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Discontinuous Galerkin and mimetic finite difference methods for coupled Stokes–Darcy flows on polygonal and polyhedral grids

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Abstract We study locally mass conservative approximations of coupled Darcy and Stokes flows on polygonal and polyhedral meshes. The discontinuous Galerkin (DG) finite element method is used in the Stokes region and the mimetic finite difference method is used in the Darcy region. DG finite element spaces are defined on polygonal and polyhedral grids by introducing lifting operators mapping mimetic degrees of freedom to functional spaces. Optimal convergence estimates for the numerical scheme are derived. Results from computational experiments supporting the theory are presented.

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1 Introduction

Coupled Stokes–Darcy flows occur in various physical processes of significant importance. Blood motion in the vessels, the interaction of ground and surface water, and engineering filtration problems are just a few examples that involve such flows. Our model consists of a fluid whose motion is governed by the Stokes equation and a porous medium saturated by the same fluid, in which the Darcy's law is valid. The two equations are coupled through transmission conditions that must be satisfied on the interface between the free fluid region and the porous medium region. These conditions are continuity of flux and normal stress, as well as slip with friction condition for the Stokes velocity known as the Beavers–Joseph–Saffman condition [6,51]. In this paper, we consider the surface–subsurface water flow as an application of the model.

There are number of stable and convergent numerical methods developed for the coupled Stokes–Darcy flow system, see e.g., [21,26,27,37,42,44,48,57]. Often it is of interest to study contaminant transport in such flows, which necessitates employing numerical schemes that conserve mass locally. In this paper we use the discontinuous Galerkin (DG) and the mimetic finite difference (MFD) methods to discretize the Stokes and Darcy equations, respectively. Both methods are locally mass conservative. We consider very general polygonal or polyhedral grids, as they allow us to model complex geometries with relatively few degrees of freedom.

The local mass conservation property of the DG method stems from the fact that discontinuous functions are used to approximate the solution on a given mesh. The original DG method was introduced in the early seventies for solving the neutron transport equation [36,46]. Since that time, several DG schemes have been introduced, including the Bassy–Rebay method [5], the interior penalty Galerkin methods [2,20,47,56], the Oden–Babuška–Baumann method [45], and the local discontinuous Galerkin (LDG) method [17]. A unifying DG framework for elliptic problems is studied in [4]. DG methods have been used to solve a wide range of problems, including compressible [5] and incompressible [29,41,48] fluid flows, magneto-hydrodynamics [55], and contaminant transport [20]. In [54], the LDG method is employed for transport coupled with Stokes–Darcy flows.

The MFD method is a relatively new discretization technique originating from the support-operator algorithms [34,52]. The method has been successfully applied to problems of continuum mechanics [43], electromagnetics [33], linear diffusion [34,39], and recently fluid dynamics [7,8]. The goal of the MFD discretization is to incorporate essential mathematical and physical principles (conservation laws, duality of operators and solution positivity) of the underlying system in the numerical model. This is achieved by approximating the differential operators in the governing equations by discrete operators that satisfy discrete versions of the fundamental identities of vector and tensor calculus. The MFD method can handle polygonal in 2-D and polyhedral in 3-D meshes with curved boundaries and possibly degenerate cells, which are well-suited to represent the irregular features of the porous medium.

For simplicial and quadrilateral meshes, an equivalence between the MFD method and the lowest order Raviart–Thomas MFE method has been established in [9] and [10], respectively. For polyhedral meshes, a relationship between the MFD method and the multipoint flux approximation (MPFA) has been studied in [40]. A strong connection between the MFD family of methods and a family of the gradient-type finite volume methods [24] and the mixed-finite volume methods [22] has been established in [23].

In this paper, we formulate the DG method on polygonal or polyhedral meshes by using one of the MFD tools, a lifting operator from mimetic degrees of freedom to a functional space. In particular, constant flux values on each edge (or face) of an element are extended into a piecewise linear function inside the element. This allows us to formulate a DG-MFD method for coupled Stokes–Darcy flows on polygonal or polyhedral meshes. The method is heterogeneous in the sense that discrete mimetic degrees of freedom in the Darcy domain are coupled with piecewise polynomial finite element spaces in the Stokes region. The meshes from the two regions may be non-matching on the interface and the continuity of flux condition is imposed through a Lagrange multiplier space. This space is defined on an interface mesh that is the trace of the mesh of the Darcy region and it is also used to approximate the normal stress on the interface. A global inf-sup condition is established that implies the well-posedness of the coupled scheme. For this we construct an interpolant in the space of DG-MFD velocities with weakly continuous normal components. We also establish optimal order convergence for the approximate velocity and pressure fields. Numerical calculations in 2-D are presented to support the theory.

The paper outline is as follows. In Sect. 2 we formulate the coupled problem. Its discretization is presented in Sect. 3. In Sect. 4 we construct some interpolants that will be used in the analysis of the method. Section 5 deals with the well-posedness of the method. Error estimates are derived in Sect. 6. In Sect. 7, we discuss some implementation details and provide results from computational tests that verify the theoretical error bounds.

2 The coupled Stokes–Darcy problem

The following model describes flow of incompressible fluid in a free fluid domain Ω_1 and a porous medium domain Ω_2 across an interface Γ_I. We assume that both Ω_1 and Ω_2 are Lipschitz polyhedral domains in $\Re^d$, $d = 2, 3$, separated by a simply connected interface Γ_I (see Fig. 1). Let

$$\Gamma_k = \partial\Omega_k \backslash \overline{\Gamma_I}, \quad k = 1, 2,$$

and $\mathbf{n}_k$ be the exterior unit normal vector to $\partial\Omega_k$. We denote the fluid velocity in domain Ω_k by $\mathbf{u}_k$, the fluid viscosity by μ, and the pressure by p_k. The stress tensor is given by

$$\mathbf{T}_1 = -p_1\mathbf{I} + 2\mu\,\mathbf{D}(\mathbf{u}_1), \quad \mathbf{D}(\mathbf{u}_1) = \frac{1}{2}(\nabla\,\mathbf{u}_1 + \nabla\,\mathbf{u}_1^T).$$

Flow in the Stokes domain is governed by the conservation of momentum and mass laws. Considering no slip boundary conditions for simplicity, we have

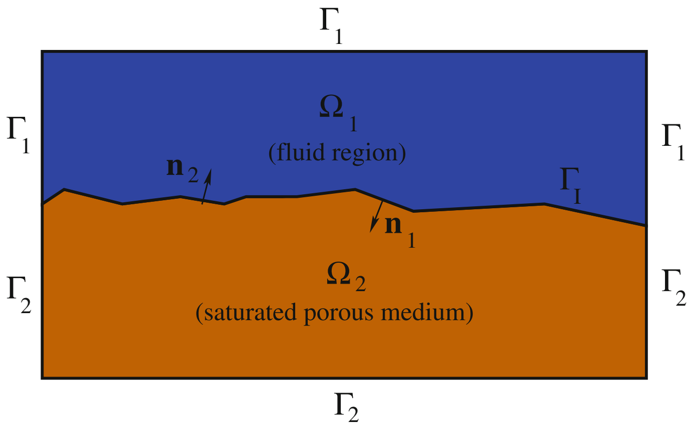

Fig. 1 A 2-D model of the coupled Stokes–Darcy flow

$$
\begin{aligned}
-\operatorname{div} \mathbf{T}_1 = \mathbf{f}_1, \quad \nabla \cdot \mathbf{u}_1 &= 0 \quad \text{in } \Omega_1, \\
\mathbf{u}_1 &= 0 \quad \text{on } \Gamma_1.
\end{aligned}
\tag{2.1}
$$

Flow in the Darcy domain is governed by Darcy's law and the conservation of mass law:

$$
\begin{aligned}
\mathbf{u}_2 = -\mathbf{K}\nabla p_2, \quad \nabla \cdot \mathbf{u}_2 &= f_2 \quad \text{in } \Omega_2, \\
\mathbf{u}_2 \cdot \mathbf{n}_2 &= 0 \quad \text{on } \Gamma_2,
\end{aligned}
\tag{2.2}
$$

where for simplicity we assume no-flow boundary conditions. In the above, $\mathbf{K}$ is a uniformly positive definite and bounded full tensor representing the rock permeability divided by the fluid viscosity.

The above problems are coupled across Γ_I through three interface conditions representing the mass conservation, the balance of normal stress, and the Beavers–Joseph–Saffman condition [6,51]:

$$
\mathbf{u}_1 \cdot \mathbf{n}_1 = -\mathbf{u}_2 \cdot \mathbf{n}_2, \tag{2.3}
$$

$$
(\mathbf{T}_1 \mathbf{n}_1) \cdot \mathbf{n}_1 = -p_2, \tag{2.4}
$$

$$
\mathbf{u}_1 \cdot \boldsymbol{\tau}_j = -2G_j(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_1) \cdot \boldsymbol{\tau}_j, \; j = 1, \cdots, d-1, \tag{2.5}
$$

where $\boldsymbol{\tau}_j$, $j = 1, \cdots, d-1$, is an orthonormal system of tangential vectors on Γ_I. Condition (2.5) models slip with friction, where $G_j = (\mu \mathbf{K} \boldsymbol{\tau}_j) \cdot \boldsymbol{\tau}_j / \alpha$ and $\alpha > 0$ is an experimentally determined friction constant. Existence of a unique weak solution to the coupled problem (2.1)–(2.5) is shown in [37].

3 Coupling of two discretization methods

In this section we describe coupling of two discretization methods, the discontinuous Galerkin (DG) method in the Stokes domain and the MFD method in the Darcy domain.

3.1 Admissible meshes

Let Ω_k^h be a partition of Ω_k, $k = 1, 2$ into polygonal in 2-D and polyhedral in 3-D elements E with diameter h_E. The meshes may be non-matching on the interface Γ_I. Let $h_k = \max_{E\in\Omega_k^h} h_E$. Hereafter, we shall use the term face, denoted by e, for both a face in 3-D and an edge in 2-D. We will denote edges in 3-D by ℓ. Let $\mathbf{x}_E$, $\mathbf{x}_e$, and $\mathbf{x}_\ell$ be the centroids of element E, face e, and edge ℓ, respectively. Let $|E|$ be the volume of E and $|e|$ be the area of face e. Let C denote a generic constant independent of h_E and E. We assume that the partitions Ω_k^h are shape-regular in the following sense.

Definition 3.1 The polygonal (polyhedral) partition Ω_k^h is shape-regular if

- Each element E has at most $N^\star$ faces, where $N^\star$ is independent of h_1 and h_2.
- Each element E is star-shaped with respect to a ball of radius $\rho^\star h_E$ centered at point $\mathbf{x}_E$, where $\rho^\star$ is independent of h_1 and h_2. Moreover, each face e of E and each edge ℓ of E in 3-D is star-shaped with respect to a ball of radius $\rho^\star h_E$ centered at the point $\mathbf{x}_e$ and $\mathbf{x}_\ell$, respectively. Thus,

$$C\, h_E^d \le |E| \le h_E^d, \quad C\, h_E^{d-1} \le |e| \le h_E^{d-1}. \tag{3.1}$$

Note that meshes with non-convex elements may be shape-regular in this sense.

Let $\mathcal{E}_k^h$ be the set of interior faces of Ω_k^h. For every face e, we define a unit normal vector $\mathbf{n}_e$ that will be fixed once and for all. If e belongs to Γ_k, we choose the outward normal to Ω_k. If e belongs to Γ_I, we choose the outward normal to Ω_2. Let $\mathbf{n}_E$ be the outward unit normal vector to E, so that $\chi_E^e \equiv \mathbf{n}_e \cdot \mathbf{n}_E$ is either 1 or -1.

3.2 Discretization in the Stokes domain

Let D be a domain in $\Re^d$ and $W^{s,p}(D)$, $s \ge 0$, $p \ge 1$, be the usual Sobolev space [1] with a norm $\|\cdot\|_{s,p,D}$ and a seminorm $|\cdot|_{s,p,D}$. The norm and the seminorm in the Hilbert spaces $H^s(D) \equiv W^{s,2}(D)$, $(H^s(D))^d$, and $(H^s(D))^{d\times d}$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively.

We extend the formulation in [29,48] on simplicial elements to general polyhedra. Let X_1 and Q_1 be Sobolev spaces for the velocity and the pressure, respectively, in the Stokes domain:

$$X_1 = \left\{\mathbf{v}_1 \in (L^2(\Omega_1))^d \ : \ \mathbf{v}_1|_E \in (W^{2,3/2}(E))^d \quad \forall E \in \Omega_1^h,\ \mathbf{v}_1 = \mathbf{0} \text{ on } \Gamma_1\right\},$$
$$Q_1 = \left\{q_1 \in L^2(\Omega_1) : \quad q_1|_E \in W^{1,3/2}(E) \quad \forall E \in \Omega_1^h\right\}.$$

The functions in X_1 and Q_1 have double valued traces on the interior element faces. The trace inequality and the Sobolev imbedding theorem imply the $q_1|_e \in L^2(e)$. For a scalar function w, we define its average $\{w\}_e$ and its jump $[w]_e$ across an interior face $e \in \mathcal{E}_1^h$ as follows:

$$\{w\}_e = \frac{1}{2}w|_{E_1} + \frac{1}{2}w|_{E_2}, \quad [w]_e = w|_{E_1} - w|_{E_2},$$

where E_1 and E_2 are two elements that share face e and such that $\mathbf{n}_e$ is directed from E_1 to E_2. For $e \in \partial\Omega_1^h$, the average and the jump are equal to the value of w. Similarly, we define the average $\{\mathbf{v}\}_e$ and the jump $[\mathbf{v}]_e$ of a vector function $\mathbf{v}$ by using the above definitions for each component of $\mathbf{v}$.

For $\mathbf{v}_1 \in X_1$, we define the elementwise gradient

$$\nabla_h \mathbf{v}_1 \in (L^2(\Omega_1))^{d\times d} : \ \nabla_h \mathbf{v}_1|_E = \nabla \mathbf{v}_1|_E \ \forall E \in \Omega_1^h.$$

We introduce the following norms:

$$\begin{aligned}
|||\mathbf{v}_1|||^2_{1,\Omega_1} &= \|\nabla_h \mathbf{v}_1\|^2_{0,\Omega_1} + \|\mathbf{v}_1\|^2_{0,\Omega_1},\\
\|\mathbf{v}_1\|^2_{X_1} &= \|\nabla_h \mathbf{v}_1\|^2_{0,\Omega_1} + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \frac{\sigma_e}{h_e}\|[\mathbf{v}_1]\|^2_{0,e} + \sum_{e\in\Gamma_I}\sum_{j=1}^{d-1}\frac{\mu}{G_j}\|\mathbf{v}_1\cdot\boldsymbol{\tau}_j\|^2_{0,e},\\
\|q_1\|_{Q_1} &= \|q_1\|_{0,\Omega_1},
\end{aligned}$$

where $\sigma_e > 0$ is a parameter that is a constant on e. The DG method is based on the bilinear forms $a_1 : X_1 \times X_1 \to \Re$ and $b_1 : X_1 \times Q_1 \to \Re$ defined as follows:

$$\begin{aligned}
a_1(\mathbf{u}_1,\ \mathbf{v}_1) =& 2\mu \sum_{E\in\Omega_1^h}\int_E \mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{v}_1)\,\mathrm{d}\,x + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1}\frac{\sigma_e}{h_e}\int_e [\mathbf{u}_1]\cdot[\mathbf{v}_1]\,\mathrm{d}\,s\\
&-2\mu\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1}\int_e \{\mathbf{D}(\mathbf{u}_1)\}\mathbf{n}_e\cdot[\mathbf{v}_1]\,\mathrm{d}\,s + 2\mu\varepsilon\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1}\int_e\{\mathbf{D}(\mathbf{v}_1)\}\mathbf{n}_e\cdot[\mathbf{u}_1]\,\mathrm{d}\,s\\
&+\sum_{e\in\Gamma_I}\sum_{j=1}^{d-1}\frac{\mu}{G_j}\int_e(\mathbf{u}_1\cdot\boldsymbol{\tau}_j)(\mathbf{v}_1\cdot\boldsymbol{\tau}_j)\,\mathrm{d}\,s, \quad \forall \mathbf{u}_1, \mathbf{v}_1 \in X_1\\
b_1(\mathbf{v},\ q) =& -\sum_{E\in\Omega_1^h}\int_E q_1 \operatorname{div}\mathbf{v}_1\,\mathrm{d}\,x + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1}\int_e \{q_1\}[\mathbf{v}_1]\cdot\mathbf{n}_e\,\mathrm{d}\,s,\\
&\forall \mathbf{v}_1 \in X_1, \forall q_1 \in Q_1.
\end{aligned}$$

The jump term involving σ_e is added for stabilization. We assume that for all faces e

$$\sigma_e \ge \sigma_0 > 0, \tag{3.2}$$

where σ_0 is chosen to be sufficiently large according to Lemma 5.3 in order to guarantee the coercivity of $a(\cdot,\cdot)$. The parameter ε controls the symmetry of the bilinear form and takes value -1, 0 or 1 for the symmetric interior penalty Galerkin (SIPG) [2,56], the incomplete interior penalty Galerkin (IIPG) [20], and the non-symmetric interior penalty Galerkin (NIPG) [45,47] methods, respectively.

Following closely the 2-D proof in Lemma 2.5 of [48], we obtain the following result.

Lemma 3.1 *The solution* $(\mathbf{u}, p) = (\mathbf{u}_1, \mathbf{u}_2; p_1, p_2)$ *to* (2.1)–(2.3) *satisfies*

$$a_1(\mathbf{u}_1, \mathbf{v}_1) + b_1(\mathbf{v}_1, p_1) + \int_{\Gamma_I} p_2 \mathbf{v}_1 \cdot \mathbf{n}_1 \, d s = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1 \, d x, \quad \forall \mathbf{v}_1 \in X_1, \tag{3.3}$$

$$b_1(\mathbf{u}_1, q_1) = 0, \quad \forall q_1 \in Q_1. \tag{3.4}$$

The case of simplicial elements has been studied extensively in the literature. Let $\mathbb{P}_r$ denote the space of polynomials of degree at most r. The DG discrete spaces X_1^h and Q_1^h for the velocity and the pressure, respectively, are defined as

$$X_1^h = \left\{ \mathbf{v}_1^h : \ \mathbf{v}_1^h|_E \in (\mathbb{P}_r(E))^d \ \ \forall E \in \Omega_1^h \right\},$$
$$Q_1^h = \left\{ q_1^h : \ q_1^h|_E \in \mathbb{P}_{r-1}(E) \ \ \forall E \in \Omega_1^h \right\}.$$

We consider the cases $r = 1, 2, 3$ in 2-D and $r = 1$ in 3-D.

To develop the lowest order ($r = 1$) DG method for general polyhedra, we follow the mimetic approach and consider a lifting operator from degrees of freedom defined on mesh faces to a functional space. For every element E and every face e of E, we associate d degrees of freedom (a vector in $\Re^d$) representing the mean velocity on e:

$$\mathbf{V}_{1,E}^e = \frac{1}{|e|} \int_e \mathbf{v}_1 \, \mathrm{d}\, s.$$

Let $X_{1,MFD}^h$ be the vector space with the above degrees of freedom. For a vector $\mathbf{V}_1 \in X_{1,MFD}^h$, let $\mathbf{V}_{1,E}$ be its restriction to element E.

On each E, we define a lifting operator $\mathcal{R}_{1,E}$ acting on a vector $\mathbf{V}_{1,E}$ and returning a function in $(H^1(E))^d$. We impose the following two properties on the lifting operator:

(L1) The mean value of the lifted function on faces e of E is equal to the prescribed degrees of freedom:

$$\frac{1}{|e|} \int_e \mathcal{R}_{1,E}(\mathbf{V}_{1,E}) \, \mathrm{d}\, s = \mathbf{V}_{1,E}^e.$$

(L2) The lifting operator is exact for linear functions. More precisely, if $\mathbf{V}_{1,E}^L$ is the vector of face mean values of a linear function $\mathbf{v}_1^L$, then

$$\mathcal{R}_{1,E}\left(\mathbf{V}_{1,E}^L\right) = \mathbf{v}_1^L.$$

Using the elemental lifting operators $\mathcal{R}_{1,E}$, we define the following finite element spaces:

$$X_{1,LIFT}^h = \left\{ \mathbf{v}^h : \ \mathbf{v}^h|_E = \mathcal{R}_{1,E}(\mathbf{V}_{1,E}), \quad \forall E \in \Omega_1^h, \ \forall\, \mathbf{V}_{1,E} \in X_{1,MFD}^h(E) \right\},$$
$$Q_{1,LIFT}^h = \left\{ q^h : \ q^h|_E \in \mathbb{P}_0(E), \quad \forall E \in \Omega_1^h \right\}.$$

When E is a tetrahedron, the lifting operator can be chosen to be the lowest order Crouzeix–Raviart finite element [19]. In this case, the DG spaces $X_1^h \times Q_1^h$ coincide with $X_{1,LIFT}^h \times Q_{1,LIFT}^h$. A constructive method for building a lifting operator for a polyhedron E is presented in Sect. 4.

The spaces $X_{1,LIFT}^h \times Q_{1,LIFT}^h$ are new DG spaces for Stokes on polygons or polyhedra. To keep the notation simple, for the rest of the paper we will denote the DG spaces for both simplicial and polyhedral elements by $X_1^h \times Q_1^h$,

Remark 3.1 Due to property (**L1**), the DG spaces on polygons and polyhedra defined above have continuous fluxes. This is desirable when the computed Stokes flow field is coupled with a transport equation.

We are now ready to formulate the DG method in Ω_1. Given an approximation $\bar{\lambda}^h$ of p_2 on Γ_I (to be defined later), the DG solution on Ω_1, $(\mathbf{u}_1^h, p_1^h) \in X_1^h \times Q_1^h$, satisfies

$$a_1\left(\mathbf{u}_1^h, \mathbf{v}_1^h\right) + b_1\left(\mathbf{v}_1^h, p_1^h\right) + \int_{\Gamma_I} \bar{\lambda}^h \mathbf{v}_1^h \cdot \mathbf{n}_1 \,\mathrm{d}s = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1^h \,\mathrm{d}x, \qquad \forall \mathbf{v}_1^h \in X_1^h, \quad (3.5)$$

$$b_1\left(\mathbf{u}_1^h, q_1^h\right) = 0, \quad \forall q_1^h \in Q_1^h. \quad (3.6)$$

Remark 3.2 An alternative approach to develop Stokes discretizations on polyhedra is to consider MFD constructions [7,13]. In [13], a discrete gradient operator is built using velocity degrees of freedom at the element vertices. In [7], the discretization of $a_1(\mathbf{u}_1, \mathbf{v}_1)$ is built algebraically in two dimensions using velocity values at vertices and normal components on edges. We do not pursue these approaches here.

3.3 Discretization in the Darcy domain

Let X_2 and Q_2 be the Sobolev spaces for the velocity and the pressure in Ω_2, respectively, defined as follows:

$$X_2 = \left\{\mathbf{v}_2 \in (L^s(\Omega_2))^d, s > 2:\ \operatorname{div} \mathbf{v}_2 \in L^2(\Omega_2), \mathbf{v}_2 \cdot \mathbf{n}_2 = 0 \text{ on } \Gamma_2\right\}, \quad Q_2 = L^2(\Omega_2).$$

We introduce the following L^2-norms:

$$\|\mathbf{v}_2\|_{X_2} = \|\mathbf{v}_2\|_{0,\Omega_2}, \quad \|q_2\|_{Q_2} = \|q_2\|_{0,\Omega_2}.$$

It is easy to see that the solution to (2.1)–(2.5) satisfies

$$\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2 \,\mathrm{d}x - \int_{\Omega_2} p_2 \operatorname{div} \mathbf{v}_2 \,\mathrm{d}x + \int_{\Gamma_I} p_2 \mathbf{v}_2 \cdot \mathbf{n}_2 \,\mathrm{d}s = 0, \qquad \forall \mathbf{v}_2 \in X_2, \quad (3.7)$$

$$\int_{\Omega_2} q_2 \operatorname{div} \mathbf{u}_2 \,\mathrm{d}x = \int_{\Omega_2} f_2\, q_2 \,\mathrm{d}x, \qquad \forall q_2 \in Q_2. \quad (3.8)$$

Note that the boundary integral in (3.7) is well defined if $p_2 \in H^1(\Omega_2)$.

We use the MFD method [15,16] to define discrete forms of (3.7)–(3.8). The first step in the MFD method is the definition of degrees of freedom. For each face e in Ω_2^h, we prescribe one degree of freedom V_2^e representing the average flux across e. Let X_2^h be the vector space with these degrees of freedom. The dimension of X_2^h is equal to the number of faces in Ω_2^h.

For any $\mathbf{v}_2 \in X_2$, we define its interpolant $\mathbf{v}_2^I \in X_2^h$ by

$$\left(\mathbf{v}_2^I\right)^e = \frac{1}{|e|} \int\limits_e \mathbf{v}_2 \cdot \mathbf{n}_e \, \mathrm{d}\, s. \tag{3.9}$$

Lemma 2.1 in [40] guarantees the existence of this integral for every $\mathbf{v}_2 \in X_2$.

For any $\mathbf{V}_2 \in X_2^h$, let $\mathbf{V}_{2,E}$ denote the vector of degrees of freedom associated only with an element E. We denote its component associated with face e by $V_{2,E}^e$.

To approximate the pressure, on each element $E \in \Omega_2^h$, we introduce one degree of freedom $P_{2,E}$ representing the average pressure on E. Let Q_2^h be the vector space with these degrees of freedom. The dimension of Q_2^h is equal to the number of elements in Ω_2^h. For any $p_2 \in Q_2$, we define its interpolant $p_2^I \in Q_2^h$ by

$$\left(p_2^I\right)_E = \frac{1}{|E|} \int\limits_E p_2 \, \mathrm{d}\, x. \tag{3.10}$$

We also need to define a discrete mimetic space for the approximation of the pressure on the interface Γ_I. This space will also serve the role of a Lagrange multiplier space for imposing the continuity of normal flux across Γ_I. For each face $e \in \Gamma_I^h = \Omega_2^h|_{\Gamma_I}$ we introduce one degree of freedom λ^e representing the average pressure on e. Let Λ_I^h be the vector space with these degrees of freedom. Note also that $\Lambda_I^h = X_2^h|_{\Gamma_I}$ and its dimension is equal to the number of faces of Γ_I.

The second step in the MFD method is to equip the discrete spaces Q_2^h, X_2^h, and Λ_I^h with inner products. The inner product in the space Q_2^h is relatively simple:

$$[\mathbf{P}, \mathbf{Q}]_{Q_2^h} = \sum_{E \in \Omega_2^h} |E| \; P_E \; Q_E, \quad \forall \mathbf{P}, \mathbf{Q} \in Q_2^h. \tag{3.11}$$

This inner product can be viewed as a mid-point quadrature rule for L^2-product of two scalar functions. The inner product in X_2^h can be defined formally as

$$[\mathbf{U},\ \mathbf{V}]_{X_2^h} = \mathbf{U}^T \, \mathbf{M}_2 \, \mathbf{V}, \quad \forall \mathbf{U}, \mathbf{V} \in X_2^h, \tag{3.12}$$

where $\mathbf{M}_2$ is a symmetric positive definite matrix. It can be viewed as a quadrature rule for the $\mathbf{K}^{-1}$-weighted L^2-product of two vector functions. The mass matrix $\mathbf{M}_2$ is assembled from element matrices $\mathbf{M}_{2,E}$:

$$\mathbf{U}^T\,\mathbf{M}_2\,\mathbf{V} = \sum_{E\in\Omega_2^h} \mathbf{U}_E^T\,\mathbf{M}_{2,E}\,\mathbf{V}_E.$$

The symmetric and positive definite matrix $\mathbf{M}_{2,E}$ induces the local inner product

$$[\mathbf{U}_E,\ \mathbf{V}_E]_{X_2^h,E} = \mathbf{U}_E^T\,\mathbf{M}_{2,E}\,\mathbf{V}_E. \tag{3.13}$$

The construction of matrix $\mathbf{M}_{2,E}$ for a general element E is at the heart of the mimetic method [16]. The inner product in Λ_I^h is defined as

$$\langle \boldsymbol{\lambda},\ \boldsymbol{\mu}\rangle_{\Lambda_I^h} = \sum_{e\in\Gamma_I^h} \lambda^e\,\mu^e\,|e|, \quad \forall\,\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_I^h. \tag{3.14}$$

Since $\mathbf{V}|_{\Gamma_I} \in \Lambda_I^h$ for every $\mathbf{V} \in X_2^h$, (3.14) can also be used to define $\langle \mathbf{V},\ \boldsymbol{\mu}\rangle_{\Lambda_I^h}$:

$$\langle \mathbf{V},\ \boldsymbol{\mu}\rangle_{\Lambda_I^h} = \sum_{e\in\Gamma_I^h} V^e\,\mu^e\,|e|, \quad \forall\,\mathbf{V} \in X_h^2,\ \boldsymbol{\mu} \in \Lambda_I^h.$$

The third step in the mimetic method is discretization of the gradient and divergence operators. The degrees of freedom have been selected to provide a simple approximation of the divergence operator. The Gauss divergence theorem naturally leads to the following formula:

$$(\mathcal{DIV}\,\mathbf{V})_E = \frac{1}{|E|}\sum_{e\subset\partial E} \chi_E^e\,V_E^e\,|e|. \tag{3.15}$$

We have a useful commutative property of the interpolants:

$$\left(\mathcal{DIV}\,\mathbf{v}^I\right)_E = \frac{1}{|E|}\int_{\partial E} \mathbf{v}\cdot\mathbf{n}_E\,\mathrm{d}\,s = \frac{1}{|E|}\int_E \mathrm{div}\,\mathbf{v}\,\mathrm{d}\,x = (\mathrm{div}\,\mathbf{v})_E^I, \quad \forall\mathbf{v} \in X_2. \tag{3.16}$$

The discrete gradient operator must be a discretization of the continuous operator $-\mathbf{K}\nabla$. To provide a compatible discretization, the mimetic method derives this discrete operator from a discrete Gauss–Green formula:

$$[\mathbf{U},\ \mathcal{GRAD}\,(\mathbf{P}, \boldsymbol{\lambda})]_{X_2^h} = [\mathcal{DIV}\,\mathbf{U},\ \mathbf{P}]_{Q_2^h} - \langle \mathbf{U},\ \boldsymbol{\lambda}\rangle_{\Lambda_I^h} \quad \forall\mathbf{U} \in X_2^h,\ \mathbf{P} \in Q_2^h,\ \boldsymbol{\lambda} \in \Lambda_I^h.$$

This equation mimics the continuous Gauss–Green formula

$$\int_{\Omega_2} \mathbf{u}\cdot\mathbf{K}^{-1}(-\mathbf{K}\nabla p)\,\mathrm{d}\,x = \int_{\Omega_2} p\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\,x - \int_{\Gamma_I} p\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}\,x, \quad \forall\mathbf{u} \in X_2,\ p \in H^1(\Omega_2).$$

Non-homogeneous velocity boundary conditions would require additional terms that represent non-zero boundary terms in the continuous Gauss–Green formula [32].

The construction of an admissible matrix $\mathbf{M}_{2,E}$ is based on the *consistency condition* (see [16] for details). Let $\mathbf{K}_E$ be the mean value of $\mathbf{K}$ on element E. We require

$$\left[\mathbf{V},\, (-\mathbb{K}_E \nabla p^l)^I\right]_{X_2^h,E} = \left[\mathcal{DIV}\,\mathbf{V},\, (p^l)^I\right]_{Q_2^h,E} - \sum_{e\in\partial E} \chi_E^e\, V_E^e \int_e p^l \,\mathrm{d}\,s, \quad \forall p^l \in \mathbb{P}_1(E). \tag{3.17}$$

The introduced inner products define the following norms:

$$|||\mathbf{P}_2|||^2_{Q_2^h} = [\mathbf{P}_2,\, \mathbf{P}_2]_{Q_2^h} \qquad \text{and} \qquad |||\mathbf{V}_2|||^2_{X_2^h} = [\mathbf{V}_2,\, \mathbf{V}_2]_{X_2^h}.$$

The Euclidean norm of algebraic vectors is denoted by $\|\cdot\|$.

Lemma 3.2 ([16]) *Under the assumptions of Definition 3.1, there exists a local inner product* (3.13) *such that*

$$\frac{1}{C}\,|E|\,\|\mathbf{V}_E\|^2 \;\le\; [\mathbf{V}_E,\, \mathbf{V}_E]_{X_2^h,E} \;\le\; C\,|E|\,\|\mathbf{V}_E\|^2, \tag{3.18}$$

where C is a constant that depends on the shape regularity of the auxiliary partition of E only.

In the following, for consistency between the DG and the mimetic notations, we will denote a vector $\mathbf{V}_2 \in X_2^h$ by $\mathbf{v}_2^h$, a vector $\mathbf{Q}_2 \in Q_2^h$ by q_2^h, and a vector $\boldsymbol{\lambda} \in \Lambda_I^h$ by λ^h. Given an approximation $\lambda^h \in \Lambda_I^h$ of p_2 on Γ_I, the mimetic approximation of (3.7)–(3.8) reads: Find $(\mathbf{u}_2^h, p_2^h) \in X_2^h \times Q_2^h$ such that

$$a_2\left(\mathbf{u}_2^h,\, \mathbf{v}_2^h\right) + b_2\left(\mathbf{v}_2^h,\, p_2^h\right) + \left\langle \mathbf{v}_2^h, \lambda^h \right\rangle_{\Lambda_I^h} = 0, \qquad \forall \mathbf{v}_2^h \in X_2^h, \tag{3.19}$$

$$b_2\left(\mathbf{u}_2^h,\, q_2^h\right) = -\left[f_2^I,\, q_2^h\right]_{Q_2^h}, \qquad \forall q_2^h \in Q_2^h, \tag{3.20}$$

where

$$a_2\left(\mathbf{u}_2^h, \mathbf{v}_2^h\right) = \left[\mathbf{u}_2^h,\, \mathbf{v}_2^h\right]_{X_2^h} \qquad \text{and} \qquad b_2\left(\mathbf{v}_2^h,\, q_2^h\right) = -\left[\mathcal{DIV}\,\mathbf{v}_2^h,\, q_2^h\right]_{Q_2^h}.$$

3.4 Discrete formulation of the coupled problem

In the two previous subsections we presented partially coupled discretizations for the Stokes and the Darcy regions, (3.5)–(3.6) and (3.19)–(3.20), respectively. The approximations $\bar{\lambda}^h$ and λ^h of p_2 on interface Γ_I are appeared in (3.5) and (3.19), respectively. We impose the continuity of normal stress (2.4) by taking $\bar{\lambda}^h$ to be the piecewise constant function on Γ_I^h satisfying

$$\bar{\lambda}^h|_e = (\lambda^h)^e, \qquad \forall e \in \Gamma_I^h.$$

We impose the flux continuity (2.3) in a weak sense, using Λ_I^h as the Lagrange multiplier space. The weak continuity is embedded in the definition of the global velocity space. More precisely, let $X^h = X_1^h \times X_2^h$, $Q^h = Q_1^h \times Q_2^h$, and

$$V^h = \left\{ \mathbf{v}^h \in X^h : \int_{\Gamma_I} \mathbf{v}_1^h \cdot \mathbf{n}_1 \, \bar{\mu}^h \, \mathrm{d}\, s + \langle \mathbf{v}_2^h, \mu^h \rangle_{\Lambda_I^h} = 0, \quad \forall \mu^h \in \Lambda_I^h \right\}. \tag{3.21}$$

We also define the composite bilinear forms

$$a\left(\mathbf{u}^h, \mathbf{v}^h\right) = a_1\left(\mathbf{u}_1^h, \mathbf{v}_1^h\right) + a_2\left(\mathbf{u}_2^h, \mathbf{v}_2^h\right), \quad \forall \mathbf{u}^h, \mathbf{v}^h \in X^h,$$
$$b\left(\mathbf{v}^h, q_h\right) = b_1\left(\mathbf{v}_1^h, q_1^h\right) + b_2\left(\mathbf{v}_2^h, q_2^h\right), \quad \forall \mathbf{v}^h \in X^h, \, q^h \in Q^h.$$

The weak formulation of the coupled problem is: find the pair $(\mathbf{u}^h, \, p^h) \in V^h \times Q^h$ such that

$$a\left(\mathbf{u}^h, \mathbf{v}^h\right) + b\left(\mathbf{v}^h, p^h\right) = \int_{\Omega_1} \mathbf{f}_1 \cdot \mathbf{v}_1^h \, \mathrm{d}\, x, \qquad \forall \mathbf{v}^h \in V^h, \tag{3.22}$$

$$b\left(\mathbf{u}^h, q^h\right) = -\left[f_2^I, q_2^h \right]_{Q_2^h}, \qquad \forall q^h \in Q^h. \tag{3.23}$$

Remark 3.3 We used a lifting operator from degrees of freedom to a functional space to define the DG spaces for the Stokes domain. A similar lifting operator can be used to define the MFD method in the Darcy domain as a finite element method.

4 Trace inequalities and interpolation results

Throughout this article, we use a few well known inequalities. The Young inequality reads:

$$ab \le \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2, \quad a, b \ge 0, \; \epsilon > 0. \tag{4.1}$$

A number of trace inequalities utilized in [48] on triangular meshes can be extended to polyhedral meshes using the auxiliary partition of an element E into shape-regular simplices. In particular, for any face e of element E, we have

$$\|\phi\|_{0,e}^2 \le C\left(h_E^{-1} \|\phi\|_{0,E}^2 + h_E |\phi|_{1,E}^2 \right), \quad \forall \phi \in H^1(E), \tag{4.2}$$

and its immediate consequence

$$\|\nabla \phi \cdot \mathbf{n}_e\|_{0,e}^2 \le C\left(h_E^{-1} \|\phi\|_{1,E}^2 + h_E |\phi|_{2,E}^2 \right), \quad \forall \phi \in H^2(E). \tag{4.3}$$

For polynomial functions, we have the trace inequality

$$\|\nabla\phi \cdot \mathbf{n}_e\|_{0,e} \leq C h_E^{-1/2} |\phi|_{1,E}, \quad \forall \phi \in \mathbb{P}_r(E). \tag{4.4}$$

For $\boldsymbol{\phi} \in (H^s(E))^2$, $0 \leq s \leq 1$, with div $\boldsymbol{\phi} \in L^2(E)$ we use Lemma 3.1 from [40] that gives

$$\|\boldsymbol{\phi} \cdot \mathbf{n}_e\|^2_{s-\frac{1}{2},e} \leq C\left(h_E^{-1}\|\boldsymbol{\phi}\|^2_{0,E} + h_E^{2s-1}\|\boldsymbol{\phi}\|^2_{s,E} + h_E\|\mathrm{div}\,\boldsymbol{\phi}\|^2_{0,E}\right). \tag{4.5}$$

The proof of the following lemma gives a constructive way for building a lifting operator.

Lemma 4.1 *For every element $E \in \Omega_h^1$, there exists a lifting operator $\mathcal{R}_{1,E}$ satisfying* (**L1**) *and* (**L2**) *such that*

$$|\mathcal{R}_{1,E}(\mathbf{V}_{1,E})|^2_{m,E} \leq C h_E^{d-2m} \|\mathbf{V}_{1,E}\|^2, \quad \forall \mathbf{V}_{1,E}, \tag{4.6}$$

where $m = 0, 1$. Moreover, the lifted function satisfies the trace inequality (4.4) *for every face e of E.*

Proof We consider an auxiliary partition of element E into simplexes. For every face e of E, we connect its centroid $\mathbf{x}_e$ with its vertices. This splits boundary ∂E into pieces t_k that are triangles in 3-D or segments in 2-D. The auxiliary simplicial partition is obtained by connecting the centroid $\mathbf{x}_E$ with the points $\mathbf{x}_e$ and the vertices of E. Due to the mesh assumptions in Definition 3.1, this is a shape regular partition.

We construct a continuous piecewise linear lifting function $\mathcal{R}_{1,E}(\mathbf{V}_{1,E})$. Property (**L1**) gives the following system of linear equations for the values of $\mathcal{R}_{1,E}(\mathbf{V}_{1,E})$ at the nodes of the auxiliary partition on ∂E:

$$\forall e \in \partial E, \quad \frac{1}{d}\sum_{t_k \in e} |t_k| \sum_{i=1}^{d} (\mathcal{R}_{1,E}(\mathbf{V}_{1,E}))(\mathbf{a}_k^i) = |e|\, \mathbf{V}^e_{1,E},$$

where $\mathbf{a}_k^i$ are the vertices of t_k. Since the unknowns associated with vertices $\mathbf{x}_e$ are not connected to each other and their number is equal to the number of equations, the matrix of this system has a full rank. Therefore, there exists a family of solutions, where the unknowns corresponding to the element centroid $\mathbf{x}_E$ and the vertices of E are free parameters.

We populate the free parameters by the values of a vector linear function $L(\mathbf{x})$ that minimizes the quadratic functional

$$\sum_{e \in E} \left| L(\mathbf{x}_e) - \mathbf{V}^e_{1,E} \right|^2.$$

This defines a continuous piecewise linear function $\mathcal{R}_{1,E}(\mathbf{V}_{1,E})(\mathbf{x})$. Property (**L1**) is satisfied by construction. Property (**L2**) also holds, since, if $\mathbf{V}^e_{1,E} = \frac{1}{|e|}\int\limits_e \mathbf{v}_1^L \,\mathrm{d}\, s$ for a linear vector $\mathbf{v}_1^L$, then $L(\mathbf{x}) = \mathbf{v}_1^L$ is the minimizer of the quadratic functional. The latter follows from the fact that for all faces e

$$\mathbf{v}_1^L(\mathbf{x}_e) = \frac{1}{|e|}\int\limits_e \mathbf{v}_1^L \,\mathrm{d}\, s = \mathbf{V}^e_{1,E}.$$

The shape regularity of E implies that the free parameters are bounded by $C\,\|\mathbf{V}_{1,E}\|$. The shape regularity of t_k and e implies that $|e|/|t_k| \le C$. Thus, the values of the lifted function at points $\mathbf{x}_e$ are bounded by the same norm. We have

$$\|\mathcal{R}_{1,E}(\mathbf{V}_{1,E})\|^2_{0,E} \le C\, h_E^d \max_{\mathbf{x}\in E} |\mathcal{R}_{1,E}(\mathbf{V}_{1,E})(\mathbf{x})|^2 \le C\, h_E^d\, \|\mathbf{V}_{1,E}\|^2.$$

The estimate for the gradient of the lifted function follows from the inverse inequality and the shape regularity of the auxiliary partition.

Finally, the shape regularity of the auxiliary partition implies that the trace inequality (4.4) holds for every t_k and hence for every face e. This proves the assertion of the lemma. □

Lemma 4.2 *Let* $\mathbf{v}_1 \in (H^1(\Omega_1))^d$. *There exists an interpolant* $\pi_h^1 : (H^1(\Omega_1))^d \to X_1^h$ *such that*

$$b_1\left(\pi_h^1(\mathbf{v}_1) - \mathbf{v}_1,\, q^h\right) = 0, \qquad \forall q^h \in Q_1^h, \tag{4.7}$$

$$\int\limits_e \left[\pi_1^h \mathbf{v}_1\right] \cdot \mathbf{w}\, d\, s = 0, \qquad \forall \mathbf{w} \in (\mathbb{P}_{r-1}(e))^d, \tag{4.8}$$

for every face $e \in \mathcal{E}_1^h \cup \Gamma_1$, *and*

$$|||\pi_1^h(\mathbf{v}_1)|||_{1,\Omega_1} \le C\|\mathbf{v}_1\|_{1,\Omega_1}. \tag{4.9}$$

The interpolant has optimal approximation properties for $\mathbf{v}_1 \in (H^s(\Omega_1))^d$, $1 \le s \le r+1$*:*

$$|\pi_1^h(\mathbf{v}_1) - \mathbf{v}_1|_{m,E} \le C h_E^{s-m} |\mathbf{v}_1|_{s,\,\delta(E)}, \qquad m = 0, 1, \tag{4.10}$$

where either $\delta(E)$ *is the union of* E *with all its closest neighbors in the case of simplices or* $\delta(E) = E$ *in the case of the lifted DG spaces on polygons and polyhedra.*

Furthermore, the following estimates hold for $\mathbf{v}_1 \in (H^s(\Omega_1))^d$, $1 \le s \le r+1$*:*

$$\|\pi_h^1(\mathbf{v}_1) - \mathbf{v}_1\|_{X_1} \le C h_1^{s-1}\, |\mathbf{v}_1|_{s,\Omega_1}, \tag{4.11}$$

$$\|\pi_h^1(\mathbf{v}_1)\|_{X_1} \le C\|\mathbf{v}_1\|_{1,\Omega_1}. \tag{4.12}$$

Proof On triangles for $r = 1, 2, 3$ and tetrahedra for $r = 1$ the existence of such an interpolant is shown in [18,19,25,47,48].

It remains to consider the case of polygonal and polyhedral meshes with $r = 1$. Let $\mathbf{v}_1 \in (H^1(\Omega_1))^d$ and let $\mathbf{V}_1$ be the corresponding vector of degrees of freedom. We introduce the interpolant π_1^h such that $\pi_1^h(\mathbf{v}_1) = \mathcal{R}_1(\mathbf{V}_1)$. Then, for every $q^h \in Q_1^h$, the lifting property **(L1)** gives

$$b_1\left(\pi_1^h(\mathbf{v}_1) - \mathbf{v}_1,\ q^h\right) = \sum_{E\in\Omega_1^h} q_E \int\limits_{\partial E} (\mathcal{R}_{1,E}(\mathbf{V}_{1,E}) - \mathbf{v}_1)\cdot \mathbf{n}_E \,\mathrm{d}\, s = 0. \tag{4.13}$$

Due to the lifting property **(L1)**, we immediately get condition (4.8) with $\mathbf{w} \in (\mathbb{P}_0(e))^d$.

To show (4.9), let $\mathbf{v}_1^c$ be the L^2-orthogonal projection of $\mathbf{v}_1$ onto the space of piecewise constant functions on Ω_1^h. Then, we have

$$\|\mathbf{v}_1^c\|_{0,E} \leq \|\mathbf{v}_1 - \mathbf{v}_1^c\|_{0,E} + \|\mathbf{v}_1\|_{0,E} \leq Ch_E|\mathbf{v}_1|_{1,E} + \|\mathbf{v}_1\|_{0,E} \leq C\|\mathbf{v}_1\|_{1,E}.$$

For every element E, the triangle inequality and the lifting properties **(L2)** and (4.6) give

$$\begin{aligned}\|\pi_1^h(\mathbf{v}_1)\|_{0,E}^2 &\leq 2\,\|\pi_1^h(\mathbf{v}_1 - \mathbf{v}_1^c)\|_{0,E}^2 + 2\,\|\mathbf{v}_1^c\|_{0,E}\\ &\leq C\left(|E| \sum_{e\in\partial E}\left(\frac{1}{|e|}\int\limits_e |\mathbf{v}_1 - \mathbf{v}_1^c|\,\mathrm{d}\,s\right)^2 + \|\mathbf{v}_1\|_{1,E}^2\right).\end{aligned}$$

Applying the trace inequality (4.2) to each component of $\mathbf{v}_1$ and using the standard approximation property of the L^2-projection, we bound each of the edge integrals:

$$\begin{aligned}\left(\int\limits_e |\mathbf{v}_1 - \mathbf{v}_1^c|\,\mathrm{d}\,s\right)^2 &\leq |e| \int\limits_e |\mathbf{v}_1 - \mathbf{v}_1^c|^2\,\mathrm{d}\,s\\ &\leq C|e|\left(h_E^{-1}\|\mathbf{v}_1 - \mathbf{v}_1^c\|_{0,E}^2 + h_E|\mathbf{v}|_{1,E}^2\right) \leq C|e|\,h_E\,|\mathbf{v}|_{1,E}^2.\end{aligned} \tag{4.14}$$

Combining the last two inequalities and using the shape regularity of E (3.1), we get

$$\|\pi_1^h(\mathbf{v}_1)\|_{0,E}^2 \leq C\left(\frac{h_E\,|E|}{|e|}|\mathbf{v}|_{1,E}^2 + \|\mathbf{v}\|_{1,E}^2\right) \leq C\|\mathbf{v}\|_{1,E}^2.$$

To bound the H^1-seminorm of $\pi_1^h(\mathbf{v}_1)$, we use (4.6) to obtain

$$|\pi_1^h(\mathbf{v}_1) - \mathbf{v}_1^c|_{1,E}^2 \leq Ch_E^{d-2}\|\mathbf{V}_{1,E} - \mathbf{V}_{1,E}^c\|^2 \leq Ch_E^{d-2}\sum_{e\in\partial E}\left(\frac{1}{|e|}\int\limits_e |\mathbf{v}_1 - \mathbf{v}_1^c|\,\mathrm{d}\,s\right)^2,$$

where $\mathbf{V}^c_{1,E}$ is the vector of degrees of freedom for the constant function $\mathbf{v}^c_1$. Combining the above inequality with (4.14), and using the shape regularity of E (3.1), we conclude that $|\pi^h_1(\mathbf{v}_1)|_{1,E} \le C|\mathbf{v}_1|_{1,E}$, which completes the proof of (4.9).

Since **(L2)** implies that π^h_1 is exact for all linear functions on E, an application of the Bramble–Hilbert lemma [12] gives (4.10).

It remains to show (4.11) and (4.12). Note that **(L1)** implies that for all faces e of E

$$\int\limits_e \left(\pi^h_1\mathbf{v}_1 - \mathbf{v}_1\right) \mathrm{d}\, s = 0, \quad \forall\, \mathbf{v}_1 \in (H^1(E))^d.$$

Therefore we can employ Lemma 3.9 of [47] to conclude that

$$\|\pi^1_h(\mathbf{v}_1) - \mathbf{v}_1\|_{X_1} \le C \left\|\nabla_h \left(\pi^1_h(\mathbf{v}_1) - \mathbf{v}_1\right)\right\|_{0,\Omega_1},$$

which, combined with (4.10), implies (4.11). The continuity bound (4.12) follows from the triangle inequality, (4.11), and the bound $\|\mathbf{v}_1\|_{X_1} \le C\|\mathbf{v}_1\|_{1,\Omega_1}$. This proves the assertion of the lemma. □

5 Stability and well-posedness of the discrete problem

In this section we prove a discrete inf-sup condition and show that the discrete problem (3.22)–(3.23) has a unique solution. Let $X = X_1 \times X_2$ and $Q = Q_1 \times Q_2$. We introduce the composite norms

$$\|q^h\|^2_{Q^h} = \|q^h_1\|^2_{0,\Omega_1} + \|q^h_2\|_{Q^h_2}, \quad \forall q^h = \left(q^h_1, q^h_2\right) \in Q^h,$$
$$\|\mathbf{v}^h\|^2_{X^h} = \|\mathbf{v}^h_1\|^2_{X_1} + |\!|\!|\mathbf{v}^h_2|\!|\!|^2_{div}, \quad \forall \mathbf{v}^h = \left(\mathbf{v}^h_1, \mathbf{v}^h_2\right) \in X^h,$$

where

$$|\!|\!|\mathbf{v}^h_2|\!|\!|^2_{div} = |\!|\!|\mathbf{v}^h_2|\!|\!|^2_{X^h_2} + |\!|\!|\mathcal{DIV}\,\mathbf{v}^h_2|\!|\!|^2_{Q^h_2}, \quad \forall \mathbf{v}^h_2 \in X^h_2.$$

Lemma 5.1 *Let* $\mathbf{v} \in (H^1(\Omega))^d$ *and* $\mathbf{v}_i = \mathbf{v}|_{\Omega_i}$, $i = 1, 2$. *Then, there exists an operator* $\pi^h : X \cap (H^1(\Omega))^d \to V^h$, $\pi^h(\mathbf{v}) = (\pi^h_1(\mathbf{v}_1),\ \pi^h_2(\mathbf{v}_2))$, *such that*

$$b(\pi^h\left(\mathbf{v}) - \mathbf{v}, q^h\right) = 0, \quad \forall q^h \in Q^h, \tag{5.1}$$

and

$$\left\|\pi^h_1(\mathbf{v}_1)\right\|_{X_1} \le C\|\mathbf{v}_1\|_{1,\Omega_1}, \quad |\!|\!|\pi^h_2(\mathbf{v}_2)|\!|\!|_{X^h_2} \le C\|\mathbf{v}\|_{1,\Omega}. \tag{5.2}$$

Proof Let π^h_1 be the operator defined in Lemma 4.2. The property (4.7) gives (5.1) for any $q^h = (q^h_1,\ 0)$. Due to (4.12), we get automatically the first inequality in (5.2). To construct $\pi^h_2(\mathbf{v}_2)$, we solve the following boundary value problem:

$$\begin{aligned} &\Delta\varphi = 0 && \text{in} \quad \Omega_2, \\ &\nabla\varphi \cdot \mathbf{n}_2 = 0 && \text{on} \quad \Gamma_2, \\ &\nabla\varphi \cdot \mathbf{n}_2 = (\mathbf{v} - \pi_1^h(\mathbf{v}_1)) \cdot \mathbf{n}_1 && \text{on} \quad \Gamma_I, \end{aligned} \tag{5.3}$$

and define $\pi_2^h(\mathbf{v}_2) = \mathbf{v}_2^I + (\nabla\varphi)^I$. By elliptic regularity [31,38],

$$\|\nabla\varphi\|_{H^\theta(\Omega_2)} \le C \left\| (\mathbf{v} - \pi_1^h(\mathbf{v}_1)) \cdot \mathbf{n}_1 \right\|_{H^{\theta-1/2}(\Gamma_I)}, \quad 0 \le \theta \le 1/2. \tag{5.4}$$

For all $q_2^h \in Q_2^h$, using definition of π_2^h and the commutative property (3.16), we get

$$\begin{aligned} b_2\left(\pi_2^h(\mathbf{v}) - \mathbf{v}_2^I,\, q_2^h\right) &= b_2\left((\nabla\varphi)^I,\, q_2^h\right) = -\left[\mathcal{DIV}\,(\nabla\varphi)^I,\, q_2^h\right]_{Q_2^h} \\ &= -\left[(\nabla\cdot\nabla\varphi)^I,\, q_2^h\right]_{Q_2^h} = 0. \end{aligned}$$

To prove the second inequality in (5.2), we start with the triangle inequality

$$\left|\!\left|\!\left| \pi_2^h(\mathbf{v}) \right|\!\right|\!\right|_{X_2^h} \le \left|\!\left|\!\left| \mathbf{v}_2^I \right|\!\right|\!\right|_{X_2^h} + \left|\!\left|\!\left| (\nabla\varphi)^I \right|\!\right|\!\right|_{X_2^h} \tag{5.5}$$

and bound every term. From the stability estimate (3.18), the trace inequality (4.2), and the shape regularity estimates (3.1), we obtain

$$\begin{aligned} \left|\!\left|\!\left| \mathbf{v}_2^I \right|\!\right|\!\right|_{X_2^h}^2 = \left[\mathbf{v}_2^I,\, \mathbf{v}_2^I\right]_{X_2^h} &\le C \sum_{E\in\Omega_2^h} \left| E \right| \sum_{e\subset\partial E} |(\mathbf{v}_2^I)_E^e| \right|^2 \\ &\le C \sum_{E\in\Omega_2^h} \sum_{e\subset\partial E} \frac{|E|}{|e|} \left(h_E^{-1} \|\mathbf{v}_2\|_{0,E}^2 + h_E |\mathbf{v}_2|_{1,E}^2 \right) \\ &\le C \sum_{E\in\Omega_2^h} \left(\|\mathbf{v}_2\|_{0,E}^2 + h_E^2 |\mathbf{v}_2|_{1,E}^2 \right) \\ &\le C \|\mathbf{v}_2\|_{1,\Omega_2}^2. \end{aligned} \tag{5.6}$$

Using the same arguments plus inequality (4.5) with $s = 1/2$, we get

$$\begin{aligned} \left|\!\left|\!\left| (\nabla\varphi)^I \right|\!\right|\!\right|_{X_2^h}^2 &\le C \sum_{E\in\Omega_2^h} |E| \sum_{e\subset\partial E} \left(\frac{1}{|e|} \int_e \nabla\varphi \cdot \mathbf{n}_e \,\mathrm{d}s \right)^2 \\ &\le C \sum_{E\in\Omega_2^h} \sum_{e\subset\partial E} \frac{|E|}{|e|} \left(h_E^{-1} \|\nabla\varphi\|_{0,E}^2 + \|\nabla\varphi\|_{\frac{1}{2},E}^2 \right) \\ &\le C \left(\|\nabla\varphi\|_{0,\Omega_2}^2 + h_2 \|\nabla\varphi\|_{\frac{1}{2},\Omega_2}^2 \right). \end{aligned} \tag{5.7}$$

To bound the first and the second term on the right hand side in (5.7) we apply (5.4) with $\theta = 0$ and $\theta = 1/2$, respectively:

$$\|\!|(\nabla\varphi)^I\|\!|^2_{X_2^h} \le C\Big(\|(\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1))\cdot\mathbf{n}_1\|^2_{-\frac{1}{2},\Gamma_I} + h_2\|(\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1))\cdot\mathbf{n}_1\|^2_{0,\Gamma_I}\Big)$$

$$\le C\,\Big\|\Big(\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1)\Big)\cdot\mathbf{n}_1\Big\|^2_{0,\Gamma_I}. \tag{5.8}$$

Using the trace inequality (4.2) for every $e \in \Gamma_I^h$ and the approximation result (4.10), we have that

$$\Big\|\Big(\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1)\Big)\cdot\mathbf{n}_1\Big\|_{L^2(e)} \le C\Big(h_E^{-1/2}\|\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1)\|_{0,E} + h_E^{1/2}|\mathbf{v}_1 - \pi_1^h(\mathbf{v}_1)|_{1,E}\Big)$$

$$\le C h_E^{s-1/2}|\mathbf{v}_1|_{s,\delta(E)}, \quad 1 \le s \le r+1.$$

Thus,

$$\|\!|(\nabla\varphi)^I\|\!|_{X_2^h} \le C h_1^{s-1/2}\|\mathbf{v}_1\|_{s,\Omega_1}, \quad 1 \le s \le r+1. \tag{5.9}$$

Combining (5.5) with estimates (5.6) and (5.9), we conclude that $\|\!|\pi_2^h(\mathbf{v})\|\!|_{X_2^h} \le C\|\mathbf{v}\|_{1,\Omega}$.

It remains to show that $\pi^h(\mathbf{v}) \in V^h$. Let $\mu_h \in \Lambda_I^h$. From definition of the inner product (3.14), definition of the interpolant (3.9), the boundary conditions in (5.3), and the regularity assumption $\mathbf{v} \in (H^1(\Omega))^d$, it follows that

$$\begin{aligned}
\Big\langle \pi_2^h\mathbf{v},\, \mu^h\Big\rangle_{\Lambda_I^h} &= \Big\langle \mathbf{v}_2^I,\, \mu^h\Big\rangle_{\Lambda_I^h} + \Big\langle (\nabla\varphi)^I,\, \mu^h\Big\rangle_{\Lambda_I^h} \\
&= \sum_{e\in\Gamma_I^h}(\mu^h)^e\int_e \mathbf{v}_2\cdot\mathbf{n}_2\,\mathrm{d}\,s + \sum_{e\in\Gamma_I^h}(\mu^h)^e\int_e \nabla\varphi\cdot\mathbf{n}_2\,\mathrm{d}\,s \\
&= \int_{\Gamma_I}\mathbf{v}_2\cdot\mathbf{n}_2\,\overline{\mu}^h\,\mathrm{d}\,s + \int_{\Gamma_I}\mathbf{v}_1\cdot\mathbf{n}_1\,\overline{\mu}^h\,\mathrm{d}\,s - \int_{\Gamma_I}\pi_1^h(\mathbf{v}_1)\cdot\mathbf{n}_1\,\overline{\mu}^h\,\mathrm{d}\,s \\
&= -\int_{\Gamma_I}\pi_1^h(\mathbf{v}_1)\cdot\mathbf{n}_1\,\overline{\mu}^h\,\mathrm{d}\,s.
\end{aligned}$$

Therefore $\pi^h(\mathbf{v}) \in V^h$. This proves the assertion of the lemma. □

Lemma 5.2 *There exists a positive constant β such that*

$$\inf_{q^h\in Q^h}\sup_{\mathbf{v}^h\in V^h}\frac{b_1(\mathbf{v}_1^h, q_1^h) + b_2(\mathbf{v}_2^h, q_2^h)}{\|\mathbf{v}^h\|_{X^h}\,\|q^h\|_{Q^h}} \ge \beta. \tag{5.10}$$

Proof For a given $q^h \in Q^h$, let us define $w \in L^2(\Omega)$ by

$$w = (w_1, w_2), \quad \text{where} \quad w_1 = -q_1^h \quad \text{and} \quad w_2|_E = -(q_2^h)_E,\ \forall E \in \Omega_2^h.$$

Note that $w_2^I = -q_2^h$ and $\|w_2\|_{0,\Omega_2} = |||q_2^h|||_{Q_2^h}$. We can construct $\mathbf{v} \in (H^1(\Omega))^d$ [28] for which

$$\operatorname{div} \mathbf{v} = w \quad \text{and} \quad \|\mathbf{v}\|_{1,\Omega} \leq C\|w\|_{0,\Omega}. \tag{5.11}$$

Let $\pi^h(\mathbf{v}) = (\pi_1^h(\mathbf{v}_1),\ \pi_2^h(\mathbf{v}_2))$ be the interpolant constructed in Lemma 5.1. Using (5.1) and the commutative property (3.16), we get

$$\begin{aligned} b_1\left(\pi_1^h(\mathbf{v}_1),\ q_1^h\right) + b_2\left(\pi_2^h(\mathbf{v}_2),\ q_2^h\right) &= b_1\left(\mathbf{v}_1,\ q_1^h\right) + b_2\left(\mathbf{v}_2^I,\ q_2^h\right) \\ &= -\int_{\Omega_1} (\operatorname{div} \mathbf{v}_1)\, q_1^h \,\mathrm{d}\,x - \left[\mathcal{DIV}\,\mathbf{v}_2^I,\ q_2^h\right]_{Q_2^h} \\ &= \left\|q_1^h\right\|_{0,\Omega_1}^2 + |||q_2^h|||_{Q_2^h}^2 = \left\|q^h\right\|_{Q^h}^2. \end{aligned} \tag{5.12}$$

The definition of π_2^h and (3.16) imply that

$$\mathcal{DIV}\left(\pi_2^h(\mathbf{v})\right) = \mathcal{DIV}\left(\mathbf{v}_2^I + (\nabla\varphi)^I\right) = (\operatorname{div} \mathbf{v}_2)^I + (\nabla\cdot\nabla\varphi)^I = -q_2^h.$$

Using estimate (5.2) from Lemma (5.1), we bound $\pi^h(\mathbf{v})$:

$$\begin{aligned} \left\|\pi^h(\mathbf{v})\right\|_X^2 &= \left\|\pi_1^h(\mathbf{v}_1)\right\|_{X_1}^2 + |||\pi_2^h(\mathbf{v}_2)|||_{X_2^h}^2 + |||\mathcal{DIV}\,(\pi_2^h(\mathbf{v}_2))|||_{Q_2^h}^2 \\ &\leq C\left(\|\mathbf{v}\|_{1,\Omega}^2 + |||q_2^h|||_{Q_2^h}^2\right) \\ &\leq C\left(\|q_1^h\|_{0,\Omega}^2 + |||q_2^h|||_{Q_2^h}^2\right) \leq C\|q^h\|_Q^2. \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13) yields

$$b_1\left(\pi_1^h(\mathbf{v}_1),\ q_1^h\right) + b_2\left(\pi_2^h(\mathbf{v}_2),\ q_2^h\right) \geq C\|\pi^h(\mathbf{v})\|_X\, \|q^h\|_Q, \tag{5.14}$$

which proves the assertion of the lemma. □

To prove that the method is well-posed we need the coercivity property established in the next lemma.

Lemma 5.3 *Assuming* (3.2)*, there exists a positive constant* α_c *dependent on* σ_0 *but independent of* h_1 *such that*

$$a_1\left(\mathbf{v}_1^h, \mathbf{v}_1^h\right) \geq \alpha_c\|\mathbf{v}_1^h\|_{X_1^h}, \quad \forall \mathbf{v}_1^h \in X_1^h. \tag{5.15}$$

Proof Let $\mathbf{v}_1^h \in X_1^h$. From the definition of $a_1(\cdot,\cdot)$ we have

$$a_1\left(\mathbf{v}_1^h, \mathbf{v}_1^h\right) = 2\mu \sum_{E\in\Omega_1^h} \int_E \mathbf{D}(\mathbf{v}_1^h) : \mathbf{D}(\mathbf{v}_1^h)\,\mathrm{d}\,x + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \frac{\sigma_e}{h_e} \int_e [\mathbf{v}_1^h]\cdot[\mathbf{v}_1^h] - 2\mu(1-\varepsilon)\,\mathrm{d}\,s$$
$$\times \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \int_e \left\{\mathbf{D}(\mathbf{v}_1^h)\,\mathbf{n}_e\right\}\cdot[\mathbf{v}_1^h]\,\mathrm{d}\,s + \sum_{e\in\Gamma_I}\sum_{j=1}^{d-1} \frac{\mu}{G_j} \int_e \left(\mathbf{v}_1^h\cdot\boldsymbol{\tau}_j\right)\left(\mathbf{v}_1^h\cdot\boldsymbol{\tau}_j\right)\mathrm{d}\,s.$$

Since $\mathbf{v}_1^h$ is continuous and piecewise linear on a shape regular auxiliary partition of E, the following Korn's holds [11]:

$$\|\nabla_h \mathbf{v}_1^h\|_{0,\Omega_1}^2 \le K_0 \left(\|\mathbf{D}_h(\mathbf{v}_1^h)\|_{0,\Omega_1}^2 + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \frac{1}{h_e} \|[\mathbf{v}_1^h]\|_{0,e}^2 \right), \quad \forall\, \mathbf{v}_1^h \in X_1^h., \tag{5.16}$$

where $\mathbf{D}_h(\mathbf{v}_1^h)$ is the elementwise deformation tensor. Thus,

$$a_1\left(\mathbf{v}_1^h, \mathbf{v}_1^h\right) \ge \frac{2\mu}{K_0} |||\mathbf{v}_1^h|||_{1,\Omega_1}^2 + \sum_{e\in\mathcal{E}_f^h\cup\Gamma_1} \frac{\sigma_e - 2\mu}{h_e} \|[\mathbf{v}_1^h]\|_{0,e}^2$$
$$-2\mu(1-\varepsilon) \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \int_e \left\{\mathbf{D}(\mathbf{v}_1^h)\,\mathbf{n}_e\right\}\cdot[\mathbf{v}_1^h]\,\mathrm{d}\,s + \sum_{e\in\Gamma_I}\sum_{j=1}^{d-1} \frac{\mu}{G_j} \|\mathbf{v}_1^h\cdot\boldsymbol{\tau}_j\|_{0,e}^2.$$

Clearly, the coercivity property holds when $\varepsilon = 1$ and $\sigma_0 = 2\mu/\alpha$ for some $0 < \alpha < 1$. To address the case when $\varepsilon = -1$ or 0, we use the trace inequality (4.4) and the Young's inequality (4.1) to estimate the third term. Let E^e be the element with face e. Then,

$$\left| \int_e \{\mathbf{D}(\mathbf{v}_1^h)\,\mathbf{n}_e\}\cdot[\mathbf{v}_1^h]\,\mathrm{d}\,s \right| \le C_1 \|\nabla \mathbf{v}_1^h\|_{E^e} \|[\mathbf{v}_1^h]\|_{0,E^e} \le \frac{C_2}{2C_3} \|\nabla \mathbf{v}_1^h\|_{0,E^e}^2$$
$$+ \frac{C_2 C_3}{2h_e} \|[\mathbf{v}_1^h]\|_{0,e}^2.$$

Then,

$$a_1\left(\mathbf{v}_1^h, \mathbf{v}_1^h\right) \ge \mu \left(\frac{2}{K_0} - \frac{C_2(1-\varepsilon)}{C_3} \right) |||\mathbf{v}_1^h|||_{1,\Omega_1}^2$$
$$+(\sigma_e - \mu(2 + C_2 C_3(1-\varepsilon))) \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1} \frac{\|[\mathbf{v}_1^h]\|_{0,e}^2}{h_e} + \sum_{e\in\Gamma_I}\sum_{j=1}^{d-1} \frac{\mu}{G_j} \|\mathbf{v}_1^h\cdot\boldsymbol{\tau}_j\|_{0,e}^2.$$

Setting $C_3 = 2K_0C_2$ ensures that the first term is positive for both $\varepsilon = 0$ and $\varepsilon = -1$. Then to control the second term it is sufficient to choose $\sigma_0 = 2\mu(1 + C_2C_3)/\alpha = 2\mu(1 + 2K_0C_2^2)/\alpha$ for some $0 < \alpha < 1$. □

Theorem 5.1 *The problem* (3.22)–(3.23) *has a unique solution.*

Proof It is sufficient to show that solution of the homogeneous problem (3.22)–(3.23) is zero. By choosing $\mathbf{v}^h = \mathbf{u}^h$ and $q^h = p^h$ we get

$$a_1\left(\mathbf{u}_1^h, \mathbf{u}_1^h\right) + a_2\left(\mathbf{u}_2^h, \mathbf{u}_2^h\right) = 0,$$

which combined with (5.15) and (3.18) implies that $\mathbf{u}^h = 0$. The remainder of (3.22) together with the inf-sup condition (5.10) imply that $p^h = 0$. □

6 Error analysis

Let the pair $(\mathbf{u}, p)$ be the solution to (2.1)–(2.3) and let $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$, $i = 1, 2$. We define functions $\tilde{\mathbf{u}} \in V^h$ and $\tilde{p} \in Q^h$ as follows:

$$\tilde{\mathbf{u}} = (\tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2) = \left(\pi_1^h(\mathbf{u}_1), \pi_2^h(\mathbf{u}_2)\right), \quad \tilde{p} = (\tilde{p}_1, \tilde{p}_2),$$

where π^h is the operator introduced in Lemma (5.1), $\tilde{p}_2 = p_2^I \in Q_2^h$ is the interpolant of p_2 introduced in (3.10) and $\tilde{p}_1$ is the L^2-projection of p_1:

$$\int_E (\tilde{p}_1 - p_1)\, q_1 \,\mathrm{d}\, x = 0, \quad \forall q_1 \in \mathbb{P}_{r-1}(E), \quad \forall E \in \Omega_1^h. \tag{6.1}$$

For any $p_1 \in H^s(\Omega_1)$ we have the approximation result:

$$\|p_1 - \tilde{p}_1\|_{m,E} \le C h_E^{s-m} |p_1|_{s,E}, \quad m = 0, 1, \quad 1 \le s \le r. \tag{6.2}$$

We also need the following approximation result [12]: for any $\phi \in H^s(E)$, $1 \le s \le 2$, there exists a linear function ϕ_E^1 such that

$$\|\phi - \phi_E^1\|_{m,E} \le C h_E^{s-m} |\phi|_{s,E}, \quad m = 0, 1. \tag{6.3}$$

Applying (4.2) to $\phi - \phi_E^1$ and using (6.3), we obtain the estimate for face e:

$$\|\phi - \phi_E^1\|_{0,e}^2 \le C\, h_E^{2s-1}\, |\phi|_{s,E}^2. \tag{6.4}$$

Similarly, (4.2) and (4.10) imply that

$$\|\mathbf{u}_1 - \tilde{\mathbf{u}}_1\|_{0,e}^2 \le C\, h_E^{2s-1}\, |\mathbf{u}_1|_{s,\delta(E)}^2, \quad 1 \le s \le r + 1. \tag{6.5}$$

Let $\overline{\mathbf{K}}$ be a piecewise constant tensor equal to $\mathbf{K}_E$ on element E. Recall that $\mathbf{K}_E$ is the mean value of $\mathbf{K}$ on E. We assume that $\mathbf{K} \in (W^{1,\infty}(E))^{d\times d}$, $\forall E \in \Omega_2^h$, and that $\max_{E\in\Omega_2^h} \|\mathbf{K}\|_{1,\infty,E}$ is uniformly bounded independently of h_2, where $||\mathbf{K}||_{1,\infty,E} = \max_{1\le i,j\le d} ||\mathbf{K}_{i,j}||_{W^{1,\infty}(E)}$. From Taylor's theorem it follows that

$$\max_{x\in E} \left|\mathbf{K}_{ij}(x) - \mathbf{K}_{E,ij}\right| \le C h_E \, \|\mathbf{K}_{ij}\|_{W^{1,\infty}(E)}. \tag{6.6}$$

6.1 Error equation

Subtracting the variational equations (3.3)–(3.4) from the discrete equations (3.22)–(3.23), we obtain

$$\begin{aligned} a_1\left(\mathbf{u}_1^h - \mathbf{u}_1,\, \mathbf{v}_1^h\right) + b_1\left(\mathbf{v}_1^h,\, p_1^h - p_1\right) - \sum_{e\in\Gamma_I^h} \int_e p_2\, \mathbf{v}_1^h \cdot \mathbf{n}_1 \,\mathrm{d}\, s & \\ + a_2\left(\mathbf{u}_2^h, \mathbf{v}_2^h\right) + b_2\left(\mathbf{v}_2^h,\, p_2^h\right) = 0, \quad \forall \mathbf{v}^h \in V^h, & \\ b_1\left(\mathbf{u}_1^h - \mathbf{u}_1,\, q_1^h\right) + b_2\left(\mathbf{u}_2^h,\, q_2^h\right) = -\left[f_2^I, q_2^h\right]_{Q_2^h}, \quad \forall q^h \in Q^h. & \end{aligned} \tag{6.7}$$

If we take $q_1^h = 0$ in the second equation, we recover the weak form of the mass balance equation for the Darcy region (3.20). Using this, plus adding and subtracting $\tilde{\mathbf{u}}_1$, $\tilde{p}_1$, and $\mathbf{u}_2^I$ in the appropriate terms of (6.7), we obtain

$$\begin{aligned} & a_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1,\, \mathbf{v}_1^h\right) + b_1\left(\mathbf{v}_1^h,\, p_1^h - \tilde{p}_1\right) + a_2\left(\mathbf{u}_2^h - \mathbf{u}_2^I, \mathbf{v}_2^h\right) + b_2\left(\mathbf{v}_2^h,\, p_2^h\right) \\ & \quad = a_1\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1,\, \mathbf{v}_1^h\right) + b_1\left(\mathbf{v}_1^h,\, p_1 - \tilde{p}_1\right) \\ & \quad\quad + \sum_{e\in\Gamma_I^h} \int_e p_2 \mathbf{v}_1^h \cdot \mathbf{n}_1 \,\mathrm{d}\, s - a_2\left(\mathbf{u}_2^I, \mathbf{v}_2^h\right), \quad \forall \mathbf{v}^h \in V^h, \\ & b_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1,\, q_1^h\right) = b_1\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1,\, q_1^h\right), \quad \forall q_h \in Q_h. \end{aligned} \tag{6.8}$$

6.2 Velocity estimate

Theorem 6.1 *Let* $(\mathbf{u}, p)$ *be the solution to* (2.1)–(2.5) *and* $(\mathbf{u}^h,\, p^h)$ *be the solution to* (3.22)–(3.23)*. Furthermore, let* $\mathbf{u}_1 \in (H^{r+1}(\Omega_1))^d$, $p_1 \in H^r(\Omega_1)$, $\mathbf{u}_2 \in (H^1(\Omega_2))^d$, *and* $p_2 \in H^2(\Omega_2)$*. Then, the following error bound holds*

$$\left\|\mathbf{u}_1^h - \mathbf{u}_1\right\|_{X_1} + |||\mathbf{u}_2^h - \mathbf{u}_2^I|||_{X_2^h} \le C\, \left(\varepsilon_1 + \varepsilon_2\right), \tag{6.9}$$

where

$$\begin{aligned} \varepsilon_1 &= h_1^r(|\mathbf{u}_1|_{r+1,\Omega_1} + |p_1|_{r,\Omega_1}) \\ \varepsilon_2 &= h_2\,(|p_2|_{1,\Omega_2} + |p_2|_{2,\Omega_2} + |\mathbf{u}_2|_{1,\Omega_2}) + h_2^{1/2}\left(h_2 h_1^{-1/2} + h_1^{1/2}\right)\|p_2\|_{1,\Omega_2}. \end{aligned}$$

Proof We choose the test functions in (6.8) to be $\mathbf{v}^h = \mathbf{u}^h - \tilde{\mathbf{u}}$ and $q^h = p^h - \tilde{p}$. The definition of $\pi_1^h(\mathbf{u}_1)$ implies that the right-hand side of the second equation in (6.8) is zero:

$$b_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1,\ p_1^h - \tilde{p}_1\right) = 0.$$

Using the commutative property (3.16) and (5.3) we conclude that

$$\begin{aligned}\mathcal{DIV}\left(\mathbf{u}_2^h - \tilde{\mathbf{u}}_2\right) &= \mathcal{DIV}\left(\mathbf{u}_2^h - \mathbf{u}_2^I - (\nabla\varphi)^I\right)\\ &= \mathcal{DIV}\,\mathbf{u}_2^h - (\operatorname{div}\mathbf{u}_2)^I - (\nabla\cdot\nabla\varphi)^I = f_2^I - f_2^I - 0 = 0.\end{aligned}$$

Inserting the last two results into the first equation in (6.8), we eliminate the terms in the left-hand side that contain the bilinear forms b_1 and b_2. Using the definition of $\tilde{\mathbf{u}}_2$, we break the third term in the left-hand side into three pieces:

$$\begin{aligned}&a_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1, \mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) + a_2\left(\mathbf{u}_2^h - \mathbf{u}_2^I,\ \mathbf{u}_2^h - \mathbf{u}_2^I\right)\\ &\quad= a_1\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1,\ \mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) + b_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1,\ p_1 - \tilde{p}_1\right)\\ &\qquad+ \sum_{e\in\Gamma_I^h}\int_e p_2\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\cdot\mathbf{n}_1\,\mathrm{d}\,s - a_2\left(\mathbf{u}_2^I, \mathbf{u}_2^h - \mathbf{u}_2^I\right) + a_2\left(\mathbf{u}_2^I, (\nabla\varphi)^I\right)\\ &\qquad+ a_2\left(\mathbf{u}_2^h - \mathbf{u}_2^I,\ (\nabla\varphi)^I\right) \equiv T_1 + T_2 + T_3 + T_4 + T_5 + T_6.\end{aligned} \tag{6.10}$$

To bound T_1, we follow the analysis of a similar term in [48]. We expand it as follows:

$$\begin{aligned}a_1\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1,\ \mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) &= 2\mu\sum_{E\in\Omega_1^h}\int_E \mathbf{D}\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1\right) : \mathbf{D}\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\mathrm{d}\,x\\ &\quad-2\mu\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\int_e \{\mathbf{D}\left(\mathbf{u}_1 - \tilde{\mathbf{u}}_1\right)\}\mathbf{n}_e\cdot\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\mathrm{d}\,s\\ &\quad+2\mu\varepsilon\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\int_e \{\mathbf{D}\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\}\mathbf{n}_e\cdot\left[\mathbf{u}_1 - \tilde{\mathbf{u}}_1\right]\mathrm{d}\,s\\ &\quad+\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\frac{\sigma_e}{h_e}\int_e\left[\mathbf{u}_1 - \tilde{\mathbf{u}}_1\right]\cdot\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\mathrm{d}\,s\\ &\quad+\sum_{e\in\Gamma_I^h}\sum_{j=1}^{d-1}\frac{\mu}{G_j}\int_e(\mathbf{u}_1 - \tilde{\mathbf{u}}_1)\cdot\boldsymbol{\tau}_j\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\cdot\boldsymbol{\tau}_j\,\mathrm{d}\,s\\ &\equiv T_{11} + T_{12} + T_{13} + T_{14} + T_{15}.\end{aligned} \tag{6.11}$$

To estimate T_{11}, we apply the Cauchy–Schwarz inequality, the Young inequality (4.1), and the approximation property (4.11):

$$
\begin{aligned}
|T_{11}| &\leq 2\mu \sum_{E\in\Omega_h^1} \|\nabla(\mathbf{u}_1 - \tilde{\mathbf{u}}_1)\|_{0,E} \left\|\nabla\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\right\|_{0,E} \\
&\leq C\, \|\nabla_h(\mathbf{u}_1 - \tilde{\mathbf{u}}_1)\|_{0,\Omega_1}^2 \; + \; \frac{1}{8}\left\|\nabla_h\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\right\|_{0,\Omega_1}^2 \\
&\leq C\, h_1^{2r}\, |\mathbf{u}_1|_{r+1,\Omega_1}^2 + \frac{1}{8}\left\|\nabla_h\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right)\right\|_{0,\Omega_1}^2 .
\end{aligned}
\tag{6.12}
$$

To bound T_{12}, we introduce the Lagrange interpolant $\mathcal{L}_h(\mathbf{u}_1)$ of degree r satisfying

$$
|\mathbf{u}_1 - \mathcal{L}_h(\mathbf{u}_1)|_{m,E} \leq C\, h_E^{s-m}\, |\mathbf{u}_1|_{s,E}, \quad 2 \leq s \leq r+1, \quad m = 0, 1, 2. \tag{6.13}
$$

Let $\delta(e)$ be the union of elements having the face e. We split T_{12} in two pieces T_{12}^a and T_{12}^b by adding and subtracting $\mathcal{L}_h(\mathbf{u}_1)$ inside the average factor $\{\cdot\}$. Using the Cauchy–Schwarz inequality, the Young inequality (4.1), the trace inequality (4.4), and (6.13), we obtain

$$
\begin{aligned}
|T_{12}^a| &= \left| \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \int_e \left\{\mathbf{D}(\mathcal{L}_h(\mathbf{u}_1) - \tilde{\mathbf{u}}_1)\right\}\mathbf{n}_e \cdot [\mathbf{u}_1^h - \tilde{\mathbf{u}}_1]\, \mathrm{d}\, s\right| \\
&\leq \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{h_e^{1/2}}{\sigma_e^{1/2}} \left\|\left\{\mathbf{D}(\mathcal{L}_h(\mathbf{u}_1) - \tilde{\mathbf{u}}_1)\right\}\mathbf{n}_e\right\|_{0,e} \frac{\sigma_e^{1/2}}{h_e^{1/2}} \left\|\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\right\|_{0,e} \\
&\leq C \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} |\mathcal{L}_h(\mathbf{u}_1) - \tilde{\mathbf{u}}_1|_{1,\delta(e)}^2 \; + \; \frac{1}{8} \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{\sigma_e}{h_e} \left\|\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\right\|_{0,e}^2 \\
&\leq C h_1^{2r} |\mathbf{u}_1|_{r+1,\Omega_1}^2 + \; \frac{1}{8} \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{\sigma_e}{h_e} \left\|\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\right\|_{0,e}^2 .
\end{aligned}
\tag{6.14}
$$

The other term is estimated similarly using the trace inequality (4.3):

$$
\begin{aligned}
|T_{12}^b| &= \left| \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \int_e \left\{\mathbf{D}(\mathbf{u}_1 - \mathcal{L}_h(\mathbf{u}_1))\right\}\mathbf{n}_e \cdot [\mathbf{u}_1^h - \tilde{\mathbf{u}}_1]\, \mathrm{d}\, s\right| \\
&\leq C \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{h_e}{\sigma_e} \left(h_e^{-1}\, |\mathbf{u}_1 - \mathcal{L}_h(\mathbf{u}_1)|_{1,\delta(e)}^2 \; + \; h_e\, |\mathbf{u}_1 - \mathcal{L}_h(\mathbf{u}_1)|_{2,\delta(e)}^2\right) \\
&\quad + \frac{1}{8} \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{\sigma_e}{h_e} \left\|\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\right\|_{0,e}^2 \\
&\leq C\, h_1^{2r}\, |\mathbf{u}_1|_{r+1,\Omega_1}^2 \; + \; \frac{1}{8} \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{\sigma_e}{h_e} \left\|\left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right]\right\|_{0,e}^2 .
\end{aligned}
\tag{6.15}
$$

We conclude that

$$|T_{12}| \le C\, h_1^{2r}\, |\mathbf{u}_1|^2_{r+1,\Omega_1} + \frac{1}{4} \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \frac{\sigma_e}{h_e} \left\| [\mathbf{u}_1^h - \tilde{\mathbf{u}}_1] \right\|^2_{0,e}. \tag{6.16}$$

For simplicial meshes, the third term in (6.11) is zero, $T_{13} = 0$, due to the continuity of $\mathbf{u}_1$ and the property (4.8). For polygonal and polyhedral meshes, we use the Cauchy–Schwarz inequality, the Young inequality (4.1), the trace inequality (4.4), and the approximation result (6.5) to obtain

$$\begin{aligned} |T_{13}| &\le 2\mu \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \left\| \mathbf{D}\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \cdot \mathbf{n}_e \right\|_{0,e} \| [\mathbf{u}_1 - \tilde{\mathbf{u}}_1] \|_{0,e} \\ &\le 2\mu \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \left(\frac{h_e}{C} \left\| \mathbf{D}\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \cdot \mathbf{n}_e \right\|^2_{0,e} + \frac{C}{h_e} \| [\mathbf{u}_1 - \tilde{\mathbf{u}}_1] \|^2_{0,e} \right) \\ &\le \frac{1}{8} \left\| \nabla_h \left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \right\|^2_{0,\Omega_1} + C h_E^2 |\mathbf{u}_1|^2_{2,\Omega_1}. \end{aligned}$$

The fourth term is bounded applying the Cauchy–Schwarz inequality, the approximation property (4.10), and the trace inequality (4.2):

$$\begin{aligned} |T_{14}| &\le C \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \frac{\sigma_e}{h_e} \| \mathbf{u}_1 - \tilde{\mathbf{u}}_1 \|^2_{0,e} + \frac{1}{8} \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \frac{\sigma_e}{h_e} \left\| \left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1 \right] \right\|^2_{0,e} \\ &\le C\, h_1^{2r}\, |\mathbf{u}_1|^2_{r+1,\Omega_1} + \frac{1}{8} \sum_{e \in \mathcal{E}_1^h \cup \Gamma_1^h} \frac{\sigma_e}{h_e} \left\| \left[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1 \right] \right\|^2_{0,e}. \end{aligned} \tag{6.17}$$

Using the same arguments, we bound the fifth term:

$$\begin{aligned} |T_{15}| &\le \sum_{e \in \Gamma_I^h} \sum_{j=1}^{d-1} \frac{\mu}{G_j} \| \mathbf{u}_1 - \tilde{\mathbf{u}}_1 \|_{0,e} \left\| \left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \cdot \boldsymbol{\tau} \right\|_{0,e} \\ &\le C\, h_1^{2r}\, |\mathbf{u}_1|^2_{r+1,\Omega_1} + \sum_{e \in \Gamma_I^h} \sum_{j=1}^{d-1} \frac{\mu}{2G_j} \left\| \left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \cdot \boldsymbol{\tau} \right\|^2_{0,e}. \end{aligned} \tag{6.18}$$

To handle the term T_2, we use the property (6.1) of the L^2-projection $\tilde{p}_1$:

$$b_1\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1,\ p_1 - \tilde{p}_1\right) = - \sum_{E \in \Omega_1^h} \int_E (p_1 - \tilde{p}_1)\, \mathrm{div} \left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\right) \mathrm{d}\, x$$

$$
\begin{aligned}
&+ \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \int \{p_1 - \tilde{p}_1\}\Big[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\Big]\cdot \mathbf{n}_e \,\mathrm{d}\, s \\
&= \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \int \{p_1 - \tilde{p}_1\}\Big[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\Big]\cdot \mathbf{n}_e \,\mathrm{d}\, s. \qquad (6.19)
\end{aligned}
$$

Thus, using the trace inequality (4.2) and the property (6.2) of the L^2 projection $\tilde{p}_1$, we get

$$
|T_2| \le C\, h_1^{2r}\, |p_1|^2_{r,\Omega_1} \;+\; \frac{1}{8} \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h} \frac{\sigma_e}{h_e} \Big\| \Big[\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\Big]\Big\|^2_{0,e}. \qquad (6.20)
$$

For the remaining terms in the error equation (6.10) we use arguments developed for the analysis of mimetic discretizations of elliptic equations [14,40]. We use the piecewise constant tensor $\overline{\mathbf{K}}$ defined at the beginning of this section.

Let p_2^1 be a discontinuous piecewise linear function defined on Ω_2^h such that (6.3) holds on every element $E \in \Omega_2^h$. Then, adding and subtracting $\overline{\mathbf{K}}\nabla p_2^1$, we obtain

$$
T_4 = a_2\Big((\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)^I,\, \mathbf{u}_2^I - \mathbf{u}_2^h\Big) - a_2\Big((\overline{\mathbf{K}}\nabla p_2^1)^I,\, \mathbf{u}_2^I - \mathbf{u}_2^h\Big) \equiv T_{41} + T_{42}. \qquad (6.21)
$$

Applying the Cauchy–Schwarz inequality, the stability assumption (3.18), and the trace inequality (4.2), we get

$$
\begin{aligned}
|T_{41}| &\le |\!|\!| (\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)^I |\!|\!|_{X_2^h}\; |\!|\!| \mathbf{u}_2^h - \mathbf{u}_2^I |\!|\!|_{X_2^h} \\
&\le C\Bigg(\sum_{E\in\Omega_2^h} |E| \sum_{e\subset\partial E} \Big| \frac{1}{|e|} \int_e (\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)\cdot \mathbf{n}_e \,\mathrm{d}\, s \Big|^2 \Bigg)^{1/2} |\!|\!| \mathbf{u}_2^h - \mathbf{u}_2^I |\!|\!|_{X_2^h} \\
&\le C\Bigg(\sum_{E\in\Omega_2^h} \Big[\|\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1\|^2_{0,E} + h_E^2\, |\mathbf{u}_2|^2_{1,E} \Big] \Bigg)^{1/2} |\!|\!| \mathbf{u}_2^h - \mathbf{u}_2^I |\!|\!|_{X_2^h}. \qquad (6.22)
\end{aligned}
$$

Using the triangle inequality and then estimates (6.6) and (6.3), we obtain

$$
\begin{aligned}
\Big\| \mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1 \Big\|^2_{0,E} &\le \Big\| \mathbf{K}\nabla(p_2 - p_2^1) \Big\|_{0,E} + \Big\| (\mathbf{K} - \overline{\mathbf{K}})\nabla p_2^1 \Big\|_{0,E} \\
&\le C\, \Big(h_E\, |p_2|_{2,E} + h_E\, \|\nabla p_2^1\|_{0,E} \Big) \\
&\le C h_E\, \Big(\, |p_2|_{2,E} + \|\nabla p_2\|_{0,E} + \|\nabla(p_2 - p_2^1)\|_{0,E} \Big) \\
&\le C h_E\, \big(|p_2|_{2,E} + |p_2|_{1,E}\big)\,.
\end{aligned}
$$

Combining the last two inequalities and applying the Young inequality (4.1), we obtain

$$|T_{41}| \leq C\, h_2^2\, (|p_2|_{1,\Omega_2} + |p_2|_{2,\Omega_2} + |\mathbf{u}_2|_{1,\Omega_2})^2 + \frac{1}{8} \left|\!\left|\!\left| \mathbf{u}_2^h - \mathbf{u}_2^I \right|\!\right|\!\right|_{X_2^h}^2. \tag{6.23}$$

The consistency condition (3.17) and continuity of p_2 allow us to rewrite T_{42} as follows:

$$\begin{aligned} T_{42} &= \sum_{E\in\Omega_2^h} \sum_{e\subset\partial E} \chi_E^e \left(\mathbf{u}_2^h - \mathbf{u}_2^I\right)_E^e \int_e p_{2,E}^1 \,\mathrm{d}\, s \\ &= \sum_{E\in\Omega_2^h} \sum_{e\subset\partial E} \chi_E^e \left(\mathbf{u}_2^h - \mathbf{u}_2^I\right)_E^e \int_e \left(p_{2,E}^1 - p_2\right) \mathrm{d}\, s + \sum_{e\in\Gamma_I^h} \chi_E^e \left(\mathbf{u}_2^h - \mathbf{u}_2^I\right)_E^e \int_e p_2 \,\mathrm{d}\, s \\ &\equiv T_{42}^a + T_{42}^b. \end{aligned} \tag{6.24}$$

We estimate T_{42}^a using (6.4) and the stability property (3.18):

$$\begin{aligned} |T_{42}^a| &\leq \sum_{E\in\Omega_2^h} \sum_{e\subset\partial E} |e|^{1/2} \left|\left(\mathbf{u}_2^h - \mathbf{u}_2^I\right)_E^e\right| \, \|p_{2,E}^1 - p_2\|_{0,e} \\ &\leq C \sum_{E\subset\Omega_2^h} h_E \left(|E| \sum_{e\subset\partial E} |(\mathbf{u}_2^h - \mathbf{u}_2^I)_E^e|^2\right)^{1/2} |p_2|_{2,E} \\ &\leq C\, h_2\, |p|_{2,\Omega_2}\, |\!|\!|\mathbf{u}_2^h - \mathbf{u}_2^I|\!|\!|_{X_2^h} \leq C\, h_2^2\, |p|_{2,\Omega_2}^2 + \frac{1}{8} \left|\!\left|\!\left| \mathbf{u}_2^h - \mathbf{u}_2^I \right|\!\right|\!\right|_{X_2^h}^2. \end{aligned} \tag{6.25}$$

The term T_{42}^b will be combined with other terms later. Now we proceed with the fifth term in the error equation. Adding and subtracting $\overline{\mathbf{K}}\nabla p_2^1$, we get

$$T_5 = a_2\left((\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)^I,\ (\nabla\varphi)^I\right) - a_2\left((\overline{\mathbf{K}}\nabla p_2^1)^I,\ (\nabla\varphi)^I\right) \equiv T_{51} + T_{52}. \tag{6.26}$$

The term T_{51} is similar to T_{41}; therefore, we use the same approach to bound it:

$$|T_{51}| \leq C\, h_2 \left(|p_2|_{1,\Omega_2} + |p_2|_{2,\Omega_2} + |\mathbf{u}_2|_{1,\Omega_2}\right) \left|\!\left|\!\left| (\nabla\varphi)^I \right|\!\right|\!\right|_{X_2^h}.$$

Using estimate (5.9), we conclude that

$$|T_{51}| \leq C\, h_2 h_1\, (|p_2|_{1,\Omega_2} + |p_2|_{2,\Omega_2} + |\mathbf{u}_2|_{1,\Omega_2})\, |\mathbf{u}_1|_{3/2,\Omega_1}. \tag{6.27}$$

For the term T_{52}, we apply estimate (6.4) and the consistency condition (3.17):

$$
\begin{aligned}
T_{52} &= -\sum_{E\in\Omega_2^h}\sum_{e\subset\partial E}\chi_E^e\left((\nabla\varphi)^I\right)_E^e\int_e p_{2,E}^1\,\mathrm{d}\,s\\
&= \sum_{E\in\Omega_2^h}\sum_{e\subset\partial E}\chi_E^e\left((\nabla\varphi)^I\right)_E^e\int_e\left(p_2-p_{2,E}^1\right)\mathrm{d}\,s-\sum_{e\in\Gamma_I^h}\chi_E^e\left((\nabla\varphi)^I\right)_E^e\int_e p_2\,\mathrm{d}\,s\\
&\equiv T_{52}^a+T_{52}^b.
\end{aligned}
\tag{6.28}
$$

Using the arguments for the terms T_{42}^a and T_{51}, the term T_{52}^a is bounded as

$$
|T_{52}^a|\le C\,h_2\,|p|_{2,\Omega_2}\left|\left|\left|(\nabla\varphi)^I\right|\right|\right|_{X_2^h}\le C\,h_2\,h_1\,|p|_{2,\Omega_2}\,|\mathbf{u}_1|_{3/2,\Omega_1}.
\tag{6.29}
$$

The term T_{52}^b will be combined with other terms later.

The sixth term in the error equation is bounded using the Cauchy–Schwarz inequality and estimate (5.9):

$$
|T_6|\le\left|\left|\left|\mathbf{u}_2^h-\mathbf{u}_2^I\right|\right|\right|_{X_2^h}\left|\left|\left|(\nabla\varphi)^I\right|\right|\right|_{X_2^h}\le\frac{1}{8}\left|\left|\left|\mathbf{u}_2^h-\mathbf{u}_2^I\right|\right|\right|_{X_2^h}^2+C\,h_1^2\,|\mathbf{u}_1|_{3/2,\Omega_1}^2.
\tag{6.30}
$$

Finally, the third term in the error equation (6.10) is combined with T_{42}^b and T_{52}^b. Let $p_2^*\in\Lambda_I^h$ such that $(p_2^*)^e$ is the L^2-projection of p_2 on $\mathbb{P}_0(e)$ and let $\bar{p}_2^*$ be the piecewise constant function on Γ_I^h satisfying

$$
\bar{p}_2^*|_e=(p_2^*)^e,\quad\forall e\in\Gamma_I^h.
$$

Because $\mathbf{u}^h-\tilde{\mathbf{u}}^h\in V^h$,

$$
\int_{\Gamma_I}\bar{p}_2^*\left(\mathbf{u}_1^h-\tilde{\mathbf{u}}_1\right)\cdot\mathbf{n}_1\,\mathrm{d}\,s+\left\langle p_2^*,\mathbf{u}_2^h-\tilde{\mathbf{u}}_2\right\rangle_{\Lambda_I^h}=0.
$$

Using the above equation, the definition of operator π_2^h and the property of the L^2 projection, we obtain

$$
\begin{aligned}
T_3+T_{42}^b+T_{52}^b&=\sum_{e\in\Gamma_I^h}\left(\int_e p_2\left(\mathbf{u}_1^h-\tilde{\mathbf{u}}_1\right)\cdot\mathbf{n}_1\,\mathrm{d}\,s+\chi_E^e\left(\mathbf{u}_2^h-\mathbf{u}_2^I-(\nabla\varphi)^I\right)_E^e\int_e p_2\,\mathrm{d}\,s\right)\\
&=\sum_{e\in\Gamma_I^h}\left(\int_e p_2(\mathbf{u}_1^h-\tilde{\mathbf{u}}_1)\cdot\mathbf{n}_1\,\mathrm{d}\,s+(\mathbf{u}_2^h-\tilde{\mathbf{u}}_2)_E^e\int_e p_2\,\mathrm{d}\,s\right)
\end{aligned}
$$

$$= \sum_{e\in\Gamma_I^h} \left(\int_e (p_2 - \bar{p}_2^*)(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1)\cdot \mathbf{n}_1 \,\mathrm{d}\, s + (\mathbf{u}_2^h - \tilde{\mathbf{u}}_2)_E^e \int_e (p_2 - (p_2^*)^e)\,\mathrm{d}\, s \right)$$
$$= \sum_{e\in\Gamma_I^h} \int_e (p_2 - \bar{p}_2^*)(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1)\cdot \mathbf{n}_1 \,\mathrm{d}\, s.$$

For each face $e \in \Gamma_I^h$ we define $\mathbf{c}^e$ to be the L^2-projection of $\mathbf{u}^h - \tilde{\mathbf{u}}$ on $\mathbb{P}_0(e)$. Let us assume that $e = E_2^e \bigcap \bigcup_{i=1}^{n_e} E_{1,i}^e$, where $E_2^e \in \Omega_2^h$, and $E_{1,i}^e \in \Omega_1^h$ for $i = 1, ..., n_e$. Using the orthogonality and approximation properties of the L^2-projection, and the trace inequality (4.2), we obtain

$$\begin{aligned}
\left|T_3 + T_{42}^b + T_{52}^b\right| &= \sum_{e\in\Gamma_I^h} \int_e (p_2 - \bar{p}_2^*|_e)\left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1 - \mathbf{c}^e\right)\cdot \mathbf{n}_1 \,\mathrm{d}\, s \\
&\le C \sum_{e\in\Gamma_I^h} h_2^{1/2} \|p_2\|_{1,E_2^e} \sum_{i=1}^{n_e} \left(h_1^{-1/2} \left\|\mathbf{u}_1^h - \tilde{\mathbf{u}}_1 - \mathbf{c}^e\right\|_{0,E_{1,i}^e} \right. \\
&\quad \left. + h_1^{1/2} |\mathbf{u}_1^h - \tilde{\mathbf{u}}_1|_{1,E_{1,i}^e} \right) \\
&\le C \sum_{e\in\Gamma_I^h} h_2^{1/2} \|p_2\|_{1,E_2^e} \sum_{i=1}^{n_e} \left(h_* h_1^{-1/2} + h_1^{1/2} \right) |\mathbf{u}_1^h - \tilde{\mathbf{u}}_1|_{1,E_{1,i}^e} \\
&\le C h_2 \left(h_* h_1^{-1/2} + h_1^{1/2} \right)^2 \|p_2\|_{1,\Omega_2}^2 + \frac{1}{8} \left\| \nabla_h \left(\mathbf{u}_1^h - \tilde{\mathbf{u}}_1 \right) \right\|_{0,\Omega_1}^2,
\end{aligned} \tag{6.31}$$

where

$$h_* = \max(h_1, h_2).$$

If $h_* = h_1$, then the terms $h_* h_1^{-1/2}$ and $h_1^{1/2}$ can be combined. Otherwise, we have the extra term $h_2 h_1^{-1/2}$. Collecting the estimates of all terms in the right hand side of error equation (6.10), using coercivity Lemma 5.3, then the triangle inequality $\|\mathbf{u}_1^h - \mathbf{u}_1\|_{X_1} \le \|\mathbf{u}_1^h - \tilde{\mathbf{u}}_1\|_{X_1} + \|\tilde{\mathbf{u}}_1 - \mathbf{u}_1\|_{X_1}$, and finally the interpolant property (4.11), we prove the assertion of the theorem. □

6.3 Pressure estimates

Theorem 6.2 *Under the assumptions of Theorem 6.1, the following error bound holds:*

$$\|p^h - p\|_{Q^h} \le C(\varepsilon_1 + \varepsilon_2) \tag{6.32}$$

where

$$\varepsilon_1 = h_1^r\,(|p_1|_{r,\Omega_1} + |\mathbf{u}_1|_{r+1,\Omega_1}),$$
$$\varepsilon_2 = h_2(|p_2|_{1,\Omega_2} + |p_2|_{2,\Omega_2} + |\mathbf{u}_2|_{1,\Omega_2}) + h_2^{1/2}\Big(h_2 h_1^{-1/2} + h_1^{1/2}\Big)\|p_2\|_{1,\Omega_2}.$$

Proof Taking $q^h = (p_1^h - \tilde{p}_1,\ p_2^h - \tilde{p}_2)$ in the inf-sup condition (5.10), we get

$$\left\|p^h - \tilde{p}\right\|_Q \le \frac{1}{\beta}\sup_{\mathbf{v}^h\in V^h}\frac{b_1(\mathbf{v}_1^h,\ p_1^h - \tilde{p}_1) + b_2(\mathbf{v}_2^h,\ p_2^h - \tilde{p}_2)}{\|\mathbf{v}^h\|_{X^h}}. \tag{6.33}$$

From (6.8), we get

$$\begin{aligned} b_1\Big(\mathbf{v}_1^h,\ p_1^h - \tilde{p}_1\Big) + b_2\Big(\mathbf{v}_2^h,\ p_2^h - \tilde{p}_2\Big) &= a_1\Big(\mathbf{u}_1 - \mathbf{u}_1^h,\ \mathbf{v}_1^h\Big) + b_1\Big(\mathbf{v}_1^h,\ p_1 - \tilde{p}_1\Big) \\ &\quad + \sum_{e\in\Gamma_I^h}\int_e p_2\,\mathbf{v}_1^h\cdot\mathbf{n}_1\,\mathrm{d}\,s - a_2\Big(\mathbf{u}_2^h,\ \mathbf{v}_2^h\Big) - b_2\Big(\mathbf{v}_2^h,\ \tilde{p}_2\Big) \\ &\equiv J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

By adding and subtracting terms, and using the consistency condition (3.17), we obtain

$$\begin{aligned} J_4 + J_5 &= -a_2\Big((\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)^I,\ \mathbf{v}_2^h\Big) + a_2\Big((\overline{\mathbf{K}}\nabla p_2^1)^I,\ \mathbf{v}_2^h\Big) \\ &\quad + \Big[\mathcal{DIV}\,\mathbf{v}_2^h,\ (p_2 - p_2^1)^I\Big]_{Q_2^h} + \Big[\mathcal{DIV}\,\mathbf{v}_2^h,\ (p_2^1)^I\Big]_{Q_2^h} - a_2\Big(\mathbf{u}_2^h - \mathbf{u}_2^I,\ \mathbf{v}_2^h\Big) \\ &= -a_2\Big((\mathbf{u}_2 + \overline{\mathbf{K}}\nabla p_2^1)^I,\ \mathbf{v}_2^h\Big) + \sum_{e\subset\partial E}\chi_E^e(\mathbf{v}_2^h)_E^e\int_e p_2^1\,\mathrm{d}\,s \\ &\quad + \Big[\mathcal{DIV}\,\mathbf{v}_2^h,\ (p_2 - p_2^1)^I\Big]_{Q_2^h} - a_2\Big(\mathbf{u}_2^h - \mathbf{u}_2^I,\ \mathbf{v}_2^h\Big) \\ &= J_6 + J_7 + J_8 + J_9. \end{aligned} \tag{6.34}$$

Thus, we need to estimate seven terms. We expand J_1 as follows:

$$\begin{aligned} J_1 = a_1(\mathbf{u}_1 - \mathbf{u}_1^h,\ \mathbf{v}_1^h) &= 2\mu\sum_{E\in\Omega_1^h}\int_E \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h) : \mathbf{D}(\mathbf{v}_1^h)\,\mathrm{d}\,x \\ &\quad - 2\mu\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\int_e \Big\{\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_1^h)\Big\}\mathbf{n}_e\cdot[\mathbf{v}_1^h]\,\mathrm{d}\,s \\ &\quad + 2\mu\varepsilon\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\int_e \Big\{\mathbf{D}(\mathbf{v}_1^h)\Big\}\mathbf{n}_e\cdot\Big[\mathbf{u}_1 - \mathbf{u}_1^h\Big]\,\mathrm{d}\,s \\ &\quad + \sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\frac{\sigma_e}{h_e}\int_e \Big[\mathbf{u}_1 - \mathbf{u}_1^h\Big]\cdot\Big[\mathbf{v}_1^h\Big]\,\mathrm{d}\,s \end{aligned}$$

$$+\sum_{e\in\Gamma_I^h}\sum_{j=1}^{d-1}\frac{\mu}{G_j}\int_e\left(\mathbf{u}_1-\mathbf{u}_1^h\right)\cdot\boldsymbol{\tau}\ \mathbf{v}_1^h\cdot\boldsymbol{\tau}\,\mathrm{d}\,s$$
$$= J_{11}+J_{12}+J_{13}+J_{14}+J_{15}. \tag{6.35}$$

From Cauchy–Schwarz inequality, we immediately get bounds for three terms:

$$|J_{11}+J_{14}+J_{15}|\le C\,\|\mathbf{u}_1-\mathbf{u}_1^h\|_{X_1}\,\|\mathbf{v}_1^h\|_{X_1}. \tag{6.36}$$

We bound J_{12} by taking similar approach as the one used for T_{12},

$$|J_{12}|\le C\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\left(\frac{h_e}{\sigma_e}\right)^{1/2}\|\nabla(\mathbf{u}_1-\mathbf{u}_1^h)\|_{0,e}\left(\frac{\sigma_e}{h_e}\right)^{1/2}\|[\mathbf{v}_1^h]\|_{0,e}$$
$$\le C\left(\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\frac{h_e}{\sigma_e}\Big(\|\nabla(\mathbf{u}_1-\tilde{\mathbf{u}}_1)\|_{0,e}^2+\|\nabla(\tilde{\mathbf{u}}_1-\mathbf{u}_1^h)\|_{0,e}^2\Big)\right)^{1/2}\|\mathbf{v}_1^h\|_{X_1}$$
$$\le C\Big(h_1^{2r}\,|\mathbf{u}_1|_{r+1,\Omega_1}^2+\|\tilde{\mathbf{u}}_1-\mathbf{u}_1^h\|_{X_1}^2\Big)^{1/2}\|\mathbf{v}_1^h\|_{X_1}. \tag{6.37}$$

To bound the term J_{13}, we use the trace inequality (4.4), and the shape regularity of element E^e with face e:

$$|J_{13}|\le C\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\left\|\left\{\mathbf{D}(\mathbf{v}_1^h)\right\}\mathbf{n}_e\right\|_{0,e}\left\|\left[\mathbf{u}_1-\mathbf{u}_1^h\right]\right\|_{0,e}$$
$$\le C\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}h_{E^e}^{-1/2}\left(\frac{h_e}{\sigma_e}\right)^{1/2}\|\nabla\mathbf{v}_1^h\|_{0,E^e}\left(\frac{\sigma_e}{h_e}\right)^{1/2}\left\|\left[\mathbf{u}_1-\mathbf{u}_1^h\right]\right\|_{0,e}$$
$$\le C\|\mathbf{v}_1^h\|_{X_1^h}\,\|\mathbf{u}_1-\mathbf{u}_1^h\|_{X_1}. \tag{6.38}$$

We proceed with J_2 by applying the trace inequality (4.2) and the property (6.2) of the L^2 projection:

$$|J_2|=\left|b_1\Big(\mathbf{v}_1^h,\ p_1-\tilde{p}_1\Big)\right|=\left|\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\int_e\{p_1-\tilde{p}_1\}\,[\mathbf{v}_1^h]\cdot\mathbf{n}_e\,\mathrm{d}\,s\right|$$
$$\le\sum_{e\in\mathcal{E}_1^h\cup\Gamma_1^h}\left(\frac{h_e}{\sigma_e}\right)^{1/2}\|\{p_1-\tilde{p}_1\}\|_{0,e}\left(\frac{\sigma_e}{h_e}\right)^{1/2}\|\mathbf{v}_1^h\|_{0,e}$$
$$\le Ch_1^r|p|_{r,\Omega_1}\,\|\mathbf{v}_1^h\|_{X_1}. \tag{6.39}$$

By combining J_3 with J_7 and repeating the steps we followed to bound T_3 and T_{42}, we get

$$|J_3+J_7| \leq C \left(h_2\,|p_2|_{2,\Omega_2}\,|||\mathbf{v}_2^h|||_{X_2^h}+h_2^{1/2}\left(h_2 h_1^{-1/2}+h_1^{1/2}\right)\|p_2\|_{1,\Omega_2}\,\|\nabla_h \mathbf{v}_1^h\|_{0,\Omega_1}\right).$$

Since J_6 is similar to T_{51}, we can write:

$$|J_6| \leq C\,h_2\,\left(|p_2|_{1,\Omega_2}+|p_2|_{2,\Omega_2}+|\mathbf{u}_2|_{1,\Omega_2}\right)\,|||\mathbf{v}_2^h|||_{X_2^h}. \tag{6.40}$$

The term J_8 is estimated by using the Cauchy–Schwartz inequality and the approximation properties (6.3):

$$|J_8| \leq C\,h_2^2\,|||\mathbf{v}_2^h|||_{div}\,|p_2|_{2,\Omega_2}. \tag{6.41}$$

Next, for the term J_9, using the Cauchy–Schwarz inequality and the velocity estimates, we find that

$$\begin{aligned}|J_9| \ &\leq C\left(h_1(|\mathbf{u}_1|_{2,\Omega_1}+|p_1|_{1,\Omega_1})+h_2(|p_2|_{1,\Omega_2}+|p_2|_{2,\Omega_2}+|\mathbf{u}_2|_{1,\Omega_2})\right.\\ &\quad\left.+\,h_2^{1/2}\left(h_2 h_1^{-1/2}+h_1^{1/2}\right)\|p_2\|_{1,\Omega_1}\right)|||\mathbf{v}_2^h|||_{X_2^h}.\end{aligned} \tag{6.42}$$

Combining all the bounds and dividing by $\|\mathbf{v}^h\|_X$ yields the assertion of the theorem. □

7 Numerical experiments

7.1 Implementation details

The global velocity space V^h, which embeds the interface continuity constraint, is not convenient for a computer program. Instead, the continuity constraints on the velocity are imposed weakly and additional variables, the Lagrange multipliers are added to the system.

For the efficient solution of Darcy's law we employ hybridization [3]. We relax the flux continuity condition on all mesh faces in the Darcy region. Two flux degrees of freedom U^e_{2,E_1} and U^e_{2,E_2} are prescribed to every interior face e and the explicit continuity condition

$$U^e_{2,E_1}+U^e_{2,E_2}=0$$

is added to the system. The new system is algebraically equivalent to the original system; however, it has a special structure that allows to eliminate efficiently the primary pressure and velocity unknowns in the Darcy region.

Each continuity constraint results in one Lagrange multiplier. We collect the Lagrange multipliers in a single vector $\mathbf{L}=(\lambda^{e_1},...,\lambda^{e_J})$, where J is the number of the mesh edges in Ω_2^h.

Let us define the block-diagonal matrix $\mathbf{M}_2 = \text{diag}\,\{\mathbf{M}_{2,E_1}, \ldots, \mathbf{M}_{2,E_N}\}$ and the velocity continuity matrix $\mathbf{C}_2 = \text{diag}\,\{|e_1|, ..., |e_J|\}$. Let $\mathbf{A_1}$ and $\mathbf{B_1}$ be the matrices associated with the bilinear forms $a_1(\cdot,\cdot)$ and $b_1(\cdot,\cdot)$, respectively. The matrix associated with the interface term is denoted by $\mathbf{C}_1$. The matrix equations are

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{0} & \mathbf{0} & \mathbf{C}_1 \\ \mathbf{B}_1^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M}_2 & \mathbf{B}_2 & \mathbf{C}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_2^T & \mathbf{0} & \mathbf{0} \\ \mathbf{C}_1^T & \mathbf{0} & \mathbf{C}_2^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{P}_1 \\ \mathbf{U}_2 \\ \mathbf{P}_2 \\ \mathbf{L} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{0} \\ \mathbf{0} \\ -\mathbf{F}_2 \\ \mathbf{0} \end{pmatrix}, \tag{7.1}$$

where $\mathbf{F}_2$ is a vector of size N consisting of the cell averages of the source term.

The first pair of equations is the matrix form of discrete Stokes problem. The second pair of equations represents elemental equations for the Darcy region. The last block equation represents continuity of Darcy velocities and no-slip boundary conditions.

The matrix of system (7.1) is symmetric. The hybridization procedure results in a block-diagonal matrix $\mathbf{B}_2$ with as many small blocks as the number of elements in Ω_2^h. Thus, the unknowns $\mathbf{U}_2$ and $\mathbf{P}_2$ may be eliminated explicitly. Changing the order of the remaining unknowns, we get the following saddle point problem:

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{C}_1 & \mathbf{B}_1 \\ \mathbf{C}_1^T & -\mathbf{A}_2 & \mathbf{0} \\ \mathbf{B}_1^T & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{L} \\ \mathbf{P}_1 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{G}_2 \\ \mathbf{0} \end{pmatrix}, \tag{7.2}$$

where

$$\mathbf{A}_2 = \mathbf{C}_2^T \left(\mathbf{M}_2^{-1} - \mathbf{M}_2^{-1}\mathbf{B}_2\,(\mathbf{B}_2^T\mathbf{M}_2^{-1}\mathbf{B}_2)^{-1}\,\mathbf{B}_2^T\mathbf{M}_2^{-1} \right) \mathbf{C}_2$$

is a symmetric positive definite (SPD) matrix. This matrix is a special approximation of the elliptic operator in the Darcy region. Note, that only $\mathbf{M}_2^{-1}$ is used in the above formula which suggests its direct calculation as described in [16].

Block-diagonal preconditioners for saddle point problems are discussed in [35,50]. A proper candidate for a preconditioner in our case could be

$$\mathbf{H} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{S} \end{pmatrix}, \tag{7.3}$$

where $\mathbf{S}$ is a suitable diagonal matrix. The analysis that Krylov space iterative methods with a preconditioner (7.3) have mesh independent convergence is beyond the scope of this article. The inversion of $\mathbf{A}_1$ and $\mathbf{A}_2$ can be performed by using one V-cycle of algebraic multigrid [53].

7.2 Three test problems

We present three computer experiments, the first two of which confirm the convergence rate of the method. The third test demonstrates the ability of the method to be applied to surface-subsurface flow problems with realistic geometries. In the first two tests the computational domain is $\Omega = \Omega_1 \cap \Omega_2$, where $\Omega_1 = [0, 1] \times [\frac{1}{2}, 1]$ and $\Omega_2 = [0, 1] \times [0, \frac{1}{2}]$. In the Stokes equation the stress tensor is taken to be

$$\mathbf{T}(\mathbf{u}_1, p_1) = -p_1\mathbf{I} + \mu\nabla\mathbf{u}_1.$$

It is easy to show that the theoretical analysis from the previous sections still applies with this choice of $\mathbf{T}(\mathbf{u}_1, p_1)$. Each convergence test uses a manufactured solution that satisfies the coupled system (2.1)–(2.3) with Dirichlet boundary conditions on $\partial\Omega$. We consider the scalar permeability field $\mathbf{K} = K\mathbf{I}$. To verify the convergence rate of the method, we solve the problem on a sequence of grids with decreasing maximum element size, using both structured and unstructured grids. We use triangles with piecewise linear velocities in the Stokes region and polygons (rectangles if structured) in the Darcy region. The grids are chosen to match on the interface. The unstructured grids are not nested - they are generated independently on each level. The structured grids are obtained by first partitioning Ω into rectangles and then dividing each rectangle in Ω_1 along its diagonal into two triangles.

In *Test 1*, the normal velocity is continuous, but the tangential velocity is discontinuous, across the interface:

$$\mathbf{u}_1 = \begin{bmatrix} (2-x)(1.5-y)(y-\xi) \\ -\dfrac{y^3}{3} + \dfrac{y^2}{2}(\xi+1.5) - 1.5\xi y - 0.5 + \sin(\omega x) \end{bmatrix},$$

$$\mathbf{u}_2 = \begin{bmatrix} \omega\cos(\omega x)y \\ \chi(y+0.5) + \sin(\omega x) \end{bmatrix},$$

$$p_1 = -\frac{\sin(\omega x)+\chi}{2K} + \mu(0.5-\xi) + \cos(\pi y), \qquad p_2 = -\frac{\chi}{K}\frac{(y+0.5)^2}{2} - \frac{\sin(\omega x)y}{K},$$

where

$$\mu = 0.1,\ K = 1,\ \alpha_0 = 0.5,\ G = \frac{\sqrt{\mu K}}{\alpha_0},\ \xi = \frac{1-G}{2(1+G)},\ \chi = \frac{-30\xi - 17}{48},\ \omega = 6.$$

In *Test 2*, the velocity field is chosen to be smooth across the interface:

$$\mathbf{u}_1 = \mathbf{u}_2 = \begin{bmatrix} \sin(\frac{x}{G}+\omega)e^{y/G} \\ -\cos(\frac{x}{G}+\omega)e^{y/G} \end{bmatrix},$$

$$p_1 = \left(\frac{G}{K} - \frac{\mu}{G}\right)\cos\left(\frac{x}{G}+\omega\right)e^{1/(2G)} + y - 0.5, \quad p_2 = \frac{G}{K}\cos\left(\frac{x}{G}+\omega\right)e^{y/G},$$

where $\omega = 1.05$ and μ, K, α_0, G are the same as in the *Test 1*.

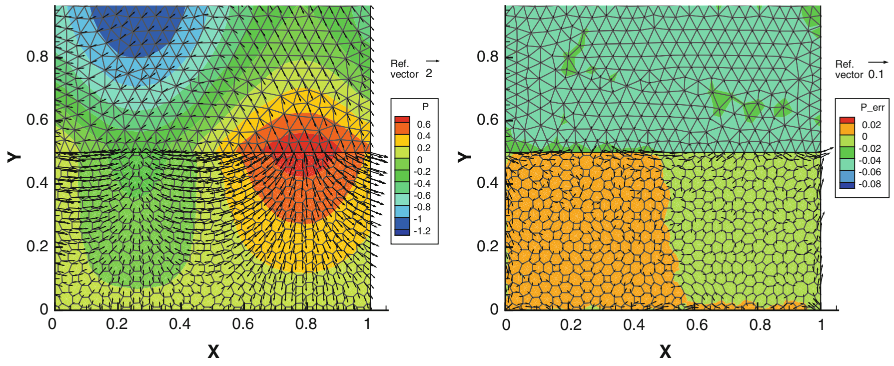

Fig. 2 *Test 1* computed solution (*left*) and error (*right*) on a mesh with $h_1 = 0.0662$, $h_2 = 0.0530$

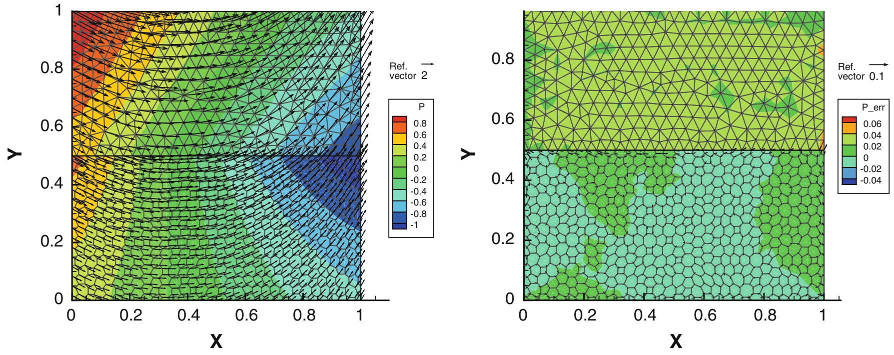

Fig. 3 *Test 2* computed solution (*left*) and error (*right*) on a mesh with $h_1 = 0.0662$, $h_2 = 0.0530$

The computed solution along with the associated numerical error on the third level of unstructured grids for the two tests are plotted in Figs. 2 and 3, respectively. The convergence rates on the unstructured grids are reported in Tables 1 and 3, respectively. The convergence rates on the structured grids are reported in Tables 2 and 4, respectively. These experimental results verify the theoretically predicted convergence rates of order one for the velocity and the pressure. The slight discrepancy in the convergence rate for the pressure in the Stokes region (Tables 1, 3) may be attributed to different shape regularity constants of the unstructured triangular meshes. Tables 1 and 3 show superconvergence of the pressure in Ω_2. Tables 2 and 4 show superconvergence of both the velocity and the pressure in Ω_2 when a rectangular mesh is used in the porous medium. It is well known that the MFD and the MFE methods for the Darcy equation alone are superconvergent on rectangular grids [10,49]. Investigation of the similar behavior for the coupled Stokes–Darcy problem is a possible topic of future work.

In *Test 3*, we present a more realistic model of coupled surface and subsurface flows. The flow domain is decomposed into two subdomains as shown on Fig. 4. The top half represents a lake or a slow flowing river (the Stokes region) and the bottom half represents an aquifer (the Darcy region). The surface fluid flows from left to right,

Table 1 Numerical errors and convergence rates for *Test 1* on unstructured grids

Elements	h_1	$\|\|\|\mathbf{u}_1 - \mathbf{u}_1^h\|\|\|_{1,\Omega_1}$	Rate	$\|p_1 - p_1^h\|_{0,\Omega_1}$	Rate
Stokes region					
44	0.2170	6.5442e−01		1.4657e−01	
164	0.1330	3.5368e−01	1.26	8.7418e−02	1.06
652	0.0662	1.8798e−01	0.91	5.5335e−02	0.66
2,468	0.0363	9.8347e−02	1.08	2.9591e−02	1.04
Elements	h_2	$\|\|\|\mathbf{u}_2^I - \mathbf{u}_2^h\|\|\|_{X_2^h}$	Rate	$\|\|\|p_2^I - p_2^h\|\|\|_{Q_2^h}$	Rate
Darcy region					
32	0.2489	1.4530e−01		2.1906e−02	
128	0.1111	5.3651e−02	1.24	5.3156e−03	1.76
512	0.0530	2.4535e−02	1.06	1.2140e−03	2.00
2,048	0.0259	1.1917e−02	1.01	2.9045e−04	2.00

Table 2 Numerical errors and convergence rates for *Test 1* on structured grids

Elements	h_1	$\|\|\|\mathbf{u}_1 - \mathbf{u}_1^h\|\|\|_{1,\Omega_1}$	Rate	$\|p_1 - p_1^h\|_{0,\Omega_1}$	Rate
Stokes region					
36	0.2357	8.4380e−01		2.8244e−01	
100	0.1414	5.0922e−01	0.99	1.7391e−01	0.95
576	0.0589	2.1303e−01	1.00	7.3116e−02	0.99
2,304	0.0295	1.0664e−01	1.00	3.6566e−02	1.00
Elements	h_2	$\|\|\|\mathbf{u}_2^I - \mathbf{u}_2^h\|\|\|_{X_2^h}$	Rate	$\|\|\|p_2^I - p_2^h\|\|\|_{Q_2^h}$	Rate
Darcy region					
18	0.2357	7.2054e−02		8.8162e−03	
50	0.1414	2.6670e−02	1.95	3.2124e−03	1.98
288	0.0589	4.6994e−03	1.98	5.5936e−04	2.00
1,152	0.0295	1.1785e−03	2.00	1.3966e−04	2.01

Table 3 Numerical errors and convergence rates for *Test 2* on unstructured grids

Elements	h_1	$\|\|\|\mathbf{u}_1 - \mathbf{u}_1^h\|\|\|_{1,\Omega_1}$	Rate	$\|p - p_1^h\|_{0,\Omega_1}$	Rate
Stokes region					
44	0.2170	5.4501e-01		1.5488e-01	
164	0.1330	2.9432e-01	1.26	6.5413e-02	1.76
652	0.0662	1.4152e-01	1.05	4.1093e-02	0.67
2,468	0.0363	7.2480e-02	1.11	2.3073e-02	0.96
Elements	h_2	$\|\|\|\mathbf{u}_2^I - \mathbf{u}_2^h\|\|\|_{X_2^h}$	Rate	$\|\|\|p_2^I - p_2^h\|\|\|_{Q_2^h}$	Rate
Darcy region					
32	0.2489	5.9883e−02		2.1452e−03	
128	0.1111	2.0731e−02	1.32	5.2424e−04	1.75
512	0.0530	9.6960e−03	1.03	1.2789e−04	1.91
2,048	0.0259	4.8383e−03	0.98	3.4431e−05	1.83

Table 4 Numerical errors and convergence rates for *Test 2* on structured grids

Elements	h_1	$\|\|\|\mathbf{u}_1 - \mathbf{u}_1^h\|\|\|_{1,\Omega_1}$	Rate	$\|p_1 - p_1^h\|_{0,\Omega_1}$	Rate
Stokes region					
36	0.2357	6.0192e−01		1.6431e−01	
100	0.1414	3.6005e−01	1.01	1.1073e−01	0.77
576	0.0589	1.4896e−01	1.01	5.1783e−02	0.87
2,304	0.0295	7.4275e−02	1.01	2.7083e−02	0.94
Elements	h_2	$\|\|\|\mathbf{u}_2^I - \mathbf{u}_2^h\|\|\|_{X_2^h}$	Rate	$\|\|\|p_2^I - p_2^h\|\|\|_{Q_2^h}$	Rate
Darcy region					
18	0.2357	3.2312e−02		3.0839e−03	
50	0.1414	1.2691e−02	1.83	1.1787e−03	1.88
288	0.0589	2.4612e−03	1.87	2.0925e−04	1.97
1,152	0.0295	6.5882e−04	1.91	5.2467e−05	2.00

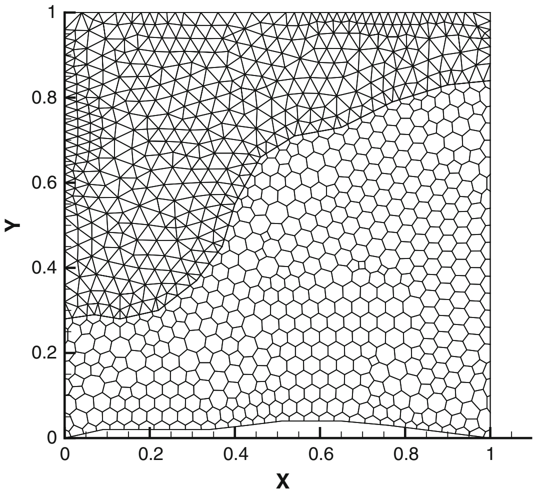

Fig. 4 *Test 3* Computational domain and mesh ($h_1 = 0.0566, h_2 = 0.0556$)

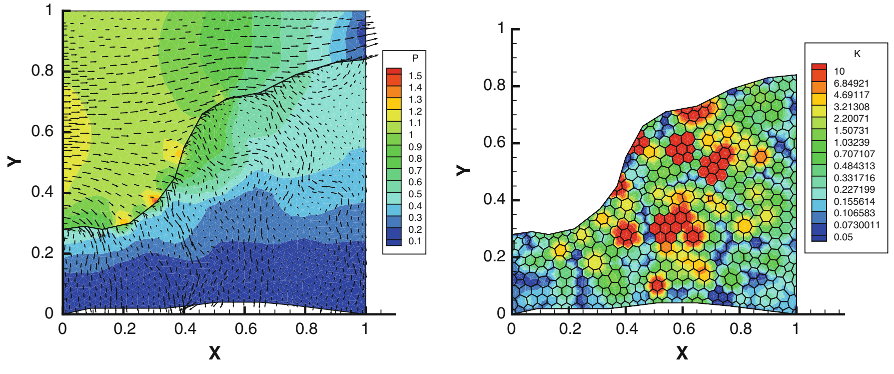

Fig. 5 *Test 3* computed solution (*left*) and permeability field (*right*)

with a parabolic inflow condition on the left boundary, no flow on the top, and zero stress on the right (outflow) boundary. No flow condition is imposed on the left and right boundaries of the aquifer. A pressure boundary condition is specified on the bottom to simulate gravity. The permeability of the porous media is heterogeneous and is shown in Fig. 5 (right). The computed pressure and velocity are shown in Fig. 5 (left). As expected, the pressure and the tangential velocity are discontinuous across the interface, while the normal velocity is continuous. After the surface fluid enters the aquifer, it does not move as fast in the tangential direction, but percolates toward the bottom.

8 Conclusion

We presented and analyzed a locally mass conservative discretization scheme for the coupled Stokes–Darcy flow problem, using the DG method and the MFD method to approximate the Stokes and Darcy equations, respectively. Traditional DG schemes employ simplicial meshes. In our approach by constructing lifting operators mapping from MFD degrees of freedom to functional spaces, we view the DG method as a MFD method, which enables us to formulate the discretization scheme in the entire domain on polygonal or polyhedral meshes. The meshes in the two regions do not have to match on the interface Γ_I and may have elements that are non-convex. To impose continuity of the normal fluxes across the interface Γ_I we defined a Lagrange multiplier space on a mesh, which was assumed to be the trace of Ω_2^h on Γ_I. It is also possible to remove the latter assumption and use mortars instead [30]. We derived optimal error estimates, which we verified by carrying out computer experiments. On unstructured meshes, we observed superconvergence of the Darcy pressure while on structured meshes we obtain superconvergence for both the pressure and the velocity in the Darcy region. Our last numerical test demonstrated the capability of the method to be applied to problems with realistic geometries. In order to take full advantage of the method, it is crucial to solve the algebraic saddle point problem (7.2) efficiently, which is a motivation to develop and study a suitable preconditioning technique as the one proposed in Sect. 7.1.

References

1. Adams, R.: Sobolev Spaces. Academic Press, New York (1975)
2. Arnold, D.: An interior penalty finite element method with discontinuous elements. SIAM J. Numer. Anal. **19**, 742–760 (1982)
3. Arnold, D., Brezzi, F.: Mixed and non-conforming finite element methods: implementation, post-processing and error estimates. Math. Model. Numer. Anal. **19**, 7–35 (1985)
4. Arnold, D., Brezzi, F., Cockburn, B., Marini, D.: Discontinuous Galerkin methods for elliptic problems. In: Cockburn, B., Karniadakis, G., Shu, C.-W. (eds.) Discontinuous Galerkin Methods. Theory, Computation and Applications, vol. 11, pp. 89–101. Springer, Berlin (2000)
5. Bassi, F., Rebay, S.: A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier–Stokes equations. J. Comput. Phys. **131**, 267–279 (1997)
6. Beavers, G., Joseph, D.: Boundary conditions at a naturally impermeable wall. J. Fluid. Mech. **30**, 197–207 (1967)

7. Beirão da Veiga, L., Gyrya, V., Lipnikov, K., Manzini, G.: Mimetic finite difference method for the Stokes problem on polygonal meshes. J. Comput. Phys. **228**, 7215–7232 (2009)
8. Beirão da Veiga, L., Lipnikov, K.: A mimetic discretization of the Stokes problem with selected edge bubbles. SIAM J. Sci. Comput. **32**, 875–893 (2010)
9. Berndt, M., Lipnikov, K., Moulton, J.D., Shashkov, M.: Convergence of mimetic finite difference discretizations of the diffusion equation. East West J. Numer. Math. **9**, 253–284 (2001)
10. Berndt, M., Lipnikov, K., Shashkov, M., Wheeler, M., Yotov, I.: Superconvergence of the velocity in mimetic finite difference methods on quadrilaterals. SIAM J. Numer. Anal. **43**, 1728–1749 (2005)
11. Brenner, S.: Korn's inequalities for piecewise H^1 vector fields. Math. Comput. **73**, 1067–1087 (2004)
12. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods. In: Texts in Applied Mathematics, Vol. 15. Springer, New York (2002)
13. Brezzi, F., Buffa, A., Lipnikov, K.: Mimetic finite differences for elliptic problems. M2AN Math. Model. Numer. Anal. **43**, 277–295 (2009)
14. Brezzi, F., Lipnikov, K., Shashkov, M.: Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes. SIAM J. Numer. Anal. **43**, 1872–1896 (2005)
15. Brezzi, F., Lipnikov, K., Shashkov, M.: Convergence of mimetic finite difference method for diffusion problems on polyhedral meshes with curved faces. Math. Mod. Meth. Appl. Sci. **16**, 275–297 (2006)
16. Brezzi, F., Lipnikov, K., Simoncini, V.: A family of mimetic finite difference methods on polygonal and polyhedral meshes. Math. Mod. Meth. Appl. Sci. **15**, 1533–1552 (2005)
17. Cockburn, B., Shu, C.-W.: The local discontinuous Galerkin method for time-dependent convection–diffusion systems. SIAM J. Numer. Anal. **35**, 2440–2463 (1998) (electronic)
18. Crouzeix, M., Falk, R.: Non conforming finite elements for the Stokes problem. Math. Comput. **52**, 437–456 (1989)
19. Crouzeix, M., Raviart, P.-A.: Conforming and nonconforming finite element methods for solving the stationary Stokes equations. Rev. Francaise Automat. Informat. Recherche Operationnelle Ser. Rouge **7**, 33–75 (1973)
20. Dawson, C., Sun, S., Wheeler, M.F.: Compatible algorithms for coupled flow and transport. Comput. Methods Appl. Mech. Eng. **193**, 2565–2580 (2004)
21. Discacciati, M., Miglio, E., Quarteroni, A.: Mathematical and numerical models for coupling surface and groundwater flows. Appl. Numer. Math. (2002)
22. Droniou, J., Eymard, R.: A mixed finite volume scheme for anisotropic diffusion problems on any grid. Numer. Math. **105**, 35–71 (2006)
23. Droniou J., Eymard R., Gallouët T., Herbin R. A unified approach to mimetic finite difference, hybrid finite volume and mixed finite volume methods. Math. Models Methods Appl. Sci. (M3AS) 20:265–295 (2010)
24. Eymard, R., Gallouët, T., Herbin, R.: Discretization of heterogeneous and anisotropic diffusion problems on general non-conforming meshes, SUSHI: a scheme using stabilisation and hybrid interfaces (2009). doi:10.1093/imanum/drn084. Published on line June 1009
25. Fortin, M., Soulie, M.: A non-conforming piecewise quadratic finite element on triangles. Internat. J. Numer. Methods Eng. **19**, 505–520 (1983)
26. Galvis, J., Sarkis, M.: Non-matching mortar discretization analysis for the coupling Stokes–Darcy equations. Electron. Trans. Numer. Anal. **26**, 350–384 (2007)
27. Gatica, G.N., Meddahi, S., Oyarzúa, R.: A conforming mixed finite-element method for the coupling of fluid flow with porous media flow. IMA J. Numer. Anal. **29**, 86–108 (2009)
28. Girault, V., Raviart, P.-A.: Finite Element Methods for Navier-Stokes Equations. In: Springer Series in Computational Mathematics, Vol. 5. Springer, Berlin (1986)
29. Girault, V., Rivière, B., Wheeler, M.F.: A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier–Stokes problems. Math. Comput. **74**, 53–84 (2005)
30. Girault, V., Yotov, I., Vassilev, D.: A mortar multiscale finite element method for Stokes–Darcy flows (In preparation)
31. Grisvard, P.: Elliptic Problems in Nonsmooth domains. Pitman, London (1985)
32. Hyman, J., Shashkov, M.: The approximation of boundary conditions for mimetic finite difference methods. Comput. Math. Appl. **36**, 79–99 (1998)
33. Hyman, J., Shashkov, M.: Mimetic discretizations for Maxwell's equations and the equations of magnetic diffusion. Progr. Electromagn. Res. **32**, 89–121 (2001)
34. Hyman, J., Shashkov, M., Steinberg, S.: The numerical solution of diffusion problems in strongly heterogeneous non-isotropic materials. J. Comput. Phys. **132**, 130–148 (1997)

35. Kuznetsov, Y.: Efficient iterative solvers for elliptic finite element problems on non-matching grids. Russian J. Numer. Anal. Math. Model. **10**, 187–211 (1995)
36. Lasaint, P., Raviart, P.-A.: On a finite element method for solving the neutron transport equation. In: Mathematical Aspects of Finite Elements in Partial Differential Equations. Math. Res. Center, Univ. of Wisconsin-Madison, Vol. 33, pp. 89–123. Academic Press, New York (1974).
37. Layton, W., Schieweck, F., Yotov, I.: Coupling fluid flow with porous media flow. SIAM J. Numer. Anal. **40**, 2195–2218 (2003)
38. Lions, J.L., Magenes, E.: Non-homogeneous boundary value problems and applications, vol. 1. Springer, New York (1972)
39. Lipnikov, K., Shashkov, M., Svyatskiy, D.: The mimetic finite difference discretization of diffusion problem on unstructured polyhedral meshes. J. Comput. Phys. **211**, 473–491 (2005)
40. Lipnikov, K., Shashkov, M., Yotov, I.: Local flux mimetic finite difference methods. Numerische Mathematik **112**, 115–152 (2009)
41. Lomtev, I., Karniadakis, G.: A discontinuous galerkin method for the Navier–Stokes equations. Int. J. Numer. Meth. Fluids **29**, 587–603 (1999)
42. Mardal, K.A., Tai, X.-C., Winther, R.: A robust finite element method for Darcy–Stokes flow. SIAM J. Numer. Anal. **40**, 1605–1631 (2002) (electronic)
43. Margolin, L., Shashkov, M., Smolarkiewicz, P.: A discrete operator calculus for finite difference approximations. Comput. Meth. Appl. Mech. Eng. **187**, 365–383 (2000)
44. Mu, M., Xu, J.: A two-grid method of a mixed stokes-darcy model for coupling fluid flow with porous media flow. SIAM J. Numer. Anal. **45**, 1801–1813 (2007)
45. Oden, J.T., Babuška, I., Baumann, C.E.: A discontinuous hp finite element method for diffusion problems. J. Comput. Phys. **146**, 491–516 (1998)
46. Reed, W., Hill, T.: Triangular mesh methods for the neutron transport equation. Tech. Report LA-UR-73-479, Los Alamos Scientific Laboratory (1973)
47. Rivière, B., Wheeler, M.F., Girault, V.: A priori error estimates for finite element methods based on discontinuous approximation spaces for elliptic problems. SIAM J. Numer. Anal. **39**, 902–931 (2001)
48. Rivière, B., Yotov, I.: Locally conservative coupling of Stokes and Darcy flows. SIAM J. Numer. Anal. **42**, 1959–1977 (2005)
49. Russell, T., Wheeler, M., Yotov, I.: Superconvergence for control-volume mixed finite element methods on rectangular grids. SIAM J. Numer. Anal. **45**, 223–235 (2007)
50. Rusten, T., Winther, R.: A preconditioned iterative methods for saddle point problems. SIAM J. Matrix Anal. Appl. **13**, 887–904 (1992)
51. Saffman, P.: On the boundary condition at the surface of a porous media. Stud. Appl. Math. **l**, 93–101 (1971)
52. Shashkov, M., Steinberg, S.: Support-operator finite difference algorithms for general elliptic problems. J. Comput. Phys. **118**, 131–151 (1995)
53. Stüben, K.: Algebraic multigrid (AMG): experiences and comparisons. Appl. Math. Comput. **13**, 419–452 (1983)
54. Vassilev, D., Yotov, D.: Coupling Stokes–Darcy flow with transport. SIAM J. Sci. Comput. **31**, 3661–3684 (2009)
55. Warburton, T., Karniadakis, G.: A discontinuous Galerkin method for the viscous MHD equations. J. Comput. Phys. **152**, 1–32 (1999)
56. Wheeler, M.F.: An elliptic collocation-finite element method with interior penalties. SIAM J. Numer. Anal. **15**, 152–161 (1978)
57. Xie, X., Xu, J., Xue, G.: Uniformly-stable finite element methods for Darcy–Stokes–Brinkman models. J. Comput. Math. **26**, 437–455 (2008)