C_\delta) \leq \beta$. □

Regarding the stationarity of simple models in $\mathcal{Y}$, see Lemma 8.3 and Proposition 14.2.

Next we will show that a model M in $\mathcal{X} \cup \mathcal{Y}$ is determined by $\sup(M \cap \kappa)$ and $\sup(M)$.

Notation 7.21. *Consider $\eta < \kappa^+$ and $\beta < \kappa$. For $\gamma < \mathrm{ot}(A_{\eta,\beta})$, let $a_{\eta,\beta,\gamma}$ be equal to the γ-th element of $A_{\eta,\beta}$. Define $\pi_\eta : \kappa \times \kappa \to \eta$ by letting $\pi_\eta(\gamma,\beta) = a_{\eta,\beta,\gamma}$ if $\gamma < \mathrm{ot}(A_{\eta,\beta})$, and 0 otherwise.*

Note that π_η is a surjection of $\kappa \times \kappa$ onto η. Also if $\xi \in A_{\eta,\beta}$, then $\xi = \pi_\eta(\mathrm{ot}(A_{\eta,\beta} \cap \xi), \beta)$.

Observe that π_η is definable in the structure $\mathcal{A}$.

Lemma 7.22. *Let $\eta < \kappa^+$ and $\beta < \kappa$. Suppose that*

$$\delta \in \lim(C_\eta) \cup A_{\eta,\beta} \cup \lim(A_{\eta,\beta})$$

and $\gamma < \mathrm{ot}(A_{\delta,\beta})$. Then $a_{\eta,\beta,\gamma} = a_{\delta,\beta,\gamma}$. So $\pi_\eta(\gamma,\beta) = \pi_\delta(\gamma,\beta)$.

Proof. By Notation 7.4(4), $A_{\delta,\beta} = A_{\eta,\beta} \cap \delta$, so clearly $a_{\eta,\beta,\gamma} = a_{\delta,\beta,\gamma}$. □

Lemma 7.23. *Let $N \in \mathcal{X} \cup \mathcal{Y}$. Then*

$$N \cap \kappa^+ = \{\pi_{\sup(N)}(\gamma,\beta) : \gamma,\beta \in N \cap \kappa\}.$$

Proof. Let $\eta := \sup(N)$. Suppose that γ and β are in $N \cap \kappa$, and we will show that $\pi_\eta(\gamma,\beta) \in N$. This is obvious if $\pi_\eta(\gamma,\beta) = 0$. So assume that $\gamma < \mathrm{ot}(A_{\eta,\beta})$ and $\pi_\eta(\gamma,\beta) = a_{\eta,\beta,\gamma}$. Since $N \in \mathcal{X} \cup \mathcal{Y}$, $\lim(C_\eta) \cap N$ is cofinal in η. As $a_{\eta,\beta,\gamma} < \eta$, we can fix $\delta \in \lim(C_\eta) \cap N$ such that $a_{\eta,\beta,\gamma} < \delta$. Since $A_{\delta,\beta} = A_{\eta,\beta} \cap \delta$, clearly $\gamma < \mathrm{ot}(A_{\delta,\beta})$. By Lemma 7.22, $\pi_\eta(\gamma,\beta) = \pi_\delta(\gamma,\beta)$. As δ, γ, and β are in N, $\pi_\delta(\gamma,\beta) \in N$ by elementarity.

Conversely, let $\xi \in N \cap \kappa^+$ be given, and we will find γ and β in $N \cap \kappa$ such that $\pi_\eta(\gamma, \beta) = \xi$. Since $\lim(C_\eta) \cap N$ is cofinal in η, we can fix $\delta \in \lim(C_\eta) \cap N$ such that $\xi < \delta$. Then ξ and δ are in N. By elementarity, there is $\beta \in N \cap \kappa$ such that $\xi \in A_{\delta,\beta}$. Let $\gamma := \mathrm{ot}(A_{\delta,\beta} \cap \xi)$. Since δ, β, and ξ are in N, $\gamma \in N$. As noted after Notation 7.21, $a_{\delta,\beta,\gamma} = \pi_\delta(\gamma, \beta) = \xi$. Since $\delta \in \lim(C_\eta)$, by Lemma 7.22, $\pi_\eta(\gamma, \beta) = \pi_\delta(\gamma, \beta) = \xi$. $\square$

Lemma 7.24. *Let M and N be in $\mathcal{X} \cup \mathcal{Y}$, and suppose that $M \cap \kappa$ and* $\sup(M)$ *are in N. Then $M \in N$.*

Proof. Since $M = f^*[M \cap \kappa^+]$, by elementarity it suffices to show that $M \cap \kappa^+ \in N$. Let $\eta := \sup(M)$. Then

$$M \cap \kappa^+ = \{\pi_\eta(\gamma, \beta) : \gamma, \beta \in M \cap \kappa\}$$

by Lemma 7.23. Since η and $M \cap \kappa$ are in N, $M \cap \kappa^+ \in N$ by elementarity. $\square$

The next topic we consider is the set $A_{\sup(M),\sup(M\cap\kappa)}$, where $M \in \mathcal{X} \cup \mathcal{Y}$.

Lemma 7.25. *Let $M \in \mathcal{X} \cup \mathcal{Y}$. Then $M \cap \kappa^+ \subseteq A_{\sup(M),\sup(M\cap\kappa)}$.*

Proof. Let $\xi \in M \cap \kappa^+$. Since $\lim(C_{\sup(M)}) \cap M$ is cofinal in $\sup(M)$, we can fix $\sigma \in \lim(C_{\sup(M)}) \cap M$ which is strictly greater than ξ. By elementarity, we can fix $\beta \in M \cap \kappa$ such that $\xi \in A_{\sigma,\beta}$. Since $\sigma \in \lim(C_{\sup(M)})$, we have that $A_{\sigma,\beta} = A_{\sup(M),\beta} \cap \sigma$. Hence $\xi \in A_{\sup(M),\beta}$. As $\beta \in M \cap \kappa$, $\beta < \sup(M \cap \kappa)$. Therefore $A_{\sup(M),\beta} \subseteq A_{\sup(M),\sup(M\cap\kappa)}$. Hence $\xi \in A_{\sup(M),\sup(M\cap\kappa)}$. $\square$

Lemma 7.26. *Let $Q \in \mathcal{Y}$. Then $Q \cap \kappa^+ = A_{\sup(Q),Q\cap\kappa}$.*

Proof. By Lemma 7.25, we have that $Q \cap \kappa^+ \subseteq A_{\sup(Q),Q\cap\kappa}$. Conversely, let $\xi \in A_{\sup(Q),Q\cap\kappa}$, and we will show that $\xi \in Q$. Since $\lim(C_{\sup(Q)}) \cap Q$ is cofinal in $\sup(Q)$, we can fix $\sigma \in \lim(C_{\sup(Q)}) \cap Q$ which is strictly greater than ξ. As $\sigma \in \lim(C_{\sup(Q)})$, $A_{\sigma,Q\cap\kappa} = A_{\sup(Q),Q\cap\kappa} \cap \sigma$. Therefore $\xi \in A_{\sigma,Q\cap\kappa}$. Since $Q \cap \kappa$ is a limit ordinal, we can fix $\beta < Q \cap \kappa$ such that $\xi \in A_{\sigma,\beta}$. Then σ and β are in Q, and hence $A_{\sigma,\beta} \in Q$. Since $|A_{\sigma,\beta}| < \kappa$ by Notation 7.4(3), $A_{\sigma,\beta} \subseteq Q$. Therefore $\xi \in Q$. $\square$

Lemma 7.27. *Let $Q \in \mathcal{Y}$ and $\eta \in \mathrm{cl}(Q)$. Then $Q \cap \eta = A_{\eta,Q\cap\kappa}$.*

Proof. By Lemma 7.26, $Q \cap \kappa^+ = A_{\sup(Q),Q\cap\kappa}$. Since $\eta \in \mathrm{cl}(Q)$,

$$\eta \in A_{\sup(Q),Q\cap\kappa} \cup \lim(A_{\sup(Q),Q\cap\kappa}).$$

By Notation 7.4(4),

$$A_{\eta,Q\cap\kappa} = A_{\sup(Q),Q\cap\kappa} \cap \eta = Q \cap \eta.$$

$\square$

Lemma 7.28. *Suppose that P_1 and P_2 are in $\mathcal{Y}$.*

(1) *If $P_1 \cap \kappa \leq P_2 \cap \kappa$ and $\eta \in \mathrm{cl}(P_1) \cap \mathrm{cl}(P_2)$, then $P_1 \cap \eta \subseteq P_2 \cap \eta$.*
(2) *If $P_1 \cap \kappa < P_2 \cap \kappa$ and $\eta \in P_1 \cap P_2 \cap \mathrm{cof}(\kappa)$, then $P_1 \cap \eta \in P_2$. In particular, $\sup(P_1 \cap \eta) \in P_2 \cap \eta$.*

Proof. By Lemma 7.27, under the assumptions of either (1) or (2), we have that

$$P_1 \cap \eta = A_{\eta, P_1 \cap \kappa}, \text{ and } P_2 \cap \eta = A_{\eta, P_2 \cap \kappa}.$$

(1) If $P_1 \cap \kappa \leq P_2 \cap \kappa$, then clearly $A_{\eta, P_1 \cap \kappa} \subseteq A_{\eta, P_2 \cap \kappa}$. Therefore $P_1 \cap \eta \subseteq P_2 \cap \eta$.

(2) If $P_1 \cap \kappa < P_2 \cap \kappa$, then $P_1 \cap \kappa \in P_2$. So $P_1 \cap \kappa$ and η are in P_2, and hence $A_{\eta, P_1 \cap \kappa} = P_1 \cap \eta$ is in P_2 by elementarity. Since η has cofinality κ, $\sup(P_1 \cap \eta) < \eta$. So $\sup(P_1 \cap \eta) \in P_2 \cap \eta$. □

Lemma 7.29. *Let $M \in \mathcal{X}$. Let $A := A_{\sup(M), \sup(M \cap \kappa)}$. Then A is closed under H^*, $A \cap \kappa = \sup(M \cap \kappa)$, and $\sup(A) = \sup(M)$.*

Proof. To show that A is closed under H^*, it suffices to show that for each $n < \omega$, A is closed under τ_n'. At the same time, we will show that $\sup(M \cap \kappa) \subseteq A$. So fix $n < \omega$, and let k be the arity of τ_n'. Let $\alpha_0, \ldots, \alpha_{k-1} \in A$ and $\beta < \sup(M \cap \kappa)$, and we will show that $\tau_n'(\alpha_0, \ldots, \alpha_{k-1})$ and β are in A. Fix $\eta_0 \in \lim(C_{\sup(M)}) \cap M$ such that $\alpha_0, \ldots, \alpha_{k-1}$ and β_M are strictly less than η_0. Then $A \cap \eta_0 = A_{\eta_0, \sup(M \cap \kappa)}$. So $\alpha_0, \ldots, \alpha_{k-1}$ are in $A_{\eta_0, \sup(M \cap \kappa)}$. Also $\beta_M \in M \cap \kappa \subseteq A$ by Lemma 7.25, so also $\beta_M \in A_{\eta_0, \sup(M \cap \kappa)}$. As $\sup(M \cap \kappa)$ is a limit ordinal, we can fix an infinite $\gamma \in M \cap \kappa$ such that $\alpha_0, \ldots, \alpha_{k-1}$ and β_M are in $A_{\eta_0, \gamma}$.

By the elementarity of M, we can fix $\eta_1 \in M$ strictly greater than η_0 such that η_1 is closed under τ_n'. Fix $\eta_2 \in \lim(C_{\sup(M)}) \cap M$ with $\eta_1 < \eta_2$. Since $\eta_0 < \eta_1$ and η_1 is closed under τ_n', for all $\gamma_0, \ldots, \gamma_{k-1}$ in $A_{\eta_0, \gamma}$, $\tau_n'(\gamma_0, \ldots, \gamma_{k-1}) < \eta_1 < \eta_2$. Define a function $h : A_{\eta_0, \gamma}^k \to \kappa$ by letting $h(\gamma_0, \ldots, \gamma_{k-1})$ be the least ordinal $\xi < \kappa$ such that $\tau_n'(\gamma_0, \ldots, \gamma_{k-1})$ and all ordinals below β_M are in $A_{\eta_2, \xi}$. Since η_0, γ, η_2, and β_M are in M, by elementarity h is in M.

Now the domain of h has size $|A_{\eta_0, \gamma}^k| \leq |\gamma| < \kappa$. So there exists a minimal $\xi < \kappa$ such that $h[A_{\eta_0, \gamma}^k] \subseteq \xi$. By elementarity, $\xi \in M \cap \kappa$. In particular, $\delta := h(\alpha_0, \ldots, \alpha_{k-1})$ is less than ξ. That means $\tau_n'(\alpha_0, \ldots, \alpha_{k-1})$ and β are in $A_{\eta_2, \delta} \subseteq A_{\eta_2, \xi}$. Since $\eta_2 \in \lim(C_{\sup(M)})$, $A_{\eta_2, \xi} = A_{\sup(M), \xi} \cap \eta_2$. So $\tau_n'(\alpha_0, \ldots, \alpha_{k-1})$ and β are in $A_{\sup(M), \xi} \subseteq A_{\sup(M), \sup(M \cap \kappa)} = A$. This completes the proof that A is closed under τ_n' for all $n < \omega$ and $\sup(M \cap \kappa) \subseteq A$. It follows that A is closed under H^*.

Now A is the union of sets of the form $A_{\delta, \beta}$, where $\delta \in \lim(C_{\sup(M)}) \cap M$ and $\beta \in M \cap \kappa$, and each such set is in M. Thus each such set $A_{\delta, \beta}$ satisfies that $\sup(A_{\delta, \beta} \cap \kappa) < \sup(M \cap \kappa)$. It follows that $\sup(A \cap \kappa) \leq \sup(M \cap \kappa)$. But we just proved that $\sup(M \cap \kappa) \subseteq A$, and therefore $\sup(M \cap \kappa) = A \cap \kappa$. By the definition of A, obviously $A \subseteq \sup(M)$. And since $M \cap \kappa^+ \subseteq A$ by Lemma 7.25, $\sup(A) = \sup(M)$. □

We conclude this section with two technical lemmas which will be useful later.

Lemma 7.30. *Let $N \in \mathcal{X}$, $a \in N$, and $\tau \in N \cap \kappa^+$. Suppose that $\mathrm{cf}(\tau) > \omega$. If $a \cap [\sup(N \cap \tau), \tau) \neq \emptyset$, then τ is a limit point of a.*

Proof. If τ is not a limit point of a, then $\sup(a \cap \tau) < \tau$. Since a and τ are in N, $\sup(a \cap \tau) \in N \cap \tau$ by elementarity. Hence $\sup(a \cap \tau) < \sup(N \cap \tau)$, which contradicts the assumption that $a \cap [\sup(N \cap \tau), \tau) \neq \emptyset$. □

Lemma 7.31. *Let $N \in \mathcal{X}$. Suppose that $\eta \in N \cap \kappa^+$ and $\beta \in N \cap \kappa$. Let $\xi \in A_{\eta, \beta} \setminus N$. Then $\xi \in A_{\xi_N, \beta}$.*

Proof. Note that since $\eta \in N$ and $\xi < \eta$, ξ_N exists. Since $\xi \notin N$, $\xi < \xi_N$. It follows that $\mathrm{cf}(\xi_N) > \omega$, since otherwise $N \cap \xi_N$ would be cofinal in ξ_N by elementarity. Also $\sup(N \cap \xi_N) \leq \xi$. Since $A_{\eta,\beta} \in N$ and $\xi \in A_{\eta,\beta} \cap [\sup(N \cap \xi_N), \xi_N)$, it follows that ξ_N is a limit point of $A_{\eta,\beta}$ by Lemma 7.30. So $A_{\xi_N,\beta} = A_{\eta,\beta} \cap \xi_N$. Since $\xi \in A_{\eta,\beta} \cap \xi_N$, $\xi \in A_{\xi_N,\beta}$. □

§8. Interaction of models past κ

The method of adequate sets, which we dealt with in Part I, handles the interaction of countable elementary substructures below κ. In this section we will show how the coherent filtration system $\vec{A}$ from Section 7 can be used to control the interaction of models between κ and κ^+.

Notation 8.1. *For M and N in $\mathcal{X} \cup \mathcal{Y}$, let $\alpha_{M,N}$ denote the ordinal* $\sup(M \cap N)$.

As we discussed in Section 1 in the paragraph after Propostion 1.29, if $M < N$, in general it does not necessarily follow that $M \cap N \in N$. The next lemma describes a situation in which this implication does hold.

Lemma 8.2. *Let M and N be in $\mathcal{X} \cup \mathcal{Y}$, where N is simple. Suppose that:*
1. *M and N are in $\mathcal{X}$ and $M < N$, or*
2. *M and N are in $\mathcal{Y}$ and $M \cap \kappa < N \cap \kappa$, or*
3. *$M \in \mathcal{X}$, $N \in \mathcal{Y}$, and* $\sup(M \cap N \cap \kappa) < N \cap \kappa$.

Then $M \cap N \in N$. In particular, $\alpha_{M,N} \in N$.

Proof. By Lemma 1.30, $M \cap N \cap \kappa \in N$. Therefore by elementarity, $\mathrm{cl}(M \cap N \cap \kappa) \in N$. We claim that
$$\mathrm{cl}(M \cap N \cap \kappa) \subseteq N \cap \kappa.$$
If $M \cap N$ is countable, then so is $\mathrm{cl}(M \cap N \cap \kappa)$. Since $\mathrm{cl}(M \cap N \cap \kappa) \in N$, it follows that $\mathrm{cl}(M \cap N \cap \kappa) \subseteq N \cap \kappa$. If $M \cap N$ is uncountable, then M and N are both in $\mathcal{Y}$. So $M \cap \kappa < N \cap \kappa$ by (2), and hence $M \cap N \cap \kappa = M \cap \kappa$. Therefore
$$\mathrm{cl}(M \cap N \cap \kappa) = (M \cap \kappa) \cup \{M \cap \kappa\},$$
which is a subset of $N \cap \kappa$.

Next we claim that
$$(N \cap \kappa^+) \setminus \alpha_{M,N} \neq \emptyset.$$
Suppose for a contradiction that $(N \cap \kappa^+) \setminus \alpha_{M,N} = \emptyset$, which means that $\sup(N) = \alpha_{M,N}$. Since N is simple, it follows that
$$\mathrm{ot}(C_{\alpha_{M,N}}) = \sup(N \cap \kappa).$$
But $\mathrm{ot}(C_{\alpha_{M,N}}) \in \mathrm{cl}(M \cap N \cap \kappa)$ by Lemma 7.17. By the first claim, it follows that $\mathrm{ot}(C_{\alpha_{M,N}}) \in N \cap \kappa$, which contradicts that $\mathrm{ot}(C_{\alpha_{M,N}}) = \sup(N \cap \kappa)$.

Let $\alpha := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$. By Lemma 7.23,
$$M \cap N \cap \kappa^+ = \{\pi_{\alpha_{M,N}}(\gamma, \beta) : \gamma, \beta \in M \cap N \cap \kappa\}.$$
We claim that for all $\gamma, \beta \in M \cap N \cap \kappa$, $\pi_{\alpha_{M,N}}(\gamma, \beta) = \pi_\alpha(\gamma, \beta)$. This is immediate if $\alpha = \alpha_{M,N}$, so assume that $\alpha_{M,N} \notin N$. Then by Lemma 7.12, $\alpha_{M,N} \in \lim(C_\alpha)$. Fix γ and β in $M \cap N \cap \kappa$.

First, assume that $\mathrm{ot}(A_{\alpha,\beta}) \leq \gamma$. Then $\pi_\alpha(\gamma, \beta) = 0$. Since $A_{\alpha_{M,N},\beta} = A_{\alpha,\beta} \cap \alpha_{M,N}$, clearly $\mathrm{ot}(A_{\alpha_{M,N},\beta}) \leq \mathrm{ot}(A_{\alpha,\beta}) \leq \gamma$. So $\pi_{\alpha_{M,N}}(\gamma, \beta) = 0$. Secondly, assume that $\gamma < \mathrm{ot}(A_{\alpha,\beta})$, so that $\pi_\alpha(\gamma, \beta) = a_{\alpha,\beta,\gamma}$. Since α, γ, and β are in N, $a_{\alpha,\beta,\gamma} \in$

$N \cap \alpha \subseteq \alpha_{M,N}$. As $A_{\alpha_{M,N},\beta} = A_{\alpha,\beta} \cap \alpha_{M,N}$, clearly $\gamma < \mathrm{ot}(A_{\alpha_{M,N},\beta})$. By Lemma 7.22, $\pi_\alpha(\gamma,\beta) = \pi_{\alpha_{M,N}}(\gamma,\beta)$.

It follows that

$$M \cap N \cap \kappa^+ = \{\pi_\alpha(\gamma,\beta) : \gamma,\beta \in M \cap N \cap \kappa\}.$$

Since α and $M \cap N \cap \kappa$ are in N, so is $M \cap N \cap \kappa^+$ by elementarity. Hence by elementarity, $M \cap N = f^*[M \cap N \cap \kappa^+] \in N$, and $\sup(M \cap N) = \alpha_{M,N} \in N$. □

Lemma 8.3. *Let $P \in \mathcal{Y}$, and assume that* $\mathrm{cf}(\sup(P)) = P \cap \kappa$. *Then P is simple.*

Proof. Since $C_{\sup(P)}$ is cofinal in $\sup(P)$,

$$P \cap \kappa = \mathrm{cf}(\sup(P)) \leq \mathrm{ot}(C_{\sup(P)}).$$

On the other hand, as $\sup(P) \in \lim(P)$, Lemma 7.17 implies that

$$\mathrm{ot}(C_{\sup(P)}) \in \mathrm{cl}(P \cap \kappa) = (P \cap \kappa) \cup \{P \cap \kappa\}.$$

Hence $\mathrm{ot}(C_{\sup(P)}) \leq P \cap \kappa$. Therefore $P \cap \kappa = \mathrm{ot}(C_{\sup(P)})$, and P is simple. □

Lemma 8.4. *Let $P \in \mathcal{Y}$, and assume that* $\mathrm{cf}(\sup(P)) = P \cap \kappa$. *If $M \in \mathcal{X}$, then $M \cap P \in P$. If $Q \in \mathcal{Y}$ and $Q \cap \kappa < P \cap \kappa$, then $Q \cap P \in P$.*

Proof. By Lemma 8.3, P is simple. Since $\mathrm{cf}(\sup(P)) = P \cap \kappa$, $P \cap \kappa$ is a regular cardinal. Obviously $\omega < P \cap \kappa$, so $P \cap \kappa$ is a regular uncountable cardinal. If $M \in \mathcal{X}$, then $\sup(M \cap P \cap \kappa) < P \cap \kappa$ since $\sup(M \cap P \cap \kappa)$ has countable cofinality. By Lemma 8.2, we are done. □

Lemma 8.5. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where $\{M, N\}$ is adequate if $N \in \mathcal{X}$, and* $\sup(M \cap N \cap \kappa) < N \cap \kappa$ *if $N \in \mathcal{Y}$. Then*

$$\lim(M) \cap \lim(N) \subseteq \alpha_{M,N} + 1.$$

Proof. Let $\eta \in \lim(M) \cap \lim(N)$, and we will show that $\eta \leq \alpha_{M,N}$. By Lemma 7.17,

$$\mathrm{ot}(C_\eta) \in \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa).$$

Since η is a limit point of M and M is countable, η has cofinality ω. Therefore $\mathrm{ot}(C_\eta)$ has cofinality ω. We claim that $\mathrm{ot}(C_\eta)$ is a limit point of $M \cap \kappa$. If $\mathrm{ot}(C_\eta) \in M$, then since $\mathrm{ot}(C_\eta)$ has countable cofinality, easily $M \cap \mathrm{ot}(C_\eta)$ is cofinal in $\mathrm{ot}(C_\eta)$ by elementarity. Hence $\mathrm{ot}(C_\eta)$ is a limit point of $M \cap \kappa$. Otherwise if $\mathrm{ot}(C_\eta) \notin M$, then since $\mathrm{ot}(C_\eta) \in \mathrm{cl}(M \cap \kappa)$, it follows immediately that $\mathrm{ot}(C_\eta)$ is in $\lim(M \cap \kappa)$.

Next we claim that

$$\mathrm{ot}(C_\eta) \in \mathrm{cl}(M \cap N \cap \kappa).$$

First, assume that $N \in \mathcal{X}$. Then by Lemma 1.20,

$$\mathrm{ot}(C_\eta) \in \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa) = \mathrm{cl}(M \cap N \cap \kappa).$$

Secondly, assume that $N \in \mathcal{Y}$. Now

$$\mathrm{ot}(C_\eta) \in \mathrm{cl}(N \cap \kappa) = (N \cap \kappa) \cup \{N \cap \kappa\}.$$

So $\mathrm{ot}(C_\eta) \leq N \cap \kappa$. Since $\sup(M \cap N \cap \kappa) < N \cap \kappa$ and $\mathrm{ot}(C_\eta)$ is a limit point of $M \cap \kappa$, it cannot be the case that $\mathrm{ot}(C_\eta) = N \cap \kappa$. Therefore $\mathrm{ot}(C_\eta) < N \cap \kappa$. By Lemma 1.31,

$$\mathrm{cl}(M \cap N \cap \kappa) = \mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa) \cap (N \cap \kappa).$$

Since $\mathrm{ot}(C_\eta)$ is in the set on the right, it is in $\mathrm{cl}(M \cap N \cap \kappa)$.

Now we claim that

$$\mathrm{ot}(C_\eta) \in \lim(M \cap N \cap \kappa).$$

As $\mathrm{ot}(C_\eta) \in \mathrm{cl}(M \cap N \cap \kappa)$, either $\mathrm{ot}(C_\eta) \in M \cap N \cap \kappa$, or $\mathrm{ot}(C_\eta) \in \lim(M \cap N \cap \kappa)$. In the latter case, we are done. In the former case, since $\mathrm{ot}(C_\eta)$ has cofinality ω, by the elementarity of $M \cap N$, clearly $M \cap N \cap \kappa$ is cofinal in $\mathrm{ot}(C_\eta)$, so again $\mathrm{ot}(C_\eta) \in \lim(M \cap N \cap \kappa)$.

Finally, we are ready to prove that $\eta \leq \alpha_{M,N}$. Suppose for a contradiction that $\alpha_{M,N} < \eta$. Since $\mathrm{ot}(C_\eta)$ is a limit point of $M \cap N \cap \kappa$, we can fix $\gamma \in M \cap N \cap \mathrm{ot}(C_\eta)$ such that $\alpha_{M,N} < c_{\eta,\gamma}$. We claim that $c_{\eta,\gamma} \in M \cap N$, which is a contradiction since $M \cap N \cap \kappa^+ \subseteq \alpha_{M,N}$. If $\eta \in M$, then obviously $c_{\eta,\gamma} \in M$ by elementarity. Otherwise $\eta \in \mathrm{cl}(M) \setminus M$. By Lemma 7.13, $\lim(C_\eta) \cap M$ is cofinal in η. So we can fix $\delta \in \lim(C_\eta) \cap M$ such that $c_{\eta,\gamma} < \delta$. Then clearly $c_{\eta,\gamma} = c_{\delta,\gamma}$, which is in M by elementarity. The proof that $c_{\eta,\gamma} \in N$ is the same. □

We now turn to address the following general issue. Suppose that M and N are in $\mathcal{X} \cup \mathcal{Y}$, and $P \in N \cap \mathcal{Y}$. Under what circumstances can we conclude that an ordinal in $M \cap P$ is in N, or is in some canonically described member of N?

The next lemma is the most frequently used result on this topic.

Lemma 8.6. *Let M and N be in $\mathcal{X}$, where $M \leq N$. Suppose that*

$$\eta \in N \cap \kappa^+ \text{ and } \beta < \sup(M \cap N \cap \kappa).$$

Then $A_{\eta,\beta} \cap M \subseteq N$. Therefore

$$A_{\eta,\sup(M \cap N \cap \kappa)} \cap M \subseteq N.$$

Proof. Let $\xi \in A_{\eta,\beta} \cap M$, and we will show that $\xi \in N$. Since $\beta < \sup(M \cap N \cap \kappa)$, we can fix $\gamma \in M \cap N \cap \kappa$ greater than β. Then $\xi \in A_{\eta,\gamma}$. So $\xi = \pi_\eta(\mathrm{ot}(A_{\eta,\gamma} \cap \xi), \gamma)$, as noted in the comments after Notation 7.21. Since $\xi \in A_{\eta,\gamma}$, $A_{\xi,\gamma} = A_{\eta,\gamma} \cap \xi$. Therefore $\mathrm{ot}(A_{\eta,\gamma} \cap \xi) = \mathrm{ot}(A_{\xi,\gamma})$. Hence $\xi = \pi_\eta(\mathrm{ot}(A_{\xi,\gamma}), \gamma)$. Since ξ and γ are in M, so is $\mathrm{ot}(A_{\xi,\gamma})$.

Since $\gamma \in M \cap N \cap \kappa$, by elementarity $c^*(\gamma) \in M \cap N \cap \kappa$. By Notation 7.4(5), since $M \leq N$, we have that

$$\mathrm{ot}(A_{\xi,\gamma}) \in M \cap c^*(\gamma) \subseteq M \cap N \cap \kappa \subseteq N.$$

So $\mathrm{ot}(A_{\xi,\gamma}) \in N \cap \kappa$. Hence η, γ, and $\mathrm{ot}(A_{\xi,\gamma})$ are in N, which implies that $\pi_\eta(\mathrm{ot}(A_{\xi,\gamma}), \gamma) = \xi$ is in N.

To show that $A_{\eta,\sup(M \cap N \cap \kappa)} \cap M \subseteq N$, let $\tau \in A_{\eta,\sup(M \cap N \cap \kappa)} \cap M$. Since $\sup(M \cap N \cap \kappa)$ is a limit ordinal, there is $\beta < \sup(M \cap N \cap \kappa)$ such that $\tau \in A_{\eta,\beta}$. By what we just proved, $A_{\eta,\beta} \cap M \subseteq N$. So $\tau \in N$. □

Lemma 8.7. *Let M and N be in $\mathcal{X}$, where $M \leq N$. Let $Q \in N \cap \mathcal{Y}$ with $Q \cap \kappa < \sup(M \cap N \cap \kappa)$. Then $Q \cap M \cap \kappa^+ \subseteq N$.*

Proof. By Lemma 7.26, $Q \cap \kappa^+ = A_{\sup(Q),Q \cap \kappa}$. By elementarity, $\sup(Q) \in N \cap \kappa^+$, and by assumption, $Q \cap \kappa < \sup(M \cap N \cap \kappa)$. By Lemma 8.6,

$$Q \cap M \cap \kappa^+ = A_{\sup(Q),Q \cap \kappa} \cap M \subseteq N.$$

□

Lemma 8.8. *Let M and N be in $\mathcal{X}$.*

(1) *If* $M \leq N$, *then*
$$A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)} \cap M \subseteq N.$$
(2) *If* $M \sim N$, *then*
$$A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)} \cap M = A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)} \cap N.$$

Proof. Note that (2) follows from (1). To prove (1), assume that $M \leq N$, and let $\xi \in A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)} \cap M$. We will show that $\xi \in N$. Fix $\beta \in M \cap N \cap \kappa$ such that $\xi \in A_{\alpha_{M,N},\beta}$. As $M \cap N \in \mathcal{X}$ and $\sup(M \cap N) = \alpha_{M,N}$, it follows that $\lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ is cofinal in $\alpha_{M,N}$. So we can fix $\delta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ which is strictly larger than ξ. Then $A_{\delta,\beta} = A_{\alpha_{M,N},\beta} \cap \delta$, and hence $\xi \in A_{\delta,\beta}$. Since $\delta \in N$, $\beta < \sup(M \cap N \cap \kappa)$, and $M \leq N$, it follows that $A_{\delta,\beta} \cap M \subseteq N$ by Lemma 8.6. So $\xi \in N$. $\square$

Lemma 8.9. *Let* $M \in \mathcal{X}$ *and* $N \in \mathcal{X} \cup \mathcal{Y}$. *Then*
$$M \cap N \cap \kappa^+ \subseteq A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}.$$

Proof. Since $M \cap N \in \mathcal{X}$ and $\sup(M \cap N) = \alpha_{M,N}$, the statement follows immediately from Lemma 7.25. $\square$

Lemma 8.10. *Let* $M \in \mathcal{X}$ *and* $N \in \mathcal{X} \cup \mathcal{Y}$. *Let* $Q \in M \cap \mathcal{Y}$, *and suppose that* $\sup(M \cap N \cap \kappa) \leq Q \cap \kappa$. *Then*
$$Q \cap N \cap \alpha_{M,N} \subseteq A_{\alpha_{M,N},Q\cap\kappa}.$$

Proof. Let $\xi \in Q \cap N \cap \alpha_{M,N}$, and we will show that $\xi \in A_{\alpha_{M,N},Q\cap\kappa}$. First assume that $\xi \in M$. Then $\xi \in M \cap N \cap \kappa^+$, so by Lemma 8.9, $\xi \in A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}$. Since $\sup(M \cap N \cap \kappa) \leq Q \cap \kappa$, it follows that $\xi \in A_{\alpha_{M,N},Q\cap\kappa}$.

Assume that ξ is not in M. Then $\xi \in Q \cap \kappa^+ = A_{\sup(Q),Q\cap\kappa}$, where $\sup(Q)$ and $Q \cap \kappa$ are in M, and $\xi \notin M$. By Lemma 7.31,
$$\xi \in A_{\xi_M,Q\cap\kappa}.$$

We claim that
$$\forall \nu \in M \cap N \cap \kappa^+ \ (\xi_M < \nu \implies \xi \in A_{\nu,Q\cap\kappa}).$$
We will prove the claim by induction. So let $\nu \in M \cap N \cap \kappa^+$ be strictly greater than ξ_M, and assume that the claim holds for all $\nu' \in M \cap N \cap \nu$.

Case 1: $\nu = \nu_0 + 1$ is a successor ordinal. Since $\nu \in M \cap N$, $\nu_0 \in M \cap N$ by elementarity. If $\xi_M < \nu_0$, then by the inductive hypothesis, $\xi \in A_{\nu_0,Q\cap\kappa}$. So
$$\xi \in A_{\nu_0,Q\cap\kappa} \cup \{\nu_0\} = A_{\nu,Q\cap\kappa}.$$
If $\nu_0 = \xi_M$, then
$$\xi \in A_{\xi_M,Q\cap\kappa} = A_{\nu_0,Q\cap\kappa} \subseteq A_{\nu,Q\cap\kappa}.$$

Case 2: ν is a limit ordinal and $\xi_M \in \lim(C_\nu)$. Then
$$A_{\xi_M,Q\cap\kappa} = A_{\nu,Q\cap\kappa} \cap \xi_M.$$
Since $\xi \in A_{\xi_M,Q\cap\kappa}$, it follows that $\xi \in A_{\nu,Q\cap\kappa}$, and we are done.

Case 3: ν is a limit ordinal and ξ_M is not in $\lim(C_\nu)$. Let $\nu' := \min(C_\nu \setminus \xi_M)$, and let $\nu'' := \sup(C_\nu \cap \nu')$. Since $\nu' = \min(C_\nu \setminus \xi_M)$, clearly $\nu'' = \sup(C_\nu \cap \xi_M)$. As ξ_M is not a limit point of C_ν, $\nu'' < \xi_M$.

We claim that $\nu' \in M \cap N$. Since ν and ξ_M are in M, $\nu' = \min(C_\nu \setminus \xi_M)$ is in M by elementarity. And as ν and ν' are in M, $\nu'' = \sup(C_\nu \cap \nu')$ is in M by elementarity. But $\nu'' < \xi_M$ and ξ_M is the least ordinal in M with $\xi \leq \xi_M$. It follows that $\nu'' < \xi$. So $\nu' = \min(C_\nu \setminus (\nu''+1)) = \min(C_\nu \setminus \xi)$. Since ν and ξ are in N, so is ν' by elementarity.

Next we claim that $\xi \in A_{\nu',Q\cap\kappa}$. This is immediate if $\xi_M = \nu'$, so assume that $\xi_M < \nu'$. Then $\xi_M < \nu' < \nu$ and $\nu' \in M \cap N \cap \kappa^+$, which imply by the inductive hypothesis that $\xi \in A_{\nu',Q\cap\kappa}$.

Let β be the least ordinal in κ such that $\nu' \in A_{\nu,\beta}$. Since ν and ν' are in $M \cap N$, it follows that $\beta \in M \cap N \cap \kappa$ by elementarity. As

$$\beta < \sup(M \cap N \cap \kappa) \leq Q \cap \kappa,$$

we have that $\nu' \in A_{\nu,Q\cap\kappa}$. And since $\nu'' = \sup(C_\nu \cap \nu')$ and

$$\nu'' < \xi < \xi_M \leq \nu',$$

$\xi \notin C_\nu$. So

$$\xi \in \nu \setminus C_\nu, \ \nu' = \min(C_\nu \setminus \xi) \in A_{\nu,Q\cap\kappa}, \text{ and } \xi \in A_{\min(C_\nu\setminus\xi),Q\cap\kappa}.$$

By Notation 7.4(7), $\xi \in A_{\nu,Q\cap\kappa}$, which completes the proof of the claim.

Since $M \cap N \in \mathcal{X}$ and $\sup(M \cap N) = \alpha_{M,N}$, it follows that $\lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ is cofinal in $\alpha_{M,N}$. Since $\xi < \alpha_{M,N}$ by assumption and $\alpha_{M,N}$ is a limit point of M, we have that $\xi_M < \alpha_{M,N}$. So we can fix $\nu \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ which is strictly greater than ξ_M. By the claim, $\xi \in A_{\nu,Q\cap\kappa}$. Since $\nu \in \lim(C_{\alpha_{M,N}})$, $A_{\nu,Q\cap\kappa} = A_{\alpha_{M,N},Q\cap\kappa} \cap \nu$. Therefore $\xi \in A_{\alpha_{M,N},Q\cap\kappa}$. $\square$

Lemma 8.11. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$. Suppose that $M < N$ in the case that $N \in \mathcal{X}$. Let $P \in M \cap \mathcal{Y}$, and suppose that $P \cap \kappa \in M \cap N \cap \kappa$ and $P \cap \alpha_{M,N}$ is bounded below $\alpha_{M,N}$.*

Define

$$\sigma := \sup(P \cap A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}).$$

Then σ satisfies:

1. *$\sigma \in M \cap N \cap \kappa^+$;*
2. *$P \cap \sigma = A_{\sigma,P\cap\kappa}$;*
3. *$P \cap (M \cap N) \cap \kappa^+ = A_{\sigma,P\cap\kappa} \cap (M \cap N)$;*
4. *$N \cap P \cap \alpha_{M,N} \subseteq A_{\sigma,P\cap\kappa}$.*

Proof. Let $\alpha := \alpha_{M,N}$ and $\delta := \sup(M \cap N \cap \kappa)$. Note that by Lemma 7.29,

$$Sk(A_{\alpha,\delta}) \cap \kappa^+ = A_{\alpha,\delta}, \ A_{\alpha,\delta} \cap \kappa = \delta, \text{ and } \sup(A_{\alpha,\delta}) = \alpha.$$

Since P and $A_{\alpha,\delta}$ are closed under successors, $P \cap A_{\alpha,\delta}$ has no maximal element. Note that since $P \cap \alpha_{M,N}$ is bounded below $\alpha_{M,N}$, we have that $\sigma < \alpha$. We claim that σ satisfies (1)–(4). Observe that since σ is a limit point of P, it follows that $P \cap \sigma = A_{\sigma,P\cap\kappa}$ by Lemma 7.27, which proves (2).

(3) We prove that

$$P \cap (M \cap N) \cap \kappa^+ = A_{\sigma,P\cap\kappa} \cap (M \cap N).$$

Let $\gamma \in P \cap (M \cap N) \cap \kappa^+$, and we will show that $\gamma \in A_{\sigma,P\cap\kappa}$. Since $\gamma \in M \cap N$, by Lemma 8.9 we have that $\gamma \in A_{\alpha,\delta}$. So $\gamma \in P \cap A_{\alpha,\delta} \subseteq \sigma$. Hence $\gamma \in P \cap \sigma = A_{\sigma,P\cap\kappa}$. Conversely,

$$A_{\sigma,P\cap\kappa} \cap (M \cap N) \subseteq A_{\sigma,P\cap\kappa} = P \cap \sigma \subseteq P.$$

(1,4) It remains to show that $\sigma \in M \cap N$ and $N \cap P \cap \alpha_{M,N} \subseteq A_{\sigma, P\cap\kappa}$. We claim that

$$\sigma = \sup(A_{\sup(P\cap\alpha), P\cap\kappa} \cap A_{\alpha,\delta}).$$

As P and α are closed under successors, $P \cap \alpha$ has no maximal element. So $\sup(P \cap \alpha)$ is a limit point of P, which by Lemma 7.27 implies that

$$P \cap \alpha = P \cap \sup(P \cap \alpha) = A_{\sup(P\cap\alpha), P\cap\kappa}.$$

Therefore

$$A_{\sup(P\cap\alpha), P\cap\kappa} \cap A_{\alpha,\delta} = P \cap \alpha \cap A_{\alpha,\delta} = P \cap A_{\alpha,\delta}.$$

Taking supremums of both sides yields the claim.

Next, we claim that $\sigma \in A_{\alpha,\delta}$. As $P \cap \alpha$ and $A_{\alpha,\delta}$ are closed under successors and $P \cap \alpha = A_{\sup(P\cap\alpha), P\cap\kappa}$, it follows that σ is a limit point of $A_{\sup(P\cap\alpha), P\cap\kappa}$ and a limit point of $A_{\alpha,\delta}$. By Notation 7.4(8) and the fact that $\sigma \in \lim(A_{\alpha,\delta}) \cap \alpha$, to show that $\sigma \in A_{\alpha,\delta}$ it suffices to show that $\mathrm{ot}(C_\sigma) < \delta$.

Since $P \cap \sigma = A_{\sigma, P\cap\kappa}$ and σ is a limit point of P, it follows that $\sigma = \sup(A_{\sigma,P\cap\kappa})$. If $P \cap \kappa < \mathrm{ot}(C_\sigma)$, then by Notation 7.4(6), it follows that $A_{\sigma,P\cap\kappa} \subseteq c_{\sigma,P\cap\kappa} < \sigma$. But this contradicts that $\sigma = \sup(A_{\sigma,P\cap\kappa})$. Hence $\mathrm{ot}(C_\sigma) \leq P \cap \kappa < \delta$, which completes the proof of the claim that $\sigma \in A_{\alpha,\delta}$.

Fix $\eta \in \lim(C_\alpha) \cap (M \cap N)$ such that $\sigma < \eta$. Then $A_{\eta,\delta} = A_{\alpha,\delta} \cap \eta$. Therefore $\sigma \in A_{\eta,\delta}$. Since $\delta = \sup(M \cap N \cap \kappa)$, we can fix $\gamma \in M \cap N \cap \kappa$ such that $\sigma \in A_{\eta,\gamma}$. Then η and γ are in $M \cap N$.

Let us show that

$$\sigma = \max(A_{\eta,\gamma} \cap \lim(P)).$$

Suppose for a contradiction that $\sigma' \in A_{\eta,\gamma} \cap \lim(P)$ and $\sigma < \sigma'$. Since $\eta \in \lim(C_\alpha)$ and $\gamma < \delta$, $\sigma' \in A_{\alpha,\delta} \cap \lim(P)$. But then by Lemma 7.27, it follows that

$$P \cap \sigma' = A_{\sigma',P\cap\kappa} \subseteq A_{\sigma',\delta} = A_{\alpha,\delta} \cap \sigma'.$$

Since σ' is a limit point of P, there is $\tau \in P \cap \sigma'$ strictly greater than σ. Then $\tau \in P \cap A_{\alpha,\delta}$, which contradicts that $\sigma = \sup(P \cap A_{\alpha,\delta})$.

Now we prove that $\sigma \in M \cap N$. Since η, γ, and P are in M, and $\sigma = \max(A_{\eta,\gamma} \cap \lim(P))$, it follows that $\sigma \in M$ by elementarity. On the other hand, $\sigma \in M \cap A_{\eta,\gamma}$, where $\eta \in N$ and $\gamma \in M \cap N \cap \kappa$. So $\sigma \in N$ by Lemma 8.6, in the case when M and N are in $\mathcal{X}$. If $N \in \mathcal{Y}$, then $\sigma \in A_{\eta,\gamma} \in N$ implies that $\sigma \in N$, since $|A_{\eta,\gamma}| < \kappa$. This proves that $\sigma \in M \cap N$.

Now we claim that $N \cap P \cap \alpha_{M,N} \subseteq \sigma$. This completes the proof, for then

$$N \cap P \cap \alpha_{M,N} \subseteq P \cap \sigma = A_{\sigma,P\cap\kappa}.$$

Suppose for a contradiction that $\pi \in N \cap P \cap \alpha_{M,N}$ and $\sigma \leq \pi$. Let $\pi_0 := \min((P \cap \kappa^+) \setminus \sigma)$, and note that $\pi_0 \leq \pi$. Since P and σ are in M, π_0 is in M by elementarity. We claim that π_0 is in N. This is immediate if $\pi_0 = \pi$, so assume that $\pi_0 < \pi$. Then $\pi \in P$ implies that $P \cap \pi = A_{\pi,P\cap\kappa}$ by Lemma 7.27. Since π and $P \cap \kappa$ are in N, so is $A_{\pi,P\cap\kappa} = P \cap \pi$. But $\pi_0 = \min((P \cap \pi) \setminus \sigma)$. Hence $\pi_0 \in N$ by elementarity. So $\pi_0 \in M \cap N \cap \alpha_{M,N}$, and therefore $\pi_0 \in A_{\alpha,\delta}$ by Lemma 8.9. So $\pi_0 \in P \cap A_{\alpha,\delta}$. Since $\sigma = \sup(P \cap A_{\alpha,\delta})$ and $P \cap A_{\alpha,\delta}$ has no maximal element as previously observed, it follows that $\pi_0 < \sigma$. But this contradicts that fact that $\sigma \leq \pi_0$. $\square$

So far in this section we have been mostly concerned about the interaction of models M and N below $\alpha_{M,N} = \sup(M \cap N)$. We now turn to analyze what happens above $\alpha_{M,N}$.

The next two lemmas state that for a simple model N, if a model does not bound N below κ, then it does not bound N above κ.

Lemma 8.12. *Let M and N be in $\mathcal{X}$, where N is simple and $\{M, N\}$ is adequate. If $R_M(N) \neq \emptyset$, then $(N \cap \kappa^+) \setminus \alpha_{M,N} \neq \emptyset$.*

Proof. Since $R_M(N)$ is nonempty, $\beta_{M,N} \leq \sup(N \cap \kappa)$. As $\alpha_{M,N}$ is a limit point of M and a limit point of N, Lemma 7.17 implies that $\mathrm{ot}(C_{\alpha_{M,N}})$ is in $\mathrm{cl}(M \cap \kappa) \cap \mathrm{cl}(N \cap \kappa)$. Hence by Lemma 1.15, $\mathrm{ot}(C_{\alpha_{M,N}}) < \beta_{M,N}$. If $(N \cap \kappa^+) \setminus \alpha_{M,N}$ is empty, then $\sup(N) = \alpha_{M,N}$. Since N is simple, it follows that $\mathrm{ot}(C_{\alpha_{M,N}}) = \sup(N \cap \kappa)$. But then $\sup(N \cap \kappa) < \beta_{M,N}$, which contradicts the first line above. $\square$

Lemma 8.13. *Let $N \in \mathcal{X}$ be simple and $Q \in \mathcal{Y}$. If $Q \cap \kappa < \sup(N \cap \kappa)$, then $\sup(N \cap Q) < \sup(N)$.*

Proof. Let $\beta := Q \cap \kappa$ and $\eta := \sup(N \cap Q)$, and assume that $\beta < \sup(N \cap \kappa)$. Since N and Q are closed under successors, η is a limit point of $N \cap Q$. In particular, η is a limit point of N. Suppose for a contradiction that $\sup(N) = \eta$. Then since N is simple, $\mathrm{ot}(C_\eta) = \sup(N \cap \kappa)$. So $\beta < \mathrm{ot}(C_\eta)$. As $N \cap Q$ is in $\mathcal{X}$ by Lemma 7.16 and $\sup(N \cap Q) = \eta$, it follows that $\lim(C_\eta) \cap (N \cap Q)$ is cofinal in η. So we can fix $\delta \in \lim(C_\eta) \cap (N \cap Q)$ such that $\beta < \mathrm{ot}(C_\eta \cap \delta) = \mathrm{ot}(C_\delta)$. But $\delta \in Q$, and therefore by elementarity, $\mathrm{ot}(C_\delta) \in Q \cap \kappa = \beta$. So $\mathrm{ot}(C_\delta) < \beta$, which is a contradiction. $\square$

We now introduce an analogue of remainder points for ordinals between $\alpha_{M,N}$ and κ^+.

Definition 8.14. *Let M and N be in $\mathcal{X} \cup \mathcal{Y}$. Define $R^+_N(M)$ as the set of ordinals η such that either:*

1. *$\eta = \min((M \cap \kappa^+) \setminus \alpha_{M,N})$ and $\alpha_{M,N} < \eta$, or*
2. *$\eta = \min((M \cap \kappa^+) \setminus \xi)$, for some $\xi \in (N \cap \kappa^+) \setminus \alpha_{M,N}$.*

Lemma 8.15. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where $\{M, N\}$ is adequate if $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ if $N \in \mathcal{Y}$. Then:*

1. *$R^+_N(M)$ is finite;*
2. *if $\eta \in R^+_N(M)$, then $\mathrm{cf}(\eta) > \omega$;*
3. *suppose that $\eta \in R^+_N(M)$, η is not equal to $\min((M \cap \kappa^+) \setminus \alpha_{M,N})$, and $\sigma := \min((N \cap \kappa^+) \setminus \sup(M \cap \eta))$; then $\sigma \in R^+_M(N)$ and $\eta = \min((M \cap \kappa^+) \setminus \sigma)$.*

Proof. (1) If $R^+_N(M)$ is not finite, then the supremum of the first ω many members of $R^+_N(M)$ is a limit point of M and a limit point of N. Hence this supremum is less than or equal to $\alpha_{M,N}$ by Lemma 8.5, which contradicts the definition of $R^+_N(M)$.

(2) is easy. (3) Note that σ exists, since otherwise η would not be in $R^+_N(M)$. Clearly $\eta = \min((M \cap \kappa^+) \setminus \sigma)$. We will show that $\sigma \in R^+_M(N)$. Since η is not equal to $\min((M \cap \kappa^+) \setminus \alpha_{M,N})$, fix $\theta \in (M \cap \eta) \setminus \alpha_{M,N}$. Then $\alpha_{M,N} \leq \theta < \sigma$, and therefore $\alpha_{M,N} < \sigma$. If $\sigma = \min((N \cap \kappa^+) \setminus \alpha_{M,N})$, then $\sigma \in R^+_M(N)$ by definition. So assume not. Then we can fix $\xi \in (N \cap \sigma) \setminus \alpha_{M,N}$. By definition, $\zeta_0 := \min((M \cap \kappa^+) \setminus \xi)$ is in $R^+_N(M)$. Since $\xi < \sigma = \min((N \cap \kappa^+) \setminus \sup(M \cap \eta))$ and $\xi \in N$, it follows that $\xi < \sup(M \cap \eta)$. Therefore $\zeta_0 < \sup(M \cap \eta) \leq \sigma$.

Let ζ_1 be the largest member of $R^+_N(M)$ which is below σ. Then ζ_1 exists since $R^+_N(M)\cap\sigma$ is finite and nonempty, as witnessed by ζ_0. We claim that $\sigma = \min((N\cap\kappa^+)\setminus\zeta_1)$, which proves that $\sigma \in R^+_M(N)$. Otherwise $\sigma_0 := \min((N\cap\kappa^+)\setminus\zeta_1)$ is strictly below σ. So $\sigma_0 < \sigma = \min((N\cap\kappa^+)\setminus\sup(M\cap\eta))$, which implies that $\sigma_0 < \sup(M\cap\eta)$. But then $\min((M\cap\eta)\setminus\sigma_0)$ is in $R^+_N(M)\cap\sigma$, and is strictly larger than ζ_1, which contradicts the maximality of ζ_1. □

Lemma 8.16. *Let M and N be in $\mathcal{X}\cup\mathcal{Y}$. Then for all $\eta\in R^+_N(M)\cup R^+_M(N)$, η is closed under H^*.*

Proof. First consider $\eta = \min((N\cap\kappa^+)\setminus\alpha_{M,N})$. Let $n<\omega$ and let k be the arity of τ'_n, and we will show that η is closed under τ'_n. Since $\eta\in N$, by elementarity it suffices to show that N models that η is closed under τ'_n. Let $\alpha_0,\ldots,\alpha_{k-1}\in N\cap\eta$. By the minimality of η, $N\cap\eta\subseteq\alpha_{M,N}$, so $\alpha_0,\ldots,\alpha_{k-1}<\alpha_{M,N}$. By the elementarity of $M\cap N$ and since $\sup(M\cap N)=\alpha_{M,N}$, there is some $\gamma\in M\cap N\cap\kappa^+$ such that $\alpha_0,\ldots,\alpha_{k-1}$ are below γ and γ is closed under τ'_n. Then

$$\tau'_n(\alpha_0,\ldots,\alpha_{k-1}) < \gamma < \alpha_{M,N} \le \eta.$$

The same proof works for $\min((M\cap\kappa^+)\setminus\alpha_{M,N})$.

Now we prove the general statement by induction on ordinals in $R^+_N(M)\cup R^+_M(N)$. Suppose that $\eta\in R^+_N(M)$, and for all $\sigma\in(R^+_N(M)\cup R^+_M(N))\cap\eta$, σ is closed under H^*. If $\eta=\min((M\cap\kappa^+)\setminus\alpha_{M,N})$, then we are done by the previous paragraph. Otherwise by Lemma 8.15(3), the ordinal

$$\sigma := \min((N\cap\kappa^+)\setminus\sup(M\cap\eta))$$

is in $R^+_M(N)$, and $\eta=\min((M\cap\kappa^+)\setminus\sigma)$. By the inductive hypothesis, σ is closed under H^*.

Let $n<\omega$, and we will show that η is closed under τ'_n. Let k be the arity of τ'_n. Since η is in M, by elementarity it suffices to show that M models that η is closed under τ'_n. Let $\alpha_0,\ldots,\alpha_{k-1}\in M\cap\eta$. Then $\alpha_0,\ldots,\alpha_{k-1}$ are strictly less than $\sup(M\cap\eta)\le\sigma$. Since σ is closed under H^*,

$$\tau'_n(\alpha_0,\ldots,\alpha_{k-1}) < \sigma < \eta.$$

The same argument works for ordinals in $R^+_M(N)$. □

§9. Canonical models

In this section we will introduce some models which are determined by canonical parameters which arise in the comparison of two models. Specifically, we consider a simple model $N\in\mathcal{X}$ and a model M in $\mathcal{X}\cup\mathcal{Y}$ which is not necessarily a member of N. The canonical models associated with M and N are models in $N\cap\mathcal{Y}$ which reflect some information about M inside N. Canonical models will be used when amalgamating side conditions or forcing conditions over a simple model N; see Sections 13 and 15.[6]

The three types of canonical models are described in Notations 9.1, 9.3, and 9.13.

[6] The idea of a canonical model is new to this paper, and does not appear in Mitchell's original proof [12].

Notation 9.1. *Let $N \in \mathcal{X}$ be simple and $P \in \mathcal{Y}$, where $P \cap \kappa < \sup(N \cap \kappa)$. Let $\beta := P \cap \kappa$ and $\eta := \sup(N \cap P)$. We let $Q(N, P)$ denote the set $Sk(A_{\eta_N,\beta_N})$.*

Note that η_N exists by Lemma 8.13. It is easy to check that if $P \in N \cap \mathcal{Y}$, then $Q(N, P) = P$.

Lemma 9.2. *Let $N \in \mathcal{X}$ be simple and $P \in \mathcal{Y}$, where $P \cap \kappa < \sup(N \cap \kappa)$. Let $\beta := P \cap \kappa$ and $\eta := \sup(N \cap P)$. Then $Q := Q(N, P)$ satisfies the following properties:*

1. *$Q \in N \cap \mathcal{Y}$;*
2. *$Q \cap \kappa = \beta_N$, $Q \cap \kappa^+ = A_{\eta_N,\beta_N}$, and $\sup(Q) = \eta_N$;*
3. *$N \cap Q \cap \kappa^+ = N \cap P \cap \kappa^+$.*

Proof. Since η_N and β_N are in N, A_{η_N,β_N} and Q are in N by elementarity. As $N \cap P$ is closed under ordinal successors, η is a limit point of $N \cap P$. Therefore $P \cap \eta = A_{\eta,\beta}$ by Lemma 7.27. And since η is a limit point of $N \cap \kappa^+$, by Lemma 7.12, $C_\eta = C_{\eta_N} \cap \eta$, and for all $\xi < \kappa$,

$$A_{\eta,\xi} = A_{\eta_N,\xi} \cap \eta \text{ and } N \cap A_{\eta_N,\xi} = N \cap A_{\eta,\xi}.$$

We claim that

$$N \cap P \cap \kappa^+ = N \cap A_{\eta_N,\beta_N}.$$

Let $\alpha \in N \cap P \cap \kappa^+$, and we will show that $\alpha \in A_{\eta_N,\beta_N}$. Then $\alpha < \eta$ by the definition of η. So

$$\alpha \in P \cap \eta = A_{\eta,\beta} = A_{\eta_N,\beta} \cap \eta.$$

Hence $\alpha \in A_{\eta_N,\beta}$. Since $\beta \leq \beta_N$, $\alpha \in A_{\eta_N,\beta_N}$. Conversely, let $\alpha \in N \cap A_{\eta_N,\beta_N}$, and we will show that $\alpha \in P$. Since η_N, β_N, and α are in N and β_N is a limit ordinal, by elementarity we can fix $\xi \in N \cap \beta_N$ such that $\alpha \in A_{\eta_N,\xi}$. Then $\alpha \in N \cap A_{\eta_N,\xi} = N \cap A_{\eta,\xi}$. Since $\xi \in N \cap \beta_N$, $\xi < \beta$. So $A_{\eta,\xi} \subseteq A_{\eta,\beta} = P \cap \eta$. Hence $\alpha \in P$.

We have proven that $N \cap P \cap \kappa^+ = N \cap A_{\eta_N,\beta_N}$. By Lemma 7.10, it follows that A_{η_N,β_N} is closed under H^*. In particular, $Q \cap \kappa^+ = A_{\eta_N,\beta_N}$. Therefore

$$N \cap Q \cap \kappa^+ = N \cap A_{\eta_N,\beta_N} = N \cap P \cap \kappa^+,$$

which proves (3).

To show that $Q \cap \kappa = \beta_N$ and $\sup(Q) = \eta_N$, it suffices to prove that N models these statements. Let $\alpha \in N \cap Q \cap \kappa$, and we will show that $\alpha < \beta_N$. Then

$$\alpha \in N \cap Q \cap \kappa = N \cap P \cap \kappa \subseteq P \cap \kappa = \beta \leq \beta_N.$$

So $\alpha < \beta_N$. Conversely, let $\alpha \in N \cap \beta_N$, and we will show that $\alpha \in Q$. Then $\alpha \in N \cap \beta_N \subseteq \beta$. So

$$\alpha \in N \cap \beta = N \cap P \cap \kappa = N \cap Q \cap \kappa.$$

So indeed $\alpha \in Q$.

Since $Q \cap \kappa^+ = A_{\eta_N,\beta_N}$, clearly $\sup(Q) \leq \eta_N$. To show that N models that $\sup(Q) = \eta_N$, let $\xi \in N \cap \eta_N$. Then $\xi < \eta = \sup(N \cap P)$. So we can fix $\sigma \in N \cap P \cap \kappa^+$ which is larger than ξ. Then $\sigma \in N \cap P \cap \kappa^+ = N \cap Q \cap \kappa^+$. So $\sigma \in Q$ and $\xi \leq \sigma$. Thus $\sup(Q) = \eta_N$. This completes the proof of (2).

To show that $Q \in \mathcal{Y}$, it suffices to prove that $\lim(C_{\eta_N}) \cap Q$ is cofinal in η_N. Again it will be enough to show that N models this statement. So let $\xi \in N \cap \eta_N$. Then $\xi < \eta = \sup(N \cap P)$. Since $N \cap P \in \mathcal{X}$ by Lemma 7.16 and $\sup(N \cap P) = \eta$, there is $\sigma \in \lim(C_\eta) \cap (N \cap P)$ with $\xi \leq \sigma$. Since $C_\eta = C_{\eta_N} \cap \eta$, $\sigma \in \lim(C_{\eta_N})$. Also $\sigma \in N \cap P \cap \kappa^+ = N \cap Q \cap \kappa^+$. So $\sigma \in \lim(C_{\eta_N}) \cap Q$ and $\xi \leq \sigma$. $\square$

Notation 9.3. *Let M and N be in $\mathcal{X}$, where N is simple. Let $\zeta \in R_M(N)$ and $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$. We let $Q(N, M, \zeta)$ denote the set $Sk(A_{\eta,\zeta})$.*

Note that η exists by Lemma 8.12.

Lemma 9.4. *Let M and N be in $\mathcal{X}$, where N is simple. Suppose that $N \leq M$ and $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$. Let $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$. Then $Q := Q(N, M, \zeta)$ satisfies the following properties:*

1. *$Q \in N \cap \mathcal{Y}$;*
2. *$Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta,\zeta}$, and $\sup(Q) = \eta$;*
3. *$N \cap Q \cap \kappa^+ = M \cap N \cap \kappa^+$.*

Proof. Since η and ζ are in N, $A_{\eta,\zeta}$ and Q are in N by elementarity. As $\alpha_{M,N}$ is a limit point of N, by Lemma 7.12, $C_{\alpha_{M,N}} = C_\eta \cap \alpha_{M,N}$, and for all $\xi < \kappa$,

$$A_{\alpha_{M,N},\xi} = A_{\eta,\xi} \cap \alpha_{M,N} \text{ and } N \cap A_{\alpha_{M,N},\xi} = N \cap A_{\eta,\xi}.$$

We claim that

$$N \cap A_{\eta,\zeta} = M \cap N \cap \kappa^+.$$

Let $\alpha \in M \cap N \cap \kappa^+$, and we will show that $\alpha \in A_{\eta,\zeta}$. By Lemma 8.9, $M \cap N \cap \kappa^+ \subseteq A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}$. Since $\zeta \in R_M(N)$, $\sup(M \cap N \cap \kappa) < \beta_{M,N} \leq \zeta$. So

$$\alpha \in M \cap N \cap \kappa^+ \subseteq A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)} \subseteq A_{\alpha_{M,N},\zeta} \subseteq A_{\eta,\zeta}.$$

Hence $\alpha \in A_{\eta,\zeta}$.

Conversely, let $\alpha \in N \cap A_{\eta,\zeta}$, and we will show that $\alpha \in M$. Since α, η, and ζ are in N and ζ is a limit ordinal, by elementarity we can fix $\gamma \in N \cap \zeta$ such that $\alpha \in A_{\eta,\gamma}$. Since $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$ and $N \leq M$, $\gamma \in N \cap \beta_{M,N} = M \cap N \cap \kappa$. So

$$\alpha \in N \cap A_{\eta,\gamma} = N \cap A_{\alpha_{M,N},\gamma} \subseteq N \cap A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}.$$

By Lemma 8.8(1), $\alpha \in M$.

We have proven that $N \cap A_{\eta,\zeta} = M \cap N \cap \kappa^+$. By Lemma 7.10, it follows that $A_{\eta,\zeta}$ is closed under H^*. In particular, $Q \cap \kappa^+ = A_{\eta,\zeta}$. Therefore

$$N \cap Q \cap \kappa^+ = N \cap A_{\eta,\zeta} = M \cap N \cap \kappa^+,$$

which proves (3).

To show that $Q \cap \kappa = \zeta$ and $\sup(Q) = \eta$, it suffices to show that N models these statements. Let $\gamma \in N \cap Q \cap \kappa$, and we will show that $\gamma < \zeta$. Then $\gamma \in N \cap Q \cap \kappa = M \cap N \cap \kappa$, so

$$\gamma < \sup(M \cap N \cap \kappa) < \beta_{M,N} \leq \zeta.$$

Conversely, let $\gamma \in N \cap \zeta$, and we will show that $\gamma \in Q$. Since $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$, $\gamma \in N \cap \beta_{M,N}$. As $N \leq M$, $N \cap \beta_{M,N} \subseteq M$, so $\gamma \in M$. Hence $\gamma \in M \cap N \cap \kappa^+ \subseteq Q$.

Since $Q \cap \kappa^+ = A_{\eta,\zeta}$, obviously $\sup(Q) \leq \eta$. To show that N models that $\sup(Q) = \eta$, let $\xi \in N \cap \eta$ be given. Since $\eta = \min((N \cap \kappa^+) \setminus \alpha_{M,N})$, $\xi < \alpha_{M,N}$. As $\alpha_{M,N} = \sup(M \cap N \cap \kappa^+)$, we can fix $\sigma \in M \cap N \cap \kappa^+$ with $\xi \leq \sigma$. Then $\sigma \in M \cap N \cap \kappa^+ \subseteq Q$. So $\xi \leq \sigma$ and $\sigma \in Q$. This completes the proof of (2).

To show that $Q \in \mathcal{Y}$, it suffices to show that N models that $\lim(C_\eta) \cap Q$ is cofinal in η. Let $\xi \in N \cap \eta$. Then $\xi < \alpha_{M,N}$. Since $M \cap N$ is in $\mathcal{X}$ and $\sup(M \cap N) = \alpha_{M,N}$, we can fix $\sigma \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ with $\xi \leq \sigma$. But $C_{\alpha_{M,N}} = C_\eta \cap \alpha_{M,N}$, so $\sigma \in \lim(C_\eta)$. Also $\sigma \in M \cap N \cap \kappa^+ \subseteq Q$. So $\sigma \in \lim(C_\eta) \cap Q$ and $\xi \leq \sigma$. $\square$

Notation 9.5. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate. Let $\zeta \in R_N(M)$. We let $Q_0(M, N, \zeta)$ denote the set $Sk(A_{\alpha_{M,N},\zeta})$.*

Lemma 9.6. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate. Let $\eta \in M \cap N \cap \kappa^+$ with $\kappa \leq \eta$. Fix $m < \omega$, and let k be the arity of τ'_m. Define $f_{m,\eta} : \kappa \to \kappa$ by letting $f_{m,\eta}(\beta)$ be the least $\beta' < \kappa$ such that $\beta \in A_{\eta,\beta'}$ and*

$$\eta \cap \tau'_m[A^k_{\eta,\beta}] \subseteq A_{\eta,\beta'}.$$

Then for all $\sigma \in R_N(M) \cup R_M(N)$, σ is closed under $f_{m,\eta}$.

Note that since $A_{\eta,\beta}$ has size less than κ by Notation 7.4(3), the set $\tau'_m[A^k_{\eta,\beta}]$ also has size less than κ. So the definition of $f_{m,\eta}$ makes sense. Also note that $f_{m,\eta}$ is definable from η in $\mathcal{A}$.

Proof. The proof is by induction on remainder points in $R_M(N) \cup R_N(M)$. For the base case, let σ be the first ordinal in $R_M(N) \cup R_N(M)$. Without loss of generality, assume that $\sigma \in R_N(M)$. Since $\eta \in M$ and $f_{m,\eta}$ is definable from η in $\mathcal{A}$, it suffices to show that M models that σ is closed under $f_{m,\eta}$. So let $\beta \in M \cap \sigma$, and we will show that $f_{m,\eta}(\beta) < \sigma$.

Since $\sigma \in R_N(M)$ and σ is the first remainder point, we have that $M \leq N$ and $\sigma = \min((M \cap \kappa) \setminus \beta_{M,N})$. As $\beta \in M \cap \sigma = M \cap \beta_{M,N}$ and $M \leq N$, $\beta \in M \cap \beta_{M,N} \subseteq N$. So $\beta \in N$. Therefore η and β are both in $M \cap N$. By elementarity, $f_{m,\eta}(\beta) \in M \cap N \cap \kappa$. But $M \cap N \cap \kappa = M \cap \beta_{M,N} \subseteq \sigma$. So $f_{m,\eta}(\beta) < \sigma$.

Now suppose that ζ is a remainder point which is greater than the least remainder point, and assume that the lemma holds for all remainder points in $R_M(N) \cup R_N(M)$ which are below ζ. Without loss of generality, assume that $\zeta \in R_N(M)$. Then by Lemma 2.2(3), $\sigma := \min((N \cap \kappa) \setminus \sup(M \cap \zeta))$ is in $R_M(N)$, and $\zeta = \min((M \cap \kappa) \setminus \sigma)$. To show that M models that ζ is closed under $f_{m,\eta}$, let $\beta \in M \cap \zeta$. Then $\beta < \sup(M \cap \zeta) < \sigma$. By the inductive hypothesis, σ is closed under $f_{m,\eta}$. So $f_{m,\eta}(\beta) < \sigma$. Since $\sigma < \zeta$, $f_{m,\eta}(\beta) < \zeta$. $\square$

Lemma 9.7. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate. Let $\sigma \in R_N(M)$. Then $Q_0 := Q_0(M, N, \sigma)$ satisfies the following properties:*

1. *$Q_0 \in \mathcal{Y}$;*
2. *$Q_0 \cap \kappa = \sigma$, $Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}$, and $\sup(Q_0) = \alpha_{M,N}$;*
3. *$M \cap N \cap \kappa^+ \subseteq Q_0$.*

Proof. Recall that $Q_0 = Q_0(M, N, \sigma) = Sk(A_{\alpha_{M,N},\sigma})$. We begin by proving that $A_{\alpha_{M,N},\sigma}$ is closed under H^*. Let $m < \omega$, and let k be the arity of τ'_m. Let $\alpha_0, \ldots, \alpha_{k-1} \in A_{\alpha_{M,N},\sigma}$, and we will show that $\tau'_m(\alpha_0, \ldots, \alpha_{k-1}) \in A_{\alpha_{M,N},\sigma}$. Since σ is a limit ordinal, we can fix $\beta < \sigma$ such that $\alpha_0, \ldots, \alpha_{k-1} \in A_{\alpha_{M,N},\beta}$.

By the elementarity of $M \cap N$, fix $\delta \in M \cap N \cap \kappa^+$ strictly greater than $\alpha_0, \ldots, \alpha_{k-1}$ such that δ is closed under τ'_m. Now fix $\eta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ strictly greater than δ and κ. Since δ is closed under τ'_m,

$$\tau'_m(\alpha_0, \ldots, \alpha_{k-1}) < \delta < \eta.$$

As $\eta \in \lim(C_{\alpha_{M,N}})$,

$$\alpha_0, \ldots, \alpha_{k-1} \in A_{\alpha_{M,N},\beta} \cap \eta = A_{\eta,\beta}.$$

So

$$\tau'_m(\alpha_0, \dots, \alpha_{k-1}) \in \eta \cap \tau'_m[A^k_{\eta,\beta}].$$

As $\beta < \sigma$ and $\sigma \in R_N(M)$, by Lemma 9.6 there is $\beta' < \sigma$ such that

$$\eta \cap \tau'_m[A^k_{\eta,\beta}] \subseteq A_{\eta,\beta'}.$$

Then

$$\tau'_m(\alpha_0, \dots, \alpha_{k-1}) \in A_{\eta,\beta'} \subseteq A_{\eta,\sigma} = A_{\alpha_{M,N},\sigma} \cap \eta \subseteq A_{\alpha_{M,N},\sigma}.$$

This proves that $A_{\alpha_{M,N},\sigma}$ is closed under H^*. In particular, $Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}$. Since $M \cap N \cap \kappa^+ \subseteq A_{\alpha_{M,N},\sup(M\cap N\cap\kappa)}$ by Lemma 8.9, and $\sup(M \cap N \cap \kappa) < \sigma$, it follows that $M \cap N \cap \kappa^+ \subseteq A_{\alpha_{M,N},\sigma} \subseteq Q_0$. In particular, since $\sup(M \cap N) = \alpha_{M,N}$, it follows that $\sup(Q_0) = \alpha_{M,N}$.

It remains to show that $Q_0 \in \mathcal{Y}$ and $Q_0 \cap \kappa = \sigma$. For the first statement, once we know that $Q_0 \cap \kappa = \sigma$, it will suffice to show that $\lim(C_{\alpha_{M,N}}) \cap Q_0$ is cofinal in $\alpha_{M,N}$. But since $M \cap N \in \mathcal{X}$, $\lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ is cofinal in $\sup(M \cap N) = \alpha_{M,N}$. And as $M \cap N \cap \kappa^+ \subseteq Q_0$, it follows that $\lim(C_{\alpha_{M,N}}) \cap Q_0$ is cofinal in $\alpha_{M,N}$.

Now we prove that $Q_0 \cap \kappa = \sigma$. First we will show that $Q_0 \cap \kappa \subseteq \sigma$. More generally, we will prove by induction on remainder points that

$$\forall \zeta \in R_M(N) \cup R_N(M),\ A_{\alpha_{M,N},\zeta} \cap \kappa \subseteq \zeta.$$

Consider the first remainder point ζ. Without loss of generality, assume that $\zeta \in R_N(M)$. Then $M \leq N$ and $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$. Let $\beta \in A_{\alpha_{M,N},\zeta} \cap \kappa$, and we will show that $\beta < \zeta$. Fix $\eta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ with $\beta < \eta$. Then $\beta \in A_{\alpha_{M,N},\zeta} \cap \eta = A_{\eta,\zeta}$. To show that $\beta < \zeta$, it suffices to show that $A_{\eta,\zeta} \cap \kappa \subseteq \zeta$. Since η and ζ are in M, it is enough to show that M models that $A_{\eta,\zeta} \cap \kappa \subseteq \zeta$.

Let $\beta' \in M \cap A_{\eta,\zeta} \cap \kappa$, and we will show that $\beta' < \zeta$. Since ζ is a limit ordinal, by elementarity we can fix $\gamma \in M \cap \zeta$ with $\beta' \in A_{\eta,\gamma}$. Since $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$ and $M \leq N$, $\gamma \in M \cap \beta_{M,N} \subseteq N$. As $M \leq N$, $\beta' \in M \cap A_{\eta,\gamma}$, $\eta \in N$, and $\gamma \in M \cap N \cap \kappa$, it follows that $\beta' \in N$ by Lemma 8.6. Hence

$$\beta' \in M \cap N \cap \kappa \subseteq \beta_{M,N} \leq \zeta.$$

So $\beta' < \zeta$.

For the inductive step, let ζ be a remainder point which is not the first remainder point. Without loss of generality, assume that $\zeta \in R_N(M)$. Then by Lemma 2.2(3), there is $\pi \in R_M(N)$ such that $\pi = \min((N \cap \kappa) \setminus \sup(M \cap \zeta))$ and $\zeta = \min((M \cap \kappa) \setminus \pi)$. Let $\beta \in A_{\alpha_{M,N},\zeta} \cap \kappa$, and we will show that $\beta < \zeta$. Fix $\eta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ with $\beta < \eta$. Then

$$\beta \in A_{\alpha_{M,N},\zeta} \cap \eta = A_{\eta,\zeta}.$$

To show that $\beta < \zeta$, it suffices to show that $A_{\eta,\zeta} \cap \kappa \subseteq \zeta$. Since η and ζ are in M, by elementarity it suffices to show that M models that $A_{\eta,\zeta} \cap \kappa \subseteq \zeta$. So let $\gamma \in M \cap A_{\eta,\zeta} \cap \kappa$, and we will show that $\gamma < \zeta$. Since ζ is a limit ordinal, by elementarity we can fix $\alpha \in M \cap \zeta$ such that $\gamma \in A_{\eta,\alpha}$. Then $\alpha < \sup(M \cap \zeta) < \pi$. So $\gamma \in A_{\eta,\alpha} \subseteq A_{\eta,\pi}$. Since $\eta \in M \cap N$ and $\pi \in R_M(N)$, the inductive hypothesis implies that

$$A_{\eta,\pi} \cap \kappa = A_{\alpha_{M,N},\pi} \cap \kappa \subseteq \pi.$$

So $\gamma < \pi < \zeta$.

This completes the induction. In particular,

$$Q_0 \cap \kappa = A_{\alpha_{M,N},\sigma} \cap \kappa \subseteq \sigma.$$

Conversely, let $\beta < \sigma$, and we will show that $\beta \in Q_0$. Fix $\eta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ with $\kappa \leq \eta$. Then by Lemma 9.6, there is $\beta' < \sigma$ such that $\beta \in A_{\eta,\beta'}$. So

$$\beta \in A_{\eta,\beta'} = A_{\alpha_{M,N},\beta'} \cap \eta \subseteq A_{\alpha_{M,N},\beta'} \subseteq A_{\alpha_{M,N},\sigma} \subseteq Q_0.$$

□

Lemma 9.8. *Let M and N be in $\mathcal{X}$, where $\{M, N\}$ is adequate and N is simple. Suppose that $\sigma \in R_N(M)$, $\zeta \in R_M(N)$, and $\zeta = \min((N \cap \kappa) \setminus \sigma)$. Let $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$. Let $Q_0 := Q_0(M, N, \sigma)$ and $Q := Q(N, M, \zeta)$. Then:*

1. *$Q \in N \cap \mathcal{Y}$;*
2. *$Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta,\zeta}$, and $\sup(Q) = \eta$;*
3. *$N \cap Q \cap \kappa^+ = N \cap Q_0 \cap \kappa^+$;*
4. *$M \cap N \cap \kappa^+ \subseteq Q$.*

Proof. We will apply Lemma 9.2 to the models N and Q_0. Let us check that the assumptions of this lemma hold, using Lemma 9.7. We know that $N \in \mathcal{X}$ is simple, $Q_0 \in \mathcal{Y}$, and

$$Q_0 \cap \kappa = \sigma < \zeta < \sup(N \cap \kappa).$$

Also, $\sup(N \cap Q_0) = \alpha_{M,N}$, since $\sup(Q_0) = \alpha_{M,N}$, $\sup(M \cap N) = \alpha_{M,N}$, and $M \cap N \cap \kappa^+ \subseteq N \cap Q_0$. Moreover,

$$\min((N \cap \kappa) \setminus (Q_0 \cap \kappa)) = \min((N \cap \kappa) \setminus \sigma) = \zeta,$$

and

$$\min((N \cap \kappa^+) \setminus \sup(N \cap Q_0)) = \min((N \cap \kappa^+) \setminus \alpha_{M,N}) = \eta.$$

By Notations 9.1 and 9.3,

$$Q(N, Q_0) = Sk(A_{\eta,\zeta}) = Q(N, M, \zeta) = Q.$$

By Lemma 9.2, we have that:

(a) $Q \in N \cap \mathcal{Y}$;
(b) $Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta,\zeta}$, and $\sup(Q) = \eta$;
(c) $N \cap Q \cap \kappa^+ = N \cap Q_0 \cap \kappa^+$.

This proves (1), (2), and (3). By Lemma 9.7(3), $M \cap N \cap \kappa^+ \subseteq Q_0$. So $M \cap N \cap \kappa^+ \subseteq N \cap Q_0 \cap \kappa^+ = N \cap Q \cap \kappa^+ \subseteq Q$, which proves (4). □

The next lemma summarizes Lemmas 9.4 and 9.8.

Lemma 9.9. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate and N is simple. Let $\zeta \in R_M(N)$, $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$, and $Q := Q(N, M, \zeta)$. Then:*

1. *$Q \in N \cap \mathcal{Y}$;*
2. *$Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta,\zeta}$, and $\sup(Q) = \eta$;*
3. *$M \cap N \cap \kappa^+ \subseteq Q$;*
4. *if $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$, then $N \cap Q \cap \kappa^+ = M \cap N \cap \kappa^+$;*
5. *if $\zeta = \min((N \cap \kappa) \setminus \sigma)$, where $\sigma \in R_N(M)$, then $N \cap Q \cap \kappa^+ = N \cap Q_0(M, N, \sigma) \cap \kappa^+$.*

Proof. Immediate from Lemmas 9.4 and 9.8. □

Let us derive some additional information about the model $Q(N, M, \zeta)$.

Lemma 9.10. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate and N is simple. Let $\zeta \in R_M(N)$, $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$, and $Q := Q(N, M, \zeta)$. Then:*

1. *if $P \in M \cap \mathcal{Y}$ and $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, then $N \cap P \cap \alpha_{M,N} \subseteq Q$;*

(2) *if* $N \leq M$, $P \in M \cap \mathcal{Y}$, *and* $P \cap \kappa < \sup(M \cap N \cap \kappa)$, *then* $N \cap P \cap \kappa^+ \subseteq Q$;
(3) *if* $M < N$ *and* $P \in M \cap N \cap \mathcal{Y}$, *then* $N \cap P \cap \kappa^+ \subseteq Q$.

Proof. Note that since $\alpha_{M,N}$ is a limit point of N, for all $\xi < \kappa$, $A_{\alpha_{M,N},\xi} = A_{\eta,\xi} \cap \alpha_{M,N}$ by Lemma 7.12.

(1) Suppose that $P \in M \cap \mathcal{Y}$ and $\sup(N \cap \zeta) < P \cap \kappa < \zeta$. Since $\beta_{M,N} \leq \zeta$,

$$M \cap N \cap \kappa = M \cap N \cap \beta_{M,N} \subseteq N \cap \zeta.$$

So $\sup(M \cap N \cap \kappa) \leq \sup(N \cap \zeta) < P \cap \kappa$. By Lemma 8.10,

$$P \cap N \cap \alpha_{M,N} \subseteq A_{\alpha_{M,N},P\cap\kappa} \subseteq A_{\alpha_{M,N},\zeta} = A_{\eta,\zeta} \cap \alpha_{M,N} \subseteq Q.$$

(2) If $N \leq M$, $P \in M \cap \mathcal{Y}$, and $P \cap \kappa < \sup(M \cap N \cap \kappa)$, then $N \cap P \cap \kappa^+ \subseteq M$ by Lemma 8.7. Hence

$$N \cap P \cap \kappa^+ \subseteq M \cap N \cap \kappa^+ \subseteq Q$$

by Lemma 9.9(3).

(3) Suppose that $M < N$ and $P \in M \cap N \cap \mathcal{Y}$. Then $\sup(P)$ and $P \cap \kappa$ are in $M \cap N \cap \kappa^+$ by elementarity, and hence in Q by Lemma 9.9(3). So $A_{\sup(P),P\cap\kappa} = P \cap \kappa^+ \in Q$ by elementarity. So $P \cap \kappa^+ \subseteq Q$. In particular, $N \cap P \cap \kappa^+ \subseteq Q$. $\square$

Finally, we consider canonical models determined by ordinals in $R^+_N(M)$.

Notation 9.11. *Let M and N be in $\mathcal{X}$, where $\{M, N\}$ is adequate and N is simple. Let $\zeta \in R_M(N)$ and $\sigma \in R^+_N(M)$. Let X be any nonempty set of $P \in M \cap \mathcal{Y}$ such that* $\sup(N \cap \zeta) < P \cap \kappa < \zeta$ *and* $P \cap N \cap [\sup(M \cap \sigma), \sigma) \neq \emptyset$. *We let P_X denote the set $Sk(\bigcup\{P \cap \sigma : P \in X\})$ and β_X denote the ordinal $P_X \cap \kappa$.*

Lemma 9.12. *Under the assumptions of Notation 9.11, the following statements hold:*

(1) $P_X \in \mathcal{Y}$;
(2) $\beta_X = \sup\{P \cap \kappa : P \in X\} < \zeta$;
(3) $P_X \cap \kappa^+ = \bigcup\{P \cap \sigma : P \in X\}$;
(4) $\sup(P_X) = \sigma$.

Proof. Let $P \in X$. Since $\sigma \in R^+_N(M)$, $\sigma \in M$ and σ has uncountable cofinality. Also $P \in M$ and $P \cap [\sup(M \cap \sigma), \sigma) \neq \emptyset$, which imply that σ is a limit point of P by Lemma 7.30. It follows that if P_1 and P_2 are in X and $P_1 \cap \kappa \leq P_2 \cap \kappa$, then $P_1 \cap \sigma \subseteq P_2 \cap \sigma$ by Lemma 7.28. Thus $\{P \cap \sigma : P \in X\}$ is a subset increasing sequence. Since each $P \cap \sigma$ is closed under H^* by Lemma 8.16, the set $\bigcup\{P \cap \sigma : P \in X\}$ is closed under H^*. Hence

$$P_X \cap \kappa^+ = Sk(\bigcup\{P \cap \sigma : P \in X\}) \cap \kappa^+ = \bigcup\{P \cap \sigma : P \in X\},$$

which proves (3).

Since σ is a limit point of $P \cap \sigma$ for each $P \in X$, obviously σ is a limit point of P_X. But $P_X \cap \kappa^+ \subseteq \sigma$, so $\sup(P_X) = \sigma$, which proves (4). Clearly

$$\beta_X = P_X \cap \kappa = \sup\{P \cap \kappa : P \in X\},$$

which is in κ. Since $P \cap \kappa < \zeta$ for all $P \in X$, it follows that $P_X \cap \kappa \leq \zeta$. But $P \cap \kappa \in M \cap \kappa$ for all $P \in X$. Therefore $P_X \cap \kappa \in \mathrm{cl}(M \cap \kappa)$, which implies that $\beta_X = P_X \cap \kappa < \zeta$ by Lemma 2.2(1), which proves (2).

To show that $P_X \in \mathcal{Y}$, it suffices to show that $\lim(C_\sigma) \cap P_X$ is cofinal in σ. Fix $P \in X$. Then it will suffice to show that $\lim(C_\sigma) \cap P$ is cofinal in σ, since this set is a subset of P_X. First, assume that $\sigma \notin P$. Then $\sigma \in \mathrm{cl}(P) \setminus P$, which implies by Lemma 7.13 that $\lim(C_\sigma) \cap P$ is cofinal in σ. Secondly, assume that $\sigma \in P$. Since σ is a limit point of P and $|P| < \kappa$, $\mathrm{cf}(\sigma) < \kappa$. So $\mathrm{ot}(C_\sigma) < \kappa$. Hence $\mathrm{ot}(C_\sigma) \in P \cap \kappa$ and $P \cap \kappa \in \kappa$, which implies that $C_\sigma \subseteq P$. As σ has uncountable cofinality, clearly $\lim(C_\sigma)$ is cofinal in σ. So $\lim(C_\sigma) \cap P$ is cofinal in σ. $\square$

Notation 9.13. *Under the assumptions of Notation 9.11, we let*

$$Q(N, M, \zeta, \sigma, X) := Sk(A_{\eta_N, \zeta}),$$

where $\eta := \sup(N \cap P_X)$.

Note that $P_X \in \mathcal{Y}$ and $\beta_X = P_X \cap \kappa < \zeta < \sup(N \cap \kappa)$ imply by Lemma 8.13 that η_N exists. Also since $\zeta = \min((N \cap \kappa) \setminus \beta_X)$, $Q(N, M, \zeta, \sigma, X)$ is equal to $Q(N, P_X)$ from Notation 9.1.

Lemma 9.14. *Let M and N be in $\mathcal{X}$, where $\{M, N\}$ is adequate and N is simple. Let $\zeta \in R_M(N)$ and $\sigma \in R^+_N(M)$.*

Let X be any nonempty set of $P \in M \cap \mathcal{Y}$ such that $\sup(N \cap \zeta) < P \cap \kappa < \zeta$ *and* $P \cap N \cap [\sup(M \cap \sigma), \sigma) \neq \emptyset$. *Let* $\eta := \sup(N \cap P_X)$ *and* $Q := Q(N, M, \zeta, \sigma, X)$. *Then:*

1. $Q \in N \cap \mathcal{Y}$;
2. $Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta_N, \zeta}$, *and* $\sup(Q) = \eta_N$;
3. $N \cap Q \cap \kappa^+ = N \cap P_X \cap \kappa^+$;
4. *for all* $P \in X$, $N \cap P \cap \sigma \subseteq Q$.

Proof. As noted above, $Q = Q(N, P_X)$. Also $\eta = \sup(N \cap P_X)$ and $\zeta = \min((N \cap \kappa) \setminus (P_X \cap \kappa))$. By Lemma 9.2:

(a) $Q \in N \cap \mathcal{Y}$;
(b) $Q \cap \kappa = \zeta$, $Q \cap \kappa^+ = A_{\eta_N, \zeta}$, and $\sup(Q) = \eta_N$;
(c) $N \cap P_X \cap \kappa^+ = N \cap Q \cap \kappa^+$.

This proves (1), (2), and (3). In particular, if $P \in X$, then

$$N \cap P \cap \sigma \subseteq N \cap P_X \cap \kappa^+ \subseteq Q,$$

which proves (4). $\square$

§10. Closure under canonical models

Fix a sequence $\langle S_\eta : \eta < \kappa^+ \rangle$, where each S_η is a subset of $\kappa \cap \mathrm{cof}(>\omega)$. Let us assume that the structure $\mathcal{A}$ from Notation 7.6 includes $\vec{S}$ as a predicate. In this section we will show that we can add canonical models to an $\vec{S}$-obedient side condition and preserve $\vec{S}$-obediency.

As stated in the comments prior to Definition 5.2, the definitions of $\vec{S}$-adequate and $\vec{S}$-obedient are made relative to a subclass of $\mathcal{Y}_0$. For the remainder of the paper, this subclass will be the set $\mathcal{Y}$ from Notation 7.8.

Lemma 10.1. *Let (A, B) be an $\vec{S}$-obedient side condition. Suppose that $N \in A$ is simple. Let $P \in B$ be such that $P \cap \kappa < \sup(N \cap \kappa)$. Let $Q := Q(N, P)$. Then $(A, B \cup \{Q\})$ is an $\vec{S}$-obedient side condition.*

See Notation 9.1 for the definition of $Q(N,P)$.

Proof. Let $\beta := P \cap \kappa$. By Lemma 9.2,

$$Q \in N \cap \mathcal{Y},\ Q \cap \kappa = \beta_N,\ \text{and}\ N \cap Q \cap \kappa^+ = N \cap P \cap \kappa^+.$$

Let us show that Q is $\vec{S}$-strong. Since $Q \in N$, it suffices to show that N models that Q is $\vec{S}$-strong. Let $\tau \in N \cap Q \cap \kappa^+$, and we will show that $Q \cap \kappa = \beta_N \in S_\tau$. But $\tau \in N \cap Q \cap \kappa^+ = N \cap P \cap \kappa^+$. Since $N \in A$, $P \in B$, and (A,B) is $\vec{S}$-obedient, it follows that $\beta_N \in S_\tau$.

Let $M \in A$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \beta_N)$. Fix $\tau \in M \cap Q \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. If $\zeta = \beta_N$, then $\zeta \in S_\tau$ because Q is $\vec{S}$-strong. Assume that $\beta_N < \zeta$, which means that $\beta_N \notin M$.

First assume that $\zeta \in R_N(M)$. Then since $Q \in N \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(M \cap \zeta) < Q \cap \kappa = \beta_N < \zeta$, it follows that $\zeta \in S_\tau$ as A is $\vec{S}$-adequate. In particular, if $\beta_{M,N} \leq \beta_N$, then $\zeta \in R_N(M)$. Suppose that $\beta_N < \beta_{M,N} \leq \zeta$. Then $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$. Since $\beta_N \in (N \cap \beta_{M,N}) \setminus M$, we have that $M < N$. So $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$ is in $R_N(M)$.

The remaining case is that $\zeta < \beta_{M,N}$. Then since $\beta_N \in (N \cap \beta_{M,N}) \setminus M$, it follows that $M < N$. So

$$M \cap \zeta \subseteq M \cap \beta_{M,N} \subseteq N.$$

As $\tau \in M \cap Q \cap \kappa^+$, $Q \in N \cap \mathcal{Y}$, and

$$Q \cap \kappa < \zeta < \sup(M \cap \beta_{M,N}) = \sup(M \cap N \cap \kappa),$$

it follows that $\tau \in N$ by Lemma 8.7. So $\tau \in N \cap Q \cap \kappa^+ = N \cap P \cap \kappa^+$. Since $M \cap \zeta \subseteq N$ and $\zeta = \min((M \cap \kappa) \setminus \beta_N)$, we have that

$$\zeta = \min((M \cap \kappa) \setminus \beta) = \min((M \cap \kappa) \setminus (P \cap \kappa)).$$

Since $M \in A$, $P \in B$, and $\tau \in M \cap P \cap \kappa^+$, it follows that $\zeta \in S_\tau$ as (A,B) is $\vec{S}$-obedient. □

Lemma 10.2. *Let (A,B) be an $\vec{S}$-obedient side condition. Let $N \in A$ be simple and $M \in A$. Suppose that $N \leq M$ and $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$. Let $Q := Q(N,M,\zeta)$. Then $(A, B \cup \{Q\})$ is an $\vec{S}$-obedient side condition.*

See Notation 9.3 for the definition of $Q(N,M,\zeta)$.

Proof. By Lemma 9.4,

$$Q \in N \cap \mathcal{Y},\ Q \cap \kappa = \zeta,\ \text{and}\ N \cap Q \cap \kappa^+ = M \cap N \cap \kappa^+.$$

First we show that Q is $\vec{S}$-strong. Since $Q \in N$, it suffices to show that N models that Q is $\vec{S}$-strong. Let $\tau \in N \cap Q \cap \kappa^+$, and we will show that $Q \cap \kappa = \zeta$ is in S_τ. Then $\tau \in N \cap Q \cap \kappa^+ = M \cap N \cap \kappa^+$. So $\tau \in M \cap N$. Since $\zeta \in R_M(N)$, $\zeta \in S_\tau$ as A is $\vec{S}$-adequate.

Now let $K \in A$, and suppose that $\theta = \min((K \cap \kappa) \setminus \zeta)$. Fix $\tau \in K \cap Q \cap \kappa^+$, and we will show that $\theta \in S_\tau$. If $\zeta = \theta$, then $\theta \in S_\tau$ since Q is $\vec{S}$-strong. So assume that $\zeta < \theta$, which means that $\zeta \notin K$.

Suppose first that $\theta \in R_N(K)$. Then since $Q \in N \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(K \cap \theta) < Q \cap \kappa = \zeta < \theta$, it follows that $\theta \in S_\tau$ as A is $\vec{S}$-adequate. In particular, if $\beta_{K,N} \leq \zeta$, then $\theta \in R_N(K)$. Suppose that $\zeta < \beta_{K,N} \leq \theta$. Then $\theta = \min((K \cap \kappa) \setminus$

$\beta_{K,N}$). Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, we have that $K < N$. So $\theta = \min((K \cap \kappa) \setminus \beta_{K,N})$ is in $R_N(K)$.

The remaining case is that $\theta < \beta_{K,N}$. We apply Lemma 2.7. We have that $\{K, M, N\}$ is adequate, $\zeta \in R_M(N)$, $\zeta \notin K$, $\theta = \min((K \cap \kappa) \setminus \zeta)$, and $\theta < \beta_{K,N}$. By Lemma 2.7, $\theta \in R_M(K)$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, it follows that $K < N$. As $Q \in N \cap \mathcal{Y}$, $K < N$, $\tau \in K \cap Q$, and

$$Q \cap \kappa = \zeta < \theta < \sup(K \cap \beta_{K,N}) = \sup(K \cap N \cap \kappa),$$

it follows that $\tau \in N$ by Lemma 8.7. So $\tau \in N \cap Q \cap \kappa^+ = M \cap N \cap \kappa^+$. Hence $\tau \in K \cap M$. Since $\theta \in R_M(K)$, it follows that $\theta \in S_\tau$ as A is $\vec{S}$-adequate. $\square$

Lemma 10.3. *Let M and N be in $\mathcal{X}$ such that $\{M, N\}$ is adequate and N is simple. Assume that $M \prec (\mathcal{A}, \mathcal{Y})$. Let $\sigma \in R_N(M)$ and $\zeta \in R_M(N)$. Then $Q_0(M, N, \sigma)$ and $Q(N, M, \zeta)$ are $\vec{S}$-strong.*

Recall that $(\mathcal{A}, \mathcal{Y})$ is the structure $\mathcal{A}$ augmented with the additional predicate $\mathcal{Y}$.

See Notations 9.3 and 9.5 for the definitions of $Q(N, M, \zeta)$ and $Q_0(M, N, \sigma)$.

Proof. The proof is by induction on remainder points in $R_M(N) \cup R_N(M)$. First consider $\zeta \in R_M(N)$. If $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$, then $Q(N, M, \zeta)$ is $\vec{S}$-strong by Lemma 10.2. So assume that $\zeta = \min((N \cap \kappa) \setminus \sigma)$, for some $\sigma \in R_N(M)$.

Let $Q := Q(N, M, \zeta)$ and $Q_0 := Q_0(M, N, \sigma)$. By the inductive hypothesis, Q_0 is $\vec{S}$-strong. And by Lemma 9.7,

$$Q_0 \cap \kappa = \sigma \text{ and } Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}.$$

To show that Q is $\vec{S}$-strong, it suffices to prove that N models that Q is $\vec{S}$-strong. Let $\tau \in N \cap Q \cap \kappa^+$, and we will show that $Q \cap \kappa \in S_\tau$. By Lemma 9.8,

$$Q \cap \kappa = \zeta \text{ and } Q \cap \kappa^+ = A_{\eta,\zeta},$$

where $\eta := \min((N \cap \kappa^+) \setminus \alpha_{M,N})$, and

$$N \cap Q \cap \kappa^+ = N \cap Q_0 \cap \kappa^+.$$

In particular, $\tau \in N \cap Q_0$. Also

$$\tau \in N \cap A_{\eta,\zeta} \subseteq \alpha_{M,N},$$

so $\tau < \alpha_{M,N}$.

Fix $\theta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ greater then τ. Then

$$\tau \in Q_0 \cap \theta = A_{\alpha_{M,N},\sigma} \cap \theta = A_{\theta,\sigma}.$$

Since θ and σ are in M, Q_0 is $\vec{S}$-strong, $Q_0 \cap \kappa = \sigma$, and $A_{\theta,\sigma} \subseteq Q_0$, by the elementarity of M we can fix an $\vec{S}$-strong model $P \in M \cap \mathcal{Y}$ such that $P \cap \kappa = \sigma$ and $A_{\theta,\sigma} \subseteq P$. Then $\tau \in N \cap P$. Since $\zeta \in R_M(N)$ and $\sup(N \cap \zeta) < \sigma = P \cap \kappa < \zeta$, it follows that $\zeta \in S_\tau$ as A is $\vec{S}$-adequate.

Now consider $\sigma \in R_N(M)$, and we will show that $Q_0 := Q_0(M, N, \sigma)$ is $\vec{S}$-strong. We first claim that for all $\theta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ and for all $\tau \in A_{\theta,\sigma}$, $\sigma \in S_\tau$. So fix $\theta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$. Since θ and σ are in M, it suffices to prove that M models that for all $\tau \in A_{\theta,\sigma}$, $\sigma \in S_\tau$. Let $\tau \in M \cap A_{\theta,\sigma}$. Since σ is a limit ordinal, by elementarity we can fix $\gamma \in M \cap \sigma$ such that $\tau \in A_{\theta,\gamma}$.

If $\sigma = \min((M \cap \kappa) \setminus \beta_{M,N})$, then $M \leq N$ and $\gamma \in M \cap \beta_{M,N} \subseteq N$. So θ is in N, $\gamma < \sup(M \cap N \cap \kappa)$, and $\tau \in M \cap A_{\theta,\gamma}$, which by Lemma 8.6 implies that $\tau \in N$. So $\tau \in M \cap N \cap \kappa^+$. As $\sigma \in R_N(M)$, it follows that $\sigma \in S_\tau$ as A is $\vec{S}$-adequate.

Otherwise there is $\zeta \in R_M(N)$ such that $\sigma = \min((M \cap \kappa) \setminus \zeta)$. By the inductive hypothesis, $Q := Q(N, M, \zeta)$ is $\vec{S}$-strong. Since $\alpha_{M,N}$ is a limit point of $M \cap N$ and $M \cap N \cap \kappa^+ \subseteq Q$ by Lemma 9.9(3), it follows that $\alpha_{M,N}$ is a limit point of Q. So

$$Q \cap \alpha_{M,N} = A_{\alpha_{M,N}, Q \cap \kappa}$$

by Lemma 7.27. By Lemma 9.9(2), $Q \cap \kappa = \zeta$. So

$$Q \cap \alpha_{M,N} = A_{\alpha_{M,N}, \zeta}.$$

Now $\gamma \in M \cap \sigma \subseteq \zeta$. Hence

$$\tau \in M \cap A_{\theta,\gamma} \subseteq M \cap A_{\theta,\zeta} = M \cap A_{\alpha_{M,N},\zeta} \cap \theta \subseteq M \cap Q.$$

So we have that $Q \in N \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup(M \cap \sigma) < \zeta = Q \cap \kappa < \sigma$, and $\tau \in M \cap Q$. Since $\sigma \in R_N(M)$, it follows that $\sigma \in S_\tau$ as A is $\vec{S}$-adequate.

This completes the proof of the claim that for all $\theta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$, for all $\tau \in A_{\theta,\sigma}$, $\sigma \in S_\tau$. Now we show that Q_0 is $\vec{S}$-strong. By Lemma 9.7, $Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}$. Let $\tau \in Q_0 \cap \kappa^+$. Then $\tau < \alpha_{M,N}$. Fix $\theta \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ which is greater than τ. Then

$$\tau \in Q_0 \cap \theta = A_{\alpha_{M,N},\sigma} \cap \theta = A_{\theta,\sigma}.$$

By the claim, $\sigma \in S_\tau$. □

Lemma 10.4. *Let (A, B) be an $\vec{S}$-obedient side condition. Suppose that $N \in A$ is simple, $M \in A$, and $M \prec (\mathcal{A}, \mathcal{Y})$. Let $\zeta \in R_M(N)$. Let $Q := Q(N, M, \zeta)$. Then $(A, B \cup \{Q\})$ is an $\vec{S}$-obedient side condition.*

Proof. If $\zeta = \min((N \cap \kappa) \setminus \beta_{M,N})$, then we are done by Lemma 10.2. So assume that $\zeta = \min((N \cap \kappa) \setminus \sigma)$, where $\sigma \in R_N(M)$. Let $Q_0 := Q_0(M, N, \sigma)$. By Lemma 9.8,

$$Q \cap \kappa = \zeta \text{ and } N \cap Q \cap \kappa^+ = N \cap Q_0 \cap \kappa^+.$$

By Lemma 9.7,

$$Q_0 \cap \kappa = \sigma \text{ and } Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}.$$

Also Q_0 and Q are $\vec{S}$-strong by Lemma 10.3.

Suppose that $K \in A$ and $\theta = \min((K \cap \kappa) \setminus \zeta)$. Let $\tau \in K \cap Q \cap \kappa^+$, and we will show that $\theta \in S_\tau$. If $\zeta = \theta$, then $\theta \in S_\tau$ since Q is $\vec{S}$-strong. So assume that $\zeta < \theta$, which means that $\zeta \notin K$.

First consider the case that $\theta \in R_N(K)$. Then since

$$\sup(K \cap \theta) < \zeta = Q \cap \kappa < \theta$$

and $Q \in N \cap \mathcal{Y}$ is $\vec{S}$-strong, it follows that $\theta \in S_\tau$ as A is $\vec{S}$-adequate. In particular, if $\beta_{K,N} \leq \zeta$, then $\theta \in R_N(K)$. Suppose that $\zeta < \beta_{K,N} \leq \theta$. Then $\theta = \min((K \cap \kappa) \setminus \beta_{K,N})$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, we have that $K < N$. So $\theta = \min((K \cap \kappa) \setminus \beta_{K,N})$ is in $R_N(K)$.

The remaining case is that $\theta < \beta_{K,N}$. We apply Lemma 2.7. We have that $\{K, M, N\}$ is adequate, $\zeta \in R_M(N)$, $\zeta \notin K$, $\theta = \min((K \cap \kappa) \setminus \zeta)$, and $\theta < \beta_{K,N}$.

By Lemma 2.7, $\theta \in R_M(K)$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, we have that $K < N$. As $Q \in N \cap \mathcal{Y}$,

$$Q \cap \kappa = \zeta < \theta < \sup(K \cap \beta_{K,N}) = \sup(K \cap N \cap \kappa),$$

and $\tau \in K \cap Q$, it follows that $\tau \in N$ by Lemma 8.7. So

$$\tau \in N \cap Q \cap \kappa^+ = N \cap Q_0 \cap \kappa^+.$$

Hence $\tau \in K \cap Q_0$. Since $Q_0 \cap \kappa^+ = A_{\alpha_{M,N},\sigma}$, it follows that $\tau < \alpha_{M,N}$.

Fix $\pi \in \lim(C_{\alpha_{M,N}}) \cap (M \cap N)$ with $\tau < \pi$. Then

$$\tau \in Q_0 \cap \pi = A_{\alpha_{M,N},\sigma} \cap \pi = A_{\pi,\sigma}.$$

Since π and σ are in M, Q_0 is $\vec{S}$-strong, $Q_0 \cap \kappa = \sigma$, and $A_{\pi,\sigma} \subseteq Q_0$, by the elementarity of M we can fix $P \in M \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $P \cap \kappa = \sigma$ and $A_{\pi,\sigma} \subseteq P$. In particular, $\tau \in P$. Since $K \cap \theta \subseteq N$ and $\zeta = \min((N \cap \kappa) \setminus \sigma)$, clearly $\theta = \min((K \cap \kappa) \setminus \sigma)$. So $\tau \in K \cap P$, $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong, and $\sup(K \cap \theta) < \sigma = P \cap \kappa < \theta$. Since $\theta \in R_M(K)$, it follows that $\theta \in S_\tau$ as A is $\vec{S}$-adequate. $\square$

Notation 10.5. *Let M and N be in $\mathcal{X}$, where $\{M, N\}$ is adequate and N is simple. Let $\zeta \in R_M(N)$ and $\sigma \in R_N^+(M)$. Let X be the set of $P \in M \cap \mathcal{Y}$ such that P is $\vec{S}$-strong,* $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, *and* $P \cap N \cap [\sup(M \cap \sigma), \sigma) \neq \emptyset$. *Assume that X is nonempty. We let $Q(N, M, \zeta, \sigma, \vec{S})$ denote the set $Q(N, M, \zeta, \sigma, X)$.*

See Notation 9.13 for the definition of $Q(N, M, \zeta, \sigma, X)$.

Lemma 10.6. *Let (A, B) be an $\vec{S}$-obedient side condition. Let M and N be in A, where N is simple. Let $\zeta \in R_M(N)$ and $\sigma \in R_N^+(M)$. Let $Q := Q(N, M, \zeta, \sigma, \vec{S})$. Then $(A, B \cup \{Q\})$ is an $\vec{S}$-obedient side condition.*

Proof. Let X be as in Notation 10.5, and let P_X be as in Notation 9.11. Then by Lemma 9.14,

$$Q \in N \cap \mathcal{Y},\ Q \cap \kappa = \zeta, \text{ and } N \cap Q \cap \kappa^+ = N \cap P_X \cap \kappa^+.$$

Let us prove that Q is $\vec{S}$-strong. Since $Q \in N$, it suffices to show that N models that Q is $\vec{S}$-strong. Fix $\tau \in N \cap Q \cap \kappa^+$, and we will show that $Q \cap \kappa = \zeta \in S_\tau$. Since $N \cap Q \cap \kappa^+ = N \cap P_X \cap \kappa^+$, we have that $\tau \in P_X$. By the definition of P_X, for some $P \in X$, $\tau \in P$. But then $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong, and $\tau \in N \cap P$. Since $\zeta \in R_M(N)$, this implies that $\zeta \in S_\tau$ as A is $\vec{S}$-adequate.

Let $K \in A_p$, and suppose that $\theta = \min((K \cap \kappa) \setminus \zeta)$. Fix $\tau \in K \cap Q \cap \kappa^+$, and we will show that $\theta \in S_\tau$. If $\theta = \zeta$, then $\theta \in S_\tau$ since Q is $\vec{S}$-strong. So assume that $\zeta < \theta$, which means that $\zeta \notin K$.

If $\theta \in R_N(K)$, then since $Q \in N \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup(K \cap \theta) < Q \cap \kappa < \theta$, and $\tau \in K \cap Q$, it follows that $\theta \in S_\tau$ as A is $\vec{S}$-adequate. In particular, if $\beta_{K,N} \leq \zeta$, then $\theta \in R_N(K)$. Suppose that $\zeta < \beta_{K,N} \leq \theta$. Then $\theta = \min((K \cap \kappa) \setminus \beta_{K,N})$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, we have that $K < N$, which implies that $\theta \in R_N(K)$.

The remaining case is that $\theta < \beta_{K,N}$. We apply Lemma 2.7. We have that $\{K, M, N\}$ is adequate, $\zeta \in R_M(N)$, $\zeta \notin K$, $\theta = \min((K \cap \kappa) \setminus \zeta)$, and $\theta < \beta_{K,N}$. By Lemma 2.7, $\theta \in R_M(K)$. Since $\zeta \in (N \cap \beta_{K,N}) \setminus K$, we have that $K < N$. As $Q \in N \cap \mathcal{Y}$,

$$Q \cap \kappa = \zeta < \theta < \sup(K \cap \beta_{K,N}) = \sup(K \cap N \cap \kappa),$$

and $\tau \in K \cap Q$, it follows that $\tau \in N$ by Lemma 8.7. So

$$\tau \in N \cap Q \cap \kappa^+ = N \cap P_X \cap \kappa^+.$$

By the definition of P_X, there is $P \in X$ such that $\tau \in P$. Since $\sup(N \cap \zeta) < P \cap \kappa < \zeta$ and $K \cap \theta \subseteq N$, clearly $\sup(K \cap \theta) < P \cap \kappa < \theta$. As $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong, $\tau \in K \cap P$, and $\theta \in R_M(K)$, it follows that $\theta \in S_\tau$ since A is $\vec{S}$-adequate. □

Definition 10.7. *Let (A, B) be an $\vec{S}$-obedient side condition. Suppose that $N \in A$ is simple. We say that (A, B) is* closed under canonical models with respect to N *if:*

1. *for all $P \in B$ with $P \cap \kappa < \sup(N \cap \kappa)$, $Q(N, P) \in B$;*
2. *for all $M \in A$ and $\zeta \in R_M(N)$, $Q(N, M, \zeta) \in B$;*
3. *for all $M \in A$, $\zeta \in R_M(N)$, and $\sigma \in R^+_N(M)$, $Q(N, M, \zeta, \sigma, \vec{S}) \in B$.*

Proposition 10.8. *Let (A, B) be an $\vec{S}$-obedient side condition such that for all $M \in A$, $M \prec (\mathcal{A}, \mathcal{Y})$. Suppose that $N \in A$ is simple. Then there exists (A, C) such that $B \subseteq C$, (A, C) is an $\vec{S}$-obedient side condition, and (A, C) is closed under canonical models with respect to N.*

Proof. First apply Lemma 10.1 finitely many times to obtain C_0 such that $B \subseteq C_0$, (A, C_0) is an $\vec{S}$-obedient side condition, and (A, C_0) satisfies property (1) of Definition 10.7. Then apply Lemmas 10.4 and 10.6 finitely many times to obtain C such that $C_0 \subseteq C$, (A, C) is an $\vec{S}$-obedient side condition, and (A, C) satisfies properties (2) and (3) of Definition 10.7. Since all of the models which are added are in N, and for all $P \in N \cap \mathcal{Y}$, $Q(N, P) = P$, it follows that (A, C) also satisfies property (1) of Definition 10.7. □

Lemma 10.9. *Suppose that (A, B) is an $\vec{S}$-obedient side condition, and $N \in A$ is simple. Assume that (A, B) is closed under canonical models with respect to N. Then:*

1. *Suppose that $P \in B$, $P \cap \kappa < \sup(N \cap \kappa)$, and $\tau \in N \cap P \cap \kappa^+$. Then there is $Q \in B \cap N$ such that $Q \cap \kappa = \min((N \cap \kappa) \setminus (P \cap \kappa))$ and $\tau \in Q$.*
2. *Suppose that $M \in A$ and $\zeta \in R_M(N)$. Then there is $Q \in B \cap N$ such that $Q \cap \kappa = \zeta$ and $M \cap N \cap \kappa^+ \subseteq Q$.*
3. *Suppose that $M \in A$, $M < N$, and $\zeta \in R_M(N)$. Then there is $Q \in B \cap N$ such that $Q \cap \kappa = \zeta$, and for all $P \in M \cap N \cap \mathcal{Y}$ which is $\vec{S}$-strong, $N \cap P \cap \kappa^+ \subseteq Q$.*
4. *Suppose that $M \in A$, $\zeta \in R_M(N)$, $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, and $\tau \in N \cap P \cap \kappa^+$. Then there is $Q \in B \cap N$ such that $Q \cap \kappa = \zeta$ and $\tau \in Q$.*

Proof. (1) Suppose that

$$P \in B,\ P \cap \kappa < \sup(N \cap \kappa), \text{ and } \tau \in N \cap P \cap \kappa^+.$$

Then $Q(N, P) \in B \cap N$. By Lemma 9.2,

$$Q(N, P) \cap \kappa = \min((N \cap \kappa) \setminus (P \cap \kappa)) \text{ and } N \cap P \cap \kappa^+ \subseteq Q(N, P).$$

In particular, $\tau \in Q(N, P)$.

(2,3) Let $M \in A$ and $\zeta \in R_M(N)$. Let $Q := Q(N, M, \zeta)$. Then $Q \in B \cap N$. By Lemma 9.9,

$$Q \cap \kappa = \zeta \text{ and } M \cap N \cap \kappa^+ \subseteq Q,$$

which proves (2). If in addition $M < N$, then by Lemma 9.10(3), for all $P \in M \cap N \cap \mathcal{Y}$, $N \cap P \cap \kappa^+ \subseteq Q$, which proves (3).

(4) Suppose that $M \in A$, $\zeta \in R_M(N)$, $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup(N \cap \zeta) < P \cap \kappa < \zeta$, and $\tau \in N \cap P \cap \kappa^+$. First assume that $\tau < \alpha_{M,N}$. Then $Q(N, M, \zeta) \in B \cap N$, and $Q(N, M, \zeta) \cap \kappa = \zeta$ by Lemma 9.9. Also by Lemma 9.10(1),

$$N \cap P \cap \alpha_{M,N} \subseteq Q(N, M, \zeta).$$

Hence $\tau \in Q(N, M, \zeta)$.

Assume that $\alpha_{M,N} \leq \tau$. Note that $\sigma := \tau_M$ exists since $\tau < \sup(P) \in M$. As $\tau \in N$, σ is in $R^+_N(M)$. So $\tau \in N \cap P \cap \sigma$. Let $Q := Q(N, M, \zeta, \sigma, \vec{S})$, which is in $B \cap N$. Then

$$Q \cap \kappa = \zeta \text{ and } N \cap P \cap \sigma \subseteq Q$$

by Lemma 9.14. In particular, $\tau \in Q$. □

§11. The main proxy lemma

Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Suppose that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$. Consider $P \in M \cap \mathcal{Y}$ such that $P \cap \kappa < \sup(M \cap N \cap \kappa)$, and assume that we are building an object in N which needs to be compatible in some sense with the model P. By Lemma 8.2, we know that $M \cap N$ is a member of N. However, when we intersect M with N, the model P will disappear if it is not in N. Thus although N sees a fragment of M, it does not necessarily see P even though $P \cap \kappa$ is in N.

Proxies are designed to handle this situation. We will define an object $p(M, N)$, called the *canonical proxy of M and N*, which is a member of N. The canonical proxy codes enough information about M that we can rebuild fragments of P inside N which can be used to avoid incompatibilities between P and the object we are constructing.[7]

Although the description and the proof of the existence of proxies is quite complicated, in practice when we use proxies we only need to appeal to a single result, called the *main proxy lemma*, which is Lemma 11.5 below. In applications of proxies, it is not necessary to understand anything else about proxies except what is contained in that lemma.

The next lemma asserts the existence of proxies. We will postpone the proof until the next section.

[7]The idea of a canonical proxy which we use in this paper is a variation of a technical device used by Mitchell for a similar purpose. In the proof of Mitchell's theorem from [12], a side condition is a pair (M, a), where M is a countable model and a is a proxy. In this paper we separate the idea of a side condition and a proxy. In contrast to [12], where proxies are present in many different parts of the proof, all applications of proxies which we give reduce to a single lemma, which is the main proxy lemma, Lemma 11.5. The idea of a canonical proxy and the main proxy lemma are new to this paper and do not appear in [12].

Lemma 11.1 (Proxy existence lemma)**.** *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$. Let $\eta^* \in R^+_N(M)$. Then there exist finite sets a and a' satisfying the following statements:*

(1) *a is a finite set of pairs of ordinals, and $a' = \{\sigma : \exists \beta\ (\beta, \sigma) \in a\}$.*
(2) *For all (β, σ) in a,*
  (a) *$\beta \in M \cap N \cap \kappa$;*
  (b) *$\sigma \in N \cap \kappa^+$ is a limit ordinal;*
  (c) *$\sup(N \cap \sigma) \leq \eta^*$;*
  (d) *if $a \neq \emptyset$, then $\min(a') = \min((N \cap \kappa^+) \setminus \sup(M \cap \eta^*))$.*
(3) *If $(\beta, \sigma) \in a$, where $\min(a') < \sigma$, and $\beta \leq \gamma < \kappa$, then:*
  (a) *$A_{\eta^*,\gamma} \cap \sup(N \cap \sigma) = A_{\sigma,\gamma} \cap \sup(N \cap \sigma)$;*
  (b) *$A_{\eta^*,\gamma} \cap N \cap \sigma = A_{\sigma,\gamma} \cap N$.*
(4) *If $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, and $P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$, then there exists $\sigma \in a'$ such that:*
  (a) *$P \cap N \cap \eta^* \subseteq \sigma$;*
  (b) *the least such σ is equal to the largest σ in a' such that for some β, $\beta \leq P \cap \kappa$ and $(\beta, \sigma) \in a$.*
(5) *Let P and σ be as in (4), and assume that $(\beta, \sigma) \in a$; then:*
  (a) *$\beta \leq P \cap \kappa$;*
  (b) *$P \cap \sup(N \cap \sigma) = A_{\sigma,P\cap\kappa} \cap \sup(N \cap \sigma)$;*
  (c) *$P \cap N \cap \eta^* = A_{\sigma,P\cap\kappa} \cap N$.*

For the remainder of this section, we will assume that the proxy existence lemma holds. We now define the canonical proxy $p(M, N)$.

A lexicographical ordering on sets of pairs of ordinals is described as follows. We identify a finite set of pairs of ordinals as a finite set of ordinals using the Gödel pairing function, and then compare any two finite sets of pairs using the lexicographical ordering on their corresponding sets of ordinals.

Definition 11.2. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$.*

Let $\eta_0, \ldots, \eta_{k-1}$ enumerate the ordinals in $R^+_N(M)$ in increasing order. Define $p(M, N)$ as the function with domain k such that for all $i < k$, $p(M, N)(i)$ is the lexicographically least set a satisfying (1)–(5) of Lemma 11.1 for $\eta^* = \eta_i$.

Note that $p(M, N)$ is a member of N.

The proof of the main proxy lemma will use the next two technical lemmas.

Lemma 11.3. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$.*

Let k be the size of $R^+_N(M)$, and assume that η^* is the i-th member of $R^+_N(M)$, where $i < k$. Let $a := p(M, N)(i)$ and $a' := \{\sigma : \exists \beta\ (\beta, \sigma) \in a\}$.

Suppose that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, $\mathrm{cf}(P \cap \kappa) > \omega$, and $P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$. Let σ be the least ordinal in a' such that $P \cap N \cap \eta^* \subseteq \sigma$, which exists by Lemma 11.1(4). Let $Q := Sk(A_{\sigma,P\cap\kappa})$. Then:

(1) *$Q \in N \cap \mathcal{Y}$;*
(2) *$Q \cap \kappa = P \cap \kappa$ and $Q \cap \kappa^+ = A_{\sigma,P\cap\kappa}$;*

(3) $Q \cap N \cap \kappa^+ = P \cap N \cap \eta^*$;

(4) $Q \cap \sup(N \cap \sigma) = P \cap \sup(N \cap \sigma)$.

In particular, $P \cap N \cap \eta^* \subseteq Q$.

Proof. Let $\theta := \sup(N \cap \sigma)$. By Lemma 11.1(5(b,c)),

$$P \cap \theta = A_{\sigma,P\cap\kappa} \cap \theta \text{ and } P \cap N \cap \eta^* = A_{\sigma,P\cap\kappa} \cap N.$$

In particular, as $A_{\sigma,P\cap\kappa} \in N$, Lemmas 7.10 and 8.16 and the second equality imply that $A_{\sigma,P\cap\kappa}$ is closed under H^*. So $Q \cap \kappa^+ = A_{\sigma,P\cap\kappa}$. Hence

$$Q \cap N \cap \kappa^+ = A_{\sigma,P\cap\kappa} \cap N = P \cap N \cap \eta^*,$$

which proves (3). Also by the first equality,

$$Q \cap \sup(N \cap \sigma) = Q \cap \theta = A_{\sigma,P\cap\kappa} \cap \theta = P \cap \theta = P \cap \sup(N \cap \sigma),$$

which proves (4).

We claim that $Q \cap \kappa = P \cap \kappa$, which proves (2). Since Q and $P \cap \kappa$ are in N, it suffices to show that N models that $Q \cap \kappa = P \cap \kappa$. So let $\alpha \in Q \cap N \cap \kappa$, and we will show that $\alpha < P \cap \kappa$. Then $\alpha \in Q \cap N \cap \kappa^+ = P \cap N \cap \eta^*$. So $\alpha \in P \cap \kappa$. Conversely, let $\alpha \in N \cap P \cap \kappa$, and we will show that $\alpha \in Q$. Then $\alpha \in P \cap N \cap \eta^* = Q \cap N \cap \kappa^+$, so $\alpha \in Q$.

To prove (1), it suffices to show that $\lim(C_{\sup(Q)}) \cap Q$ is cofinal in $\sup(Q)$. Since $Q \cap \kappa^+ = A_{\sigma,P\cap\kappa}$, $\sup(Q) \leq \sigma$. Also note that since $P \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$ and P and η^* are in M, η^* is a limit point of P by Lemma 7.30.

Case 1: $\theta < \sigma$. Since $\mathrm{cf}(Q \cap \kappa) = \mathrm{cf}(P \cap \kappa) > \omega$, it suffices by Lemma 7.15 to show that $\mathrm{cf}(\sup(Q)) > \omega$. Since $\sigma = \min((N \cap \kappa^+) \setminus \theta)$, σ has uncountable cofinality. So if $\sup(Q) = \sigma$, then we are done.

Otherwise by elementarity, $\sup(Q) \in N \cap \sigma \subseteq \theta$. By (4), $Q \cap \kappa^+ = Q \cap \theta = P \cap \theta$. It follows that $\sup(Q) = \sup(P \cap \theta)$, which is a limit point of P below θ. Since η^* is a limit point of P and $\sup(Q) < \theta \leq \eta^*$ by Lemma 11.1(2(c)), if $\sup(Q)$ has countable cofinality then $\sup(Q) \in P$ by Lemma 7.14. But then $\sup(Q) \in P \cap \theta = Q \cap \theta$, which is impossible. Therefore $\sup(Q)$ has uncountable cofinality.

Case 2: $\theta = \sigma$. Then σ is a limit point of N, and in particular, σ has cofinality ω. By Lemma 11.1(2(c)), $\sup(N \cap \sigma) = \sigma \leq \eta^*$. Since σ has cofinality ω and η^* has uncountable cofinality, it follows that $\sigma < \eta^*$.

We claim that $\sup(P \cap N \cap \sigma) < \sigma$. Suppose for a contradiction that $\sup(P \cap N \cap \sigma) = \sigma$. Then σ is a limit point of P. As η^* is a limit point of P, $\sigma < \sup(P)$. Since σ has cofinality ω and $\mathrm{cf}(P \cap \kappa) > \omega$, Lemma 7.14 implies that $\sigma \in P$. So $\sigma \in N \cap P \cap \eta^*$, which contradicts that $N \cap P \cap \eta^* \subseteq \sigma$.

To show that $Q \in \mathcal{Y}$, by Lemma 7.15 it suffices to show that $\mathrm{cf}(\sup(Q)) > \omega$. Since $\theta = \sigma$, by (4) we have that $Q \cap \sigma = P \cap \sigma$. Therefore $Q \cap N \cap \sigma = P \cap N \cap \sigma$. By the claim,

$$\sup(Q \cap N \cap \sigma) = \sup(P \cap N \cap \sigma) < \sigma.$$

If $\sup(Q) = \sigma$, then since $Q \in N$ and σ is a limit point of N, it is easy to argue by elementarity that $Q \cap N \cap \kappa^+$ is cofinal in σ, which is false. Therefore $\sup(Q) < \sigma = \theta$. Since $Q \cap \theta = P \cap \theta$, it follows that $\sup(Q) = \sup(Q \cap \theta) = \sup(P \cap \theta)$, which is a limit point of P below θ. Since η^* is a limit point of P and $\sup(Q) < \theta \leq \eta^*$ by Lemma 11.1(2(c)), if $\sup(Q)$ has countable cofinality then $\sup(Q) \in P$ by Lemma

7.14. But then $\sup(Q) \in P \cap \theta = Q \cap \theta$, which is impossible. Therefore $\sup(Q)$ has uncountable cofinality. $\square$

Lemma 11.4. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$.*

Let k be the size of $R^+_N(M)$, and assume that η^* is the i-th member of $R^+_N(M)$, where $i < k$. Let $a := p(M,N)(i)$ and $a' := \{\sigma : \exists\beta\ (\beta,\sigma) \in a\}$.

Suppose that $(\beta,\sigma) \in a$, where $\min(a') < \sigma$. Assume that $Q \in N \cap \mathcal{Y}$ is such that $\beta \le Q \cap \kappa$, $Q \cap \kappa \in M \cap N \cap \kappa$, $\mathrm{cf}(Q \cap \kappa) > \omega$, $Q \cap \kappa^+ = A_{\sigma,Q\cap\kappa}$, and $Q \cap N \cap [\sup(M \cap \eta^*), \sigma) \neq \emptyset$.

Let $P := Sk(A_{\eta^*,Q\cap\kappa})$. Then:

1. *$P \in M \cap \mathcal{Y}$;*
2. *$P \cap \kappa = Q \cap \kappa$, $P \cap \kappa^+ = A_{\eta^*,Q\cap\kappa}$, and $\sup(P) = \eta^*$;*
3. *$P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$;*
4. *$P \cap M \cap \kappa^+ = Q \cap M \cap \kappa^+$.*

Proof. Let $\gamma := Q \cap \kappa$ and $\theta := \sup(N \cap \sigma)$. By Lemma 11.1(3),

$$A_{\eta^*,\gamma} \cap \theta = A_{\sigma,\gamma} \cap \theta \text{ and } A_{\eta^*,\gamma} \cap N \cap \sigma = A_{\sigma,\gamma} \cap N.$$

We claim that

$$A_{\eta^*,\gamma} \cap M = A_{\sigma,\gamma} \cap M.$$

Let $\alpha \in A_{\eta^*,\gamma} \cap M$, and we will show that $\alpha \in A_{\sigma,\gamma}$. Then $\alpha \in M \cap \eta^*$. By Lemma 11.1(2(d)), $\sup(M \cap \eta^*) \le \min(a') < \sigma$. Since $\min(a') \in N$, $\sup(M \cap \eta^*) < \sup(N \cap \sigma)$. So

$$\alpha < \sup(M \cap \eta^*) < \sup(N \cap \sigma) = \theta.$$

Hence

$$\alpha \in A_{\eta^*,\gamma} \cap \theta = A_{\sigma,\gamma} \cap \theta,$$

so $\alpha \in A_{\sigma,\gamma}$.

Conversely, let $\alpha \in A_{\sigma,\gamma} \cap M$, and we will show that $\alpha \in A_{\eta^*,\gamma}$. Since $Q \cap \kappa^+ = A_{\sigma,\gamma}$, $\alpha \in Q \cap M \cap \kappa^+$. We claim that $\alpha \in N$. If $N \in \mathcal{Y}$, then $\alpha \in Q \in N$ implies that $\alpha \in N$. Suppose that $N \in \mathcal{X}$. Then $M < N$, $Q \in N \cap \mathcal{Y}$, and $Q \cap \kappa = \gamma \in M \cap N \cap \kappa$, which implies by Lemma 8.7 that $Q \cap M \cap \kappa^+ \subseteq N$. In particular, $\alpha \in N$. Hence in either case,

$$\alpha \in A_{\sigma,\gamma} \cap N = A_{\eta^*,\gamma} \cap N \cap \sigma.$$

So $\alpha \in A_{\eta^*,\gamma}$.

We have proven that $A_{\eta^*,\gamma} \cap M = A_{\sigma,\gamma} \cap M = Q \cap M \cap \kappa^+$. Since $A_{\eta^*,\gamma}$ is in M, it follows that $A_{\eta^*,\gamma}$ is closed under H^* by Lemma 7.10. In particular, $P \cap \kappa^+ = A_{\eta^*,\gamma}$. So

$$P \cap M \cap \kappa^+ = A_{\eta^*,\gamma} \cap M = A_{\sigma,\gamma} \cap M = Q \cap M \cap \kappa^+,$$

which proves (4).

Next we claim that $P \cap \kappa = \gamma$ and $\sup(P) = \eta^*$, which proves (2). Since P, γ, and η^* are in M, it suffices to show that M models these statements. Let $\alpha \in P \cap M \cap \kappa$, and we will show that $\alpha < \gamma$. Then $\alpha \in P \cap M \cap \kappa^+ = Q \cap M \cap \kappa^+$. So $\alpha \in Q \cap \kappa = \gamma$. Conversely, let $\alpha \in M \cap \gamma$, and we will show that $\alpha \in P \cap \kappa$. Then

$$\alpha \in M \cap \gamma = M \cap Q \cap \kappa \subseteq M \cap Q \cap \kappa^+ = M \cap P \cap \kappa^+.$$

So $\alpha \in P$.

Now $Q \cap N \cap [\sup(M \cap \eta^*), \sigma)$ is nonempty by assumption, so fix τ in this intersection. Then

$$\tau \in Q \cap \kappa^+ \cap N = A_{\sigma,\gamma} \cap N = A_{\eta^*,\gamma} \cap N \cap \sigma.$$

So $\tau \in A_{\eta^*,\gamma} = P \cap \kappa^+$. Hence $P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$, which proves (3). By Lemma 7.30, it follows that η^* is a limit point of P. Since $\sup(P \cap \kappa^+) = \sup(A_{\eta^*,\gamma}) \leq \eta^*$, we have that $\sup(P) \leq \eta^*$. As η^* is a limit point of P, it follows that $\sup(P) = \eta^*$, finishing the proof of (2). In particular, as the ordinals $P \cap \kappa = Q \cap \kappa$ and η^* both have uncountable cofinality, it follows that $P \in \mathcal{Y}$ by Lemma 7.15, which proves (1). $\square$

We are now ready to prove the main lemma on proxies. This lemma contains all the information about proxies that we will need for applications.

Lemma 11.5 (Main proxy lemma). *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$. Let $\eta^* \in R^+_N(M)$.*

Suppose that $M' \in N \cap \mathcal{X}$ and $N' \in N \cap (\mathcal{X} \cup \mathcal{Y})$, where N' is simple. Assume that $M' < N'$ in the case that $N' \in \mathcal{X}$, $\sup(M' \cap N' \cap \kappa) < N' \cap \kappa$ in the case that $N' \in \mathcal{Y}$, and $N \in \mathcal{X}$ iff $N' \in \mathcal{X}$. Suppose that $M \cap N = M' \cap N'$ and $p(M, N) = p(M', N')$.

Assume that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, $\mathrm{cf}(P \cap \kappa) > \omega$, and $\tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*)$. Then:

1. *There is $Q \in N' \cap \mathcal{Y}$ such that $Q \cap \kappa = P \cap \kappa$ and $Q \cap N \cap \kappa^+ = P \cap N \cap \eta^*$; in particular, $\tau \in Q$.*
2. *If $\tau \in N'$, then there is $P' \in M' \cap \mathcal{Y}$ such that $P' \cap \kappa = P \cap \kappa$, $P' \cap N' \cap \kappa^+ = Q \cap N' \cap \kappa^+$, and $P' \cap M' \cap \kappa^+ = Q \cap M' \cap \kappa^+$; in particular, $\tau \in P'$.*
3. *If $N \in \mathcal{Y}$, then there is $P' \in M' \cap \mathcal{Y}$ such that $P' \cap \kappa = P \cap \kappa$ and $\tau \in P'$.*

Moreover, if $\vec{S}$ is given and P is $\vec{S}$-strong, then the models Q and P' described in (1), (2), and (3) are also $\vec{S}$-strong.

Proof. Let k be the size of $R^+_N(M)$, and fix $i < k$ such that η^* is the i-th member of $R^+_N(M)$. Let $a := p(M, N)(i)$ and $a' := \{\sigma : \exists \beta\ (\beta, \sigma) \in a\}$.

(1) Let σ be the least ordinal in a' such that $P \cap N \cap \eta^* \subseteq \sigma$, which exists by Lemma 11.1(4). By Lemma 11.1(2(d)), $\min(a') = \min((N \cap \kappa^+) \setminus \sup(M \cap \eta^*))$, which is strictly less than σ since $P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$ and $P \cap N \cap \eta^* \subseteq \sigma$. Fix β such that $(\beta, \sigma) \in a$. By Lemma 11.1(5(a)), $\beta \leq P \cap \kappa$.

We apply Lemma 11.3. Note that all of the assumptions of this lemma are satisfied. Let $Q := Sk(A_{\sigma,P\cap\kappa})$. Then by Lemma 11.3,

(a) $Q \in N \cap \mathcal{Y}$;
(b) $Q \cap \kappa = P \cap \kappa$ and $Q \cap \kappa^+ = A_{\sigma,P\cap\kappa}$;
(c) $Q \cap N \cap \kappa^+ = P \cap N \cap \eta^*$;
(d) $Q \cap \sup(N \cap \sigma) = P \cap \sup(N \cap \sigma)$.

Since $p(M, N) = p(M', N')$, it follows that $p(M, N) \in N'$, and so in particular, $\sigma \in N'$. And $Q \cap \kappa = P \cap \kappa \in M \cap N \cap \kappa = M' \cap N' \cap \kappa \subseteq N'$. So σ and $Q \cap \kappa$ are in N'. Therefore $A_{\sigma,Q\cap\kappa}$ and Q are in N' by elementarity. Properties (a), (b), and (c) above imply (1).

(2) Assume that $\tau \in N'$. We apply Lemma 11.4 to M' and N'. Let η_0^* be the i-th member of $R^+_{N'}(M')$. Let $a_0 := p(M',N')(i)$. Note that $a_0 = p(M,N)(i) = a$ and $a_0' = a'$.

Let us check that the assumptions of Lemma 11.4 are satisfied. Since $p(M,N) = p(M',N')$, $(\beta,\sigma) \in p(M',N')(i) = a_0$, and $\min(a_0') = \min(a') < \sigma$. We know that $Q \in N' \cap \mathcal{Y}$, $\beta \le P \cap \kappa = Q \cap \kappa$, $Q \cap \kappa = P \cap \kappa \in M \cap N \cap \kappa = M' \cap N' \cap \kappa$, $\mathrm{cf}(Q \cap \kappa) = \mathrm{cf}(P \cap \kappa) > \omega$, and $Q \cap \kappa^+ = A_{\sigma,Q\cap\kappa}$.

It remains to show that

$$Q \cap N' \cap [\sup(M' \cap \eta_0^*), \sigma) \ne \emptyset.$$

Since $\tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*)$ and

$$P \cap N \cap \eta^* \subseteq N \cap \sigma \subseteq \sup(N \cap \sigma),$$

it follows that $\tau < \sigma$, and

$$\tau \in P \cap \sup(N \cap \sigma) = Q \cap \sup(N \cap \sigma)$$

by property (d) above. Also $\tau \in N'$ by assumption. So

$$\tau \in Q \cap N' \cap [\sup(M \cap \eta^*), \eta^*),$$

and therefore $\min(a') = \min((N \cap \kappa^+) \setminus \sup(M \cap \eta^*)) \le \tau$. But then

$$\sup(M' \cap \eta_0^*) \le \min((N' \cap \kappa^+) \setminus \sup(M' \cap \eta_0^*)) = \min(a_0') = \min(a') \le \tau.$$

So $\tau \in Q \cap N' \cap [\sup(M' \cap \eta_0^*), \sigma)$. This completes the verification of the assumptions of Lemma 11.4.

Let $P' := Sk(A_{\eta_0^*,Q\cap\kappa})$. Then by Lemma 11.4,

(i) $P' \in M' \cap \mathcal{Y}$;
(ii) $P' \cap \kappa = Q \cap \kappa$, $P' \cap \kappa^+ = A_{\eta_0^*,Q\cap\kappa}$, and $\sup(P') = \eta_0^*$;
(iii) $P' \cap N' \cap [\sup(M' \cap \eta_0^*), \eta_0^*) \ne \emptyset$;
(iv) $P' \cap M' \cap \kappa^+ = Q \cap M' \cap \kappa^+$.

In particular, $P' \cap \kappa = P \cap \kappa$. It remains to prove that

$$P' \cap N' \cap \kappa^+ = Q \cap N' \cap \kappa^+.$$

We apply Lemma 11.3 to M', N', and P'. Note that the assumptions of Lemma 11.3 are obviously satisfied, except for the claim that σ is the least ordinal in a_0' such that $P' \cap N' \cap \eta_0^* \subseteq \sigma$. So let σ' be the least ordinal in a_0' such that $P' \cap N' \cap \eta_0^* \subseteq \sigma'$. Then by Lemma 11.1(4(b)), σ' is the largest ordinal in a_0' such that for some γ, $\gamma \le P' \cap \kappa$ and $(\gamma, \sigma') \in a_0$. Now σ is the least ordinal in $a' = a_0'$ such that $N \cap P \cap \eta^* \subseteq \sigma$, so again by Lemma 11.1(4(b)), σ is the largest ordinal in $a' = a_0'$ such that for some γ, $\gamma \le P \cap \kappa = P' \cap \kappa$ and $(\gamma, \sigma) \in a = a_0$. So σ and σ' satisfy the same definition, and hence $\sigma = \sigma'$. So indeed σ is the least ordinal in a_0' such that $P' \cap N' \cap \eta_0^* \subseteq \sigma$.

Since $\sigma = \sigma'$ and $P \cap \kappa = P' \cap \kappa$, $Q = Sk(A_{\sigma,P\cap\kappa}) = Sk(A_{\sigma',P'\cap\kappa})$. By Lemma 11.3, $Q \cap N' \cap \kappa^+ = P' \cap N' \cap \eta_0^*$. But $P' = Sk(A_{\eta_0^*,Q\cap\kappa})$, and therefore $P' \cap \eta_0^* = P' \cap \kappa^+$. So $Q \cap N' \cap \kappa^+ = P' \cap N' \cap \kappa^+$.

(3) If $N \in \mathcal{Y}$, then $N' \in \mathcal{Y}$. So $Q \in N'$ implies that $Q \subseteq N'$. Hence $\tau \in N'$. So we are done by (2).

Finally, the last statement follows from the properties of Q and P' described in (1) and (2) together with Lemma 5.6. $\square$

The main proxy lemma was concerned with the case that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, $\eta^* \in R^+_N(M)$, and $\tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*)$. Another case which often occurs in the same contexts is that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, and $\tau \in P \cap N \cap \alpha_{M,N}$. This situation is handled by the next two lemmas.

Lemma 11.6. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$.*

Suppose that $M' \in N \cap \mathcal{X}$ and $N' \in N \cap (\mathcal{X} \cup \mathcal{Y})$, where N' is simple. Assume that $M' < N'$ in the case that $N' \in \mathcal{X}$, $\sup(M' \cap N' \cap \kappa) < N' \cap \kappa$ in the case that $N' \in \mathcal{Y}$, and $N \in \mathcal{X}$ iff $N' \in \mathcal{X}$. Suppose that $M \cap N = M' \cap N'$ and $p(M,N) = p(M',N')$.

Assume that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, $\mathrm{cf}(P \cap \kappa) > \omega$, $P \cap \alpha_{M,N}$ is unbounded in $\alpha_{M,N}$, and $\tau_0 \in P \cap N \cap \alpha_{M,N}$. Let $\eta^* := \min((M \cap \kappa^+) \setminus \alpha_{M,N})$. Then:

1. *There is $Q \in N' \cap \mathcal{Y}$ such that $Q \cap \kappa = P \cap \kappa$ and $Q \cap N \cap \kappa^+ = P \cap N \cap \eta^*$; in particular, $\tau_0 \in Q$.*
2. *If $\tau_0 \in N'$, then there is $P' \in M' \cap \mathcal{Y}$ such that $P' \cap \kappa = P \cap \kappa$ and $\tau_0 \in P'$.*

Moreover, if $\vec{S}$ is given and P is $\vec{S}$-strong, then the models Q and P' described in (1) and (2) are also $\vec{S}$-strong.

Proof. Since $\alpha_{M,N} \in N$ by Lemma 8.2, $\alpha_{M,N} \notin M$. But $\alpha_{M,N} \leq \sup(P)$ and $\sup(P) \in M$. It follows that $\alpha_{M,N} < \sup(P)$. Consequently, the ordinal $\eta^* = \min((M \cap \kappa^+) \setminus \alpha_{M,N})$ exists and is greater than $\alpha_{M,N}$. Therefore $\eta^* \in R^+_N(M)$. Since $\alpha_{M,N}$ is a limit point of the countable set $M \cap N$, it follows that $\mathrm{cf}(\alpha_{M,N}) = \omega$. As $\mathrm{cf}(P \cap \kappa) > \omega$, we have that $\alpha_{M,N} \in P$ by Lemma 7.14. So $\alpha_{M,N} \in N \cap P \cap [\sup(M \cap \eta^*), \eta^*)$.

We apply the main proxy lemma, Lemma 11.5, letting $\tau = \alpha_{M,N}$. Then the first statement of (1) above follows from Lemma 11.5(1). Since $\tau_0 < \alpha_{M,N}$, $\tau_0 \in N \cap P \cap \eta^* \subseteq Q$. For (2), we have that

$$\alpha_{M,N} = \sup(M \cap N) = \sup(M' \cap N'),$$

which is in N' by Lemma 8.2. By Lemma 11.5(2), there is $P' \in M' \cap \mathcal{Y}$ such that $P' \cap \kappa = P \cap \kappa$ and $P' \cap N' \cap \kappa^+ = Q \cap N' \cap \kappa^+$. Assume that $\tau_0 \in N'$. Then

$$\tau_0 \in Q \cap N' \cap \kappa^+ = P' \cap N' \cap \kappa^+ \subseteq P'.$$

Therefore $\tau_0 \in P'$. □

Lemma 11.7. *Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$.*

Suppose that $P \in M \cap \mathcal{Y}$, $P \cap \kappa \in M \cap N \cap \kappa$, $P \cap \alpha_{M,N}$ is bounded below $\alpha_{M,N}$, and $\tau \in P \cap N \cap \alpha_{M,N}$. Then there is $P' \in M \cap N \cap \mathcal{Y}$ such that $P' \cap \kappa = P \cap \kappa$ and $\tau \in P'$. Moreover, if $\vec{S}$ is given and P is $\vec{S}$-strong, then P' is $\vec{S}$-strong.

Proof. Let $\alpha := \alpha_{M,N}$ and $\delta := \sup(M \cap N \cap \kappa)$. Define

$$\sigma := \sup(P \cap A_{\alpha,\delta}).$$

By Lemma 8.11, σ satisfies:

(a) $\sigma \in M \cap N \cap \kappa^+$;
(b) $P \cap \sigma = A_{\sigma,P\cap\kappa}$;

(c) $P \cap (M \cap N) \cap \kappa^+ = A_{\sigma,P\cap\kappa} \cap (M \cap N)$;
(d) $N \cap P \cap \alpha_{M,N} \subseteq A_{\sigma,P\cap\kappa}$.

Since σ and $P \cap \kappa$ are in $M \cap N$, Lemma 7.10 and (c) imply that $A_{\sigma,P\cap\kappa}$ is closed under H^*. Let $P' := Sk(A_{\sigma,P\cap\kappa})$. Then P' is in $M \cap N$.

By (b), we have that

$$P' \cap \kappa = A_{\sigma,P\cap\kappa} \cap \kappa = P \cap \sigma \cap \kappa = P \cap \kappa.$$

By (d), $\tau \in A_{\sigma,P\cap\kappa} \subseteq P'$. It remains to show that $P' \in \mathcal{Y}$. It suffices to show that $\lim(C_{\sup(P')}) \cap P'$ is cofinal in $\sup(P')$.

Since P and $A_{\alpha,\delta}$ are closed under successors, $P \cap A_{\alpha,\delta}$ has no maximal element. As σ is a limit point of P,

$$\sup(P') = \sup(A_{\sigma,P\cap\kappa}) = \sup(P \cap \sigma) = \sigma.$$

Also $P' \cap \kappa^+ = A_{\sigma,P\cap\kappa} = P \cap \sigma$. So it is enough to show that $\lim(C_\sigma) \cap P$ is cofinal in σ.

Now σ is a limit point of P, and therefore has cofinality less than κ. If $\sigma \notin P$, then $\sigma \in \mathrm{cl}(P) \setminus P$, so by Lemma 7.13, $\lim(C_\sigma) \cap P$ is cofinal in σ and we are done. Otherwise $\sigma \in P$. By the definition of σ, σ is not in $A_{\alpha,\delta}$. Now $A_{\alpha,\delta}$ is closed under H^* by Lemma 7.29. So $Q := Sk(A_{\alpha,\delta})$ is an elementary substructure of $\mathcal{A}$ with $Q \cap \kappa^+ = A_{\alpha,\delta}$. Let $\sigma' := \min((Q \cap \kappa^+) \setminus \sigma)$, which exists since $\sigma < \alpha$. Then σ' has uncountable cofinality, which implies that $\lim(C_{\sigma'})$ is cofinal in σ'. Also by the elementarity of Q, σ is a limit point of $C_{\sigma'}$, and therefore

$$C_\sigma = C_{\sigma'} \cap \sigma.$$

Again by the elementarity of Q, $\lim(C_{\sigma'}) \cap Q$ is cofinal in $\sup(Q \cap \sigma') = \sigma$. In particular, $\lim(C_\sigma)$ is cofinal in σ.

Since $\sigma \in P$ and σ has cofinality less than κ, $\mathrm{ot}(C_\sigma) \in P \cap \kappa$, and therefore $C_\sigma \subseteq P$. Hence $\lim(C_\sigma) \cap P = \lim(C_\sigma)$, and this set is cofinal in σ as observed above.

Finally, assume that P is $\vec{S}$-strong. Then $P' \cap \kappa = P \cap \kappa$, $P' \in M \cap N$, and by (b),

$$P' \cap (M \cap N) \cap \kappa^+ = A_{\sigma,P\cap\kappa} \cap (M \cap N) \subseteq A_{\sigma,P\cap\kappa} \subseteq P.$$

So P' is $\vec{S}$-strong by Lemma 5.6. □

§12. The proxy construction

Let $M \in \mathcal{X}$ and $N \in \mathcal{X} \cup \mathcal{Y}$, where N is simple. Assume that $M < N$ in the case that $N \in \mathcal{X}$, and $\sup(M \cap N \cap \kappa) < N \cap \kappa$ in the case that $N \in \mathcal{Y}$. Let $\eta^* \in R^+_N(M)$. We will prove that there exist sets a and a' satisfying properties (1)–(5) of Lemma 11.1.[8]

We recall a well-ordering on finite sets of ordinals which was used in [12]. For finite sets of ordinals x and y, define $x \prec y$ if $x \neq y$ and $\max(x \triangle y) \in y$.

Lemma 12.1. *The relation $\prec$ is a well-ordering of $[On]^{<\omega}$.*

[8]Our proof of the proxy existence lemma is based on the construction of Mitchell [12, Lemma 3.46]. We point out that there is a mistake in Mitchell's construction. The problem arises in the case when the ordinal η from that proof is defined as $\max(\lim(C_\alpha) \cap \overline{X})$, and η happens to have dropped below $\sup(M')$. In this case, there appears to be no reason why recursion hypothesis (1c) can be maintained. This problem was discovered by Gilton, and later corrected by Krueger.

Proof. It is obvious that $\prec$ is irreflexive and total. For transitivity, let $x \prec y \prec z$, and we will show that $x \prec z$. Let $\alpha := \max(x \triangle y)$, $\beta := \max(y \triangle z)$, and $\gamma := \max(x \triangle z)$. Then $\alpha \in y \setminus x$ and $\beta \in z \setminus y$. We will show that $\gamma \in z$. Suppose for a contradiction that $\gamma \notin z$, so that $\gamma \in x$.

The following statements can be easily proved: (1) α, β, and γ are distinct; (2) $\alpha \in z$ implies that $\alpha < \gamma$; (3) $\alpha \notin z$ implies that $\alpha < \beta$; (4) $\beta \in x$ implies that $\beta < \alpha$; (5) $\beta \notin x$ implies that $\beta < \gamma$; (6) $\gamma \in y$ implies that $\gamma < \beta$; and (7) $\gamma \notin y$ implies that $\gamma < \alpha$. Now one can easily check by inspection that any Boolean combination of the statements $\alpha \in z$, $\beta \in x$, and $\gamma \in y$ yields a contradiction. For example, suppose that $\alpha \in z$, $\beta \in x$, and $\gamma \in y$. Then (2), (4), and (6) imply that $\alpha < \gamma$, $\beta < \alpha$, and $\gamma < \beta$, which in turn imply that $\alpha < \gamma < \beta < \alpha$, which is absurd. The other possibilities are ruled out in a similar manner. This completes the proof that $\prec$ is transitive.

To show that $\prec$ is a well-ordering, suppose for a contradiction that $\langle x_n : n < \omega \rangle$ is a $\prec$-decreasing sequence of finite sets of ordinals. We define by induction an increasing sequence $\langle k_n : n < \omega \rangle$ of integers and a $\subseteq$-decreasing sequence $\langle A_n : n < \omega \rangle$ of infinite subsets of ω as follows. Let $k_0 = 0$ and $A_0 = \omega$.

Assume that k_n and A_n are defined, where A_n is an infinite subset of ω. Let k_{n+1} be the least integer in A_n strictly greater than k_n. Now for all $r \in A_n$ with $r > k_{n+1}$, we have that $x_r \prec x_{k_{n+1}}$, and hence $\max(x_r \triangle x_{k_{n+1}}) \in x_{k_{n+1}}$. Since $x_{k_{n+1}}$ is finite and A_n is infinite, we can find an infinite subset A_{n+1} of $A_n \setminus (k_{n+1}+1)$ such that for all $r, s \in A_{n+1}$, $\max(x_r \triangle x_{k_{n+1}}) = \max(x_s \triangle x_{k_{n+1}})$.

This completes the construction. For each n, let $\alpha_n := \max(x_{k_n} \triangle x_{k_{n+1}})$. We claim that $\langle \alpha_n : n < \omega \rangle$ is a descending sequence of ordinals, which gives a contradiction. Let $n < \omega$. Since $x_{k_{n+1}} \prec x_{k_n}$, $\alpha_n \in x_{k_n} \setminus x_{k_{n+1}}$. So clearly $\alpha_n \neq \alpha_{n+1}$. Suppose for a contradiction that $\alpha_n < \alpha_{n+1}$. Then by the maximality of α_n, α_{n+1} cannot be in $x_{k_n} \triangle x_{k_{n+1}}$, and therefore must be in $x_{k_n} \cap x_{k_{n+1}}$. But by construction, $\max(x_{k_n} \triangle x_{k_{n+2}}) = \alpha_n$. Therefore α_{n+1} must be in $x_{k_{n+2}}$, since otherwise it is in $x_{k_n} \triangle x_{k_{n+2}}$ but larger than α_n. This contradicts that $\alpha_{n+1} = \max(x_{k_{n+1}} \triangle x_{k_{n+2}})$ is in $x_{k_{n+1}} \setminus x_{k_{n+2}}$. $\square$

We will define by induction two sequences of sets $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$. The induction stops when $b_n = \emptyset$. Each a_k and b_k will be a finite set of pairs of ordinals. We let $a'_k := \{\sigma : \exists \beta\ (\beta, \sigma) \in a_k\}$ and $b'_k := \{\eta : \exists \beta\ (\beta, \eta) \in b_k\}$.

By construction, for each k, b'_{k+1} will be equal to $(b'_k \setminus \{\eta\}) \cup x$, where $\eta = \min(b'_k)$ and x is a finite subset of η. In particular, $\max(b'_k \triangle b'_{k+1})$ will be equal to η, which is in b'_k, and hence $b'_{k+1} \prec b'_k$. Therefore the sequence of b'_k's is $\prec$-descending, and so must terminate with the empty set after finitely many steps.

When defining these sequences, we will maintain the following inductive hypotheses:

(A) For all $(\beta, \sigma) \in a_k$, $\beta \in M \cap N \cap \kappa$, $\sigma \in N \cap \kappa^+$ is a limit ordinal, and $\sup(N \cap \sigma) \leq \eta^*$. The least member of a'_k, if it exists, is equal to $\min((N \cap \kappa^+) \setminus \sup(M \cap \eta^*))$. For each $\sigma \in a'_k$, there is a unique β with $(\beta, \sigma) \in a_k$.

(B) For all $(\beta, \eta) \in b_k$, $\beta \in M \cap N \cap \kappa$ and $\eta \leq \eta^*$ is a limit ordinal. If $\eta_0 < \eta_1$ are successive elements of b'_k, then $N \cap [\eta_0, \eta_1) \neq \emptyset$. For each $\eta \in b'_k$, there is a unique β with $(\beta, \eta) \in b_k$.

(C) If $b_k \neq \emptyset$, then $a_k \neq \emptyset$ and $\max(a'_k) < \min(b'_k)$.

(D) If $(\beta, \sigma) \in a_k$ and $\min(a'_k) < \sigma$, then for all γ with $\beta \leq \gamma < \kappa$,
$$A_{\eta^*,\gamma} \cap \sup(N \cap \sigma) = A_{\sigma,\gamma} \cap \sup(N \cap \sigma).$$

(E) If $(\beta, \eta) \in b_k$, then for all γ with $\beta \leq \gamma < \kappa$,
$$A_{\eta,\gamma} = A_{\eta^*,\gamma} \cap \eta.$$

(F) If $(\beta, \eta) \in b_k$, then whenever $P \in M \cap \mathcal{Y}$ is such that
$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\eta^-, \eta) \neq \emptyset,$$
where η^- is the largest ordinal in $a'_k \cup b'_k$ less than η, then $\beta \leq P \cap \kappa$.

(G) Whenever $P \in M \cap \mathcal{Y}$ is such that
$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset,$$
then $P \cap N \cap \eta^* \subseteq \max(a'_k \cup b'_k)$.

(H) Suppose that $P \in M \cap \mathcal{Y}$ and τ satisfy that
$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } \tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*).$$
Let $\sigma := \min((a'_k \cup b'_k) \setminus (\tau + 1))$, which exists by (G), and assume that $\sigma \in a'_k$. Fix β with $(\beta, \sigma) \in a_k$. Then:
(i) $\beta \leq P \cap \kappa$;
(ii) $P \cap \sup(N \cap \sigma) = A_{\sigma, P\cap\kappa} \cap \sup(N \cap \sigma)$.

(I) If $(\beta, \sigma) \in a_k \cup b_k$, where $\min(a'_k) < \sigma$, then $P := Sk(A_{\eta^*,\beta})$ satisfies that
$$P \in M \cap \mathcal{Y},\ P \cap \kappa = \beta,\ P \cap \kappa^+ = A_{\eta^*,\beta}, \text{ and } P \cap N \cap [\sigma^-, \eta^*) \neq \emptyset,$$
where σ^- is the largest member of $a'_k \cup b'_k$ less than σ.

Note that since $\alpha_{M,N}$ is a limit point of M and $\alpha_{M,N} < \eta^*$, it follows that $\alpha_{M,N} \leq \sup(M \cap \eta^*)$.

Suppose that P is as in (G), and σ is the least ordinal in $a'_k \cup b'_k$ such that $P \cap N \cap \eta^* \subseteq \sigma$. By the minimality of σ, we can fix $\tau \in P \cap N \cap \eta^*$ such that $\sigma^- \leq \tau$, where σ^- is the greatest member of $a'_k \cup b'_k$ less than σ. Since N and P are closed under successors, $\tau + 1 \in P \cap N \cap \eta^*$. As $P \cap N \cap \eta^* \subseteq \sigma$, it follows that $\sigma = \min((a'_k \cup b'_k) \setminus (\tau + 1))$.

In the arguments which follow, we will frequently consider models $P \in M \cap \mathcal{Y}$ such that $P \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$, for example, in (G) and (H). Note that by Lemma 7.30, for any such P, η^* is a limit point of P. Therefore by Lemma 7.27, $P \cap \eta^* = A_{\eta^*, P\cap\kappa}$.

Assume that $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are sequences satisfying properties (A)–(I), where n is the least integer such that $b_n = \emptyset$. Let us show that the sets $a := a_n$ and $a' := \{\sigma : \exists \beta\ (\beta, \sigma) \in a\}$ satisfy properties (1)–(5) in the conclusion of Lemma 11.1.

(1) is immediate, and (2) follows from (A). (3(a)) follows from (D). For (3(b)), let us prove that the equation
$$A_{\eta^*,\gamma} \cap N \cap \sigma = A_{\sigma,\gamma} \cap N$$

follows from the equation

$$A_{\eta^*,\gamma} \cap \sup(N \cap \sigma) = A_{\sigma,\gamma} \cap \sup(N \cap \sigma),$$

which holds by (3(a)). Let $\xi \in A_{\eta^*,\gamma} \cap N \cap \sigma$, and we will show that $\xi \in A_{\sigma,\gamma}$. Then $\xi < \sup(N \cap \sigma)$, so by the last equation, $\xi \in A_{\sigma,\gamma}$. Conversely, let $\xi \in A_{\sigma,\gamma} \cap N$, and we will show that $\xi \in A_{\eta^*,\gamma}$. Then $\xi \in N \cap \sigma$, so $\xi < \sup(N \cap \sigma)$. By the last equation, $\xi \in A_{\eta^*,\gamma}$.

(4) Suppose that

$$P \in M \cap \mathcal{Y},\ P \cap \kappa \in M \cap N \cap \kappa, \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset.$$

By (G) and the fact that $b_n' = \emptyset$,

$$P \cap N \cap \eta^* \subseteq \max(a').$$

Let $\sigma \in a'$ be the least ordinal such that $P \cap N \cap \eta^* \subseteq \sigma$. Fix β such that $(\beta, \sigma) \in a$. Define

$$X := \{(\beta', \sigma') \in a : \beta' \leq P \cap \kappa\} \text{ and } X' := \{\sigma' : \exists \beta'\ (\beta', \sigma') \in X\}.$$

We will prove that $\sigma = \max(X')$, which completes the proof of (4).

By the minimality of σ, clearly there is $\tau \in N \cap P \cap [\sup(M \cap \eta^*), \eta^*)$ such that $\sigma = \min(a' \setminus (\tau + 1))$. By (H), $\beta \leq P \cap \kappa$. It follows that $(\beta, \sigma) \in X$, and so $\sigma \in X'$.

Suppose for a contradiction that there is $\sigma' \in X'$ which is larger than σ. Fix β' with $(\beta', \sigma') \in X$. Then $\sigma \leq (\sigma')^-$, where $(\sigma')^-$ is the largest member of a' which is less than σ'. By (I),

$$A_{\eta^*,\beta'} \cap N \cap [(\sigma')^-, \eta^*) \neq \emptyset.$$

Since $\sigma \leq (\sigma')^-$, it follows that

$$A_{\eta^*,\beta'} \cap N \cap [\sigma, \eta^*) \neq \emptyset.$$

As $\beta' \leq P \cap \kappa$ by the definition of X,

$$A_{\eta^*,\beta'} \subseteq A_{\eta^*,P\cap\kappa} = P \cap \eta^*.$$

But then $P \cap N \cap [\sigma, \eta^*) \neq \emptyset$, which contradicts that $P \cap N \cap \eta^* \subseteq \sigma$.

(5) Suppose that $P \in M \cap \mathcal{Y}$ satisfies that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset,$$

σ is the least ordinal in a' such that $P \cap N \cap \eta^* \subseteq \sigma$, and $(\beta, \sigma) \in a$. By the minimality of σ, we can fix $\tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*)$ such that $\sigma = \min(a' \setminus (\tau + 1))$. Then (5(a,b)) follow immediately from (H(i,ii)). For (5(c)), $P \cap N \cap \eta^* = P \cap N \cap \sigma$, and

$$P \cap N \cap \sigma = P \cap \sup(N \cap \sigma) \cap N = A_{\sigma,P\cap\kappa} \cap \sup(N \cap \sigma) \cap N = A_{\sigma,P\cap\kappa} \cap N.$$

We now turn to proving that there exist sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ satisfying properties (A)–(I), where n is the least integer such that $b_n = \emptyset$.

First we consider the base case. Let β be the least ordinal in $M \cap N \cap \kappa$ for which there exists $P \in M \cap \mathcal{Y}$ such that

$$P \cap \kappa = \beta \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset.$$

If there is no such β, then let $a_0 = \emptyset$ and $b_0 = \emptyset$, and we are done.

Suppose that β exists. Then obviously $N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset$. Define

$$a_0 := \{(0, \min((N \cap \kappa^+) \setminus \sup(M \cap \eta^*)))\} \text{ and } b_0 := (\beta, \eta^*).$$

In the case that $a_0 = b_0 = \emptyset$, the inductive hypotheses are all vacuously true. In the other case, the inductive hypotheses are all either vacuously true or trivial.

Now we handle the induction step. Assume that $k < \omega$ and a_k and b_k have been defined and satisfy the inductive hypotheses. If $b_k = \emptyset$, then we are done. Assume that b_k is nonempty. Then by (C), $a_k \neq \emptyset$. Let η be the least member of b'_k, and let β be the unique ordinal such that $(\beta, \eta) \in b_k$. By (A) and (B), $\max(a'_k)$ and η are limit ordinals. By (C), $\max(a'_k) < \eta$, and in particular, $\omega < \eta$.

First consider the easy case that $\eta = \eta_0 + \omega$ for some limit ordinal η_0. Let $a_{k+1} := a_k$. If $\max(a'_k) < \eta_0$, then let $b_{k+1} := (b_k \setminus \{(\beta, \eta)\}) \cup \{(\beta, \eta_0)\}$. Suppose that $\eta_0 \leq \max(a'_k)$. Since $\max(a'_k)$ is a limit ordinal and $\max(a'_k) < \eta$, clearly $\max(a'_k) = \eta_0$. In this case, let $b_{k+1} := b_k \setminus \{(\beta, \eta)\}$. All of the inductive hypotheses can be easily checked, using Notation 7.2(4) and the fact that if $P \in \mathcal{Y}$ and $P \cap N \cap [\eta_0, \eta) \neq \emptyset$, then by the elementarity of $P \cap N$, $\eta \in P \cap N$.

From now on we will assume that η is a limit of limit ordinals. In particular, every ordinal in C_η is a limit ordinal by Notation 7.2(3).

Define
$$\theta := \sup(\lim(C_\eta) \cap \mathrm{cl}(N)).$$
We split the definition of a_{k+1} and b_{k+1} into two cases.

Case 1: $\theta = \sup(N)$.

Note that since $\max(a'_k) \in N$ and $\sup(N) = \theta$, it follows that $\max(a'_k) < \theta$.

Claim 1: $\eta = \max(b'_k)$. Suppose for a contradiction that there is $\eta' \in b'_k$ greater than η. Fix β' with $(\beta', \eta') \in b_k$. Then $\eta \leq (\eta')^-$, where $(\eta')^-$ is the largest ordinal in $a'_k \cup b'_k$ less than η'. By (I), $A_{\eta^*,\beta'} \cap N \cap [(\eta')^-, \eta^*) \neq \emptyset$. Fix τ in this intersection. Then $\theta \leq \eta \leq (\eta')^- \leq \tau$ and $\tau \in N$, which contradicts that $\theta = \sup(N)$.

Since N is simple, $\mathrm{ot}(C_\theta) = \sup(N \cap \kappa)$. Let $\xi := \sup(M \cap N \cap \kappa)$, which is in $N \cap \kappa$ by Lemma 1.30. Define $\sigma := c_{\theta,\xi}$. Since $\theta = \sup(N)$ and $\xi \in N$, $\sigma \in N$ by Lemma 7.19. As ξ has countable cofinality, so does σ. In particular, $N \cap \sigma$ is cofinal in σ.

Claim 2: $A_{\theta,\xi} = A_{\sigma,\xi}$ and $\sup(A_{\sigma,\xi}) = \sigma$. As ξ is a limit ordinal, $\sigma = c_{\theta,\xi} \in \lim(C_\theta)$. Therefore $A_{\sigma,\xi} = A_{\theta,\xi} \cap \sigma$. In particular, $A_{\sigma,\xi} \subseteq A_{\theta,\xi}$. On the other hand, since $\xi < \sup(N \cap \kappa) = \mathrm{ot}(C_\theta)$, it follows that $A_{\theta,\xi} \subseteq c_{\theta,\xi} = \sigma$ by Notation 7.4(6). So $A_{\theta,\xi} \subseteq A_{\theta,\xi} \cap \sigma = A_{\sigma,\xi}$. This proves that $A_{\theta,\xi} = A_{\sigma,\xi}$.

Since $\xi = \sup(M \cap N \cap \kappa)$, by the elementarity of $M \cap N$, ξ is a limit of limit ordinals. Therefore $\sigma = c_{\theta,\xi}$ is a limit of $\lim(C_\theta) \cap \sigma$. For any ordinal $\zeta \in \lim(C_\theta) \cap \sigma$, the fact that $\zeta \in \lim(C_\theta) \cap c_{\theta,\xi}$ and $\xi < \mathrm{ot}(C_\theta)$ implies by Notation 7.4(6) that $\zeta \in A_{\theta,\xi}$. So $\lim(C_\theta) \cap \sigma$ is cofinal in σ and is a subset of $A_{\theta,\xi}$. Hence $\sup(A_{\sigma,\xi}) = \sup(A_{\theta,\xi}) = \sigma$.

We now define a_{k+1} and b_{k+1}. Let $b_{k+1} := \emptyset$. If $\sigma \leq \max(a'_k)$, then let $a_{k+1} := a_k$. If $\max(a'_k) < \sigma$, then let $a_{k+1} := a_k \cup \{(\beta, \sigma)\}$.

We prove that the inductive hypotheses are maintained. First, consider the case when $\sigma \leq \max(a'_k)$. Then $a_{k+1} = a_k$ and $b_{k+1} = \emptyset$. Inductive hypotheses (A), (B), (C), (D), (E), (F), and (I) are all either vacuously true, or follow immediately from the inductive hypotheses.

For (G) and (H), suppose that $P \in M \cap \mathcal{Y}$ satisfies that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset.$$

By inductive hypothesis (G) and Claim 1,

$$P \cap N \cap \eta^* \subseteq \max(a'_k \cup b'_k) = \eta.$$

If $P \cap N \cap \eta^* \subseteq \max(a'_k)$, then this proves (G) for $k+1$, and in that case (H) follows immediately from the inductive hypotheses.

Otherwise $P \cap N \cap [\max(a'_k), \eta) \neq \emptyset$. Let us show that this is impossible. Since $\max(a'_k)$ is the predecessor of η in $a'_k \cup b'_k$, inductive hypothesis (F) implies that $\beta \leq P \cap \kappa$. Inductive hypothesis (E) then implies that

$$A_{\eta, P\cap\kappa} = A_{\eta^*, P\cap\kappa} \cap \eta.$$

As η^* is a limit point of P,

$$P \cap \eta = A_{\eta^*, P\cap\kappa} \cap \eta = A_{\eta, P\cap\kappa}.$$

Since $\theta \in \lim(C_\eta)$,

$$A_{\theta, P\cap\kappa} = A_{\eta, P\cap\kappa} \cap \theta = P \cap \theta.$$

As $\theta = \sup(N)$,

$$P \cap N \cap \eta = P \cap N \cap \theta = A_{\theta, P\cap\kappa} \cap N.$$

And as $P \cap \kappa \in M \cap N \cap \kappa$ and $\xi = \sup(M \cap N \cap \kappa)$, it follows that $P \cap \kappa \leq \xi$, so

$$A_{\theta, P\cap\kappa} \subseteq A_{\theta,\xi} = A_{\sigma,\xi}$$

by Claim 2. So

$$P \cap N \cap \eta = A_{\theta, P\cap\kappa} \cap N \subseteq A_{\sigma,\xi} \subseteq \sigma \leq \max(a'_k).$$

This contradicts the initial assumption that $P \cap N \cap [\max(a'_k), \eta) \neq \emptyset$.

Secondly, consider the case that $\max(a'_k) < \sigma$. Then $a_{k+1} = a_k \cup \{(\beta, \sigma)\}$ and $b_{k+1} = \emptyset$. We prove that the inductive hypotheses are maintained. Inductive hypotheses (A), (B), (C), (E), and (F) are all either vacuously true, or follow immediately from the inductive hypotheses. It remains to show (D), (G), (H), and (I).

(D) By inductive hypothesis (D), we only need to check that (D) holds for $k+1$ in the case of (β, σ). As noted in the paragraph before Claim 2, $N \cap \sigma$ is cofinal in σ. Also observe that since $\theta \in \lim(C_\eta)$, $\sigma = c_{\theta,\xi} = c_{\eta,\xi}$ is a limit point of C_η.

Let $\beta \leq \gamma < \kappa$ be given. Since $(\beta, \eta) \in b_k$, inductive hypothesis (E) implies that $A_{\eta,\gamma} = A_{\eta^*,\gamma} \cap \eta$. Since σ is a limit point of C_η and $\sup(N \cap \sigma) = \sigma$, it follows that $A_{\sigma,\gamma} \cap \sup(N \cap \sigma) = A_{\sigma,\gamma} \cap \sigma = A_{\sigma,\gamma} = A_{\eta,\gamma} \cap \sigma = (A_{\eta^*,\gamma} \cap \eta) \cap \sigma = A_{\eta^*,\gamma} \cap \sigma = A_{\eta^*,\gamma} \cap \sup(N \cap \sigma)$, which proves (D).

(G) Suppose that $P \in M \cap \mathcal{Y}$ satisfies that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset.$$

By inductive hypothesis (G) and Claim 1,

$$P \cap N \cap \eta^* \subseteq \max(a'_k \cup b'_k) = \eta.$$

If $P \cap N \cap \eta^* \subseteq \max(a'_k)$, then since $\max(a'_k) \leq \max(a'_{k+1})$, we are done.

So assume that there exists $\tau \in (P \cap N \cap \eta^*) \setminus \max(a'_k)$. We will show that $P \cap N \cap \eta^* \subseteq \sigma$, which completes the proof since $\sigma = \max(a'_{k+1})$. As $P \cap N \cap \eta^* \subseteq \eta$, it follows that $\tau \in P \cap N \cap [\max(a'_k), \eta)$. Since $\max(a'_k)$ is the largest member of $a'_k \cup b'_k$ less than η, inductive hypothesis (F) implies that $\beta \leq P \cap \kappa$. Inductive hypothesis (E) then implies that

$$A_{\eta, P\cap\kappa} = A_{\eta^*, P\cap\kappa} \cap \eta.$$

But $\theta \in \lim(C_\eta)$ and $\theta = \sup(N)$, so

$$P \cap N \cap \eta = A_{\eta^*, P\cap\kappa} \cap N \cap \eta = A_{\eta, P\cap\kappa} \cap N = A_{\eta, P\cap\kappa} \cap N \cap \theta = A_{\theta, P\cap\kappa} \cap N.$$

Hence $P \cap N \cap \eta \subseteq A_{\theta, P\cap\kappa}$. Since $P \cap \kappa \in M \cap N \cap \kappa$, $P \cap \kappa < \sup(M \cap N \cap \kappa) = \xi$. So by Claim 2,

$$P \cap N \cap \eta \subseteq A_{\theta, P\cap\kappa} \subseteq A_{\theta,\xi} = A_{\sigma,\xi} \subseteq \sigma.$$

Since $P \cap N \cap \eta^* \subseteq \eta$ as noted above, we have that

$$P \cap N \cap \eta^* = P \cap N \cap \eta \subseteq \sigma = \max(a'_{k+1}).$$

(H) Suppose that $P \in M \cap \mathcal{Y}$ and τ satisfy that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } \tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*).$$

Let $\sigma' = \min(a'_{k+1} \setminus (\tau + 1))$. Fix β' with $(\beta', \sigma') \in a_{k+1}$. If $\sigma' < \sigma$, then clearly $\sigma' = \min(a'_k \setminus (\tau + 1))$, so (i) and (ii) follow from inductive hypothesis (H).

Suppose that $\sigma' = \sigma$, which means that $\max(a'_k) \leq \tau + 1$. Then $\beta' = \beta$. Since $\max(a'_k)$ is the largest ordinal in $a'_k \cup b'_k$ less than η, inductive hypothesis (F) implies that $\beta \leq P \cap \kappa$, which proves (i). By inductive hypothesis (E),

$$A_{\eta, P\cap\kappa} = A_{\eta^*, P\cap\kappa} \cap \eta.$$

Since $\sigma \in \lim(C_\eta)$,

$$P \cap \sigma = A_{\eta^*, P\cap\kappa} \cap \sigma = A_{\eta, P\cap\kappa} \cap \sigma = A_{\sigma, P\cap\kappa} = A_{\sigma, P\cap\kappa} \cap \sigma.$$

As $\sigma = \sup(N \cap \sigma)$, we have that $P \cap \sup(N \cap \sigma) = A_{\sigma, P\cap\kappa} \cap \sup(N \cap \sigma)$, which proves (ii).

(I) By inductive hypothesis (I), it suffices to consider (β, σ). Let $P := Sk(A_{\eta^*,\beta})$. Since $(\beta, \eta) \in b_k$ and $\max(a'_k)$ is the largest ordinal in $a'_k \cup b'_k$ less than η, inductive hypothesis (I) implies that $P \in M \cap \mathcal{Y}$, $P \cap \kappa = \beta$, $P \cap \kappa^+ = A_{\eta^*,\beta}$, and $P \cap N \cap [\max(a'_k), \eta^*) \neq \emptyset$. Since $\max(a'_k)$ is also the largest ordinal in $a'_{k+1} \cup b'_{k+1}$ less than σ, we are done.

Case 2: $\theta < \sup(N)$.

Let $\sigma' := \min((N \cap \kappa^+) \setminus \theta)$, which exists by Case 2. If $\sigma' \leq \max(a'_k)$, then let $\sigma := \max(a'_k)$ and $a_{k+1} := a_k$. If $\max(a'_k) < \sigma'$, then let $\sigma := \sigma'$ and $a_{k+1} := a_k \cup \{(\beta, \sigma)\}$.

Define A as the set of ordinals of the form $\min(C_\eta \setminus (\xi + 1))$, where for some $P \in M \cap \mathcal{Y}$ with $P \cap \kappa \in M \cap N \cap \kappa$, $\xi \in P \cap N \cap [\sigma, \eta)$.[9]

[9]The set A could be empty. In fact, it is possible for example that $\theta = \eta^*$ and $\sigma = \min((N \cap \kappa^+) \setminus \eta^*)$, so that $\eta < \sigma$. In this case we interpret $[\sigma, \eta)$ to be the empty set, so that A is empty as well.

Note that every ordinal in A is a limit ordinal, since C_η consists of limit ordinals, and is strictly greater than σ. Suppose that $\gamma_0 < \gamma_1$ are in A. Then for some $\xi \in P \cap N \cap [\sigma, \eta)$, $\gamma_1 = \min(C_\eta \setminus (\xi + 1))$. So $\gamma_0 \leq \xi < \xi + 1 < \gamma_1$. In particular, $N \cap [\gamma_0, \gamma_1) \neq \emptyset$.

Claim: A is finite. Suppose for a contradiction that A is infinite. Fix an increasing sequence $\langle \gamma_n : n < \omega \rangle$ from A. Then $\sigma < \gamma_0$, and for each n, $\gamma_n \in C_\eta$ and $N \cap [\gamma_n, \gamma_{n+1}) \neq \emptyset$. It follows that the ordinal $\sup\{\gamma_n : n < \omega\}$ is in $\lim(C_\eta) \cap \mathrm{cl}(N)$, and yet is greater than σ and hence θ. This contradicts the definition of θ.

For each $\delta \in A$, define β_δ as the least ordinal in $M \cap N \cap \kappa$ such that for some $P \in M \cap \mathcal{Y}$, $P \cap \kappa = \beta_\delta$, and there is $\xi \in P \cap N \cap [\sigma, \eta)$ such that $\delta = \min(C_\eta \setminus (\xi + 1))$. Note that β_δ exists by the definition of A. Also as $\max(a_k') \leq \sigma$, $P \cap N \cap [\max(a_k'), \eta) \neq \emptyset$. Therefore since $\max(a_k')$ is the largest ordinal in $a_k' \cup b_k'$ less than η, inductive hypothesis (F) implies that $\beta \leq P \cap \kappa = \beta_\delta$.

Define $b_{k+1} := (b_k \setminus \{(\beta, \eta)\}) \cup \{(\beta_\delta, \delta) : \delta \in A\}$.

We verify the inductive hypotheses. Hypotheses (A) and (B) are straightforward to check.

(C) We know that $a_{k+1} \neq \emptyset$ and $\max(a_{k+1}') = \sigma$. If A is nonempty, then

$$\max(a_{k+1}') = \sigma < \min(A) = \min(b_{k+1}').$$

If A is empty and b_{k+1} is nonempty, then $\min(b_{k+1}')$ is the least member of b_k' greater than η. So if $\max(a_{k+1}') = \max(a_k')$, then by inductive hypothesis (C),

$$\max(a_{k+1}') = \max(a_k') < \min(b_k') = \eta < \min(b_{k+1}').$$

Suppose that $\max(a_k') < \sigma$. By inductive hypothesis (B), we have that $N \cap [\eta, \min(b_{k+1}')) \neq \emptyset$. Since $\theta \leq \eta$, $N \cap [\theta, \min(b_{k+1}')) \neq \emptyset$. As $\sigma = \min((N \cap \kappa^+) \setminus \theta)$, this implies that $\max(a_{k+1}') = \sigma < \min(b_{k+1}')$.

(D) By inductive hypothesis (D), it suffices to consider (β, σ) in the case where $\max(a_k') < \sigma$ and $a_{k+1} = a_k \cup \{(\beta, \sigma)\}$. So $\sigma = \min((N \cap \kappa^+) \setminus \theta)$, and therefore $\theta = \sup(N \cap \sigma)$. Let $\beta \leq \gamma < \kappa$. Then by Lemma 7.12(2),

$$A_{\theta,\gamma} = A_{\sigma,\gamma} \cap \theta.$$

Since $(\beta, \eta) \in b_k$, inductive hypothesis (E) implies that

$$A_{\eta,\gamma} = A_{\eta^*,\gamma} \cap \eta.$$

And since $\theta \in \lim(C_\eta)$,

$$A_{\theta,\gamma} = A_{\eta,\gamma} \cap \theta.$$

Therefore

$$A_{\sigma,\gamma} \cap \theta = A_{\theta,\gamma} = A_{\eta,\gamma} \cap \theta = A_{\eta^*,\gamma} \cap \theta.$$

Since $\sup(N \cap \sigma) = \theta$, this proves (D).

(E) Consider $(\beta_\delta, \delta) \in b_{k+1}$, where $\delta \in A$. Fix $P \in M \cap \mathcal{Y}$ and ξ such that $P \cap \kappa = \beta_\delta$, $\xi \in P \cap N \cap [\sigma, \eta)$, and $\delta = \min(C_\eta \setminus (\xi + 1))$. Let $\beta_\delta \leq \gamma < \kappa$, and we will show that

$$A_{\delta,\gamma} = A_{\eta^*,\gamma} \cap \delta.$$

As observed above, $\beta \leq \beta_\delta \leq \gamma$. Since $(\beta, \eta) \in b_k$, by inductive hypothesis (E),

$$A_{\eta,\beta_\delta} = A_{\eta^*,\beta_\delta} \cap \eta \text{ and } A_{\eta,\gamma} = A_{\eta^*,\gamma} \cap \eta.$$

As $\xi \in N \cap P$, by elementarity $\xi + 1 \in N \cap P$. Hence

$$\xi + 1 \in P \cap \eta = A_{\eta^*,P\cap\kappa} \cap \eta = A_{\eta^*,\beta_\delta} \cap \eta = A_{\eta,\beta_\delta}.$$

So $\xi + 1 \in A_{\eta,\beta_\delta} \setminus C_\eta$. By Notation 7.4(7), $\min(C_\eta \setminus (\xi + 1)) = \delta$ is in A_{η,β_δ}. Since $\beta_\delta \leq \gamma$, $\delta \in A_{\eta,\gamma}$. Therefore

$$A_{\delta,\gamma} = A_{\eta,\gamma} \cap \delta = (A_{\eta^*,\gamma} \cap \eta) \cap \delta = A_{\eta^*,\gamma} \cap \delta,$$

proving (E).

(F) Let $(\gamma, \zeta) \in b_{k+1}$. Then either $(\gamma, \zeta) = (\beta_\delta, \delta)$ for some $\delta \in A$, or $(\gamma, \zeta) \in b_k$ and $\eta < \zeta$.

Case a: $(\gamma, \zeta) = (\beta_\delta, \delta)$ for some $\delta \in A$. Suppose that $P \in M \cap \mathcal{Y}$ satisfies that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\delta^-, \delta) \neq \emptyset,$$

where δ^- is the greatest member of $a'_{k+1} \cup b'_{k+1}$ which is less than δ. Then clearly $\sigma = \max(a'_{k+1}) \leq \delta^-$. So if we fix $\xi \in P \cap N \cap [\delta^-, \delta)$, then $\sigma \leq \xi$ and $\delta = \min(C_\eta \setminus (\xi + 1))$. By the minimality of β_δ, $\beta_\delta \leq P \cap \kappa$.

Case b: $(\gamma, \zeta) \in b_k$ and $\eta < \zeta$. If ζ is not the least element of b'_k greater than η, then the greatest ordinal in $a'_k \cup b'_k$ less than ζ is equal to the greatest ordinal in $a'_{k+1} \cup b'_{k+1}$ less than ζ. In that case, (F) follows easily from inductive hypothesis (F).

Suppose that ζ is the least member of b'_k greater than η. Then the greatest member of $a'_{k+1} \cup b'_{k+1}$ less than ζ, which we denote by ζ^-, is equal to either $\max(A)$ if A is nonempty, or σ if A is empty.

Assume that $P \in M \cap \mathcal{Y}$ satisfies that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\zeta^-, \zeta) \neq \emptyset.$$

We will show that $\gamma \leq P \cap \kappa$. If $P \cap N \cap [\eta, \zeta) \neq \emptyset$, then since η is the greatest member of $a'_k \cup b'_k$ less than ζ, $\gamma \leq P \cap \kappa$ by inductive hypothesis (F).

Otherwise $P \cap N \cap [\zeta^-, \eta) \neq \emptyset$. Fix ξ in this intersection. Then $\xi + 1$ is also in this intersection, by the elementarity of $P \cap N$ and because η is a limit ordinal. By the definition of A, $\min(C_\eta \setminus (\xi+1))$ is in A. So A is nonempty, and hence $\zeta^- = \max(A)$. Yet $\min(C_\eta \setminus (\xi + 1))$ is in A and is strictly greater than $\zeta^- = \max(A)$, which is a contradiction.

(G) Let $P \in M \cap \mathcal{Y}$ satisfy that

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } P \cap N \cap [\sup(M \cap \eta^*), \eta^*) \neq \emptyset.$$

By inductive hypothesis (G),

$$P \cap N \cap \eta^* \subseteq \max(a'_k \cup b'_k) = \max(b'_k).$$

If $\max(b'_k) = \max(b'_{k+1})$, then we are done. Otherwise η is equal to $\max(b'_k)$, so $P \cap N \cap \eta^* \subseteq \eta$. If $P \cap N \cap \eta^*$ is not a subset of $\max(a'_{k+1} \cup b'_{k+1})$, then there is $\xi \in P \cap N \cap \eta^*$ such that $\max(a'_{k+1}) = \sigma \leq \xi$, and also $\max(A) = \max(b'_{k+1}) \leq \xi$ if A is nonempty. So $\xi \in P \cap N \cap [\sigma, \eta)$, which implies that $\min(C_\eta \setminus (\xi + 1))$ is in A. So A is nonempty, and $\max(A) \leq \xi < \min(C_\eta \setminus (\xi + 1)) \in A$, which is impossible.

(H) Suppose that $P \in M \cap \mathcal{Y}$ and τ satisfy

$$P \cap \kappa \in M \cap N \cap \kappa \text{ and } \tau \in P \cap N \cap [\sup(M \cap \eta^*), \eta^*).$$

Let

$$\sigma^* := \min((a'_{k+1} \cup b'_{k+1}) \setminus (\tau + 1)),$$

and assume that $\sigma^* \in a'_{k+1}$. Fix β^* with $(\beta^*, \sigma^*) \in a_{k+1}$.

If $\sigma^* \in a'_k$, then the conclusion of (H) follows immediately from inductive hypothesis (H) for a_k. Otherwise we are in the case that $\max(a'_k) < \sigma$ and $\sigma^* = \sigma$. Hence also $\beta^* = \beta$. Clearly $\min((a'_k \cup b'_k) \setminus (\tau + 1))$ is equal to η. Since $(\beta, \eta) \in b_k$ and $\max(a'_k)$ is the greatest member of $a'_k \cup b'_k$ less than η, inductive hypothesis (F) implies that $\beta \leq P \cap \kappa$, proving (H(i)).

By inductive hypothesis (E),

$$A_{\eta, P\cap\kappa} = A_{\eta^*, P\cap\kappa} \cap \eta = P \cap \eta.$$

Since $\theta \leq \eta$,

$$A_{\eta, P\cap\kappa} \cap \theta = P \cap \theta.$$

As $\theta \in \lim(C_\eta)$,

$$A_{\theta, P\cap\kappa} = A_{\eta, P\cap\kappa} \cap \theta.$$

By Lemma 7.12(2),

$$A_{\theta, P\cap\kappa} = A_{\sigma, P\cap\kappa} \cap \theta.$$

So

$$P \cap \theta = A_{\eta, P\cap\kappa} \cap \theta = A_{\theta, P\cap\kappa} = A_{\sigma, P\cap\kappa} \cap \theta.$$

Since $\sup(N \cap \sigma) = \theta$, it follows that

$$P \cap \sup(N \cap \sigma) = A_{\sigma, P\cap\kappa} \cap \sup(N \cap \sigma),$$

which proves (H(ii)).

(I) Let $(\gamma, \zeta) \in a_{k+1} \cup b_{k+1}$, where $\min(a'_{k+1}) = \min(a'_k) < \zeta$. If $(\gamma, \zeta) \in a_k$, then the conclusion of (I) follows from inductive hypothesis (I). If $(\gamma, \zeta) \in a_{k+1} \setminus a_k$, then $(\gamma, \zeta) = (\beta, \sigma)$ and $\max(a'_k) < \sigma$. Let $P := Sk(A_{\eta^*,\beta})$. Since $(\beta, \eta) \in b_k$, by inductive hypothesis (I) we know that

$$P \in M \cap \mathcal{Y}, \ P \cap \kappa = \beta, \text{ and } P \cap \kappa^+ = A_{\eta^*,\beta}.$$

Also since $\max(a'_k)$ is the greatest member of $a'_k \cup b'_k$ less than η, inductive hypothesis (I) implies that

$$P \cap N \cap [\max(a'_k), \eta^*) \neq \emptyset.$$

But the greatest member of $a'_{k+1} \cup b'_{k+1}$ less than σ is also equal to $\max(a'_k)$, so we are done.

Suppose that $(\gamma, \zeta) \in b_k$ and $\eta < \zeta$. If ζ is not the second element of b'_k, then (I) follows immediately from inductive hypothesis (I). Suppose that ζ is the second element of b'_k. Then η is the greatest member of $a'_k \cup b'_k$ less than ζ. Let $P := Sk(A_{\eta^*,\gamma})$. By inductive hypothesis (I),

$$P \in M \cap \mathcal{Y}, \ P \cap \kappa = \gamma, \ P \cap \kappa^+ = A_{\eta^*,\gamma}, \text{ and } P \cap N \cap [\eta, \eta^*) \neq \emptyset.$$

Let ζ^- denote the largest member of $a'_{k+1} \cup b'_{k+1}$ less than ζ, and we will show that $P \cap N \cap [\zeta^-, \eta^*) \neq \emptyset$. If $\zeta^- \leq \eta$, then this follows immediately from the fact that $P \cap N \cap [\eta, \eta^*) \neq \emptyset$. If A is nonempty, then clearly $\zeta^- = \max(A) < \eta$, and we are done.

Suppose that A is empty. If $\min((N\cap\kappa^+)\setminus\theta) \le \max(a'_k)$, then $\zeta^- = \max(a'_k) < \eta$, and we are done. Suppose that $\max(a'_k) < \sigma = \min((N\cap\kappa^+)\setminus\theta)$. By inductive hypothesis (B), $N\cap[\eta,\zeta) \neq \emptyset$. Since $\theta \le \eta$, $N\cap[\theta,\zeta) \neq \emptyset$. As $\sigma = \min((N\cap\kappa^+)\setminus\theta)$, it follows that $\sigma < \zeta$, and so clearly $\zeta^- = \sigma$. If $\sigma \le \eta$, then we are done. Otherwise $\sigma = \min((N\cap\kappa^+)\setminus\eta)$. We know that $P\cap N\cap[\eta,\eta^*) \neq \emptyset$. But the first member of this intersection must be greater than or equal to $\min((N\cap\kappa^+)\setminus\eta) = \sigma$. Hence $P\cap N\cap[\sigma,\eta^*) \neq \emptyset$, and we are done since $\sigma = \zeta^-$.

In the final case, assume that (γ,ζ) is equal to (β_δ,δ), for some $\delta \in A$. By the definition of β_δ, there exists $Q \in M\cap\mathcal{Y}$ and ξ such that

$$Q\cap\kappa = \beta_\delta,\ \xi \in Q\cap N\cap[\sigma,\eta),\ \text{and}\ \delta = \min(C_\eta\setminus(\xi+1)).$$

So $Q\cap\eta^* = A_{\eta^*,\beta_\delta}$. Since Q and η^* are closed under H^*, it follows that A_{η^*,β_δ} is closed under H^*.

We will show that $P := Sk(A_{\eta^*,\beta_\delta})$ satisfies the conclusions of (I). Since A_{η^*,β_δ} is closed under H^*,

$$P\cap\kappa^+ = A_{\eta^*,\beta_\delta} = Q\cap\eta^*.$$

Hence also

$$P\cap\kappa = Q\cap\kappa = \beta_\delta.$$

Since η^* is a limit point of Q,

$$\sup(P) = \sup(Q\cap\eta^*) = \eta^*.$$

As η^* and β_δ are in M, so is P.

To show that $P \in \mathcal{Y}$, it suffices to show that $\lim(C_{\eta^*})\cap P$ is cofinal in η^*. Since $Q\cap\eta^* = P\cap\eta^*$, η^* is a limit point of Q. If η^* is not in Q, then $\eta^* \in \mathrm{cl}(Q)\setminus Q$. By Lemma 7.13,

$$\lim(C_{\eta^*})\cap Q = \lim(C_{\eta^*})\cap P$$

is cofinal in η^*. Otherwise $\eta^* \in Q$. Since η^* is a limit point of Q, $\mathrm{cf}(\eta^*) < \kappa$. Therefore $\mathrm{ot}(C_{\eta^*}) \in Q\cap\kappa$ by elementarity. Hence $C_{\eta^*} \subseteq Q$ by elementarity. Since η^* has uncountable cofinality, $\lim(C_{\eta^*})$ is cofinal in η^*. So

$$\lim(C_{\eta^*})\cap P = \lim(C_{\eta^*})\cap Q = \lim(C_{\eta^*})$$

is cofinal in η^*.

Let δ^- denote the largest member of $a'_{k+1}\cup b'_{k+1}$ which is less than δ. Then either $\delta^- = \sigma$ if $\delta = \min(A)$, or else δ^- is the largest member of A which is less than δ. In the first case, the ordinal ξ, which is in $Q\cap N\cap[\sigma,\eta)$, is a witness to the fact that $Q\cap N\cap[\delta^-,\eta^*) \neq \emptyset$. In the second case, $\xi+1$ must be greater than δ^-, since otherwise $\delta = \min(C_\eta\setminus(\xi+1)) \le \delta^-$. So $\xi+1$ is a witness to the fact that $Q\cap N\cap[\delta^-,\eta^*) \neq \emptyset$. In either case, since $Q\cap\eta^* \subseteq P$, $P\cap N\cap[\delta^-,\eta^*) \neq \emptyset$.

§13. Amalgamation of side conditions

We are now in a position to prove amalgamation results for $\vec{S}$-obedient side conditions over simple models in $\mathcal{X}$, strong models in $\mathcal{Y}$, and transitive models. The proofs of these results will use almost the entirety of the technology developed in the paper thus far. In Part III, the amalgamation results we present here will be used to prove the existence of strongly generic conditions.

Proposition 13.1. *Let (A, B) be an $\vec{S}$-obedient side condition, where $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$. Suppose that $N \in A$ is simple, and (A, B) is closed under canonical models with respect to N. Assume that for all $M \in A$, if $M < N$ then $M \cap N \in A$.*

Let (C, D) be an $\vec{S}$-obedient side condition, where $C \subseteq \mathcal{X}$ and $D \subseteq \mathcal{Y}$, such that

$$A \cap N \subseteq C \subseteq N \text{ and } B \cap N \subseteq D \subseteq N.$$

Also assume that $N' \in C$ is simple, and for all $M \in A$, if $M < N$, then there is M' in C such that

$$M' < N', \ M \cap N = M' \cap N', \text{ and } p(M, N) = p(M', N').$$

Then $(A \cup C, B \cup D)$ is an $\vec{S}$-obedient side condition.

Proof. First note that for all $M \in A$, if $M < N$ then $M \cap N \in C$. For since N is simple, $M \cap N \in N$ by Lemma 8.2. So $M \cap N \in A \cap N \subseteq C$.

Consider $M < N$ in A. Since $M \cap N = M' \cap N'$ and $M' < N'$, it follows by Lemma 1.19(2) that

$$M \cap \beta_{M,N} = M \cap N \cap \kappa = M' \cap N' \cap \kappa = M' \cap \beta_{M',N'}.$$

So $M \cap \beta_{M,N} = M' \cap \beta_{M',N'}$. Also by Lemma 1.19(3),

$$\beta_{M,N} = \min(\Lambda \setminus \sup(M \cap N \cap \kappa)) = \min(\Lambda \setminus \sup(M' \cap N' \cap \kappa)) = \beta_{M',N'}.$$

So $\beta_{M,N} = \beta_{M',N'}$.

To show that $(A \cup C, B \cup D)$ is $\vec{S}$-obedient, we verify properties (1), (2), and (3) of Definition 5.3. (2) is immediate.

(3) Let $M \in C$ and $P \in B$. Let $\beta := P \cap \kappa$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \beta)$. Fix $\tau \in M \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. If $\beta = \zeta$, then $\zeta \in S_\tau$ since P is $\vec{S}$-strong. So assume that $\beta < \zeta$, which means that $\beta \notin M$. If $P \in N$ then $P \in B \cap N \subseteq D$, so $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Assume that $P \notin N$. Since $M \in C$ and $C \subseteq N$, $M \in N$. Therefore ζ and τ are in N. Hence $P \cap \kappa = \beta < \zeta < \sup(N \cap \kappa)$. Let $\xi := \min((N \cap \kappa) \setminus \beta)$. Since $M \subseteq N$, $\zeta = \min((M \cap \kappa) \setminus \xi)$. By Lemma 10.9(1), there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \xi$ and $\tau \in Q$. Then $\zeta = \min((M \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in M \cap Q \cap \kappa^+$. So $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Let $M \in A$ and $P \in D$. Let $\beta := P \cap \kappa$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \beta)$. Fix $\tau \in M \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. If $\beta = \zeta$, then $\zeta \in S_\tau$ since P is $\vec{S}$-strong. So assume that $\beta < \zeta$, which means that $\beta \notin M$.

Since $P \in D$ and $D \subseteq N$, $P \in N$. So $\beta = P \cap \kappa$ is in N by elementarity.

Case 1: $\beta_{M,N} \leq \beta$. Since $\beta \in N$, $\zeta = \min((M \cap \kappa) \setminus \beta)$ is in $R_N(M)$. As $P \in N \cap \mathcal{Y}$ is $\vec{S}$-strong, $\tau \in M \cap P \cap \kappa^+$, and

$$\sup(M \cap \zeta) < P \cap \kappa = \beta < \zeta,$$

it follows that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate.

Case 2: $\beta < \beta_{M,N} \leq \zeta$. Then $\zeta = \min((M \cap \kappa) \setminus \beta_{M,N})$. Since $\beta \in (N \cap \beta_{M,N}) \setminus M$, it follows that $M < N$. Therefore $\zeta \in R_N(M)$. As $P \in N \cap \mathcal{Y}$ is $\vec{S}$-strong, $\tau \in M \cap P \cap \kappa^+$, and

$$\sup(M \cap \zeta) < P \cap \kappa = \beta < \zeta,$$

we have that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate.

Case 3: $\beta < \zeta < \beta_{M,N}$. Since $\beta \in (N \cap \beta_{M,N}) \setminus M$, it follows that $M < N$. As $M < N$, $P \in N \cap \mathcal{Y}$, and

$$P \cap \kappa < \zeta < \sup(M \cap \beta_{M,N}) = \sup(M \cap N \cap \kappa),$$

it follows that $M \cap P \cap \kappa^+ \subseteq N$ by Lemma 8.7. In particular, $\tau \in M \cap N \cap \kappa^+$. As $\zeta < \beta_{M,N}$ and $M \cap \beta_{M,N} = M \cap N \cap \kappa$, $\zeta = \min((M \cap N \cap \kappa) \setminus \beta)$. But $M \cap N \in C$, $P \in D$, and $\tau \in (M \cap N) \cap P \cap \kappa^+$. So $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

(1) Now we prove that $A \cup C$ is $\vec{S}$-adequate. By Proposition 1.29, $A \cup C$ is adequate. Let $M \in A$ and $L \in C$. Then $L \in N$. We will prove that the remainder points in $R_M(L)$ and $R_L(M)$ are as required.

First, consider $\zeta \in R_L(M)$. Then by Lemmas 2.4 and 2.5, either (1) $M < N$, $\zeta < \beta_{M,N}$, and $\zeta \in R_L(M \cap N)$, or (2) $\beta_{M,N} \leq \zeta$ and $\zeta \in R_N(M)$.

Case 1: $M < N$, $\zeta < \beta_{M,N}$, and $\zeta \in R_L(M \cap N)$. Recall that L and $M \cap N$ are in C. Fix $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $L \in N$, $\tau \in N$. So $\tau \in L \cap (M \cap N)$. Since $\zeta \in R_L(M \cap N)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Suppose that $P \in L \cap \mathcal{Y}$ is $\vec{S}$-strong,

$$\sup(M \cap \zeta) < P \cap \kappa < \zeta, \text{ and } \tau \in M \cap P \cap \kappa^+.$$

We will show that $\zeta \in S_\tau$. Since $P \in L$ and $L \in N$, $P \in N$. So $P \in N \cap \mathcal{Y}$, $M < N$, and

$$P \cap \kappa < \zeta < \sup(M \cap \beta_{M,N}) = \sup(M \cap N \cap \kappa).$$

By Lemma 8.7, $M \cap P \cap \kappa^+ \subseteq N$. In particular, $\tau \in N$. So $\tau \in (M \cap N) \cap P \cap \kappa^+$. Since $\zeta < \beta_{M,N}$ and $M < N$, we have that $M \cap \zeta = M \cap N \cap \zeta$. Therefore

$$\sup((M \cap N) \cap \zeta) = \sup(M \cap \zeta) < P \cap \kappa < \zeta.$$

So $\zeta \in R_L(M \cap N)$, $P \in L \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup((M \cap N) \cap \zeta) < P \cap \kappa < \zeta$, and $\tau \in (M \cap N) \cap P \cap \kappa^+$. It follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Case 2: $\beta_{M,N} \leq \zeta$ and $\zeta \in R_N(M)$. Fix $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $L \in N$, $\tau \in N$. So $\tau \in M \cap N \cap \kappa^+$. As $\zeta \in R_N(M)$, it follows that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate.

Suppose that $P \in L \cap \mathcal{Y}$ is $\vec{S}$-strong and $\sup(M \cap \zeta) < P \cap \kappa < \zeta$. Fix $\tau \in M \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $P \in L$ and $L \in N$, $P \in N$. So $P \in N \cap \mathcal{Y}$ is $\vec{S}$-strong. Since $\zeta \in R_N(M)$, $\sup(M \cap \zeta) < P \cap \kappa < \zeta$, and $\tau \in M \cap P$, it follows that $\zeta \in S_\tau$ since A is $\vec{S}$-adequate.

This completes the proof that the ordinals in $R_L(M)$ are as required.

Now consider $\zeta \in R_M(L)$. Then by Lemmas 2.4 and 2.5, either $M < N$ and $\zeta \in R_{M \cap N}(L)$, or there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$. Since $\zeta \in L$ and $L \in N$, $\zeta \in N$.

Let $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $L \in N$, $\tau \in N$. So $\tau \in L \cap (M \cap N)$. First, assume that $M < N$ and $\zeta \in R_{M \cap N}(L)$. Since $\zeta \in R_{M \cap N}(L)$ and $\tau \in L \cap (M \cap N)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Secondly, assume that there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$. Since $\xi \in R_M(N)$ and $\tau \in M \cap N \cap \kappa^+$, by Lemma 10.9(2) there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \xi$ and $\tau \in Q$. Then $\zeta = \min((L \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in L \cap Q \cap \kappa^+$, which implies that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Suppose that $P \in M \cap \mathcal{Y}$ is $\vec{S}$-strong,
$$\sup(L \cap \zeta) < P \cap \kappa < \zeta, \text{ and } \tau \in L \cap P \cap \kappa^+.$$
We will prove that $\zeta \in S_\tau$. Note that since $\tau \in L$ and $L \in N$, $\tau \in N$.

Case 1: $\beta_{M,N} \leq P \cap \kappa$. Let $\theta := \min((N \cap \kappa) \setminus (P \cap \kappa))$. Note that θ exists since $\zeta \in N$. Also $\zeta = \min((L \cap \kappa) \setminus \theta)$. Since $P \cap \kappa \in (M \cap \kappa) \setminus \beta_{M,N}$, it follows that $\theta \in R_M(N)$ and
$$\sup(N \cap \theta) < P \cap \kappa < \theta.$$
Also $\tau \in N \cap P \cap \kappa^+$. By Lemma 10.9(4), there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \theta$ and $\tau \in Q$. But then $\zeta = \min((L \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in L \cap Q \cap \kappa^+$, which implies that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Case 2: $P \cap \kappa < \beta_{M,N}$ and $N \leq M$. Recall that either $M < N$ and $\zeta \in R_{M \cap N}(L)$, or there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$. Since $N \leq M$, we are in the second case.

Subcase 2(a): $P \cap \kappa < \sup(N \cap \beta_{M,N})$. Since $N \leq M$, $\sup(N \cap \beta_{M,N}) = \sup(M \cap N \cap \kappa)$. Therefore $P \cap \kappa < \sup(M \cap N \cap \kappa)$. Since $\tau \in N \cap P \cap \kappa^+$, it follows that $\tau \in M$ by Lemma 8.7. So $\tau \in M \cap N \cap \kappa^+$. Since $\xi \in R_M(N)$, by Lemma 10.9(2) there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \xi$ and $\tau \in Q$. Since $\zeta = \min((L \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in L \cap Q \cap \kappa^+$, it follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Subcase 2(b): $\sup(N \cap \beta_{M,N}) \leq P \cap \kappa$. Since $\sup(N \cap \beta_{M,N})$ has countable cofinality and $P \cap \kappa$ has uncountable cofinality, we have that $\sup(N \cap \beta_{M,N}) < P \cap \kappa$. Let $\delta := \min((N \cap \kappa) \setminus \beta_{M,N})$, which exists since $\zeta \in N$. As $N \leq M$, $\delta \in R_M(N)$. Also
$$\sup(N \cap \delta) = \sup(N \cap \beta_{M,N}) < P \cap \kappa < \delta$$
and $\tau \in N \cap P \cap \kappa^+$. By Lemma 10.9(4), there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \delta$ and $\tau \in Q$. Since $\sup(N \cap \delta) < P \cap \kappa < \delta$, we have that $\delta = \min((N \cap \kappa) \setminus (P \cap \kappa))$. As $L \subseteq N$ and $\sup(L \cap \zeta) < P \cap \kappa < \zeta$, clearly $\zeta = \min((L \cap \kappa) \setminus \delta)$. So $\zeta = \min((L \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in L \cap Q \cap \kappa^+$. It follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Case 3: $P \cap \kappa < \beta_{M,N}$ and $M < N$. Then $P \cap \kappa \in M \cap \beta_{M,N} = M \cap N \cap \kappa$.

Subcase 3(a): $\tau < \alpha_{M,N}$ and $P \cap \alpha_{M,N}$ is bounded below $\alpha_{M,N}$. Then $\tau \in P \cap N \cap \alpha_{M,N}$. By Lemma 11.7, there is $P' \in M \cap N \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $P' \cap \kappa = P \cap \kappa$ and $\tau \in P'$.

Recall that either $\zeta \in R_{M \cap N}(L)$, or there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$. Suppose first that $\zeta \in R_{M \cap N}(L)$. Since $P' \in M \cap N \cap \mathcal{Y}$ is $\vec{S}$-strong,

$\sup(L \cap \zeta) < P \cap \kappa = P' \cap \kappa < \zeta$, and $\tau \in L \cap P' \cap \kappa^+$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Now suppose that there is $\xi \in R_M(N)$ such that $\zeta = \min((L \cap \kappa) \setminus \xi)$. Then by Lemma 10.9(3), there is $Q \in B \cap N \subseteq D$ such that $Q \cap \kappa = \xi$ and $N \cap P' \cap \kappa^+ \subseteq Q$. So $\zeta = \min((L \cap \kappa) \setminus (Q \cap \kappa))$ and $\tau \in L \cap Q \cap \kappa^+$. It follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Case 3(b): Either $\tau < \alpha_{M,N}$ and $P \cap \alpha_{M,N}$ is unbounded in $\alpha_{M,N}$, or $\alpha_{M,N} \leq \tau$. In the first case, we apply Lemma 11.6 letting $\tau_0 = \tau$ to get that there exists $Q \in N' \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $Q \cap \kappa = P \cap \kappa$ and $\tau \in Q$. In the second case, we apply the main proxy lemma, Lemma 11.5. Since $\tau \in P$ and $P \in M$,

$$\alpha_{M,N} \leq \tau < \sup(P) < \sup(M).$$

Let $\eta := \min((M \cap \kappa^+) \setminus \tau)$, which is in $R^+_N(M)$. Note that the assumptions of Lemma 11.5 for $\eta^* = \eta$ are satisfied. By Lemma 11.5(1), there is $Q \in N' \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $Q \cap \kappa = P \cap \kappa$ and $\tau \in Q$. In either case, we have that $Q \in N' \cap \mathcal{Y}$ is $\vec{S}$-strong, $Q \cap \kappa = P \cap \kappa$, and $\tau \in Q$.

Let us note that if $\zeta \in R_{N'}(L)$, then $\zeta \in S_\tau$ and we are done. For $Q \in N' \cap \mathcal{Y}$ is $\vec{S}$-strong,

$$\sup(L \cap \zeta) < P \cap \kappa = Q \cap \kappa < \zeta,$$

and $\tau \in L \cap Q \cap \kappa^+$. It follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Subcase 3(b(i)): $\beta_{L,N'} \leq \zeta$. We claim that $\zeta \in R_{N'}(L)$, which finishes the proof. If $\beta_{L,N'} \leq P \cap \kappa$, then $P \cap \kappa = Q \cap \kappa \in (N' \cap \kappa) \setminus \beta_{L,N'}$. Therefore $\zeta = \min((L \cap \kappa) \setminus (P \cap \kappa))$ is in $R_{N'}(L)$. Suppose on the other hand that $P \cap \kappa < \beta_{L,N'}$. Then $\zeta = \min((L \cap \kappa) \setminus \beta_{L,N'})$. Since $P \cap \kappa \in (N' \cap \beta_{L,N'}) \setminus L$, we have that $L < N'$. So $\zeta \in R_{N'}(L)$.

Subcase 3(b(ii)): $\zeta < \beta_{L,N'}$. In particular, since $Q \cap \kappa = P \cap \kappa < \zeta$ and $\zeta \in L \cap \beta_{L,N'}$, it follows that

$$Q \cap \kappa < \sup(L \cap \beta_{L,N'}) < \beta_{L,N'}.$$

As $Q \cap \kappa \in (N' \cap \beta_{L,N'}) \setminus L$, we have that $L < N'$. So $L < N'$, $Q \in N' \cap \mathcal{Y}$, and

$$Q \cap \kappa < \sup(L \cap \beta_{L,N'}) = \sup(L \cap N' \cap \kappa).$$

By Lemma 8.7, $Q \cap L \cap \kappa^+ \subseteq N'$. Since $\tau \in L \cap Q \cap \kappa^+$, it follows that $\tau \in N'$. By Lemma 11.5(2) in the case that $\alpha_{M,N} \leq \tau$, and by Lemma 11.6(2) in the case that $\tau < \alpha_{M,N}$, there is $P' \in M' \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $P' \cap \kappa = P \cap \kappa$ and $\tau \in P'$.

By Lemmas 1.27(1) and 2.6(1), either $\beta_{L,M} = \beta_{L,M \cap N} = \beta_{L,M'}$, or $\beta_{M,N} < \beta_{L,M'}$. Suppose first that $\beta_{L,M} = \beta_{L,M'}$. We claim that

$$\beta_{L,M'} \leq P \cap \kappa.$$

Suppose for a contradiction that $P \cap \kappa < \beta_{L,M'} = \beta_{L,M}$. Since Λ is cofinal in $P \cap \kappa$ by the elementarity of P, and $\sup(L \cap \zeta) < P \cap \kappa$, we can find $\pi \in \Lambda \cap P \cap \kappa$ such that $\sup(L \cap \zeta) < \pi$. Then $\pi < \beta_{L,M}$. By Lemma 1.19(5),

$$L \cap M \cap [\pi, \beta_{L,M}) \neq \emptyset.$$

Fix ξ in this intersection. As $\zeta \in R_M(L)$, $\beta_{L,M} \leq \zeta$. So $\xi \in L$ and

$$\sup(L \cap \zeta) < \pi \leq \xi < \beta_{L,M} \leq \zeta.$$

But $\sup(L \cap \zeta) < \xi < \zeta$ and $\xi \in L$ is obviously impossible. Hence indeed $\beta_{L,M'} \leq P \cap \kappa$.

So $P \cap \kappa = P' \cap \kappa \in (M' \cap \kappa) \setminus \beta_{L,M'}$. Since $\zeta = \min((L \cap \kappa) \setminus (P' \cap \kappa))$, we have that $\zeta \in R_{M'}(L)$. As $P' \in M' \cap \mathcal{Y}$ is $\vec{S}$-strong, $\sup(L \cap \zeta) < P' \cap \kappa < \zeta$, and $\tau \in L \cap P' \cap \kappa^+$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

The other alternative is that $\beta_{M,N} < \beta_{L,M'}$. We will show that this is impossible. So assume that $\beta_{M,N} < \beta_{L,M'}$. We claim that $L < M'$. For $P \cap \kappa = P' \cap \kappa \in M'$. Also

$$P \cap \kappa < \beta_{M,N} < \beta_{L,M'}.$$

So

$$P \cap \kappa \in (M' \cap \beta_{L,M'}) \setminus L,$$

which implies that $L < M'$.

Next we claim that $\beta_{M,N} \leq \zeta$. Suppose for a contradiction that $\zeta < \beta_{M,N}$. Then $\zeta \in L \cap \beta_{M,N} \subseteq L \cap \beta_{L,M'}$, and $L \cap \beta_{L,M'} \subseteq M'$ since $L < M'$. So

$$\zeta \in M' \cap \beta_{M,N} = M' \cap \beta_{M',N'} = M \cap \beta_{M,N}.$$

Hence $\zeta \in M$. But this is not true since $\zeta \in R_M(L)$.

Since $P \cap \kappa < \beta_{M,N} \leq \zeta$ and $\sup(L \cap \zeta) < P \cap \kappa$, clearly $\zeta = \min((L \cap \kappa) \setminus \beta_{M,N})$. Since $\beta_{M,N} < \beta_{L,M'}$ and $\beta_{M,N} \in \Lambda$, by Lemma 1.19(5) we have that $L \cap M' \cap [\beta_{M,N}, \beta_{L,M'})$ is nonempty. Since $\zeta = \min((L \cap \kappa) \setminus \beta_{M,N})$, it follows that $\zeta < \beta_{L,M'}$. As $L < M'$, $\zeta \in M'$.

Now we will get a contradiction. By Subcase 3(b(ii)), $\zeta < \beta_{L,N'}$. So $\zeta \in L \cap \beta_{L,N'}$. Since $L < M' < N'$, we have that $L < N'$. Hence $\zeta \in N'$. So $\zeta \in M' \cap N' \cap \kappa$, which implies that $\zeta < \beta_{M',N'} = \beta_{M,N}$. But $\beta_{M,N} \leq \zeta$, and we have a contradiction. □

Proposition 13.2. *Let (A, B) be an $\vec{S}$-obedient side condition, where $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$. Suppose that $P \in B$ satisfies that* $\mathrm{cf}(\sup(P)) = P \cap \kappa$. *Assume that for all $M \in A$, $M \cap P \in A$, and for all $Q \in B$, if $Q \cap \kappa < P \cap \kappa$ then $Q \cap P \in B$.*

Let (C, D) be an $\vec{S}$-obedient side condition, where $C \subseteq \mathcal{X}$ and $D \subseteq \mathcal{Y}$, such that

$$A \cap P \subseteq C \subseteq P \text{ and } B \cap P \subseteq D \subseteq P.$$

In addition, assume that there exists $P' \in D$ such that $\mathrm{cf}(\sup(P')) = P' \cap \kappa$, *and for all $M \in A$, there exists $M' \in C$ such that*

$$M \cap P = M' \cap P' \text{ and } p(M, P) = p(M', P').$$

Then $(A \cup C, B \cup D)$ is an $\vec{S}$-obedient side condition.

Proof. By Lemma 8.3, P and P' are simple. Note that for all $M \in A$, $M \cap P \in C$. For $M \cap P \in P$ by Lemma 8.4, and so $M \cap P \in A \cap P \subseteq C$. Similarly, for all $Q \in B$ with $Q \cap \kappa < P \cap \kappa$, $Q \cap P \in D$. For $Q \cap P \in P$ by Lemma 8.4, and hence $Q \cap P \in B \cap P \subseteq D$.

Let $\beta := P \cap \kappa$ and $\beta' := P' \cap \kappa$. Consider $M \in A$. Then by our assumptions,

$$M \cap \beta = M \cap P \cap \kappa = M' \cap P' \cap \kappa = M' \cap \beta'.$$

So $M \cap \beta = M' \cap \beta'$. In particular, since β' has uncountable cofinality, $\sup(M \cap \beta) < \beta'$.

To show that $(A \cup C, B \cup D)$ is $\vec{S}$-obedient, we verify properties (1), (2), and (3) of Definition 5.3. (2) is immediate.

(3) Let $M \in A$ and $Q \in D$. Let $\theta := Q \cap \kappa$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \theta)$. Let $\tau \in Q \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$.

Case 1: $P \cap \kappa \leq \zeta$. Since $Q \in P$, $\theta < P \cap \kappa$, so $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$. As $\tau \in Q$ and $Q \in P$, $\tau \in P$. So $\tau \in P \cap M \cap \kappa^+$. Therefore $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

Case 2: $\zeta < P \cap \kappa$. Then $\zeta \in M \cap P \cap \kappa$. Since $M \cap P \cap \kappa = M \cap \beta$ is an initial segment of $M \cap \kappa$, $\zeta = \min((M \cap P \cap \kappa) \setminus \theta)$. As $\tau \in Q$ and $Q \in P$, $\tau \in P$. So $\tau \in Q \cap (M \cap P) \cap \kappa^+$. Since $M \cap P \in C$, it follows that $\zeta \in S_\tau$ as (C, D) is $\vec{S}$-obedient.

Let $M \in C$ and $Q \in B$. Let $\theta := Q \cap \kappa$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \theta)$. Fix $\tau \in Q \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\zeta \in M$ and $M \in P$, $\zeta \in P \cap \kappa$. Hence $Q \cap \kappa < P \cap \kappa$. So by our assumptions, $Q \cap P \in D$. As $\tau \in M$ and $M \in P$, $\tau \in P$. So $\tau \in (Q \cap P) \cap M \cap \kappa^+$. Since $Q \cap P \cap \kappa = Q \cap \kappa = \theta$,

$$\zeta = \min((M \cap \kappa) \setminus (Q \cap P \cap \kappa)).$$

It follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

(1) The set $A \cup C$ is adequate by Proposition 1.35. Let $M \in A$ and $L \in C$. Then $L \in P$. We will prove that the remainder points in $R_M(L)$ and $R_L(M)$ are as required.

Consider $\zeta \in R_L(M)$. Then by Lemma 2.9, either $\zeta \in R_L(M \cap P)$ or $\zeta = \min((M \cap \kappa) \setminus \beta)$.

Case 1: $\zeta \in R_L(M \cap P)$. Fix $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in P$, $\tau \in P$. So $\tau \in L \cap (M \cap P) \cap \kappa^+$. Since $\zeta \in R_L(M \cap P)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Suppose that $Q \in L \cap \mathcal{Y}$ is $\vec{S}$-strong and

$$\sup(M \cap \zeta) < Q \cap \kappa < \zeta.$$

Fix $\tau \in Q \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $M \cap P \cap \kappa = M \cap \beta$ is an initial segment of $M \cap \kappa$ and $\zeta \in M \cap P \cap \kappa$,

$$\sup(M \cap P \cap \zeta) = \sup(M \cap \zeta) < Q \cap \kappa < \zeta.$$

As $Q \in L$ and $L \in P$, $Q \in P$. And since $\tau \in Q$ and $Q \in P$, $\tau \in P$. So $\tau \in Q \cap (M \cap P) \cap \kappa^+$. Since $\zeta \in R_L(M \cap P)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Case 2: $\zeta = \min((M \cap \kappa) \setminus \beta)$. Fix $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in P$, $\tau \in P$. So $\tau \in P \cap M \cap \kappa^+$. Since $M \in A$ and $P \in B$, it follows that $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

Suppose that $Q \in L \cap \mathcal{Y}$ is $\vec{S}$-strong and

$$\sup(M \cap \zeta) < Q \cap \kappa < \zeta.$$

Fix $\tau \in Q \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. As $Q \in L$ and $L \in P$, $Q \in P$. And since $\tau \in Q$ and $Q \in P$, $\tau \in P$. So $\tau \in M \cap P \cap \kappa^+$. As $M \in A$ and $P \in B$, it follows that $\zeta \in S_\tau$ since (A, B) is $\vec{S}$-obedient.

Consider $\zeta \in R_M(L)$. By Lemma 2.9, $\zeta \in R_{M \cap P}(L)$. Fix $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in P$, $\tau \in P$. So $\tau \in L \cap (M \cap P) \cap \kappa^+$. As $\zeta \in R_{M \cap P}(L)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Suppose that $Q \in M \cap \mathcal{Y}$ is $\vec{S}$-strong and

$$\sup(L \cap \zeta) < Q \cap \kappa < \zeta.$$

Fix $\tau \in Q \cap L \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\zeta \in L$ and $L \in P$, $\zeta \in P \cap \kappa$. Therefore $Q \cap \kappa \in M \cap P \cap \kappa$. As $\tau \in L$ and $L \in P$, $\tau \in P$. So $\tau \in Q \cap P \cap \kappa^+$.

Case 1: $\tau < \alpha_{M,P}$ and $Q \cap \alpha_{M,P}$ is bounded below $\alpha_{M,P}$. Then $\tau \in Q \cap P \cap \alpha_{M,P}$. By Lemma 11.7, there is $Q' \in M \cap P \cap \mathcal{Y}$ which is $\vec{S}$-strong such that $Q' \cap \kappa = Q \cap \kappa$ and $\tau \in Q'$. So

$$\sup(L \cap \zeta) < Q' \cap \kappa = Q \cap \kappa < \zeta$$

and $\tau \in Q' \cap L \cap \kappa^+$. Since $\zeta \in R_{M \cap P}(L)$ and $Q' \in (M \cap P) \cap \mathcal{Y}$ is $\vec{S}$-strong, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Case 2: Either $\tau < \alpha_{M,P}$ and $Q \cap \alpha_{M,P}$ is unbounded in $\alpha_{M,P}$, or $\alpha_{M,P} \leq \tau$. In the first case, we apply Lemma 11.6(1) to get $Q^* \in P' \cap \mathcal{Y}$ such that $\tau \in Q^*$. In the second case, we apply the main proxy lemma, Lemma 11.5. Assuming $\alpha_{M,P} \leq \tau$, let $\eta := \min((M \cap \kappa^+) \setminus \tau)$. Note that η exists since $\sup(Q) \in M$ and $\tau < \sup(Q)$. Also

$$\tau \in Q \cap P \cap [\sup(M \cap \eta), \eta).$$

By Lemma 11.5(1) there is $Q^* \in P' \cap \mathcal{Y}$ such that $\tau \in Q^*$.

Thus in either case, there is $Q^* \in P' \cap \mathcal{Y}$ such that $\tau \in Q^*$. As $\tau \in Q^*$ and $Q^* \in P'$, $\tau \in P'$. Also by Lemma 11.6(2) in the first case, and Lemma 11.5(3) in the second case, there is $Q' \in M' \cap \mathcal{Y}$ such that Q' is $\vec{S}$-strong, $Q' \cap \kappa = Q \cap \kappa$, and $\tau \in Q'$.

By Lemma 2.10(2), either $\beta_{L,M} = \beta_{L,M'}$ or $\beta' < \beta_{L,M'}$. First, suppose that $\beta_{L,M} = \beta_{L,M'}$. Since $\zeta \in R_{M \cap P}(L)$, Lemma 2.10(3) implies that $\zeta \in R_{M'}(L)$. But $Q' \in M' \cap \mathcal{Y}$ is $\vec{S}$-strong,

$$\sup(L \cap \zeta) < Q' \cap \kappa = Q \cap \kappa < \zeta,$$

and $\tau \in Q' \cap L \cap \kappa^+$. It follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Secondly, assume that $\beta' < \beta_{L,M'}$. Since

$$Q \cap \kappa \in M \cap P \cap \kappa = M \cap \beta = M' \cap \beta',$$

it follows that

$$Q \cap \kappa < \beta' < \beta_{L,M'}.$$

As $Q \cap \kappa = Q' \cap \kappa \in (M' \cap \beta_{L,M'}) \setminus L$, we have that $L < M'$.

We claim that $\beta' \leq \zeta$. Otherwise since $L < M'$,

$$\zeta \in L \cap \beta' \subseteq L \cap \beta_{L,M'} \subseteq M'.$$

So $\zeta \in M' \cap \beta' = M \cap \beta$. Hence $\zeta \in M$, which contradicts that $\zeta \in R_M(L)$.

Since

$$\sup(L \cap \zeta) < Q \cap \kappa < \beta' \leq \zeta,$$

we have that $\zeta = \min((L \cap \kappa) \setminus \beta')$. As noted above, $\tau \in P'$. So $\tau \in L \cap P' \cap \kappa^+$, and

$$\zeta = \min((L \cap \kappa) \setminus \beta') = \min((L \cap \kappa) \setminus (P' \cap \kappa)).$$

It follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient. $\square$

Proposition 13.3. *Let (A, B) be an $\vec{S}$-obedient side condition, where $A \subseteq \mathcal{X}$ and $B \subseteq \mathcal{Y}$. Suppose that $X \prec \mathcal{A}$ is such that $|X| = \kappa$, $X \cap \kappa^+ \in \kappa^+$, and $X^{<\kappa} \subseteq X$. Let $\theta := X \cap \kappa^+$.*

Let $M_0, \ldots, M_{k-1}$ and $P_0, \ldots, P_{m-1}$ enumerate the members of A and B respectively. For each $i < k$, let $\langle Q^i_n : n < \omega \rangle$ enumerate the $\vec{S}$-strong models in $M_i \cap \mathcal{Y}$.

Let (C, D) be an $\vec{S}$-obedient side condition, where $C \subseteq \mathcal{X}$ and $D \subseteq \mathcal{Y}$, such that

$$A \cap X \subseteq C \subseteq X \text{ and } B \cap X \subseteq D \subseteq X.$$

Assume that $M'_0, \ldots, M'_{k-1}$, $P'_0, \ldots, P'_{m-1}$, θ', and $\langle R^i_n : n < \omega \rangle$ for $i < k$ satisfy the following properties:

1. *$\theta' \in \theta \cap \mathrm{cof}(\kappa)$;*
2. *for all $i < k$, $M'_i \in C$ and $M_i \cap \theta = M'_i \cap \theta'$;*
3. *for all $j < m$, $P'_j \in D$ and $P_j \cap \theta = P'_j \cap \theta'$;*
4. *for all $i < k$, $\langle R^i_n : n < \omega \rangle$ enumerates the $\vec{S}$-strong models in $M'_i \cap \mathcal{Y}$, and for all $n < \omega$, $Q^i_n \cap \theta = R^i_n \cap \theta'$.*

Then $(A \cup C, B \cup D)$ is an $\vec{S}$-obedient side condition.

Proof. To show that $(A \cup C, B \cup D)$ is $\vec{S}$-obedient, we verify properties (1), (2), and (3) of Definition 5.3. (2) is immediate.

(3) Let $M \in C$ and $P \in B$. Fix $j < m$ such that $P = P_j$, and let $P' := P'_j$. Let $\beta := P \cap \kappa$, and suppose that $\zeta := \min((M \cap \kappa) \setminus \beta)$. Fix $\tau \in M \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in M$ and $M \in X$, $\tau \in X \cap \kappa^+ = \theta$. So $\tau \in P \cap \theta = P' \cap \theta'$. Also $P' \cap \kappa = P \cap \kappa = \beta$. Hence $\tau \in M \cap P' \cap \kappa^+$ and $\zeta = \min((M \cap \kappa) \setminus (P' \cap \kappa))$. It follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

Let $M \in A$ and $P \in D$. Fix $i < k$ such that $M = M_i$, and let $M' := M'_i$. Let $\beta := P \cap \kappa$, and suppose that $\zeta = \min((M \cap \kappa) \setminus \beta)$. Fix $\tau \in M \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in P$ and $P \in X$, $\tau \in X \cap \kappa^+ = \theta$. Hence $\tau \in M \cap \theta = M' \cap \theta'$. Also $M \cap \kappa = M' \cap \kappa$, so $\zeta = \min((M' \cap \kappa) \setminus \beta)$. Since $\tau \in M' \cap P \cap \kappa^+$, it follows that $\zeta \in S_\tau$ since (C, D) is $\vec{S}$-obedient.

(1) The set $A \cup C$ is adequate by Proposition 1.38. Let $M \in A$ and $L \in C$. We will prove that the remainder points in $R_M(L)$ and $R_L(M)$ are as required. Fix $i < k$ such that $M = M_i$, and let $M' := M'_i$.

Consider $\zeta \in R_L(M)$. Since $M \cap \kappa = M' \cap \kappa$, $\zeta \in R_L(M')$ by Lemma 2.11. Let $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in X$, $\tau \in X \cap \kappa^+ = \theta$. Hence $\tau \in M \cap \theta = M' \cap \theta'$. Therefore $\tau \in L \cap M'$. Since $\zeta \in R_L(M')$, it follows that $\zeta \in S_\tau$ as C is $\vec{S}$-adequate.

Suppose that $P \in L \cap \mathcal{Y}$ is $\vec{S}$-strong and

$$\sup(M \cap \zeta) < P \cap \kappa < \zeta.$$

Let $\tau \in P \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $P \in L$ and $L \in X$, $P \in X$. And as $\tau \in P$ and $P \in X$, $\tau \in X \cap \kappa^+ = \theta$. Hence $\tau \in M \cap \theta = M' \cap \theta'$. Therefore $\tau \in P \cap M' \cap \kappa^+$. Also since $M \cap \kappa = M' \cap \kappa$,

$$\sup(M' \cap \zeta) = \sup(M \cap \zeta) < P \cap \kappa < \zeta.$$

As $\zeta \in R_L(M')$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Now consider $\zeta \in R_M(L)$. Since $M \cap \kappa = M' \cap \kappa$, $\zeta \in R_{M'}(L)$ by Lemma 2.11. Let $\tau \in L \cap M \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in X$, $\tau \in X \cap \kappa^+ = \theta$. Hence $\tau \in M \cap \theta = M' \cap \theta'$. Therefore $\tau \in L \cap M'$. Since $\zeta \in R_{M'}(L)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate.

Let $P \in M \cap \mathcal{Y}$ be $\vec{S}$-strong, and assume that

$$\sup(L \cap \zeta) < P \cap \kappa < \zeta.$$

Let $\tau \in L \cap P \cap \kappa^+$, and we will show that $\zeta \in S_\tau$. Since $\tau \in L$ and $L \in X$, $\tau \in X \cap \kappa^+ = \theta$. Fix $n < \omega$ such that $P = Q^i_n$. Then

$$\tau \in P \cap \theta = Q^i_n \cap \theta = R^i_n \cap \theta'.$$

So $\tau \in R^i_n \cap L \cap \kappa^+$. Now $R^i_n \in M' \cap \mathcal{Y}$ is $\vec{S}$-strong, and

$$\sup(L \cap \zeta) < P \cap \kappa = Q^i_n \cap \kappa = R^i_n \cap \kappa < \zeta.$$

Since $\zeta \in R_{M'}(L)$, it follows that $\zeta \in S_\tau$ since C is $\vec{S}$-adequate. $\square$

Part 3. Mitchell's Theorem

§14. The ground model

With the general development of side conditions from Parts I and II at our disposal, we now begin our proof of Mitchell's theorem. We start by describing the ground model over which we will force a generic extension satisfying that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$.

We will use the same notation which was introduced at the beginning of Parts I and II, together with some additional assumptions. Recall that $\kappa \geq \omega_2$ is regular, $2^\kappa = \kappa^+$, and $\Box_\kappa$. Also the cardinal λ from Part I is equal to κ^+. In addition, we will assume that κ is a greatly Mahlo cardinal, and the thin stationary set T^* from Notation 1.4 is equal to $P_{\omega_1}(\kappa)$.

Define a sequence of sets $\langle S_\xi : \xi < \kappa^+\rangle$ inductively as follows. Let S_0 denote the set of inaccessible cardinals less than κ. Let $\delta < \kappa^+$, and suppose that S_ξ has been defined for all $\xi < \delta$. If $\delta = \delta_0 + 1$, then let $\alpha \in S_\delta$ if α is inaccessible, $\alpha \in S_{\delta_0}$, and $S_{\delta_0} \cap \alpha$ is stationary in α. If δ is a limit ordinal, then let $\alpha \in S_\delta$ if α is inaccessible, and for all $\xi \in A_{\delta,\alpha}$, $\alpha \in S_\xi$. Let $\vec{S} := \langle S_\xi : \xi < \kappa^+\rangle$.

The fact that κ is greatly Mahlo implies that for all $\delta < \kappa^+$, S_δ is stationary in κ. In fact, it is easily seen that this consequence is actually equivalent to κ being greatly Mahlo. See [1, Definition 4.2] for more information about greatly Mahlo cardinals.

Notation 14.1. *For the remainder of Part III, the structure $\mathcal{A}$ from Notation 7.6 will be equal to*

$$(H(\kappa^+), \in, \trianglelefteq, \kappa, T^*, \pi^*, C^*, \Lambda, \mathcal{Y}_0, f^*, \vec{C}, \vec{A}, c^*, \vec{S}).$$

In Proposition 7.20, we proved that the set of simple models in $\mathcal{X}$ is stationary in $P_{\omega_1}(H(\kappa^+))$. We will prove in Proposition 15.3 that most simple models in $\mathcal{X}$ have strongly generic conditions. The next proposition describes the kind of models in $\mathcal{Y}$ which will have strongly generic conditions.

Proposition 14.2. *There are stationarily many $P \in P_\kappa(H(\kappa^+))$ such that $P \in \mathcal{Y}$, P is $\vec{S}$-strong, and* $\mathrm{cf}(\sup(P)) = P \cap \kappa$.

Proof. Let $F : H(\kappa^+)^{<\omega} \to H(\kappa^+)$. Fix X which is an elementary substructure of $\mathcal{A}$ of size κ such that X is closed under F, and $\tau := X \cap \kappa^+$ has cofinality κ. Note that $X = Sk(X \cap \kappa^+) = Sk(\tau)$. Since τ is the union of the increasing and continuous sequence of sets $\{A_{\tau,i} : i < \kappa\}$, it follows that X is the union of the increasing and continuous sequence of sets $\{Sk(A_{\tau,i}) : i < \kappa\}$.

For all infinite $\beta < \kappa$, $|A_{\tau,\beta}| \leq |\beta| < \kappa$ by Notation 7.4(3). Since τ has cofinality κ, $\sup(A_{\tau,\beta}) < \tau$, and hence $\sup(A_{\tau,\beta}) \in X$. Fix a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $A_{\tau,\alpha}$ is closed under H^*, $A_{\tau,\alpha} \cap \kappa = \alpha$, $Sk(A_{\tau,\alpha})$ is closed under F, and for all $\beta < \alpha$, $\sup(A_{\tau,\beta}) \in A_{\tau,\alpha}$. As S_τ is stationary in κ, we can fix $\alpha \in \lim(C) \cap S_\tau$. Let $P := Sk(A_{\tau,\alpha})$.

We claim that $P \in \mathcal{Y}$, P is $\vec{S}$-strong, $\mathrm{cf}(\sup(P)) = P \cap \kappa$, and P is closed under F. The last statement follows from the fact that $\alpha \in C$. Since $A_{\tau,\alpha}$ is closed under H^*, $P \cap \kappa^+ = A_{\tau,\alpha}$. In particular, since $\alpha \in C$, $P \cap \kappa = \alpha$. As $\alpha \in S_\tau$, α is inaccessible. After we show that $\mathrm{cf}(\sup(P)) = \alpha$, it will follow that $P \in \mathcal{Y}$ by Lemma 7.15.

To show that P is $\vec{S}$-strong, let $\sigma \in P \cap \kappa^+$. Then $\sigma \in P \cap \kappa^+ = A_{\tau,\alpha}$. Since $\alpha \in S_\tau$ and τ is a limit ordinal, for all $\pi \in A_{\tau,\alpha}$, $\alpha \in S_\pi$. In particular, $\alpha \in S_\sigma$.

It remains to show that $\mathrm{cf}(\sup(P)) = \alpha$. For $\alpha_0 < \alpha_1$ in $C \cap \alpha$,

$$\sup(A_{\tau,\alpha_0}) \in A_{\tau,\alpha_1} \subseteq P$$

by the definition of C. Since $P \cap \kappa$ is a limit point of C,

$$A_{\tau,\alpha} = \bigcup\{A_{\tau,\beta} : \beta \in C \cap \alpha\}.$$

Therefore

$$\sup(P) = \sup(A_{\tau,\alpha}) = \sup\{\sup(A_{\tau,\beta}) : \beta \in C \cap \alpha\},$$

which is the supremum of a strictly increasing sequence. Since α is in S_τ, α is inaccessible, so $C \cap \alpha$ has order type α. Hence $\sup(P)$ has cofinality equal to $P \cap \kappa = \alpha$. □

§15. The forcing poset

We now define and analyze the forcing poset which will force that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$.

Definition 15.1. *Let $\mathbb{P}$ be the forcing poset consisting of pairs $p = (A_p, B_p)$ satisfying:*

(1) *$A_p \subseteq \mathcal{X}$, and for all $M \in A_p$, $M \prec (\mathcal{A}, \mathcal{Y})$;*

(2) $B_p \subseteq \mathcal{Y}$;
(3) (A_p, B_p) *is an* $\vec{S}$*-obedient side condition.*

Let $q \leq p$ *if* $A_p \subseteq A_q$ *and* $B_p \subseteq B_q$.

The rest of this section is devoted to proving amalgamation results for $\mathbb{P}$, which in turn yield the existence of strongly generic conditions.

Lemma 15.2. *Let* $N \in \mathcal{X}$ *with* $N \prec (\mathcal{A}, \mathcal{Y})$. *Then* $q_N := (\{N\}, \emptyset)$ *is in* $\mathbb{P}$, *and for all* $p \in N \cap \mathbb{P}$, p *and* q_N *are compatible.*

Proof. Immediate from Lemma 5.4(1). □

Proposition 15.3. *Let* $N \in \mathcal{X}$ *be simple such that* $N \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. *Let* $q_N := (\{N\}, \emptyset)$. *Then* q_N *is a universal strongly* N*-generic condition.*

See Section 3 for a discussion of universal strongly generic conditions.

Proof. By Lemma 15.2, q_N is compatible with all conditions in $N \cap \mathbb{P}$. So it suffices to show that q_N is strongly N-generic. Let $r_0 \leq q_N$ be given. We will find a condition v in $N \cap \mathbb{P}$ such that for all $w \leq v$ in $N \cap \mathbb{P}$, r_0 and w are compatible.

Let $M_0, \ldots, M_{k-1}$ list the models M in $A_{r_0} \setminus N$ such that $M < N$. Note that by Lemma 8.2, $M_i \cap N \in N$ for all $i < k$.

By finitely many applications of Lemmas 5.5(1) and 7.16, together with the fact that $N \prec (\mathcal{A}, \mathcal{Y})$, the pair

$$r_1 := (A_{r_0} \cup \{M_0 \cap N, \ldots, M_{k-1} \cap N\}, B_{r_0})$$

is a condition below r_0.

By Proposition 10.8, there is a condition $r \leq r_1$ such that $A_r = A_{r_1}$, and (A_r, B_r) is closed under canonical models with respect to N. Note that the assumptions of the first paragraph of Proposition 13.1 hold for $A = A_r$ and $B = B_r$.

The objects r, N, and $M_0, \ldots, M_{k-1}$ witness that the following statement holds in $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$:

There exist v, N', and $M_0', \ldots, M_{k-1}'$ satisfying:

(1) $v \in \mathbb{P}$;
(2) $A_r \cap N \subseteq A_v$, $B_r \cap N \subseteq B_v$, and $M_0', \ldots, M_{k-1}'$ and N' are in A_v;
(3) N' is simple;
(4) for all $i < k$, $M_i' < N'$, $M_i \cap N = M_i' \cap N'$, and $p(M_i, N) = p(M_i', N')$.

The parameters which appear in the above statement, namely $A_r \cap N$, $B_r \cap N$, and for $i < k$, $M_i \cap N$ and $p(M_i, N)$, are all members of N. By the elementarity of N, there are v, N', and $M_0', \ldots, M_{k-1}'$ in N which satisfy the same statement.

We will show that for all $w \leq v$ in $N \cap \mathbb{P}$, w is compatible with r, and hence is compatible with r_0 since $r \leq r_0$. This will complete the proof.

So fix $w \leq v$ in $N \cap \mathbb{P}$. We claim that the pair

$$(A_r \cup A_w, B_r \cup B_w)$$

is in $\mathbb{P}$. Note that the assumptions of the second paragraph of Proposition 13.1 hold for $C = A_w$ and $D = B_w$. So by Proposition 13.1, $(A_r \cup A_w, B_r \cup B_w)$ is an $\vec{S}$-obedient side condition. So this pair is a condition in $\mathbb{P}$, and it is obviously below r and w. □

Corollary 15.4. *The forcing poset $\mathbb{P}$ satisfies the ω_1-covering property. In particular, it preserves ω_1.*

Proof. By Proposition 7.20, the set of $N \in P_{\omega_1}(H(\kappa^+))$ such that $N \in \mathcal{X}$ and N is simple is stationary. Hence there are stationarily many such N with $N \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. Any such N has a universal strongly N-generic condition by Proposition 15.3. By Corollary 3.10, $\mathbb{P}$ has the ω_1-covering property. □

Next we prove that many models in $\mathcal{Y}$ have strongly generic conditions.

Lemma 15.5. *Let $P \in \mathcal{Y}$ be $\vec{S}$-strong. Let $q_P := (\emptyset, \{P\})$. Then q_P is in $\mathbb{P}$, and for all $p \in P \cap \mathbb{P}$, p and q_P are compatible.*

Proof. Immediate from Lemma 5.4(2). □

Proposition 15.6. *Let $P \in \mathcal{Y}$ be $\vec{S}$-strong such that* $\mathrm{cf}(\sup(P)) = P \cap \kappa$ *and $P \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. Let $q_P := (\emptyset, \{P\})$. Then q_P is a universal strongly P-generic condition.*

Proof. By Lemma 15.5, q_P is compatible with all members of $P \cap \mathbb{P}$. So it suffices to show that q_P is strongly P-generic. Let $r_0 \leq q_P$ be given. We will find a condition $v \in P \cap \mathbb{P}$ such that for all $w \leq v$ in $P \cap \mathbb{P}$, r_0 and w are compatible.

By finitely many applications of Lemmas 5.5(2), 5.5(3), and 7.16, together with the fact that $P \prec (\mathcal{A}, \mathcal{Y})$, there is a condition $r \leq r_0$ such that

$$A_r = A_{r_0} \cup \{P \cap M : M \in A_{r_0}\},$$

and

$$B_r = B_{r_0} \cup \{P \cap Q : Q \in B_{r_0},\ Q \cap \kappa < P \cap \kappa\}.$$

Then the assumptions of the first paragraph of Proposition 13.2 hold for $A = A_r$ and $B = B_r$.

Let $\beta := P \cap \kappa$. Let $M_0, \ldots, M_{k-1}$ list the members of A_r.

The objects r, P, β, and $M_0, \ldots, M_{k-1}$ witness that the following statement holds in $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$:

There exist v, P', β', and $M'_0, \ldots, M'_{k-1}$ satisfying:

(1) $v \in \mathbb{P}$;
(2) $A_r \cap P \subseteq A_v$, $B_r \cap P \subseteq B_v$, $M'_0, \ldots, M'_{k-1}$ are in A_v, and $P' \in B_v$;
(3) $P' \cap \kappa = \beta'$ and $\mathrm{cf}(\sup(P')) = \beta'$;
(4) for all $i < k$, $M_i \cap P = M'_i \cap P'$ and $p(M_i, P) = p(M'_i, P')$.

The parameters appearing in the statement above, namely, $A_r \cap P$, $B_r \cap P$, and for $i < k$, $M_i \cap P$ and $p(M_i, P)$, are all members of P. By the elementarity of P, we can fix v, P', β', and $M'_0, \ldots, M'_{k-1}$ in P which satisfy the same statement.

For each $M \in A_r$, let M' denote M'_i, where $i < k$ and $M = M_i$.

We will show that for all $w \leq v$ in $P \cap \mathbb{P}$, w is compatible with r, and hence is compatible with r_0 since $r \leq r_0$. This will complete the proof.

So fix $w \leq v$ in $P \cap \mathbb{P}$. We claim that the pair

$$(A_r \cup A_w, B_r \cup B_w)$$

is in $\mathbb{P}$. Note that the assumptions of the second paragraph of Proposition 13.2 hold for $C = A_w$ and $D = B_w$. So by Proposition 13.2, $(A_r \cup A_w, B_r \cup B_w)$ is an

$\vec{S}$-obedient side condition. So this pair is a condition in $\mathbb{P}$, and it is obviously below r and w. $\square$

Corollary 15.7. *The forcing poset $\mathbb{P}$ has the κ-covering property. In particular, $\mathbb{P}$ forces that κ is a regular cardinal.*

Proof. By Proposition 14.2, there are stationarily many P in $\mathcal{Y}$ such that P is $\vec{S}$-strong and $\mathrm{cf}(\sup(P)) = P \cap \kappa$. Therefore there are stationarily many such P with $P \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. By Proposition 15.6, any such P has a universal strongly P-generic condition. Hence by Corollary 3.10, $\mathbb{P}$ has the κ-covering property. $\square$

Finally, we prove that for most transitive models, the empty condition is a strongly generic condition.

Proposition 15.8. *Suppose that X is an elementary substructure of $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$ of size κ such that $X \cap \kappa^+ \in \kappa^+$ and $X^{<\kappa} \subseteq X$. Then the pair $(\emptyset, \emptyset)$ is a strongly X-generic condition.*

Proof. Let $\theta := X \cap \kappa^+$. Since $X^{<\kappa} \subseteq X$, θ has cofinality κ.

Let D be a dense subset of $\mathbb{P} \cap X$, and we will show that D is predense in $\mathbb{P}$. Let p be a condition.

Let $M_0, \ldots, M_{k-1}$ and $P_0, \ldots, P_{m-1}$ enumerate the members of A_p and B_p respectively. Note that since $X^{<\kappa} \subseteq X$, for any model K on either of these lists, $K \cap \theta \in X$. For each $i < k$, let $\langle Q_n^i : n < \omega \rangle$ enumerate the $\vec{S}$-strong models in $M_i \cap \mathcal{Y}$. Since $X^{<\kappa} \subseteq X$, for each $n < \omega$, $Q_n^i \cap \theta \in X$. Therefore the sequence $\langle Q_n^i \cap \theta : n < \omega \rangle$ is in X.

Note that the assumptions of the first and second paragraphs of Proposition 13.3 hold for $A = A_p$ and $B = B_p$.

The objects p, θ, $M_0, \ldots, M_{k-1}$, $P_0, \ldots, P_{m-1}$, and $\langle Q_n^i : n < \omega \rangle$ for $i < k$ witness that $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$ satisfies the following statement:

There exist v, θ', $M_0', \ldots, M_{k-1}'$, $P_0', \ldots, P_{m-1}'$, and $\langle R_n^i : n < \omega \rangle$ for $i < k$ such that:

(1) $v \in \mathbb{P}$;
(2) $M_0', \ldots, M_{k-1}'$ are in A_v and $P_0', \ldots, P_{m-1}'$ are in B_v;
(3) $\mathrm{cf}(\theta') = \kappa$;
(4) $M_i \cap \theta = M_i' \cap \theta'$ and $P_j \cap \theta = P_j' \cap \theta'$ for $i < k$ and $j < m$;
(5) for all $i < k$, $\langle R_n^i : n < \omega \rangle$ enumerates the $\vec{S}$-strong models in $M_i' \cap \mathcal{Y}$, and for all $n < \omega$, $Q_n^i \cap \theta = R_n^i \cap \theta'$.

The parameters which appear in the above statement, namely κ, $M_i \cap \theta$ for $i < k$, $P_j \cap \theta$ for $j < m$, and $\langle Q_n^i \cap \theta : n < \omega \rangle$ for $i < k$ are all members of X. By the elementarity of X, we can fix v, θ', $M_0', \ldots, M_{k-1}'$, $P_0', \ldots, P_{m-1}'$, and $\langle R_n^i : n < \omega \rangle$ for $i < k$ in X which satisfy the same statement.

For each M in A_p, let M' denote M_i', where $i < k$ and $M = M_i$. For each P in B_p, let P' denote P_j', where $j < m$ and $P = P_j$.

Since D is a dense subset of $\mathbb{P} \cap X$, we can fix $w \leq v$ in D. Let us show that w and p are compatible. This proves that D is predense in $\mathbb{P}$, finishing the proof. It suffices to show that the pair

$$(A_p \cup A_w, B_p \cup B_w)$$

is a condition. Note that the assumptions of the third paragraph of Proposition 13.3 hold for $C = A_w$ and $D = B_w$. So by Proposition 13.3, $(A_p \cup A_w, B_p \cup B_w)$ is an $\vec{S}$-obedient side condition. Therefore this pair is in $\mathbb{P}$, and it is obviously below p and w. $\square$

Corollary 15.9. *The forcing poset $\mathbb{P}$ is κ^+-c.c.*

Proof. Since $(\emptyset, \emptyset)$ is the maximum element of $\mathbb{P}$, by Proposition 3.11 it suffices to show that there are stationarily many X in $P_{\kappa^+}(H(\kappa^+))$ for which $(\emptyset, \emptyset)$ is strongly X-generic. By Proposition 15.8, it suffices to show that there are stationarily many X in $P_{\kappa^+}(H(\kappa^+))$ such that $X \cap \kappa^+ \in \kappa^+$ and $X^{<\kappa} \subseteq X$. But this follows easily from the fact that $\kappa^{<\kappa} = \kappa$. $\square$

§16. The final argument

We now complete the proof of Mitchell's theorem. We begin by noting that the forcing poset $\mathbb{P}$ has the desired effect on cardinal structure.

Proposition 16.1. *The forcing poset $\mathbb{P}$ preserves ω_1, collapses κ to become ω_2, and is κ^+-c.c.*

Proof. Immediate from Proposition 3.12, Lemma 15.2, and Corollaries 15.4, 15.7, and 15.9. $\square$

Next we will show that we can apply the factorization theorem, Theorem 6.4.

It is easy to see that $\mathbb{P}$ has greatest lower bounds. Namely, if (A, B) and (C, D) are in $\mathbb{P}$ and are compatible, then $(A \cup C, B \cup D)$ is the greatest lower bound of (A, B) and (C, D).

Lemma 16.2. *The forcing poset $\mathbb{P}$ satisfies property $*(\mathbb{P}, \mathbb{P})$.*

See Definition 6.2 for the definition of $*$.

Proof. Let p, q, and r be pairwise compatible conditions in $\mathbb{P}$. Then $q \wedge r = (A_q \cup A_r, B_q \cup B_r)$, $p \wedge q = (A_p \cup A_q, B_p \cup B_q)$, and $p \wedge r = (A_p \cup A_r, B_p \cup B_r)$. To see that p is compatible with $q \wedge r$, it suffices to show that

$$(A_p \cup A_q \cup A_r, B_p \cup B_q \cup B_r)$$

is an $\vec{S}$-obedient side condition. But looking over the requirements of being $\vec{S}$-obedient, any violation of these requirements involves an incompatibility between two objects appearing in the components of the pair, and hence would lead to a violation of the same requirement for one of the triples $p \wedge q$, $p \wedge r$, or $q \wedge r$. $\square$

Proposition 16.3. *Let $Q \in \mathcal{Y}$ be $\vec{S}$-strong such that $\mathrm{cf}(\sup(Q)) = Q \cap \kappa$ and $Q \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. Let $q_Q := (\emptyset, \{Q\})$. Let G be a generic filter on $\mathbb{P}$ which contains q_Q. Then $G \cap Q$ is a V-generic filter on $\mathbb{P} \cap Q$, and $V[G] = V[G \cap Q][H]$, where H is a $V[G \cap Q]$-generic filter on $(\mathbb{P}/q_Q)/(G \cap Q)$. Moreover, the pair $(V[G \cap Q], V[G])$ satisfies the ω_1-approximation property.*

Proof. By Proposition 15.6, q_Q is a universal strongly Q-generic condition. By Propositions 7.20 and 15.3, there are stationarily many models in $P_{\omega_1}(H(\kappa)^+)$ which have universal strongly generic conditions. By Lemma 16.2, $\mathbb{P}$ satisfies property $*(\mathbb{P}, \mathbb{P})$. So the assumptions of Theorem 6.4 are satisfied, and we are done. $\square$

We will need the following technical lemma about names.

Lemma 16.4. *Suppose that $Q \in \mathcal{Y}$ with $Q \prec (H(\kappa^+), \in, \mathbb{P})$, and q is a strongly Q-generic condition. Let G be a V-generic filter on $\mathbb{P}$ which contains q. Let $\dot{a} \in Q$ be a nice $\mathbb{P}$-name for a set of ordinals, and suppose that $\dot{a}^G$ is a subset of $Q \cap \kappa$. Then $\dot{a}^G \in V[G \cap Q]$.*

Proof. Note that since q is strongly Q-generic, $G \cap Q$ is a V-generic filter on $\mathbb{P} \cap Q$ by Lemma 3.3.

Let $\alpha := Q \cap \kappa$. Since $\dot{a}$ is a nice name, for each $\gamma < \alpha$ there is a unique antichain A_γ such that $(p, \check{\gamma}) \in \dot{a}$ iff $p \in A_\gamma$. Since $\dot{a} \in Q$, by elementarity each A_γ is in Q.

We claim that for all $\gamma < \alpha$,

$$\gamma \in \dot{a}^G \text{ iff } A_\gamma \cap G \cap Q \neq \emptyset.$$

Since $\dot{a}^G \subseteq Q \cap \kappa = \alpha$, it follows that $\dot{a}^G$ is definable in $V[G \cap Q]$ from the sequence $\langle A_\gamma : \gamma < \alpha \rangle$ and the set $G \cap Q$. Therefore $\dot{a}^G \in V[G \cap Q]$, which finishes the proof.

If $p \in A_\gamma \cap G \cap Q$, then $(p, \check{\gamma}) \in \dot{a}$ by the choice of A_γ. Since $p \in G$, it follows that $(\check{\gamma})^G = \gamma$ is in $\dot{a}^G$. This shows that $A_\gamma \cap G \cap Q \neq \emptyset$ implies that $\gamma \in \dot{a}^G$.

Conversely, assume that $\gamma \in \dot{a}^G$. Then by the choice of A_γ, we can fix $p \in G \cap A_\gamma$. So to show that $A_\gamma \cap G \cap Q$ is nonempty, it suffices to show that $p \in Q$.

Since $A_\gamma \in Q$ is an antichain, by elementarity there is a maximal antichain $A \in Q$ with $A_\gamma \subseteq A$. Let D be the dense set of $u \in \mathbb{P}$ such that for some $s \in A$, $u \leq s$. By elementarity, $D \in Q$, and therefore $D \cap Q$ is dense in $\mathbb{P} \cap Q$ by elementarity. Since q is strongly Q-generic, $D \cap Q$ is predense below q.

As $q \in G$ and $D \cap Q$ is predense below q, we can fix $u \in G \cap D \cap Q$. By elementarity and the definition of D, there is $s \in A \cap Q$ such that $u \leq s$. Since $u \in G$, $s \in G$. Now $p \in A_\gamma$ and $A_\gamma \subseteq A$, so $p \in A$. Also $s \in A$. Since s and p are both in G, they are compatible. But A is an antichain, so $s = p$. Since $s \in Q$, $p \in Q$. □

Proposition 16.5. *Let $\tau < \kappa^+$ be an ordinal with cofinality κ which is closed under H^*. Let $\dot{Y}$ and $\dot{D}_\tau$ be $\mathbb{P}$-names such that $\mathbb{P}$ forces*

$$\dot{Y} = \{P : \exists p \in \dot{G}\ (P \in B_p)\} \text{ and } \dot{D}_\tau = \{P \cap \kappa : P \in \dot{Y},\ \tau \in P\}.$$

Suppose that $\beta < \kappa$ is an ordinal with uncountable cofinality, and p is a condition which forces that β is a limit point of $\dot{D}_\tau$. Let $Q := Sk(A_{\tau,\beta})$. Then:

1. *$Q \in \mathcal{Y}$;*
2. *$Q \cap \kappa = \beta$ and $Q \cap \kappa^+ = A_{\tau,\beta}$;*
3. *$\beta \in S_{\tau+1}$;*
4. *$\mathrm{cf}(\sup(Q)) = \beta$;*
5. *Q is $\vec{S}$-strong;*
6. *p forces that Q is in $\dot{Y}$.*

Proof. Define

$$Z^* := \{P \in \mathcal{Y} : P \text{ is } \vec{S}\text{-strong},\ P \cap \kappa < \beta,\ \tau \in P\},$$

and

$$Z := \{P \cap \tau : P \in Z^*\}.$$

Note that by Lemma 7.28, if P_1 and P_2 are in Z^* and $P_1 \cap \kappa \leq P_2 \cap \kappa$, then $P_1 \cap \tau \subseteq P_2 \cap \tau$.

We claim that for all $\gamma < \beta$, there is $P \in Z^*$ such that $\gamma < P \cap \kappa$. Namely, since p forces that β is a limit point of $\dot{D}_\tau$, there is $q \leq p$ and $P \in B_q$ such that $\tau \in P$ and $\gamma < P \cap \kappa < \beta$. Since $P \in B_q$, it follows that $P \in \mathcal{Y}$ is $\vec{S}$-strong. So $P \in Z^*$ and $\gamma < P \cap \kappa$, proving the claim. Consequently, $(\bigcup Z) \cap \kappa = \beta$.

Next we claim that $\bigcup Z = A_{\tau,\beta}$. First, suppose that $P \in Z^*$, and we will show that $P \cap \tau \subseteq A_{\tau,\beta}$. Since $\tau \in P$, it follows that $P \cap \tau = A_{\tau,P\cap\kappa}$ by Lemma 7.27, which is a subset of $A_{\tau,\beta}$ since $P \cap \kappa < \beta$. This shows that $\bigcup Z \subseteq A_{\tau,\beta}$. Conversely, let $\xi \in A_{\tau,\beta}$, and we will show that $\xi \in \bigcup Z$. Since β is a limit ordinal, we can fix $\gamma < \beta$ such that $\xi \in A_{\tau,\gamma}$. By the first claim, there is $P \in Z^*$ such that $\gamma < P \cap \kappa$. Since $\gamma < P \cap \kappa$ and $\xi \in A_{\tau,\gamma}$, it follows that $\xi \in A_{\tau,P\cap\kappa}$. But $\tau \in P$, so $A_{\tau,P\cap\kappa} = P \cap \tau$ by Lemma 7.27. Hence $\xi \in P \cap \tau$. As $P \cap \tau \in Z$, we have that $\xi \in \bigcup Z$.

Since τ is closed under H^*, every set in Z is closed under H^*. As Z is a $\subseteq$-chain, $\bigcup Z = A_{\tau,\beta}$ is also closed under H^*. In particular, $Q \cap \kappa^+ = A_{\tau,\beta}$. As noted above,

$$Q \cap \kappa = A_{\tau,\beta} \cap \kappa = (\bigcup Z) \cap \kappa = \beta.$$

This proves (2).

By Lemma 7.28(2), if P_1 and P_2 are in Z^* and $P_1 \cap \kappa < P_2 \cap \kappa$, then $\sup(P_1 \cap \tau) \in P_2 \cap \tau$, and hence $\sup(P_1 \cap \tau) < \sup(P_2 \cap \tau)$. In particular, since $A_{\tau,\beta} = \bigcup Z$,

$$\sup(A_{\tau,\beta}) = \sup\{\sup(P \cap \tau) : P \in Z^*\}.$$

Since β has uncountable cofinality, $\sup(A_{\tau,\beta}) = \sup(Q)$ has uncountable cofinality. It follows that $Q \in \mathcal{Y}$ by Lemma 7.15.

Now we prove that $Q \cap \kappa = \beta$ is in $S_{\tau+1}$. Fix M in $\mathcal{X}$ such that p, β, and τ are in M. Then $q := (A_p \cup \{M\}, B_p)$ is in $\mathbb{P}$ and $q \leq p$. Since β has uncountable cofinality, $\sup(M \cap \beta) < \beta$. As q forces that β is a limit point of $\dot{D}_\tau$, we can fix $r \leq q$ and $P \in B_r$ such that $\sup(M \cap \beta) < P \cap \kappa < \beta$ and $\tau \in P$. It follows that $\beta = \min((M \cap \kappa) \setminus (P \cap \kappa))$. As $\tau \in M \cap P$, $\tau + 1 \in M \cap P$ by elementarity. So $M \in A_r$, $P \in B_r$, and $\tau + 1 \in M \cap P \cap \kappa^+$, which implies that $\beta = \min((M \cap \kappa) \setminus (P \cap \kappa))$ is in $S_{\tau+1}$ by the fact that (A_r, B_r) is $\vec{S}$-obedient.

Since $\beta \in S_{\tau+1}$, β is inaccessible, and in particular is regular. Therefore the ordinal $\sup(A_{\tau,\beta}) = \sup\{\sup(P \cap \tau) : P \in Z^*\}$ is the supremum of a sequence of ordinals of order type β. It follows that $\sup(Q)$ has cofinality β. So $\mathrm{cf}(\sup(Q)) = Q \cap \kappa$.

Now we show that Q is $\vec{S}$-strong. Since $\beta \in S_{\tau+1}$, $\beta \in S_\tau$. As τ is a limit ordinal, for all $\xi \in A_{\tau,\beta}$, $\beta \in S_\xi$. So if $\xi \in Q \cap \kappa^+ = A_{\tau,\beta}$, then $Q \cap \kappa = \beta \in S_\xi$, which shows that Q is $\vec{S}$-strong.

It remains to show that p forces that Q is in $\dot{Y}$. It suffices to prove that for all $q \leq p$, there is $r \leq q$ such that $Q \in B_r$. So let $q \leq p$. We claim that $(A_q, B_q \cup \{Q\})$ is a condition below q.

Since Q is $\vec{S}$-strong, to prove that $(A_q, B_q \cup \{Q\})$ is an $\vec{S}$-obedient side condition, it suffices to show that if $M \in A_q$ and $\zeta = \min((M \cap \kappa) \setminus \beta)$, then for all $\sigma \in M \cap Q \cap \kappa^+$, $\zeta \in S_\sigma$.

Let $\sigma \in M \cap Q \cap \kappa^+$. Since $\sigma \in Q \cap \kappa^+ = A_{\tau,\beta}$ and β is a limit ordinal, we can fix $\gamma < \beta$ such that $\sigma \in A_{\tau,\gamma}$. By increasing γ if necessary, also assume that

$\sup(M \cap \beta) < \gamma$. As q forces that β is a limit point of $\dot{D}_\tau$, we can fix $s \leq q$ and $P \in B_s$ such that $\gamma < P \cap \kappa < \beta$ and $\tau \in P$. So

$$A_{\tau,\gamma} \subseteq A_{\tau,P\cap\kappa} = P \cap \tau.$$

In particular, $\sigma \in P$. Since $\zeta = \min((M \cap \kappa) \setminus \beta)$, we have that

$$\sup(M \cap \zeta) = \sup(M \cap \beta) < \gamma < P \cap \kappa < \beta \leq \zeta.$$

So $M \in A_s$, $P \in B_s$, $\sigma \in M \cap P \cap \kappa^+$, and $\zeta = \min((M \cap \kappa) \setminus (P \cap \kappa))$. Therefore $\zeta \in S_\sigma$ since (A_s, B_s) is an $\vec{S}$-obedient side condition. □

Lemma 16.6. *Let $\tau < \kappa^+$ be an ordinal of cofinality κ, and let $\beta \in S_{\tau+1}$. Suppose that $Q \in \mathcal{Y}$ is $\vec{S}$-strong, $Q \cap \kappa = \beta$, $Q \cap \kappa^+ = A_{\tau,\beta}$, and $\mathrm{cf}(\sup(Q)) = \beta$. Then the set*

$$\{P \in Q \cap \mathcal{Y} : P \textit{ is } \vec{S}\textit{-strong},\ \mathrm{cf}(\sup(P)) = P \cap \kappa\}$$

is stationary in $P_\beta(Q)$.

Proof. Let $F : Q^{<\omega} \to Q$, and we will find $P \in Q \cap \mathcal{Y}$ such that $\mathrm{cf}(\sup(P)) = P \cap \kappa$, P is $\vec{S}$-strong, and P is closed under F. Since $Q \in \mathcal{Y}$, $Q \cap \kappa^+ = A_{\tau,\beta}$ is closed under H^*. As $Q \cap \kappa^+$ is the union of the increasing and continuous chain $\{A_{\tau,i} : i < \beta\}$, there exists a club $C \subseteq \beta$ such that for all $\alpha \in C$, $A_{\tau,\alpha}$ is closed under H^* and $A_{\tau,\alpha} \cap \kappa = \alpha$. Then Q is the union of the increasing and continuous chain $\{Sk(A_{\tau,\alpha}) : \alpha \in C\}$. Fix a club $D \subseteq C$ such that for all $\alpha \in D$, $Sk(A_{\tau,\alpha})$ is closed under F.

For each $\alpha \in D$, let $Q_\alpha := Sk(A_{\tau,\alpha})$. We claim that for all $\alpha \in D$, $Q_\alpha \in Q$. Since $|Q_\alpha \cap \kappa^+| = |A_{\tau,\alpha}| < |\alpha|^+ < \beta$ and $\mathrm{cf}(\sup(Q)) = \beta$, it follows that $A_{\tau,\alpha}$ is a bounded subset of $Q \cap \kappa^+ = A_{\tau,\beta}$. Also

$$\mathrm{cf}(\sup(Q_\alpha)) \leq \alpha < \beta = \mathrm{cf}(Q \cap \kappa).$$

By Lemma 7.14, $\sup(Q_\alpha) \in Q$. But $A_{\tau,\alpha} = A_{\sup(A_{\tau,\alpha}),\alpha}$ by coherence, and since $\sup(A_{\tau,\alpha})$ and α are in Q, so is $A_{\tau,\alpha}$ by elementarity. Hence $Q_\alpha \in Q$ by elementarity.

Fix a club $E \subseteq \lim(D)$ such that for all $\alpha \in E$, for all $\gamma \in \alpha \cap D$, $Q_\gamma \in Q_\alpha$. In particular, for all $\alpha \in E$, since $Q_\alpha = \bigcup\{Q_\gamma : \gamma \in \alpha \cap D\}$, it follows that $\mathrm{cf}(\sup(Q_\alpha)) = \mathrm{cf}(\mathrm{ot}(\alpha \cap D))$. So if $\alpha \in E$ is regular, then $\mathrm{cf}(\sup(Q_\alpha)) = \alpha$.

Since $\beta \in S_{\tau+1}$, $S_\tau \cap \beta$ is stationary in β. So we can fix $\alpha \in E \cap S_\tau$. To finish the proof, it suffices to show that $Q_\alpha = Sk(A_{\tau,\alpha})$ is in $Q \cap \mathcal{Y}$, Q_α is $\vec{S}$-strong, $\mathrm{cf}(\sup(Q_\alpha)) = Q_\alpha \cap \kappa = \alpha$, and Q_α is closed under F.

We know that Q_α is closed under F by the definition of D. We previously observed that $Q_\alpha \in Q$, and since $\alpha \in E$ is regular, $\mathrm{cf}(\sup(Q_\alpha)) = \alpha$. In particular, $Q_\alpha \in \mathcal{Y}$ by Lemma 7.15. To see that Q_α is $\vec{S}$-strong, let $\xi \in Q_\alpha \cap \kappa^+ = A_{\tau,\alpha}$. Since $\alpha \in S_\tau$ and τ is a limit ordinal, $\alpha \in S_\eta$ for all $\eta \in A_{\tau,\alpha}$. In particular, $Q_\alpha \cap \kappa = \alpha \in S_\xi$. □

Lemma 16.7. *Suppose that $Q \in \mathcal{Y}$ is $\vec{S}$-strong and $Q \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. Let $\beta := Q \cap \kappa$. Suppose that the set*

$$\{P \in Q \cap \mathcal{Y} : P \textit{ is } \vec{S}\textit{-strong},\ \mathrm{cf}(\sup(P)) = P \cap \kappa\}$$

is stationary in $P_\beta(Q)$. Then the forcing poset $\mathbb{P} \cap Q$ forces that β is a regular cardinal.

Proof. Let $\gamma < \beta$, and let $\dot{f}$ be a $(\mathbb{P} \cap Q)$-name for a function from γ to β. Fix a condition $p \in \mathbb{P} \cap Q$, and we will find $q \leq p$ in $\mathbb{P} \cap Q$ which forces that $\dot{f}$ is bounded in β. Let F be the set of triples (u, i, ξ) such that $u \in \mathbb{P} \cap Q$ and $u \Vdash_{\mathbb{P} \cap Q} \dot{f}(i) = \xi$.

Let $\{g_n : n < \omega\}$ be a set of definable Skolem functions for the structure $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$. Since $Q \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$, Q is closed under g_n for all $n < \omega$. By the assumption of the lemma, we can fix $P \in Q \cap \mathcal{Y}$ such that P is $\vec{S}$-strong, $\mathrm{cf}(\sup(P)) = P \cap \kappa$, P is closed under g_n for all $n < \omega$, and $P \prec (Q, \in, \mathbb{P} \cap Q, p, \gamma, F)$. In particular, $P \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$. As $p \in P$, Proposition 15.6 implies that $q := (A_p, B_p \cup \{P\})$ is a strongly P-generic condition below p.

We claim that
$$q \Vdash_{\mathbb{P} \cap Q} \mathrm{ran}(\dot{f}) \subseteq P \cap \kappa.$$
Since $P \cap \kappa < Q \cap \kappa = \beta$, this completes the proof. Let $i < \gamma$, and we will show that
$$q \Vdash_{\mathbb{P} \cap Q} \dot{f}(i) \in P \cap \kappa.$$

Let D be the set of $s \in \mathbb{P} \cap P$ such that for some $\xi \in P \cap \kappa$, $(s, i, \xi) \in F$. We claim that D is dense in $\mathbb{P} \cap P$. So let $u \in \mathbb{P} \cap P$ be given. Then $u \in \mathbb{P} \cap Q$. Since $\dot{f}$ is a $(\mathbb{P} \cap Q)$-name for a function from γ to β, there is $v \leq u$ and $\xi < Q \cap \kappa$ such that $v \Vdash_{\mathbb{P} \cap Q} \dot{f}(i) = \xi$, and hence $(v, i, \xi) \in F$. Since $P \prec (Q, \in, \mathbb{P} \cap Q, p, \gamma, F)$ and u and i are in P, by elementarity there is $v \in P$ and $\xi \in P \cap \kappa$ such that $v \leq u$ and $(v, i, \xi) \in F$. Then $v \leq u$ and $v \in D$.

Since q is strongly P-generic, D is predense in $\mathbb{P}$ below q. Let $r \leq q$ in $Q \cap \mathbb{P}$ decide the value of $\dot{f}(i)$ to be ξ, and we will show that $\xi \in P$. Then $r \leq q$ is in $\mathbb{P}$, and $(r, i, \xi) \in F$. Since D is predense in $\mathbb{P}$ below q, for some $u \in D$, r and u are compatible in $\mathbb{P}$. By the elementarity of Q, $\mathbb{P} \cap Q$ is closed under greatest lower bounds, and therefore r and u are also compatible in $\mathbb{P} \cap Q$. Since $u \in D$, by the definition of D there is $\xi' \in P \cap \kappa$ such that $(u, i, \xi') \in F$. But $(r, i, \xi) \in F$ and $(u, i, \xi') \in F$ imply, by the compatibility of r and u in $\mathbb{P} \cap Q$, that $\xi = \xi'$. Since $\xi' \in P$, it follows that $\xi \in P$. $\square$

Recall that for a sequence $\vec{a} = \langle a_i : i < \omega_2 \rangle$ of countable sets, $S_{\vec{a}}$ is the set of limit ordinals $\alpha < \omega_2$ for which there exists a club $c \subseteq \alpha$ with order type $\mathrm{cf}(\alpha)$ such that for all $\beta < \alpha$, there is $i < \alpha$ with $c \cap \beta = a_i$. A set S is in the approachability ideal $I[\omega_2]$ iff there exists such a sequence $\vec{a}$ and a club D with $S \cap D \subseteq S_{\vec{a}}$. In particular, if $I[\omega_2]$ contains a stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$, then for some sequence $\vec{a}$, $S_{\vec{a}} \cap \mathrm{cof}(\omega_1)$ is stationary. We will show that this last statement fails in any generic extension by $\mathbb{P}$.

Theorem 16.8. *The forcing poset $\mathbb{P}$ forces that there is no stationary subset of $\omega_2 \cap \mathrm{cof}(\omega_1)$ in the approachability ideal $I[\omega_2]$.*

Proof. Suppose for a contradiction that p is a condition, $\vec{a} = \langle \dot{a}_i : i < \kappa \rangle$ is a sequence of $\mathbb{P}$-names for countable subsets of κ, and p forces that $\dot{S}_{\vec{a}} \cap \mathrm{cof}(\omega_1)$ is stationary. Without loss of generality, assume that each $\dot{a}_i$ is a nice name, which means that for some sequence of antichains $\langle A^i_\alpha : \alpha < \kappa \rangle$ of $\mathbb{P}$, $\dot{a}_i$ is equal to the set of pairs $\{(p, \check{\alpha}) : p \in A^i_\alpha,\ \alpha < \kappa\}$. As $\mathbb{P}$ is κ^+-c.c., each name $\dot{a}_i$ is a member of $H(\kappa^+)$. Fix $\gamma < \kappa^+$ such that for all $i < \kappa$, $\dot{a}_i$ is in $f^*[\gamma]$, where $f^* : \kappa^+ \to H(\kappa^+)$ is the bijection described in Notation 7.1.

Let M be an elementary substructure of $(\mathcal{A}, \mathcal{Y}, \mathbb{P})$ such that $|M| = \kappa$, $M \cap \kappa^+ \in \kappa^+ \cap \mathrm{cof}(\kappa)$, and $\gamma < M \cap \kappa^+$. Let $\tau := M \cap \kappa^+$. Since $\gamma < M \cap \kappa^+$ and M is closed under f^*, it follows that for all $i < \kappa$, $\dot{a}_i$ is in M. Fix $\mathbb{P}$-names $\dot{Y}$ and $\dot{D}_\tau$ such that $\mathbb{P}$ forces

$$\dot{Y} = \{P : \exists p \in \dot{G}\ (P \in B_p)\} \text{ and } \dot{D}_\tau = \{P \cap \kappa : P \in \dot{Y},\ \tau \in P\}.$$

An easy observation which follows from Lemma 15.5 is that $\dot{D}_\tau$ is forced to be cofinal in κ. Therefore $\lim(\dot{D}_\tau)$ is forced to be club in κ.

For each $\alpha < \kappa$, let $Q_\alpha := Sk(A_{\tau,\alpha})$. Since τ is the union of the increasing and continuous sequence $\{A_{\tau,\alpha} : \alpha < \kappa\}$, clearly $M = Sk(\tau)$ is the union of the increasing and continuous sequence $\{Q_\alpha : \alpha < \kappa\}$. Let E be a club subset of κ such that for all $\alpha \in E$, $Q_\alpha \cap \kappa = \alpha$, $Q_\alpha \cap \kappa^+ = A_{\tau,\alpha}$, $Q_\alpha \prec (\mathcal{A}, \mathcal{Y}, \mathbb{P})$, and for all $i < \alpha$, $\dot{a}_i \in Q_\alpha$.

Clearly p forces that $E \cap \lim(\dot{D}_\tau)$ is club in κ. Since p forces that $\dot{S}_{\vec{a}} \cap \mathrm{cof}(\omega_1)$ is stationary, we can fix $q \leq p$ and $\alpha < \kappa$ such that q forces that α is in $E \cap \lim(\dot{D}_\tau) \cap \dot{S}_{\vec{a}} \cap \mathrm{cof}(\omega_1)$. Since q forces that $\mathrm{cf}(\alpha) = \omega_1$, clearly α has uncountable cofinality. Let $Q := Q_\alpha$. Then by Proposition 16.5, $Q \in \mathcal{Y}$ is $\vec{S}$-strong, $Q \cap \kappa = \alpha \in S_{\tau+1}$, $Q \cap \kappa^+ = A_{\tau,\alpha}$, $\mathrm{cf}(\sup(Q)) = Q \cap \kappa = \alpha$, and q forces that Q is in $\dot{Y}$. By extending q if necessary, we can assume without loss of generality that $Q \in B_q$.

By Lemma 16.6, the set

$$\{P \in Q \cap \mathcal{Y} : P \text{ is } \vec{S}\text{-strong},\ \mathrm{cf}(\sup(P)) = P \cap \kappa\}$$

is stationary in $P_\alpha(Q)$. By Lemma 16.7, the forcing poset $\mathbb{P} \cap Q$ forces that α is a regular cardinal. Since $Q \in B_q$, clearly $q \leq q_Q := (\emptyset, \{Q\})$.

Let G be a V-generic filter on $\mathbb{P}$ containing q, and we will get a contradiction by considering the generic extension $V[G]$. Since $q \leq q_Q$, it follows that $q_Q \in G$. By Proposition 16.3, $V[G]$ can be factored as

$$V[G] = V[G \cap Q][H],$$

where $G \cap Q$ is a V-generic filter on $\mathbb{P} \cap Q$, H is a $V[G \cap Q]$-generic filter on $(\mathbb{P}/q_Q)/(G \cap Q)$, and the pair $(V[G \cap Q], V[G])$ has the ω_1-approximation property.

As α is in $S_{\vec{a}} \cap \mathrm{cof}(\omega_1)$, in $V[G]$ there is a club $c \subseteq \alpha$ with order type ω_1 such that for all $\beta < \alpha$, there is $i < \alpha$ such that $c \cap \beta = \dot{a}_i^G$. For any such β and i, $\dot{a}_i \in Q$, and $\dot{a}_i^G = c \cap \beta$ is a subset of $Q \cap \kappa = \alpha$. By Lemma 16.4, it follows that $\dot{a}_i^G \in V[G \cap Q]$. So for all $\beta < \alpha$, $c \cap \beta \in V[G \cap Q]$.

By Lemma 6.1, $c \in V[G \cap Q]$. But since c has order type ω_1, it follows that α has cofinality ω_1 in $V[G \cap Q]$. Now $\alpha \in S_{\tau+1}$, and in particular, α is inaccessible in V, but α is not regular in $V[G \cap Q]$. However, we previously observed that $\mathbb{P} \cap Q$ forces that α is a regular cardinal, so we have a contradiction. $\square$

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