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# Mortar multiscale finite element methods for Stokes–Darcy flows

**Vivette Girault · Danail Vassilev · Ivan Yotov**

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**Abstract** We investigate mortar multiscale numerical methods for coupled Stokes and Darcy flows with the Beavers–Joseph–Saffman interface condition. The domain is decomposed into a series of subdomains (coarse grid) of either Stokes or Darcy type. The subdomains are discretized by appropriate Stokes or Darcy finite elements. The solution is resolved locally (in each coarse element) on a fine scale, allowing for non-matching grids across subdomain interfaces. Coarse scale mortar finite elements are introduced on the interfaces to approximate the normal stress and impose weakly continuity of the normal velocity. Stability and a priori error estimates in terms of the fine subdomain scale $h$ and the coarse mortar scale $H$ are established for fairly general grid configurations, assuming that the mortar space satisfies a certain inf-sup condition. Several examples of such spaces in two and three dimensions are given. Numerical experiments are presented in confirmation of the theory.

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V. Girault
UPMC-Paris 6 and CNRS, UMR 7598, 75230 Paris Cedex 05, France
e-mail: girault@ann.jussieu.fr

V. Girault
Department of Mathematics, TAMU, College Station, TX 77843, USA

D. Vassilev
Mathematics Research Institute, College of Engineering, Mathematics and Physical Sciences,
University of Exeter, Exeter EX4 4QF, UK
e-mail: D.Vassilev@exeter.ac.uk

I. Yotov (✉)
Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA
e-mail: yotov@math.pitt.edu

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## 1 Introduction

Mathematical and numerical modeling of coupled Stokes and Darcy flows has become a topic of significant interest in recent years. Such coupling occurs in many applications, including surface water-groundwater interaction, flows through vuggy or fractured porous media, industrial filters, fuel cells, and cardiovascular flows. The most commonly used model is based on the experimentally derived Beavers–Joseph–Saffman interface condition [10,55], a slip with friction condition for the Stokes flow with a friction coefficient that depends on the permeability of the adjacent porous media. Existence and uniqueness of a weak solution has been studied in [24,46]. Numerous stable and convergent numerical methods have been developed, see, e.g., [24,28,29,32,33,44,46,49,54] for methods based on different numerical discretizations suitable for each region, and [4,9,19,45,47,59] for approaches employing unified finite elements. The full Beavers–Joseph condition was considered in [2,20]. A coupling of Stokes–Darcy flows with transport was analyzed in [57]. The nonlinear system of coupled Navier–Stokes and Darcy flows has been studied in [8,26,35].

In this paper we develop multiscale mortar methods for multi-domain non-matching grid discretizations of Stokes–Darcy flows in two and three dimensions. Non-matching grids provide flexibility in meshing complex geometries with relatively simple locally constructed subdomain grids that are suitable for the choice of subdomain discretization methods. Mortar finite elements play the role of Lagrange multipliers to impose weakly interface conditions. In [46], a Lagrange multiplier approximating the normal stress was introduced to impose continuity of the normal velocity for discretizations involving mixed finite element methods for Darcy and conforming Stokes elements. With a choice of the Lagrange multiplier space as the normal trace of the Darcy velocity space, the analysis in [46] applied to non-matching grids on the Stokes–Darcy interface, although this was not explicitly noted. A similar choice was considered in subsequent mortar discretizations for Stokes–Darcy flows [14,29,54]. Mortar methods for mixed finite element discretizations for Darcy have been studied in [5,6,52,60]. The analysis in these papers allows for the mortar grid to be different from the traces of the subdomain grids with the assumption that the mortar space satisfies a suitable solvability condition that limits the number of mortar degrees of freedom. Mortar discretizations for Stokes have been developed in [11,12]. There, the mortar grid was chosen to be the trace of one of the neighboring subdomain grids, similar to the choice in mortar methods for conforming Galerkin discretizations for second order elliptic problems [13].

In this work we allow for non-matching grid interfaces of Stokes–Darcy, Stokes–Stokes, and Darcy–Darcy types. We develop multiscale discretizations, where the subdomains are discretized on a fine scale $h$ and the mortar space is discretized on a coarse scale $H$. Our method is based on saddle-point formulations in both regions and employs inf-sup stable mixed finite elements for Darcy and conforming elements for Stokes. The mortars approximate different physical variables and are used to impose different matching conditions depending on the type of interface. On Stokes–Stokes interfaces, the mortar functions represent the entire stress vector and impose weak continuity of the entire velocity vector. On Stokes–Darcy and Darcy–Darcy interfaces, the mortars approximate the normal stress, which is just the pressure in the Darcy region, and impose weak continuity of the normal velocity. The mortar spaces are assumed to satisfy suitable inf-sup conditions, allowing for very general subdomain and mortar grid configurations. We consider approximations of different polynomial degrees on the three types of interfaces and the two types of subdomains. The mortar spaces can be continuous or discontinuous, the latter providing localized mass conservation across interfaces. Our method is more general than existing Stokes–Stokes mortar methods [11,12] and Stokes–Darcy mortar methods [14,29,46,54]. On Darcy–Darcy interfaces, our condition is closely related to the solvability condition considered in [5,6,52,60].

The stability and convergence analysis relies on the construction of a bounded global interpolant in the space of weakly continuous velocities that also preserves the velocity divergence in the usual discrete sense. This is done in two steps, starting from suitable local interpolants and correcting them to satisfy the interface matching conditions. The correction step requires the existence of bounded mortar interpolants. This is a very general condition that can be easily satisfied in practice. We present two examples in 2-D and one example in 3-D that satisfy this solvability condition. Our error analysis shows that the global velocity and pressure errors are bounded by the fine scale local approximation error and the coarse scale non-conforming error. Since the polynomial degrees on subdomains and interfaces may differ, one can choose higher order mortar polynomials to balance the fine scale and the coarse scale error terms and obtain fine scale asymptotic convergence. The dependence of the stability and convergence constants on the subdomain size is explicitly determined. In particular, the stability and fine scale convergence constants do not depend on the size of subdomains, while the coarse scale non-conforming error constants deteriorate when the subdomain size goes to zero. This is to be expected, as the relative effect of the non-conforming error becomes more significant in such regime. However, this dependence can be made negligible by choosing higher order mortar polynomials, as mentioned above.

Our multiscale Stokes–Darcy formulation can be viewed as an extension of the mortar multiscale mixed finite element (MMMFE) method for Darcy developed in [6]. The MMMFE method provides an alternative to other multiscale methods in the literature such as the variational multiscale method [3,41] and the multiscale finite element method [22,40]. All three methods utilize a divide and conquer approach: solve relatively small fine scale subdomain problems that are only coupled on the coarse scale through a reduced number of degrees of freedom. The mortar multiscale approach is more flexible as it allows for employment of a posteriori error estimation to adaptively refine the mortar grids where necessary to improve the global accuracy [6]. Following the non-overlapping domain decomposition approach from [37], it can be shown that the global Stokes–Darcy problem can be reduced to a positive definite coarse scale interface problem [58]. The latter can be solved using a preconditioned Krylov space solver requiring Stokes or Darcy subdomain solves at each iteration. An alternative more efficient implementation for MMMFE discretizations for Darcy was developed in [31], where a multiscale flux basis giving the interface flux response for each coarse scale mortar degree of freedom is precomputed. The multiscale flux basis is used to replace the solution of subdomain problems by a simple linear combination. The application of this methodology to the Stokes–Darcy interface problem will be discussed in a forthcoming paper. We should mention that there have been a number of papers in the literature studying domain decomposition methods for the Stokes–Darcy problem, primarily in the two-subdomain case, see, e.g., [21,25,27,30,39].

### 1.1 Notation and preliminaries

Let $\Omega$ be a bounded, connected Lipschitz domain of $\mathbb{R}^n$, $n = 2, 3$, with boundary $\partial\Omega$ and exterior unit normal vector $\boldsymbol{n}$, and let $\Gamma$ be a part of $\partial\Omega$ with positive $n-1$ measure: $|\Gamma| > 0$. We do not assume that $\Gamma$ is connected, but if it is not connected, we assume that it has a finite number of connected components. In the case when $n = 3$, we also assume that $\Gamma$ is itself Lipschitz. Let

$$H^1_{0,\Gamma}(\Omega) = \{v \in H^1(\Omega);\ v|_\Gamma = 0\}.$$

Poincaré's inequality in $H^1_{0,\Gamma}(\Omega)$ reads: There exists a constant $\mathcal{P}_\Gamma$ depending only on $\Omega$ and $\Gamma$ such that

$$\forall v \in H^1_{0,\Gamma}(\Omega), \quad \|v\|_{L^2(\Omega)} \le \mathcal{P}_\Gamma |v|_{H^1(\Omega)}. \tag{1.1}$$

The norms and spaces are made precise later on. The formula (1.1) is a particular case of a more general result (cf. [15,50]):

**Proposition 1.1** *Let $\Omega$ be a bounded, connected Lipschitz domain of $\mathbb{R}^n$ and let $\Phi$ be a seminorm on $H^1(\Omega)$ satisfying:*

1) *There exists a constant $P_1$ such that*

$$\forall v \in H^1(\Omega), \quad \Phi(v) \le P_1 \|v\|_{H^1(\Omega)}. \tag{1.2}$$

2) *The condition $\Phi(c) = 0$ for a constant function $c$ holds if and only if $c = 0$. Then there exists a constant $P_2$ depending only on $\Omega$, such that*

$$\forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega)} \le P_2\big(|v|^2_{H^1(\Omega)} + \Phi(v)^2\big)^{\frac{1}{2}}. \tag{1.3}$$

We recall Korn's first inequality: There exists a constant $C_1$ depending only on $\Omega$ and $\Gamma$ such that

$$\forall \boldsymbol{v} \in H^1_{0,\Gamma}(\Omega)^n, \quad |\boldsymbol{v}|_{H^1(\Omega)} \le C_1 \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega)}, \tag{1.4}$$

where $\boldsymbol{D}(\boldsymbol{v})$ is the deformation rate tensor, also called the symmetric gradient tensor:

$$\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\big(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T\big).$$

This is a particular case of the following more general result (see (1.6) in [16]):

**Proposition 1.2** *Let $\Omega$ be a bounded, connected Lipschitz domain of $\mathbb{R}^n$ and let $\Phi$ be a seminorm on $H^1(\Omega)^n$ satisfying:*

1) *There exists a constant $C_2$ such that*

$$\forall \boldsymbol{v} \in H^1(\Omega)^n, \quad \Phi(\boldsymbol{v}) \leq C_2 \|\boldsymbol{v}\|_{H^1(\Omega)}. \tag{1.5}$$

2) *The condition $\Phi(\boldsymbol{m}) = 0$ for a rigid-body motion $\boldsymbol{m}$ holds if and only if $\boldsymbol{m}$ is a constant vector. Then there exists a constant $C_3$ depending only on $\Omega$, such that*

$$\forall \boldsymbol{v} \in H^1(\Omega)^n, \quad |\boldsymbol{v}|_{H^1(\Omega)} \leq C_3 \big(\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega)} + \Phi(\boldsymbol{v})^2\big)^{\frac{1}{2}}. \tag{1.6}$$

In particular, Proposition 1.2 implies that there exists a constant $C_\Omega$, only depending on $\Omega$ such that (see (1.9) in [16]),

$$\forall \boldsymbol{v} \in H^1(\Omega)^n, \quad |\boldsymbol{v}|_{H^1(\Omega)} \leq C_\Omega \left( \|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega)} + \left| \int_\Omega \mathbf{curl}\, \boldsymbol{v} \right|^2 \right)^{\frac{1}{2}}, \tag{1.7}$$

where $|\cdot|$ denotes the Euclidean vector norm.

For any non-negative integer $m$, recall the classical Sobolev space (cf. [1] or [50])

$$H^m(\Omega) = \left\{ v \in L^2(\Omega);\ \partial^k v \in L^2(\Omega)\, \forall\, |k| \leq m \right\},$$

equipped with the following seminorm and norm (for which it is a Hilbert space)

$$|v|_{H^m(\Omega)} = \left( \sum_{|k|=m} \int_\Omega |\partial^k v|^2\, d\boldsymbol{x} \right)^{\frac{1}{2}}, \quad \|v\|_{H^m(\Omega)} = \left( \sum_{0 \leq |k| \leq m} |v|^2_{H^k(\Omega)} \right)^{\frac{1}{2}}.$$

This definition is extended to any real number $s = m + s'$ for an integer $m \geq 0$ and $0 < s' < 1$ by defining in dimension $n$ the fractional semi-norm and norm

$$|v|_{H^s(\Omega)} = \left( \sum_{|k|=m} \int_\Omega \int_\Omega \frac{|\partial^k v(\boldsymbol{x}) - \partial^k v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{n+2s'}}\, d\boldsymbol{x}\, d\boldsymbol{y} \right)^{\frac{1}{2}}, \quad \|v\|_{H^s(\Omega)} = \left( \|v\|^2_{H^m(\Omega)} + |v|^2_{H^s(\Omega)} \right)^{\frac{1}{2}}.$$

In particular, we shall frequently use the fractional Sobolev spaces $H^{\frac{1}{2}}(\Gamma)$ and $H_{00}^{\frac{1}{2}}(\Gamma)$ for a Lipschitz surface $\Gamma$ when $n = 3$ or curve when $n = 2$ with the following seminorms and norms:

$$|v|_{H^{\frac{1}{2}}(\Gamma)} = \left(\int_\Gamma \int_\Gamma \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^n} d\boldsymbol{x} d\boldsymbol{y}\right)^{\frac{1}{2}} \quad \|v\|_{H^{\frac{1}{2}}(\Gamma)} = \left(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{\frac{1}{2}}(\Gamma)}\right)^{\frac{1}{2}}, \tag{1.8}$$

$$|v|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \left(|v|^2_{H^{\frac{1}{2}}(\Gamma)} + \int_\Gamma \frac{|v(\boldsymbol{x})|^2}{\mathrm{d}_{\partial\Gamma}(\boldsymbol{x})} d\boldsymbol{x}\right)^{\frac{1}{2}} \quad \|v\|_{H^{\frac{1}{2}}_{00}(\Gamma)} = \left(\|v\|^2_{L^2(\Gamma)} + |v|^2_{H^{\frac{1}{2}}_{00}(\Gamma)}\right)^{\frac{1}{2}}, \tag{1.9}$$

where $\mathrm{d}_{\partial\Gamma}(\boldsymbol{x})$ denotes the distance from $\boldsymbol{x}$ to $\partial\Gamma$. When $\Gamma$ is a subset of $\partial\Omega$ with positive $n-1$ measure, $H^{\frac{1}{2}}_{00}(\Gamma)$ is the space of traces of all functions of $H^1_{0,\partial\Omega\setminus\Gamma}(\Omega)$. The above norms (1.8) and (1.9) are not equivalent except when $\Gamma$ is a closed surface or curve. The dual space of $H^{\frac{1}{2}}(\Gamma)$ is denoted by $H^{-\frac{1}{2}}(\Gamma)$.

In addition to these spaces, we shall use the Hilbert space

$$H(\mathrm{div};\Omega) = \left\{\boldsymbol{v} \in L^2(\Omega)^n;\ \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\right\},$$

equipped with the graph norm

$$\|\boldsymbol{v}\|_{H(\mathrm{div};\Omega)} = \left(\|\boldsymbol{v}\|^2_{L^2(\Omega)} + \|\mathrm{div}\,\boldsymbol{v}\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

The normal trace $\boldsymbol{v} \cdot \boldsymbol{n}$ of a function $\boldsymbol{v}$ of $H(\mathrm{div};\Omega)$ on $\partial\Omega$ belongs to $H^{-\frac{1}{2}}(\partial\Omega)$ (cf. [34]). The same result holds when $\Gamma$ is a part of $\partial\Omega$ and is a closed surface. But if $\Gamma$ is not a closed surface, then $\boldsymbol{v} \cdot \boldsymbol{n}$ belongs to the dual of $H^{\frac{1}{2}}_{00}(\Gamma)$. When $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, we use the space

$$H_0(\mathrm{div};\Omega) = \{\boldsymbol{v} \in H(\mathrm{div};\Omega);\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

## 2 Problem statement

### 2.1 Coupled Stokes and Darcy systems

Let $\Omega$ be partitioned into two non-overlapping regions: the region of the Darcy flow, $\Omega_d$, and the region of the Stokes flow, $\Omega_s$, each one possibly non-connected, but with a finite number of connected components, and with Lipschitz-continuous boundaries $\partial\Omega_d$ and $\partial\Omega_s$:

$$\overline{\Omega} = \overline{\Omega}_d \cup \overline{\Omega}_s.$$

Let $\Gamma_d = \partial\Omega_d \cap \partial\Omega$, $\Gamma_s = \partial\Omega_s \cap \partial\Omega$, $\Gamma_{sd} = \partial\Omega_d \cap \partial\Omega_s$. The unit normal vector on $\Gamma_{sd}$ exterior to $\Omega_d$, respectively $\Omega_s$, is denoted by $\boldsymbol{n}_d$, respectively $\boldsymbol{n}_s$. In dimension three, we assume that $\Gamma_d$, $\Gamma_s$, and $\Gamma_{sd}$ also have Lipschitz-continuous boundaries. Let $\boldsymbol{f}_d$ be the gravity force in $\Omega_d$, $\boldsymbol{f}_s$ a given body force in $\Omega_s$, let $\nu_d > 0$, respectively $\nu_s > 0$, be the constant viscosity coefficient of the Darcy, respectively Stokes flow, let $\boldsymbol{K}$ be the rock permeability tensor in $\Omega_d$, let $q_d$ be an external source or sink term in $\Omega_d$, and let $\alpha > 0$ be the slip coefficient in the Beavers–Joseph–Saffman law [10,55], determined by experiment. As far as the data are concerned, we assume on one hand that $\boldsymbol{K}$ is bounded, symmetric and uniformly positive definite in $\Omega_d$: there exist two constants, $\lambda_{\min} > 0$ and $\lambda_{\max} > 0$ such that

$$\forall \boldsymbol{x} \in \Omega_d, \forall \boldsymbol{\chi} \in \mathbb{R}^n, \quad \lambda_{\min}|\boldsymbol{\chi}|^2 \le \boldsymbol{K}(\boldsymbol{x})\boldsymbol{\chi} \cdot \boldsymbol{\chi} \le \lambda_{\max}|\boldsymbol{\chi}|^2, \tag{2.1}$$

and we assume on the other hand, that the source $q_d$ satisfies the solvability condition

$$\int_{\Omega_d} q_d \, d\boldsymbol{x} = 0. \tag{2.2}$$

The fluid velocity and pressure in $\Omega_d$, respectively $\Omega_s$, are denoted by $\boldsymbol{u}_d$ and $p_d$, respectively by $\boldsymbol{u}_s$ and $p_s$. The stress tensor of the Stokes flow is denoted by $\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)$,

$$\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s) = -p_s \boldsymbol{I} + 2\nu_s \boldsymbol{D}(\boldsymbol{u}_s).$$

In the Darcy region $\Omega_d$, the pair $(\boldsymbol{u}_d, p_d)$ satisfies

$$\nu_d \boldsymbol{K}^{-1}\boldsymbol{u}_d + \nabla p_d = \boldsymbol{f}_d \quad \text{in } \Omega_d, \tag{2.3}$$

$$\operatorname{div} \boldsymbol{u}_d = q_d \quad \text{in } \Omega_d, \tag{2.4}$$

$$\boldsymbol{u}_d \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_d. \tag{2.5}$$

In the Stokes region $\Omega_s$, the pair $(\boldsymbol{u}_s, p_s)$ satisfies

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s) \equiv -2\nu_s \operatorname{div} \boldsymbol{D}(\boldsymbol{u}_s) + \nabla p_s = \boldsymbol{f}_s \quad \text{in } \Omega_s, \tag{2.6}$$

$$\operatorname{div} \boldsymbol{u}_s = 0 \quad \text{in } \Omega_s, \tag{2.7}$$

$$\boldsymbol{u}_s = \boldsymbol{0} \quad \text{on } \Gamma_s. \tag{2.8}$$

These operators suggest to look for $(\boldsymbol{u}_d, p_d)$ in $H(\text{div}; \Omega_d) \times H^1(\Omega_d)$ and $(\boldsymbol{u}_s, p_s)$ in $H^1(\Omega_s)^n \times L^2(\Omega_s)$. The Darcy and Stokes flows are coupled on $\Gamma_{sd}$ through the following *interface conditions*

$$\boldsymbol{u}_s \cdot \boldsymbol{n}_s + \boldsymbol{u}_d \cdot \boldsymbol{n}_d = 0 \quad \text{on } \Gamma_{sd}, \tag{2.9}$$

$$-\big(\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s\big) \cdot \boldsymbol{n}_s \equiv p_s - 2\nu_s\big(\boldsymbol{D}(\boldsymbol{u}_s)\boldsymbol{n}_s\big) \cdot \boldsymbol{n}_s = p_d \quad \text{on } \Gamma_{sd}, \tag{2.10}$$

$$-\frac{\sqrt{K_l}}{\nu_s \alpha}\big(\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s\big) \cdot \boldsymbol{\tau}_l \equiv -\frac{\sqrt{K_l}}{\alpha} 2\big(\boldsymbol{D}(\boldsymbol{u}_s)\boldsymbol{n}_s\big) \cdot \boldsymbol{\tau}_l = \boldsymbol{u}_s \cdot \boldsymbol{\tau}_l, \ \text{on } \Gamma_{sd}, \ 1 \le l \le n-1, \tag{2.11}$$

where $\boldsymbol{\tau}_l$, $1 \le l \le n-1$ is an orthogonal system of unit tangent vectors on $\Gamma_{sd}$ and $K_l = \big(\boldsymbol{K}\boldsymbol{\tau}_l\big)\cdot\boldsymbol{\tau}_l$. Conditions (2.9) and (2.10) incorporate continuity of flux and normal stress, respectively. Condition (2.11) is known as the Beavers–Joseph–Saffman law [10,42,55] describing slip with friction, where $\sqrt{K_l}/\alpha$ is a friction coefficient.

## 2.2 First variational formulation

For any functions $\varphi_d$ defined in $\Omega_d$ and $\varphi_s$ defined in $\Omega_s$, it is convenient to define $\varphi$ in $\Omega$ by $\varphi|_{\Omega_d} = \varphi_d$ and $\varphi|_{\Omega_s} = \varphi_s$. With this notation, regarding the data, we assume that $\boldsymbol{f} \in L^2(\Omega)^n$, we extend $q_d$ by zero in $\Omega_s$, i.e. we set $q_s = 0$ and owing to (2.2), the extended function $q$ belongs to $L^2_0(\Omega)$. Before setting problem (2.3)–(2.11) into an equivalent variational formulation, it is useful to interpret the interface conditions (2.10)–(2.11). First we observe from the regularity of $\boldsymbol{f}_s$ that each row of $\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)$ belongs to $H(\text{div};\Omega_s)$; hence $\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s$ belongs to $H^{-\frac{1}{2}}(\partial\Omega_s)^n$, and in particular is well-defined as an element of the dual of $H^{\frac{1}{2}}_{00}(\Gamma_{sd})^n$, which is a distribution space on $\Gamma_{sd}$. But without further information, it cannot be multiplied directly by the normal or tangent vectors, since the boundary is only Lipschitz-continuous. To bypass this difficulty, following [35] we define the function on $\Gamma_{sd}$

$$\boldsymbol{g} = -p_d\boldsymbol{n}_s - \sum_{l=1}^{n-1} \frac{\nu_s\alpha}{\sqrt{K_l}}(\boldsymbol{u}_s\cdot\boldsymbol{\tau}_l)\boldsymbol{\tau}_l, \tag{2.12}$$

and replace (2.10)–(2.11) by the condition

$$\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s = \boldsymbol{g} \quad \text{on } \Gamma_{sd}. \tag{2.13}$$

As the traces of $p_d$ and of all components of $\boldsymbol{u}_s$ on $\Gamma_{sd}$ belong to $H^{\frac{1}{2}}(\Gamma_{sd})$, Sobolev's imbeddings [1] imply that $\boldsymbol{g}$ belongs to $L^r(\Gamma_{sd})^n$ for any finite $r$ when $n = 2$ and $r = 4$ when $n = 3$. Hence condition (2.13) makes sense. Let us check that it implies (2.10)–(2.11). First, prescribing (2.13) guarantees that $\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s$ belongs at least to $L^4(\Gamma_{sd})^n$ and thus can be multiplied by the normal or tangent vectors. Then by virtue of this regularity, (2.12), (2.13) are equivalent to:

$$\big(\big(\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s\big)\cdot\boldsymbol{n}_s\big)\boldsymbol{n}_s + \sum_{l=1}^{n-1}\big(\big(\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s\big)\cdot\boldsymbol{\tau}_l\big)\boldsymbol{\tau}_l = -p_d\boldsymbol{n}_s - \sum_{l=1}^{n-1}\frac{\nu_s\alpha}{\sqrt{K_l}}(\boldsymbol{u}_s\cdot\boldsymbol{\tau}_l)\boldsymbol{\tau}_l,$$

and therefore, by identifying on both sides the components of the normal and tangent vectors (that forms an orthonormal set), we derive (2.10)–(2.11). Hence (2.13) is the interpretation of (2.10)–(2.11).

Now, let $(\boldsymbol{u}_d, p_d) \in H(\text{div};\Omega_d)\times H^1(\Omega_d)$ and $(\boldsymbol{u}_s, p_s) \in H^1(\Omega_s)^n \times L^2(\Omega_s)$ be a solution of (2.3)–(2.11). In order to set (2.3)–(2.11) in variational form, we take the scalar product of (2.3) and (2.6) respectively with any test function $\boldsymbol{v}_d$ in $H^1(\Omega_d)^n$ satisfying $\boldsymbol{v}_d\cdot\boldsymbol{n} = 0$ on $\Gamma_d$, and any $\boldsymbol{v}_s$ in $H^1(\Omega_s)^n$ satisfying $\boldsymbol{v}_s = \boldsymbol{0}$ on $\Gamma_s$. Then we apply Green's formula in $\Omega_d$ and $\Omega_s$. This yields

$$\nu_d \int_{\Omega_d} \boldsymbol{K}^{-1}\boldsymbol{u}_d \cdot \boldsymbol{v}_d - \int_{\Omega_d} p_d \operatorname{div} \boldsymbol{v}_d + \int_{\Gamma_{sd}} p_d \boldsymbol{v}_d \cdot \boldsymbol{n}_d = \int_{\Omega_d} \boldsymbol{f}_d \cdot \boldsymbol{v}_d, \tag{2.14}$$

$$2\,\nu_s \int_{\Omega_s} \boldsymbol{D}(\boldsymbol{u}_s) : \boldsymbol{D}(\boldsymbol{v}_s) - \int_{\Omega_s} p_s \operatorname{div} \boldsymbol{v}_s - \langle \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s, \boldsymbol{v}_s \rangle_{\Gamma_{sd}} = \int_{\Omega_s} \boldsymbol{f}_s \cdot \boldsymbol{v}_s, \tag{2.15}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_{sd}}$ denotes the duality pairing between $H_{00}^{\frac{1}{2}}(\Gamma_{sd})^n$ and its dual space. The validity of (2.14) and (2.15) follows from the above considerations. By summing (2.14) and (2.15), by using the fact that $\boldsymbol{n}_d = -\boldsymbol{n}_s$, and by applying (2.13), the term on the interface, say $I$, reads

$$I = -\langle \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s, \boldsymbol{v}_s \rangle_{\Gamma_{sd}} - \int_{\Gamma_{sd}} p_d \boldsymbol{v}_d \cdot \boldsymbol{n}_s = -\int_{\Gamma_{sd}} \boldsymbol{g} \cdot \boldsymbol{v}_s - \int_{\Gamma_{sd}} p_d \boldsymbol{v}_d \cdot \boldsymbol{n}_s.$$

Then the expression (2.12) for $\boldsymbol{g}$ yields

$$I = \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} \frac{\nu_s \alpha}{\sqrt{K_l}} (\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l) + \int_{\Gamma_{sd}} p_d [\boldsymbol{v} \cdot \boldsymbol{n}], \tag{2.16}$$

where the jump $[\boldsymbol{v} \cdot \boldsymbol{n}]$ is defined by

$$[\boldsymbol{v} \cdot \boldsymbol{n}] = \boldsymbol{v}_s \cdot \boldsymbol{n}_s + \boldsymbol{v}_d \cdot \boldsymbol{n}_d.$$

Finally, we eliminate this jump by enforcing *strongly* the transmission condition (2.9) on the test function $\boldsymbol{v}$. In view of the interior terms in (2.14) and (2.15) and what remains in (2.16), we see that we can reduce the regularity of our functions and work in the space

$$\tilde{X} = \{\boldsymbol{v} \in H(\text{div}; \Omega);\ \boldsymbol{v}_s \in H^1(\Omega_s)^n, \boldsymbol{v}|_{\Gamma_s} = \boldsymbol{0}, (\boldsymbol{v} \cdot \boldsymbol{n})|_{\Gamma_d} = 0\}, \tag{2.17}$$

which is a Hilbert space equipped with the norm

$$\forall \boldsymbol{v} \in \tilde{X}, \quad \|\boldsymbol{v}\|_{\tilde{X}} = \left(\|\boldsymbol{v}\|^2_{H(\text{div};\Omega)} + |\boldsymbol{v}_s|^2_{H^1(\Omega_s)}\right)^{\frac{1}{2}}. \tag{2.18}$$

Note that the restriction of $\boldsymbol{v} \cdot \boldsymbol{n}$ on $\Gamma_{sd}$ belongs at least to $L^4(\Gamma_{sd})$. Let us denote $W = L_0^2(\Omega)$ with the norm $\|w\|_W = \|w\|_{L^2(\Omega)}$. Then we propose the following variational formulation: Find $(\boldsymbol{u}, p) \in \tilde{X} \times W$ such that

$$\forall \boldsymbol{v} \in \tilde{X}, \quad \nu_d \int_{\Omega_d} \boldsymbol{K}^{-1}\boldsymbol{u}_d \cdot \boldsymbol{v}_d + 2\,\nu_s \int_{\Omega_s} \boldsymbol{D}(\boldsymbol{u}_s) : \boldsymbol{D}(\boldsymbol{v}_s) - \int_{\Omega} p \operatorname{div} \boldsymbol{v}$$
$$+ \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} \frac{\nu_s \alpha}{\sqrt{K_l}} (\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}, \tag{2.19}$$

$$\forall w \in W, \quad \int_{\Omega} w \operatorname{div} \boldsymbol{u} = \int_{\Omega_d} w\, q_d. \tag{2.20}$$

**Lemma 2.1** *For any data $\boldsymbol{f}$ in $L^2(\Omega)^n$ and $q_d$ in $L^2_0(\Omega_d)$, any solution $(\boldsymbol{u}, p) \in \tilde{X} \times W$ of problem (2.19)–(2.20) satisfies $p_d \in H^1(\Omega_d)$ and solves (2.3)–(2.11). Conversely, any solution of (2.3)–(2.11), $(\boldsymbol{u}, p) \in \tilde{X} \times W$ with $p_d \in H^1(\Omega_d)$, solves (2.19)–(2.20).*

*Proof* It stems from the above considerations that any solution $(\boldsymbol{u}_d, p_d) \in H(\text{div}; \Omega_d) \times H^1(\Omega_d)$ and $(\boldsymbol{u}_s, p_s) \in H^1(\Omega_s)^n \times L^2(\Omega_s)$ of (2.3)–(2.11) is such that $(\boldsymbol{u}, p)$ belongs to $\tilde{X} \times L^2(\Omega)$ and solves (2.19)–(2.20). Moreover as $\Omega$ is connected, (2.19) only defines $p$ up to an additive constant and this constant can be chosen so that $p$ belongs to $W$.

Conversely, let $(\boldsymbol{u}, p) \in \tilde{X} \times W$ solve (2.19)–(2.20), and denote its restriction to $\Omega_d$ and $\Omega_s$ as above. By choosing smooth test functions with compact support first in $\Omega_d$ and next in $\Omega_s$, we immediately derive that $(\boldsymbol{u}_d, p_d)$ is a solution of (2.3)–(2.5) and $(\boldsymbol{u}_s, p_s)$ is a solution of (2.6)–(2.8). Furthermore, since both $\boldsymbol{f}_d$ and $\nu_d \boldsymbol{K}^{-1}\boldsymbol{u}_d$ belong to $L^2(\Omega_d)^n$, (2.3) implies that $p_d \in H^1(\Omega_d)$.

It remains to recover the transmission conditions (2.9)–(2.11). First, (2.9) is a consequence of the definition (2.17) of $\tilde{X}$. Next, we recover (2.14) and (2.15) by taking the scalar product of (2.3) and (2.6) with a function $\boldsymbol{v} \in \tilde{X}$ that is smooth in $\Omega_d$ and in $\Omega_s$, and by applying Green's formula in both regions. By comparing with (2.19), this gives

$$\int_{\Gamma_{sd}} p_d \boldsymbol{v}_d \cdot \boldsymbol{n}_d - \langle \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s, \boldsymbol{v}_s\rangle_{\Gamma_{sd}} = \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} \frac{\nu_s \alpha}{\sqrt{K_l}} (\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l).$$

By taking into account the orientation of the normal, the regularity of $\boldsymbol{v}$, and the definition (2.12) of $\boldsymbol{g}$, this is equivalent to:

$$\langle \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s, \boldsymbol{v}_s\rangle_{\Gamma_{sd}} = -\int_{\Gamma_{sd}} p_d \boldsymbol{v}_s \cdot \boldsymbol{n}_s - \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} \frac{\nu_s \alpha}{\sqrt{K_l}} (\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l) = \int_{\Gamma_{sd}} \boldsymbol{g} \cdot \boldsymbol{v}_s.$$

As the trace space of $\boldsymbol{v}_s$ on $\Gamma_{sd}$ is large enough, this implies (2.13).

## 2.3 Existence and uniqueness of the solution

For any functions $\boldsymbol{u}_d$, $\boldsymbol{v}_d$ in $L^2(\Omega_d)^n$ and $\boldsymbol{u}_s$, $\boldsymbol{v}_s$ in $H^1(\Omega_s)^n$, we define the bilinear form

$$\tilde{a}(\boldsymbol{u}, \boldsymbol{v}) = \nu_d \int_{\Omega_d} \boldsymbol{K}^{-1}\boldsymbol{u}_d \cdot \boldsymbol{v}_d + 2\,\nu_s \int_{\Omega_s} \boldsymbol{D}(\boldsymbol{u}_s) : \boldsymbol{D}(\boldsymbol{v}_s) + \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} \frac{\nu_s \alpha}{\sqrt{K_l}} (\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l), \tag{2.21}$$

and for any functions $\boldsymbol{v}_d \in H(\text{div}; \Omega_d)$, $\boldsymbol{v}_s \in H(\text{div}; \Omega_s)$ and $w \in L^2(\Omega)$, we define the bilinear form

$$\tilde{b}(\boldsymbol{v}, w) = -\int_{\Omega_d} w \, \text{div} \, \boldsymbol{v}_d - \int_{\Omega_s} w \, \text{div} \, \boldsymbol{v}_s. \tag{2.22}$$

Note that $\tilde{a}(\cdot, \cdot)$ is continuous on $\tilde{X} \times \tilde{X}$ and $\tilde{b}(\cdot, \cdot)$ is continuous on $\tilde{X} \times L^2(\Omega)$:

$$\begin{aligned}
\forall(\boldsymbol{u}, \boldsymbol{v}) \in \tilde{X} \times \tilde{X},\ |\tilde{a}(\boldsymbol{u}, \boldsymbol{v})| \le\ & \frac{\nu_d}{\lambda_{\min}} \|\boldsymbol{u}_d\|_{L^2(\Omega_d)} \|\boldsymbol{v}_d\|_{L^2(\Omega_d)} \\
& +2\,\nu_s \|\nabla \boldsymbol{u}_s\|_{L^2(\Omega_s)} \|\nabla \boldsymbol{v}_s\|_{L^2(\Omega_s)} \\
& + \sum_{l=1}^{n-1} \frac{\nu_s \alpha}{\sqrt{\lambda_{\min}}} \|\boldsymbol{u}_s \cdot \boldsymbol{\tau}_l\|_{L^2(\Gamma_{sd})} \|\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l\|_{L^2(\Gamma_{sd})}, \\
\forall(\boldsymbol{v}, w) \in \tilde{X} \times L^2(\Omega),\ |\tilde{b}(\boldsymbol{v}, w)| \le\ & \|\boldsymbol{v}\|_{\tilde{X}} \|w\|_{L^2(\Omega)}.
\end{aligned} \tag{2.23}$$

Then (2.19)–(2.20) has the familiar form : Find $(\boldsymbol{u}, p) \in \tilde{X} \times W$ such that

$$\forall \boldsymbol{v} \in \tilde{X}, \quad \tilde{a}(\boldsymbol{u}, \boldsymbol{v}) + \tilde{b}(\boldsymbol{v}, p) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}, \tag{2.24}$$

$$\forall w \in W, \quad \tilde{b}(\boldsymbol{u}, w) = -\int_{\Omega_d} w \, q_d. \tag{2.25}$$

Next, we set

$$\tilde{X}_0 = \{\boldsymbol{v} \in \tilde{X};\ \text{div}\, \boldsymbol{v} = 0\}, \tag{2.26}$$

and more generally, for a given function $g \in W$, we define the affine variety

$$\tilde{X}_g = \{\boldsymbol{v} \in \tilde{X};\ \text{div}\, \boldsymbol{v} = g\}. \tag{2.27}$$

Then we consider the reduced problem : Find $\boldsymbol{u} \in \tilde{X}_q$ such that

$$\forall \boldsymbol{v} \in \tilde{X}_0, \quad \tilde{a}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}, \tag{2.28}$$

recall that $q$ is $q_d$ extended by zero on $\Omega_s$. It is well known [18,34] that showing equivalence between problems (2.28) and (2.24)–(2.25) amounts to proving the following inf-sup condition.

**Lemma 2.2** *There exists a constant* $\beta > 0$ *such that*

$$\forall w \in W, \quad \sup_{\boldsymbol{v} \in \tilde{X}} \frac{\tilde{b}(\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{\tilde{X}}} \ge \beta \|w\|_W. \tag{2.29}$$

*Proof* Let $w \in W$. The inf-sup condition between $H_0^1(\Omega)^n$ and $L_0^2(\Omega)$ implies that there exists a function $\boldsymbol{v} \in H_0^1(\Omega)^n$ such that

$$\operatorname{div} \boldsymbol{v} = w \quad \text{in } \Omega \quad \text{and} \quad |\boldsymbol{v}|_{H^1(\Omega)} \leq \frac{1}{\kappa}\|w\|_{L^2(\Omega)},$$

where $\kappa$ depends only on $\Omega$; see for example [34] or [18]. Then $\boldsymbol{v}$ belongs to $\tilde{X}$ and a straightforward computation yields (2.29) with $\beta \geq \kappa/\big(\mathcal{P}_{\partial\Omega}^2 + 2\big)^{\frac{1}{2}}$, where $\mathcal{P}_{\partial\Omega}$ is the Poincaré constant in (1.1).

Lemma 2.2 implies that (2.28) and (2.24)–(2.25) are equivalent in the following sense.

**Proposition 2.1** *Let $(\boldsymbol{f}, q_d)$ be given in $L^2(\Omega)^n \times L_0^2(\Omega_d)$. Let $(\boldsymbol{u}, p) \in \tilde{X} \times W$ be a solution of (2.24)–(2.25). Then $\boldsymbol{u}$ solves (2.28). Conversely, let $\boldsymbol{u} \in \tilde{X}_{q_d}$ be a solution of (2.28). Then there exists a unique $p$ in $W$ such that $(\boldsymbol{u}, p)$ satisfies (2.24)–(2.25).*

Now, let us prove that (2.28), and hence (2.24)–(2.25), is well-posed. This relies on the ellipticity of $\tilde{a}(\cdot,\cdot)$ on $\tilde{X}_0$. If $\Omega_s$ is connected and $|\Gamma_s| > 0$, a partial ellipticity result for $\tilde{a}(\cdot,\cdot)$ follows directly from Korn's inequality (1.4):

$$\forall \boldsymbol{v} \in \tilde{X}, \quad \tilde{a}(\boldsymbol{v}, \boldsymbol{v}) \geq 2\frac{\nu_s}{C_1^2}|\boldsymbol{v}_s|_{H^1(\Omega_s)}^2 + \frac{\nu_d}{\lambda_{\max}}\|\boldsymbol{v}_d\|_{L^2(\Omega_d)}^2, \tag{2.30}$$

with the constant $C_1$ of (1.4). If $\Omega_s$ is connected and $|\Gamma_s| = 0$, then $\Gamma_{sd} = \partial\Omega_s$ up to a set of zero measure, and proving the analogue of (2.30) makes use of (1.7) and the tangential components on $\Gamma_{sd}$. Indeed, we have formally

$$\text{a.e. on } \partial\Omega_s, \ \big|(\boldsymbol{v}_s \times \boldsymbol{n}_s)(\boldsymbol{x})\big| \leq \sum_{l=1}^{n-1} |(\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l)(\boldsymbol{x})|, \tag{2.31}$$

and therefore

$$\int_{\Omega_s} \mathbf{curl}\, \boldsymbol{v}_s = -\int_{\Gamma_{sd}} \boldsymbol{v}_s \times \boldsymbol{n}_s \ \Rightarrow \ \Big|\int_{\Omega_s} \mathbf{curl}\, \boldsymbol{v}_s\Big| \leq \int_{\Gamma_{sd}} \Big(\sum_{l=1}^{n-1} |\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l|\Big).$$

Hence (1.7) and a straightforward manipulation yield for all $\boldsymbol{v}_s$ in $H^1(\Omega_s)^n$:

$$\begin{aligned} 2\nu_s\|\boldsymbol{D}(\boldsymbol{v}_s)\|_{L^2(\Omega_s)}^2 &+ \sum_{l=1}^{n-1} \frac{\nu_s \alpha}{\sqrt{K_l}}\|\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l\|_{L^2(\Gamma_{sd})}^2 \\ &\geq \frac{\nu_s}{C_\Omega^2} \min\left(2, \frac{\alpha}{\sqrt{\lambda_{\max}}|\Gamma_{sd}|}\right)|\boldsymbol{v}_s|_{H^1(\Omega_s)}^2. \end{aligned} \tag{2.32}$$

As a consequence (2.30) is replaced by

$$\forall \boldsymbol{v} \in \tilde{X}, \quad \tilde{a}(\boldsymbol{v}, \boldsymbol{v}) \geq \frac{\nu_d}{\lambda_{\max}} \|\boldsymbol{v}_d\|^2_{L^2(\Omega_d)} + \frac{\nu_s}{C_\Omega^2} \min\left(2, \frac{\alpha}{\sqrt{\lambda_{\max}}|\Gamma_{sd}|}\right) |\boldsymbol{v}_s|^2_{H^1(\Omega_s)}. \tag{2.33}$$

Finally, if $\Omega_s$ is not connected, the analogue of (2.30) holds on all connected components of $\Omega_s$ that are adjacent to $\Gamma_s$ and the analogue of (2.33) holds on all connected components of $\Omega_s$ that are not adjacent to $\Gamma_s$.

It remains to establish that the mapping:

$$\boldsymbol{v} \mapsto |\boldsymbol{v}|_{\tilde{X}_0} = \left(|\boldsymbol{v}_s|^2_{H^1(\Omega_s)} + \|\boldsymbol{v}_d\|^2_{L^2(\Omega_d)}\right)^{\frac{1}{2}} \tag{2.34}$$

is a norm on $\tilde{X}_0$ equivalent to $\|\boldsymbol{v}\|_{\tilde{X}}$. This is the object of the next lemma.

**Lemma 2.3** *There exists a constant $C_4$ such that*

$$\forall \boldsymbol{v} \in \tilde{X}_0, \quad \|\boldsymbol{v}_s\|_{L^2(\Omega_s)} \leq C_4\left(|\boldsymbol{v}_s|^2_{H^1(\Omega_s)} + \|\boldsymbol{v}_d\|^2_{L^2(\Omega_d)}\right)^{\frac{1}{2}}. \tag{2.35}$$

*Proof* Let us assume that $\Omega_s$ is connected; the case when $\Omega_s$ is not connected is treated as above. If $|\Gamma_s| > 0$, (2.35) follows from Poincaré's inequality (1.1) applied in $\Omega_s$ and does not require the norm of $\boldsymbol{v}_d$ in the right-hand side. When $|\Gamma_s| = 0$, the proof of (2.35) is a variant of the proof of Peetre–Tartar's Lemma [51]. Let us recall its argument. Assume that (2.35) is not true. Then there exists a sequence $(\boldsymbol{v}^m)$ in $\tilde{X}_0$ such that

$$\lim_{m\to\infty} \|\boldsymbol{v}_d^m\|_{L^2(\Omega_d)} = \lim_{m\to\infty} |\boldsymbol{v}_s^m|_{H^1(\Omega_s)} = 0 \ \text{ and } \ \forall m, \quad \|\boldsymbol{v}_s^m\|_{L^2(\Omega_s)} = 1.$$

As $\tilde{X}_0$ is reflexive, this implies that there exists a function $\boldsymbol{v} \in \tilde{X}_0$ such that

$$\lim_{m\to\infty} \boldsymbol{v}^m = \boldsymbol{v} \quad \text{weakly in } \tilde{X}.$$

Moreover $\boldsymbol{v}_s = \boldsymbol{c}$, a constant vector, and $\boldsymbol{v}_d = \boldsymbol{0}$. Then the fact that $\boldsymbol{v}$ belongs to $\tilde{X}$ implies that $\boldsymbol{c} \cdot \boldsymbol{n}_s = 0$ on $\Gamma_{sd}$, and since $\Gamma_{sd}$ coincides a.e. with $\partial\Omega_s$, this implies that $\boldsymbol{c} = \boldsymbol{0}$. Thus

$$\lim_{m\to\infty} \boldsymbol{v}_s^m = \boldsymbol{0} \quad \text{weakly in } H^1(\Omega_s)^n,$$

hence

$$\lim_{m\to\infty} \boldsymbol{v}_s^m = \boldsymbol{0} \quad \text{strongly in } L^2(\Omega_s)^n.$$

This contradicts the fact that for all $m$ $\|\boldsymbol{v}_s^m\|_{L^2(\Omega_s)} = 1$.

Therefore (2.35) combined with either (2.30) or (2.33) yields the ellipticity of $\tilde{a}(\cdot, \cdot)$.

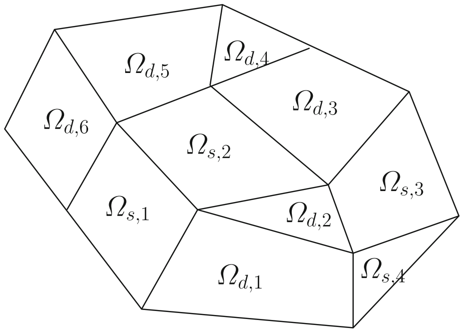



**Fig. 1** Domain decomposition with multiple connected components

**Lemma 2.4** *There exists a constant* $C_5 > 0$ *such that*

$$\forall \boldsymbol{v} \in \tilde{X}_0, \quad \tilde{a}(\boldsymbol{v}, \boldsymbol{v}) \geq C_5 \|\boldsymbol{v}\|_{\tilde{X}}^2. \tag{2.36}$$

With the continuity of $\tilde{a}(\cdot, \cdot)$ and $\tilde{b}(\cdot, \cdot)$, the ellipticity of $\tilde{a}(\cdot, \cdot)$ on $\tilde{X}_0$, and the inf-sup condition (2.29), the Babuška–Brezzi's theory [7,17] implies immediately that (2.24)–(2.25) is well-posed.

**Theorem 2.1** *Problem* (2.24)–(2.25) *has a unique solution* $(\boldsymbol{u}, p) \in \tilde{X} \times W$ *and there exists a constant* $C$ *that depends only on* $\Omega_d$, $\Omega_s$, $\lambda_{\min}$, $\lambda_{\max}$, $\alpha$, $\nu_d$, *and* $\nu_s$, *such that*

$$\|\boldsymbol{u}\|_{\tilde{X}} + \|p\|_{L^2(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|q_d\|_{L^2(\Omega_d)}\big). \tag{2.37}$$

In turn, the well-posedness of problem (2.3)–(2.11) stems from Lemma 2.1.

## 2.4 Domain decomposition of the Darcy and Stokes regions

Let $\Omega_s$, respectively $\Omega_d$, be decomposed into $M_s$, respectively $M_d$, *non-overlapping*, open Lipschitz subdomains:

$$\overline{\Omega}_s = \cup_{i=1}^{M_s} \overline{\Omega}_{s,i}, \quad \overline{\Omega}_d = \cup_{i=1}^{M_d} \overline{\Omega}_{d,i}.$$

Set $M = M_d + M_s$; according to convenience we can also number the subdomains with a single index $i$, $1 \leq i \leq M$, the Darcy subdomains running from $M_s + 1$ to $M$. An example of a domain decomposition with multiple connected components is depicted in Fig. 1. Let $\boldsymbol{n}_i$ denote the outward unit normal vector on $\partial\Omega_i$. For $1 \leq i \leq M$, let the boundary interfaces be denoted by $\Gamma_i$, with possibly zero measure:

$$\Gamma_i = \partial\Omega_i \cap \partial\Omega,$$

and for $1 \leq i < j \leq M$, let the interfaces between subdomains be denoted by $\Gamma_{ij}$, again with possibly zero measure:

$$\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j.$$

In addition to $\Gamma_{sd}$, let $\Gamma_{dd}$, respectively $\Gamma_{ss}$, denote the set of interfaces between subdomains of $\Omega_d$, respectively $\Omega_s$. Then, assuming that the solution $(\boldsymbol{u}, p)$ of (2.3)–(2.11) is slightly smoother, we can obtain an equivalent formulation by writing individually (2.3)–(2.11) in each subdomain $\Omega_i$, $1 \le i \le M$, and complementing these systems with the following interface conditions

$$[uu_d \cdot \boldsymbol{n}] = 0, \quad [p_d] = 0 \quad \text{on } \Gamma_{dd}, \tag{2.38}$$

$$[\boldsymbol{u}_s] = \boldsymbol{0}, \quad [\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}] = \boldsymbol{0} \quad \text{on } \Gamma_{ss}, \tag{2.39}$$

where the jumps on an interface $\Gamma_{ij}$, $1 \le i < j \le M$, are defined as

$$[\boldsymbol{v} \cdot \boldsymbol{n}] = \boldsymbol{v}_i \cdot \boldsymbol{n}_i + \boldsymbol{v}_j \cdot \boldsymbol{n}_j, \quad [\boldsymbol{\sigma n}] = \boldsymbol{\sigma}_i \boldsymbol{n}_i + \boldsymbol{\sigma}_j \boldsymbol{n}_j, \quad [v] = (v_i - v_j)|_{\Gamma_{ij}},$$

using the notation $v_i = v|_{\Omega_i}$. The smoothness requirement on the solution is meant to ensure that the jumps $[\boldsymbol{u}_d \cdot \boldsymbol{n}]$, respectively $[\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}]$, are well-defined on each interface of $\Gamma_{dd}$, respectively $\Gamma_{ss}$.

Finally, let us prescribe weakly the interface conditions (2.38), (2.39), and (2.9) by means of Lagrange multipliers, usually called *mortars*. For this, it is convenient to attribute a unit normal vector $\boldsymbol{n}_{ij}$ to each interface $\Gamma_{ij}$ of positive measure, directed from $\Omega_i$ to $\Omega_j$ (recall that $i < j$). The basic velocity spaces are:

$$\begin{aligned} X_d &= \{\boldsymbol{v} \in L^2(\Omega_d)^n;\ \boldsymbol{v}_{d,i} := \boldsymbol{v}|_{\Omega_{d,i}} \in H(\text{div}; \Omega_{d,i}), 1 \le i \le M_d, \\ &\qquad \boldsymbol{v} \cdot \boldsymbol{n}_{ij} \in H^{-\frac{1}{2}}(\Gamma_{ij}), \Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_d\}, \\ X_s &= \{\boldsymbol{v} \in L^2(\Omega_s)^n;\ \boldsymbol{v}_{s,i} := \boldsymbol{v}|_{\Omega_{s,i}} \in H^1(\Omega_{s,i})^n, 1 \le i \le M_s, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_s\}, \end{aligned} \tag{2.40}$$

and the mortar spaces are:

$$\forall \Gamma_{ij} \in \Gamma_{ss},\ \Lambda_{ij} = H^{-\frac{1}{2}}(\Gamma_{ij})^n, \quad \forall \Gamma_{ij} \in \Gamma_{sd} \cup \Gamma_{dd},\ \Lambda_{ij} = H^{\frac{1}{2}}(\Gamma_{ij}). \tag{2.41}$$

Then we replace $\tilde{X}$ (see (2.17)) by

$$X = \{\boldsymbol{v} \in L^2(\Omega)^n;\ \boldsymbol{v}_d := \boldsymbol{v}|_{\Omega_d} \in X_d, \boldsymbol{v}_s := \boldsymbol{v}|_{\Omega_s} \in X_s\}, \tag{2.42}$$

we keep $W = L^2_0(\Omega)$ for the pressure, and we define the mortar spaces

$$\begin{aligned} \Lambda_s &= \{\boldsymbol{\lambda} \in \left(\mathcal{D}'(\Gamma_{ss})\right)^n;\ \boldsymbol{\lambda}|_{\Gamma_{ij}} \in H^{-\frac{1}{2}}(\Gamma_{ij})^n \text{ for all } \Gamma_{ij} \in \Gamma_{ss}\}, \\ \Lambda_{sd} &= \{\lambda \in L^2(\Gamma_{sd});\ \lambda|_{\Gamma_{ij}} \in H^{\frac{1}{2}}(\Gamma_{ij}) \text{ for all } \Gamma_{ij} \in \Gamma_{sd}\}, \\ \Lambda_d &= \{\lambda \in L^2(\Gamma_{dd});\ \lambda|_{\Gamma_{ij}} \in H^{\frac{1}{2}}(\Gamma_{ij}) \text{ for all } \Gamma_{ij} \in \Gamma_{dd}\}. \end{aligned} \tag{2.43}$$

We equip these spaces with broken norms:

$$|||\boldsymbol{v}|||_{X_d} = \left( \sum_{i=1}^{M_d} \|\boldsymbol{v}\|^2_{H(\mathrm{div};\Omega_{d,i})} \right)^{\frac{1}{2}}, \quad |||\boldsymbol{v}|||_{X_s} = \left( \sum_{i=1}^{M_s} \|\boldsymbol{v}\|^2_{H^1(\Omega_{s,i})} \right)^{\frac{1}{2}},$$

$$|||\boldsymbol{v}|||_{X} = \left( |||\boldsymbol{v}|||^2_{X_d} + |||\boldsymbol{v}|||^2_{X_s} \right)^{\frac{1}{2}}, \quad |||\boldsymbol{\lambda}|||_{\Lambda_s} = \left( \sum_{\Gamma_{ij}\in\Gamma_{ss}} \|\boldsymbol{\lambda}\|^2_{H^{-\frac{1}{2}}(\Gamma_{ij})} \right)^{\frac{1}{2}},$$

$$|||\lambda|||_{\Lambda_{sd}} = \left( \sum_{\Gamma_{ij}\in\Gamma_{sd}} \|\lambda\|^2_{H^{\frac{1}{2}}(\Gamma_{ij})} \right)^{\frac{1}{2}}, \quad |||\lambda|||_{\Lambda_d} = \left( \sum_{\Gamma_{ij}\in\Gamma_{dd}} \|\lambda\|^2_{H^{\frac{1}{2}}(\Gamma_{ij})} \right)^{\frac{1}{2}}.$$

Note that in most geometrical situations, $X_d$ (and hence $X$) is not complete for the above norm, but none of the subsequent proofs require its completeness.

The matching condition between subdomains is weakly enforced through the following bilinear forms:

$$\begin{aligned}
&\forall \boldsymbol{v} \in X_s, \forall \boldsymbol{\mu} \in \Lambda_s,\ b_s(\boldsymbol{v},\boldsymbol{\mu}) = \sum_{\Gamma_{ij}\in\Gamma_{ss}} \langle [\boldsymbol{v}], \boldsymbol{\mu}\rangle_{\Gamma_{ij}},\\
&\forall \boldsymbol{v} \in X_d, \forall \mu \in \Lambda_d,\ b_d(\boldsymbol{v},\mu) = \sum_{\Gamma_{ij}\in\Gamma_{dd}} \langle [\boldsymbol{v}\cdot\boldsymbol{n}], \mu\rangle_{\Gamma_{ij}},\\
&\forall \boldsymbol{v} \in X, \forall \mu \in \Lambda_{sd},\ b_{sd}(\boldsymbol{v},\mu) = \sum_{\Gamma_{ij}\in\Gamma_{sd}} \langle [\boldsymbol{v}\cdot\boldsymbol{n}], \mu\rangle_{\Gamma_{ij}}.
\end{aligned} \tag{2.44}$$

For the velocity and pressure in the Darcy and Stokes regions, we use the following bilinear forms:

$$\begin{aligned}
&\forall (\boldsymbol{u},\boldsymbol{v}) \in X_s\times X_s,\ a_{s,i}(\boldsymbol{u},\boldsymbol{v}) = 2\,\nu_s \int_{\Omega_{s,i}} \boldsymbol{D}(\boldsymbol{u}_{s,i}) : \boldsymbol{D}(\boldsymbol{v}_{s,i})\\
&\qquad + \sum_{l=1}^{n-1} \int_{\partial\Omega_{s,i}\cap\Gamma_{sd}} \frac{\nu_s\alpha}{\sqrt{K_l}} (\boldsymbol{u}_s\cdot\boldsymbol{\tau}_l)(\boldsymbol{v}_s\cdot\boldsymbol{\tau}_l),\ 1\le i\le M_s,\\
&\forall (\boldsymbol{u},\boldsymbol{v}) \in X_d\times X_d,\ a_{d,i}(\boldsymbol{u},\boldsymbol{v}) = \nu_d \int_{\Omega_{d,i}} \boldsymbol{K}^{-1}\boldsymbol{u}_{d,i}\cdot\boldsymbol{v}_{d,i},\ 1\le i\le M_d,\\
&\forall \boldsymbol{v} \in X, \forall w \in L^2(\Omega),\ b_i(\boldsymbol{v},w) = -\int_{\Omega_i} w \operatorname{div} \boldsymbol{v}_i,\ 1\le i\le M.
\end{aligned} \tag{2.45}$$

Then we set

$$\forall (\boldsymbol{u},\boldsymbol{v}) \in X\times X, \quad a(\boldsymbol{u},\boldsymbol{v}) = \sum_{i=1}^{M_s} a_{s,i}(\boldsymbol{u},\boldsymbol{v}) + \sum_{i=1}^{M_d} a_{d,i}(\boldsymbol{u},\boldsymbol{v}),$$

$$\forall(\boldsymbol{v}, w) \in X \times L^2(\Omega), \quad b(\boldsymbol{v}, w) = \sum_{i=1}^{M} b_i(\boldsymbol{v}, w).$$

The second variational formulation reads: Find $(\boldsymbol{u}, p, \lambda_{sd}, \lambda_d, \boldsymbol{\lambda}_s) \in X \times W \times \Lambda_{sd} \times \Lambda_d \times \Lambda_s$ such that

$$\begin{aligned}
&\forall \boldsymbol{v} \in X,\ a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_{sd}(\boldsymbol{v}, \lambda_{sd}) + b_d(\boldsymbol{v}, \lambda_d) + b_s(\boldsymbol{v}, \boldsymbol{\lambda}_s) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v},\\
&\forall w \in W,\ b(\boldsymbol{u}, w) = -\int_{\Omega_d} w\, q_d, \qquad (2.46)\\
&\forall \mu \in \Lambda_{sd},\ b_{sd}(\boldsymbol{u}, \mu) = 0,\ \forall \mu \in \Lambda_d,\ b_d(\boldsymbol{u}, \mu) = 0,\ \forall \boldsymbol{\mu} \in \Lambda_s,\ b_s(\boldsymbol{u}, \boldsymbol{\mu}) = 0.
\end{aligned}$$

It remains to prove that (2.46) is equivalent to (2.3)–(2.11) when the solution is sufficiently smooth. Since we know from Theorem 2.1 that (2.3)–(2.11) has a unique solution, equivalence will also establish that (2.46) is uniquely solvable.

**Theorem 2.2** *Assume that the solution* $(\boldsymbol{u}, p) \in \tilde{X} \times W$ *of* (2.3)–(2.11) *with* $p_d \in H^1(\Omega_d)$ *satisfies*

$$\forall \Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd},\ (\boldsymbol{u}_d \cdot \boldsymbol{n}_d)|_{\Gamma_{ij}} \in H^{-\frac{1}{2}}(\Gamma_{ij}),\ \forall \Gamma_{ij} \in \Gamma_{ss},$$
$$(\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_s)|_{\Gamma_{ij}} \in H^{-\frac{1}{2}}(\Gamma_{ij})^n.$$

*Then* (2.46) *is equivalent to* (2.3)–(2.11).

*Proof* The argument is similar to that used in proving Lemma 2.1. Let $(\boldsymbol{u}, p)$ be a solution of (2.3)–(2.11) satisfying the above regularity. Take the scalar product in each $\Omega_i$ of (2.3) and (2.6) with a test function $\boldsymbol{v}$ in $X$, apply Green's formula and add the corresponding equations. In view of (2.16) and the regularity of $(\boldsymbol{u}, p)$, this gives:

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) - \sum_{\Gamma_{ij} \in \Gamma_{ss}} \langle \boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)\boldsymbol{n}_{ij}, [\boldsymbol{v}] \rangle_{\Gamma_{ij}} + \sum_{\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}} \langle p_d, [\boldsymbol{v} \cdot \boldsymbol{n}] \rangle_{\Gamma_{ij}} = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}. \qquad (2.47)$$

We recover the first equation in (2.46) by defining

$$\begin{aligned}
&\forall \Gamma_{ij} \in \Gamma_{ss}, \quad \boldsymbol{\lambda}_s|_{\Gamma_{ij}} = -\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)|_{\Gamma_{ij}} \boldsymbol{n}_{ij},\\
&\forall \Gamma_{ij} \in \Gamma_{dd},\ \lambda_d|_{\Gamma_{ij}} = p_d|_{\Gamma_{ij}}, \quad \forall \Gamma_{ij} \in \Gamma_{sd},\ \lambda_{sd}|_{\Gamma_{ij}} = p_d|_{\Gamma_{ij}},
\end{aligned} \qquad (2.48)$$

and the remaining equations follow from the regularity of $(\boldsymbol{u}, p)$.

Conversely, let $(\boldsymbol{u}, p, \lambda_{sd}, \lambda_d, \boldsymbol{\lambda}_s)$ in $X \times W \times \Lambda_{sd} \times \Lambda_d \times \Lambda_s$ be a solution of (2.46). By choosing smooth test functions with compact support in each $\Omega_i$ we recover the interior equations (2.3), (2.4), (2.6), (2.7) in each subdomain. On one hand, we easily derive from the last equation of (2.46) that $\boldsymbol{u}$ has no jump through the interfaces of $\Gamma_{ss}$. Hence $\boldsymbol{u} \in H^1(\Omega_s)^n$. On the other hand, we pick an index $i$ with $1 \leq i \leq M_s$ and an interface $\Gamma_{ij}$ in $\Gamma_{ss}$, we take a function $\boldsymbol{v}$ in $X$, smooth in $\Omega_i$, zero outside $\overline{\Omega}_i$, and zero on $\partial\Omega_i \backslash \Gamma_{ij}$. By taking the scalar product of (2.6) in $\Omega_i$ with $\boldsymbol{v}$, applying Green's formula, comparing with (2.46), and using the same process in $\Omega_j$, we find

$$\boldsymbol{\lambda}_s|_{\Gamma_{ij}} = -\boldsymbol{\sigma}(\boldsymbol{u}_{s,i}, p_{s,i})|_{\Gamma_{ij}}\boldsymbol{n}_{ij} = -\boldsymbol{\sigma}(\boldsymbol{u}_{s,j}, p_{s,j})|_{\Gamma_{ij}}\boldsymbol{n}_{ij}.$$

As $\boldsymbol{\lambda}_s$ is single-valued, this implies that $\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s)|_{\Gamma_{ij}}\boldsymbol{n}_{ij}$ has no jump through $\Gamma_{ij}$. This is true for all interfaces in $\Gamma_{ss}$. Therefore (2.6) is satisfied in $\Omega_s$. By applying a similar process to the interfaces of $\Gamma_{dd}$, we derive first that $(\boldsymbol{u}_d, p_d)$ belongs to $H(\text{div}; \Omega_d) \times H^1(\Omega_d)$ and next that (2.3) is satisfied in $\Omega_d$. Finally, the third equation in (2.46) implies that $\boldsymbol{u}$ belongs to $H(\text{div}; \Omega)$, therefore $\boldsymbol{u}$ is in $X$ and the pair $(\boldsymbol{u}, p)$ solves (2.19)–(2.20); by virtue of Lemma 2.1, it also solves (2.3)–(2.11).

## 3 Discretization

### 3.1 Meshes and discrete spaces

In view of discretization, we assume from now on that $\Omega$ and all its subdomains $\Omega_i$, $1 \leq i \leq M$, have polygonal or polyhedral boundaries. Since none of the subdomains overlap, they form a mesh, $\mathcal{T}_d$ of $\Omega_d$ and $\mathcal{T}_s$ of $\Omega_s$, and the union of these meshes constitutes a mesh $\mathcal{T}_\Omega$ of $\Omega$. Furthermore, we suppose that this mesh satisfies the following assumptions:

**Hypothesis 3.1** 1. *$\mathcal{T}_\Omega$ is conforming, i.e. it has no hanging nodes.*
2. *The subdomains of $\mathcal{T}_\Omega$ can take at most $L$ different geometrical shapes, where $L$ is a fixed integer independent of $M$.*
3. *$\mathcal{T}_\Omega$ is shape-regular in the sense that there exists a real number $\sigma$, independent of $M$ such that*

$$\forall i, 1 \leq i \leq M, \quad \frac{\text{diam}(\Omega_i)}{\text{diam}(B_i)} \leq \sigma, \tag{3.1}$$

*where* $\text{diam}(\Omega_i)$ *is the diameter of* $\Omega_i$ *and* $\text{diam}(B_i)$ *is the diameter of the largest ball contained in* $\Omega_i$. *Without loss of generality, we can assume that* $\text{diam}(\Omega_i) \leq 1$.

As each subdomain $\Omega_i$ is polygonal or polyhedral, it can be entirely triangulated. Let $h > 0$ denote a discretization parameter, and for each $h$, let $\mathcal{T}_i^h$ be a regular family of partitions of $\Omega_i$ made of triangles or tetrahedra $T$ in the Stokes region and triangles, tetrahedra, parallelograms, or parallelepipeds in the Darcy region, with *no matching requirement* at the subdomains interfaces. Thus the meshes are independent and the parameter $h < 1$ is allowed to vary with $i$, but to reduce the notation, unless necessary, we do not indicate its dependence on $i$. By regular, we mean that there exists a real number $\sigma_0$, independent of $i$ and $h$ such that

$$\forall i, 1 \leq i \leq M, \quad \forall T \in \mathcal{T}_i^h, \; \frac{h_T}{\rho_T} \leq \sigma_0, \tag{3.2}$$

where $h_T$ is the diameter of $T$ and $\rho_T$ is the diameter of the largest ball contained in $T$.

In addition we assume that each element of $\mathcal{T}_i^h$ has at least one vertex in $\Omega_i$. For the interfaces, let $H > 0$ be another discretization parameter and for each $H$ and each $i < j$, let $\mathcal{T}_{ij}^H$ denote a regular family of partitions of $\Gamma_{ij}$ into segments, triangles or parallelograms of diameter bounded by $H$. These partitions may not match the traces of the subdomain grids.

On these meshes, we define the following finite element spaces. In the Stokes region, for each $\Omega_{s,i}$, let $(X_{s,i}^h, W_{s,i}^h) \subset H^1(\Omega_{s,i})^n \times L^2(\Omega_{s,i})$ be a pair of finite element spaces satisfying a local uniform inf-sup condition for the divergence. More precisely, setting $X_{0,s,i}^h = X_{s,i}^h \cap H_0^1(\Omega_{s,i})^n$ and $W_{0,s,i}^h = W_{s,i}^h \cap L_0^2(\Omega_{s,i})$, we assume that there exists a constant $\beta_s^\star > 0$, independent of $h$ and the diameter of $\Omega_{s,i}$, such that

$$\forall i, 1 \leq i \leq M_s, \quad \inf_{w^h \in W_{0,s,i}^h} \sup_{\boldsymbol{v}^h \in X_{0,s,i}^h} \frac{\int_{\Omega_{s,i}} w^h \operatorname{div} \boldsymbol{v}^h}{|\boldsymbol{v}^h|_{H^1(\Omega_{s,i})} \|w^h\|_{L^2(\Omega_{s,i})}} \geq \beta_s^\star. \tag{3.3}$$

In addition, since $X_{0,s,i}^h \subset H_0^1(\Omega_{s,i})^n$, it satisfies a Korn inequality: There exists a constant $\alpha^\star > 0$, independent of $h$ and the diameter of $\Omega_{s,i}$, such that

$$\forall i, 1 \leq i \leq M_s, \; \forall \boldsymbol{v}^h \in X_{0,s,i}^h, \quad \|\boldsymbol{D}(\boldsymbol{v}^h)\|_{L^2(\Omega_{s,i})} \geq \alpha^\star |\boldsymbol{v}^h|_{H^1(\Omega_{s,i})}. \tag{3.4}$$

There are well-known examples of pairs satisfying (3.3) (cf. [34]), such as the mini-element, the Bernardi–Raugel element, or the Taylor-Hood element. Similarly, in the Darcy region, for each $\Omega_{d,i}$, let $(X_{d,i}^h, W_{d,i}^h) \subset H(\mathrm{div}; \Omega_{d,i}) \times L^2(\Omega_{d,i})$ be a pair of mixed finite element spaces satisfying a uniform inf-sup condition for the divergence. More precisely, setting $X_{0,d,i}^h = X_{d,i}^h \cap H_0(\mathrm{div}; \Omega_{d,i})$ and $W_{0,d,i}^h = W_{d,i}^h \cap L_0^2(\Omega_{d,i})$, we assume that there exists a constant $\beta_d^\star > 0$ independent of $h$ and the diameter of $\Omega_{d,i}$, such that

$$\forall i, 1 \leq i \leq M_d, \; \inf_{w^h \in W_{0,d,i}^h} \sup_{\boldsymbol{v}^h \in X_{0,d,i}^h} \frac{\int_{\Omega_{d,i}} w^h \operatorname{div} \boldsymbol{v}^h}{\|\boldsymbol{v}^h\|_{H(\mathrm{div};\Omega_{d,i})} \|w^h\|_{L^2(\Omega_{d,i})}} \geq \beta_d^\star. \tag{3.5}$$

Furthermore, we assume that

$$\forall i, 1 \leq i \leq M_d, \; \forall \boldsymbol{v}^h \in X_{d,i}^h, \; \operatorname{div} \boldsymbol{v}^h \in W_{d,i}^h. \tag{3.6}$$

Again, there are well-known examples of pairs satisfying (3.5) and (3.6) (cf. [18] or [34]), such as the Raviart–Thomas elements, the Brezzi–Douglas–Marini elements, the Brezzi–Douglas–Fortin–Marini elements, the Brezzi–Douglas–Durán–Fortin elements, or the Chen–Douglas elements. Then we discretize straightforwardly $X_d$ and $X_s$ by

$$X_d^h = \{\boldsymbol{v} \in L^2(\Omega_d)^n;\ \boldsymbol{v}|_{\Omega_{d,i}} \in X_{d,i}^h, 1 \le i \le M_d, \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_d\},$$
$$X_s^h = \{\boldsymbol{v} \in L^2(\Omega_s)^n;\ \boldsymbol{v}|_{\Omega_{s,i}} \in X_{s,i}^h, 1 \le i \le M_s, \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_s\},$$

and we set

$$W_d^h = \{w \in L^2(\Omega_d);\ w|_{\Omega_{d,i}} \in W_{d,i}^h\},\ W_s^h = \{w \in L^2(\Omega_s);\ w|_{\Omega_{s,i}} \in W_{s,i}^h\},$$
$$X^h = \{\boldsymbol{v} \in L^2(\Omega)^n;\ \boldsymbol{v}|_{\Omega_d} \in X_d^h, \boldsymbol{v}|_{\Omega_s} \in X_s^h\},$$
$$W^h = \{w \in L_0^2(\Omega);\ w|_{\Omega_d} \in W_d^h, w|_{\Omega_s} \in W_s^h\}.$$

The finite elements regularity implies that $X_d^h \subset X_d$, $X_s^h \subset X_s$ and $X^h \subset X$. Of course, $W^h \subset W$.

In the mortar region, we take finite element spaces $\Lambda_s^H$, $\Lambda_d^H$, and $\Lambda_{sd}^H$. These spaces consist of continuous or discontinuous piecewise polynomials. We will allow for varying polynomial degrees on different types of interfaces. Although the mortar meshes and the subdomain meshes so far are unrelated, we need compatibility conditions between $\Lambda_s^H$, $\Lambda_{sd}^H$ and $\Lambda_d^H$ on one hand, and $X_d^h$ and $X_s^h$ on the other hand. The following set of conditions is fairly crude and will be sharpened further on.

1. For all $\Gamma_{ij} \in \Gamma_{ss} \cup \Gamma_{sd}$, $i < j$, and for all $\boldsymbol{v} \in \tilde{X}$, there exists $\boldsymbol{v}^h \in X_{s,i}^h$, $\boldsymbol{v}^h = \boldsymbol{0}$ on $\partial\Omega_{s,i} \backslash \Gamma_{ij}$ satisfying

$$\int_{\Gamma_{ij}} \boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \boldsymbol{v} \cdot \boldsymbol{n}_{ij}. \tag{3.7}$$

2. For all $\Gamma_{ij} \in \Gamma_{ss}$, $i < j$, and for all $\boldsymbol{v} \in \tilde{X}$, there exists $\boldsymbol{v}^h \in X_{s,j}^h$, $\boldsymbol{v}^h = \boldsymbol{0}$ on $\partial\Omega_{s,j} \backslash \Gamma_{ij}$ satisfying

$$\forall \boldsymbol{\mu}^H \in \Lambda_s^H, \quad \int_{\Gamma_{ij}} \boldsymbol{\mu}^H \cdot \boldsymbol{v}^h = \int_{\Gamma_{ij}} \boldsymbol{\mu}^H \cdot \boldsymbol{v}. \tag{3.8}$$

3. For all $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$, $i < j$, and for all $\boldsymbol{v} \in \tilde{X}$, there exists $\boldsymbol{v}^h \in X_{d,j}^h$, $\boldsymbol{v}^h \cdot \boldsymbol{n}_j = \boldsymbol{0}$ on $\partial\Omega_{d,j} \backslash \Gamma_{ij}$ satisfying

$$\forall \mu^H \in \Lambda_d^H, \ \forall \mu^H \in \Lambda_{sd}^H, \quad \int_{\Gamma_{ij}} \mu^H \boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \mu^H \boldsymbol{v} \cdot \boldsymbol{n}_{ij}. \tag{3.9}$$

Condition (3.7) is very easy to satisfy in practice and it trivially holds true for all examples of Stokes spaces considered in this paper. Conditions (3.8) and (3.9) state that the mortar space is controlled by the traces of the subdomain velocity spaces. Both conditions are easier to satisfy for coarser mortar grids. Condition (3.8) is more general than previously considered in the literature for mortar discretizations of the Stokes equations [11,12]. Examples for which (3.8) holds are given in the Appendix:

Sect. 7.1 in 2-D and Sect. 7.2 in 3-D. The condition (3.9) on $\Gamma_{dd}$ is closely related to the mortar condition for Darcy flow in [5,6,52,60] and on $\Gamma_{sd}$, it is more general than existing mortar discretizations for Stokes–Darcy flows on [14,29,46,54]. It is discussed in more detail in Sect. 4.4.

**Lemma 3.1** *Under assumptions* (3.8) *and* (3.9)*, the only solution* $\left(\lambda_{sd}^H, \lambda_d^H, \boldsymbol{\lambda}_s^H\right)$ *in* $\Lambda_{sd}^H \times \Lambda_d^H \times \Lambda_s^H$ *to the system*

$$\forall \boldsymbol{v}^h \in X^h, \quad b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s^H) + b_d(\boldsymbol{v}^h, \lambda_d^H) + b_{sd}(\boldsymbol{v}^h, \lambda_{sd}^H) = 0 \tag{3.10}$$

*is the zero solution.*

*Proof* Consider any $\Gamma_{ij} \in \Gamma_{ss}$ with $i < j$; the proof for the other interfaces being the same. Take an arbitrary $\boldsymbol{v}$ in $H_0^1(\Omega)^n$ and $\boldsymbol{v}^h$ associated with $\boldsymbol{v}$ by (3.8), extended by zero outside $\Omega_{s,j}$. Then on one hand,

$$\int_{\Gamma_{ij}} \boldsymbol{\lambda}_s^H \cdot \boldsymbol{v} = \int_{\Gamma_{ij}} \boldsymbol{\lambda}_s^H \cdot \boldsymbol{v}^h = b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s^H),$$

and on the other hand,

$$b_d(\boldsymbol{v}^h, \lambda_d^H) = b_{sd}(\boldsymbol{v}^h, \lambda_{sd}^H) = 0.$$

Therefore

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^n, \quad \int_{\Gamma_{ij}} \boldsymbol{\lambda}_s^H \cdot \boldsymbol{v} = 0,$$

thus implying that $\boldsymbol{\lambda}_s^H = \mathbf{0}$.

Finally, we define

$$\begin{gathered} V_d^h = \{\boldsymbol{v} \in X_d^h;\ \forall \mu \in \Lambda_d^H, b_d(\boldsymbol{v}, \mu) = 0\},\ \ V_s^h = \{\boldsymbol{v} \in X_s^h;\ \forall \boldsymbol{\mu} \in \Lambda_s^H, b_s(\boldsymbol{v}, \boldsymbol{\mu}) = 0\}, \\ V^h = \{\boldsymbol{v} \in X^h;\ \boldsymbol{v}|_{\Omega_d} \in V_d^h, \boldsymbol{v}|_{\Omega_s} \in V_s^h, \forall \mu \in \Lambda_{sd}^H, b_{sd}(\boldsymbol{v}, \mu) = 0\}, \\ Z^h = \{\boldsymbol{v} \in V^h;\ \forall w \in W^h, b(\boldsymbol{v}, w) = 0\}. \end{gathered} \tag{3.11}$$

## 3.2 Variational formulations and uniform stability of the discrete problem

The discrete version of the second variational formulation (2.46) is: Find $(\boldsymbol{u}^h, p^h, \lambda_{sd}^H, \lambda_d^H, \boldsymbol{\lambda}_s^H) \in X^h \times W^h \times \Lambda_{sd}^H \times \Lambda_d^H \times \Lambda_s^H$ such that

$$\forall \boldsymbol{v}^h \in X^h,\ a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) + b_{sd}(\boldsymbol{v}^h, \lambda_{sd}^H) + b_d(\boldsymbol{v}^h, \lambda_d^H) + b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s^H) = \int_{\Omega} \boldsymbol{f} \cdot \boldsymbol{v}^h,$$

$$\forall w^h \in W^h,\ b(\boldsymbol{u}^h, w^h) = -\int_{\Omega_d} w^h\, q_d, \tag{3.12}$$
$$\forall \mu^H \in \Lambda_{sd}^H,\ b_{sd}(\boldsymbol{u}^h, \mu^H) = 0,\ \forall \mu^H \in \Lambda_d^H,\ b_d(\boldsymbol{u}^h, \mu^H) = 0,$$
$$\forall \boldsymbol{\mu}^H \in \Lambda_s^H,\ b_s(\boldsymbol{u}^h, \boldsymbol{\mu}^H) = 0.$$

The last three equations of (3.12) state that $\boldsymbol{u}^h \in V^h$. Therefore, we can extract from (3.12) the following reduced formulation: Find $\boldsymbol{u}^h \in V^h$, $p^h \in W^h$ such that

$$\begin{aligned} \forall \boldsymbol{v}^h \in V^h, \quad a(\boldsymbol{u}^h, \boldsymbol{v}^h) + b(\boldsymbol{v}^h, p^h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^h, \\ \forall w^h \in W^h, \quad b(\boldsymbol{u}^h, w^h) &= -\int_{\Omega_d} w^h\, q_d. \end{aligned} \tag{3.13}$$

**Lemma 3.2** *Problems* (3.12) *and* (3.13) *are equivalent.*

*Proof* Clearly, (3.12) implies (3.13). Conversely, if the pair $(\boldsymbol{u}^h, p^h)$ solves (3.13), existence of $\lambda_{sd}^H, \lambda_d^H, \boldsymbol{\lambda}_s^H$ such that all these variables satisfy (3.12) is an easy consequence of Lemma 3.1 and an algebraic argument.

In view of this equivalence, it suffices to analyze problem (3.13). From the Babuška–Brezzi's theory, uniform stability of the solution of (3.13) stems from an ellipticity property of the bilinear form $a$ in $Z^h$ and an inf-sup condition of the bilinear form $b$. Let us prove an ellipticity property of the bilinear form $a$, valid when $n = 2, 3$. For this, we make the following assumptions on the mortar spaces:

**Hypothesis 3.2** 1. *On each* $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$, $\Lambda_d^H|_{\Gamma_{ij}}$ *and* $\Lambda_{sd}^H|_{\Gamma_{ij}}$ *contain at least* $\mathbb{P}_0$.
2. *On each* $\Gamma_{ij} \in \Gamma_{ss}$, *on each hyperplane* $F \subset \Gamma_{ij}$, $\Lambda_s^H|_F$ *contains at least* $\mathbb{P}_0^n$.
3. *On each* $\Gamma_{ij} \in \Gamma_{ss}$, $\Lambda_s^H|_{\Gamma_{ij}}$ *contains at least* $\mathbb{P}_1^n$.

The second assumption guarantees that $\boldsymbol{n}_{ij} \in \Lambda_s^H|_{\Gamma_{ij}}$; it follows from the third assumption when $\Gamma_{ij}$ is flat. The third assumption is solely used for deriving a discrete Korn inequality; it can be relaxed, as we shall see in Sect. 7.2. The first two assumptions imply that all functions $\boldsymbol{v}^h$ in $V^h$ satisfy

$$\sum_{i=1}^{M} \int_{\Omega_i} \operatorname{div} \boldsymbol{v}^h = \sum_{i=1}^{M} \int_{\partial\Omega_i} \boldsymbol{v}^h \cdot \boldsymbol{n}_i = \sum_{i<j} \int_{\Gamma_{ij}} [\boldsymbol{v}^h \cdot \boldsymbol{n}] = 0.$$

Therefore, the zero mean-value restriction on the functions of $W^h$ can be relaxed. Thus the condition $\boldsymbol{v}^h \in Z^h$ and (3.6) imply in particular that $\operatorname{div} \boldsymbol{v}_d^h = 0$ in $\Omega_{d,i}$, $1 \le i \le M_d$. Hence

$$\forall \boldsymbol{v}^h \in Z^h, \quad \|\boldsymbol{v}_d^h\|_{X_d} = \|\boldsymbol{v}_d^h\|_{L^2(\Omega_d)}. \tag{3.14}$$

First, we treat the simpler case when $|\Gamma_s| > 0$ and $\Omega_s$ is connected.

**Lemma 3.3** *Let $|\Gamma_s| > 0$ and $\Omega_s$ be connected. Then under Hypothesis 3.2, we have*

$$\forall \boldsymbol{v}^h \in Z^h, \quad a(\boldsymbol{v}^h, \boldsymbol{v}^h) \geq \frac{\nu_d}{\lambda_{\max}} \|\boldsymbol{v}_d^h\|_{X_d}^2 + 2\frac{\nu_s}{C^2} \|\boldsymbol{v}_s^h\|_{X_s}^2, \tag{3.15}$$

*where the constant $C$ only depends on the shape regularity of $\mathcal{T}_s$.*

*Proof* Since $\boldsymbol{v}_s^h \in V_s^h$ and $\mathbb{P}_1^n \in \Lambda_{ss}^H|_{\Gamma_{ij}}$ for each $\Gamma_{ij} \in \Gamma_{ss}$, then $P_1[\boldsymbol{v}_s^h] = \mathbf{0}$, where $P_1$ is the orthogonal projection on $\mathbb{P}_1^n$ for the $L^2$ norm on each $\Gamma_{ij}$. Thus, as $\Omega_s$ is connected and $|\Gamma_s| > 0$, inequality (1.12) in [16] gives

$$\forall \boldsymbol{v}_s^h \in V_s^h, \quad \sum_{i=1}^{M_s} |\boldsymbol{v}_s^h|_{H^1(\Omega_{s,i})}^2 \leq C^2 \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}_s^h)\|_{L^2(\Omega_{s,i})}^2, \tag{3.16}$$

where the constant $C$ only depends on the shape regularity of $\mathcal{T}_s$. Hence we have the analogue of (2.30):

$$\forall \boldsymbol{v}^h \in Z^h, \quad a(\boldsymbol{v}^h, \boldsymbol{v}^h) \geq \frac{\nu_d}{\lambda_{\max}} \|\boldsymbol{v}_d^h\|_{X_d}^2 + 2\frac{\nu_s}{C^2} \sum_{i=1}^{M_s} |\boldsymbol{v}_s^h|_{H^1(\Omega_{s,i})}^2. \tag{3.17}$$

Finally the above argument permits to apply formula (1.3) in [15] in order to recover the full norm of $X_s$ in the right-hand side of (3.17). In fact, it is enough that $\mathbb{P}_0^n \in \Lambda_{ss}^H|_{\Gamma_{ij}}$ for each $\Gamma_{ij} \in \Gamma_{ss}$.

Now we turn to the case when $\Omega_s$ is connected and $|\Gamma_s| = 0$, consequently $\Gamma_{sd} = \partial\Omega_s$, up to a set of zero measure.

**Lemma 3.4** *Let $|\Gamma_s| = 0$ and $\Omega_s$ be connected, i.e. $\Gamma_{sd} = \partial\Omega_s$. Then under Hypothesis 3.2, we have*

$$\forall \boldsymbol{v}^h \in Z^h, \quad a(\boldsymbol{v}^h, \boldsymbol{v}^h) \geq \frac{\nu_d}{\lambda_{\max}} \|\boldsymbol{v}_d^h\|_{X_d}^2 + \frac{\nu_s}{C^2} \min\left(2, \frac{\alpha}{\sqrt{\lambda_{\max}}|\Gamma_{sd}|}\right) \|\boldsymbol{v}_s^h\|_{X_s}^2, \tag{3.18}$$

*where the constant $C$ only depends on the shape regularity of $\mathcal{T}_s$.*

*Proof* All constants in this proof only depend on the shape regularity of $\mathcal{T}_s$. Since (3.14) holds, it suffices to derive a lower bound for $a_s(\boldsymbol{v}_s^h, \boldsymbol{v}_s^h)$. To begin with, as $\boldsymbol{v}_s^h \in V_s^h$, we apply Theorem 4.2 if $n = 2$ or 5.2 if $n = 3$ in [16]:

$$\forall \boldsymbol{v}_s^h \in V_s^h, \quad \sum_{i=1}^{M_s} |\boldsymbol{v}_s^h|_{H^1(\Omega_{s,i})}^2 \leq C^2 \left( \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}_s^h)\|_{L^2(\Omega_{s,i})}^2 + \left(\Phi(\boldsymbol{v}_s^h)\right)^2 \right),$$

where the functional $\Phi$ is a suitable seminorm. Let us choose

$$\forall \boldsymbol{v} \in X_s, \quad \Phi(\boldsymbol{v}) = \left| \int_{\Gamma_{sd}} \boldsymbol{v} \times \boldsymbol{n}_s \right|. \tag{3.19}$$

Clearly, $\Phi$ is a seminorm on $X_s$. Next, considering that

$$\forall \boldsymbol{v} \in H^1(\Omega_s)^n, \quad \Phi(\boldsymbol{v}) = \left| \int_{\Omega_s} \mathbf{curl}\, \boldsymbol{v} \right|,$$

it is easy to check that if $\boldsymbol{m}$ is a rigid body motion, then $\Phi(\boldsymbol{m}) = 0$ if and only if $\boldsymbol{m}$ is a constant vector. Finally, observing that $\Phi$ behaves exactly like the functional $\Phi_2$ of Example 2.4 in [16], we see that $\Phi$ satisfies all assumptions of Theorem 4.2 or 5.3 in [16]. Thus

$$\forall \boldsymbol{v}_s^h \in V_s^h, \quad \sum_{i=1}^{M_s} |\boldsymbol{v}_s^h|^2_{H^1(\Omega_{s,i})} \le C^2 \left( \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}_s^h)\|^2_{L^2(\Omega_{s,i})} + \left| \int_{\Gamma_{sd}} \boldsymbol{v}_s^h \times \boldsymbol{n}_s \right|^2 \right). \tag{3.20}$$

In order to recover the full norm of $X_s$ in the left-hand side of (3.20), we apply Theorem 5.1 in [15] with the functional

$$\Phi(\boldsymbol{v}) = \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} |\boldsymbol{v}_s \cdot \boldsymbol{\tau}_l|.$$

Again, $\Phi$ is a seminorm on $X_s$ and since $\Gamma_{sd}$ is a closed curve or surface, the condition $\Phi(\boldsymbol{c}) = 0$ for a constant vector $\boldsymbol{c}$ implies that $\boldsymbol{c} = \boldsymbol{0}$. The remaining assumptions of this theorem easily follow by observing that $\Phi$ has the same behavior as the functional $\Phi_1$ of Example 4.2 in [15]. This yields

$$\forall \boldsymbol{v}_s^h \in V_s^h, \quad \|\boldsymbol{v}_s^h\|^2_{L^2(\Omega_s)} \le C^2 \left( \sum_{i=1}^{M_s} |\boldsymbol{v}_s^h|^2_{H^1(\Omega_{s,i})} + \left( \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} |\boldsymbol{v}_s^h \cdot \boldsymbol{\tau}_l| \right)^2 \right). \tag{3.21}$$

By combining (3.20) and (3.21), we obtain

$$\forall \boldsymbol{v}_s^h \in V_s^h,\ \|\!|\boldsymbol{v}_s^h|\!\|^2_{X_s} \le C^2 \left( \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}_s^h)\|^2_{L^2(\Omega_{s,i})} + \left| \int_{\Gamma_{sd}} \boldsymbol{v}_s^h \times \boldsymbol{n}_s \right|^2 + \left( \sum_{l=1}^{n-1} \int_{\Gamma_{sd}} |\boldsymbol{v}_s^h \cdot \boldsymbol{\tau}_l| \right)^2 \right). \tag{3.22}$$

Then (3.18) follows from (3.22) by arguing as in deriving (2.31).

The case when $\Omega_s$ is not connected follows from Lemmas 3.3 or 3.4 applied to each connected component of $\Omega_s$ according that it is or is not adjacent to $\Gamma_s$.

To control the bilinear form $b$ in $\Omega_s$, we make the following assumption: There exists a linear approximation operator $\Theta_s^h : H_0^1(\Omega)^n \mapsto V_s^h$ satisfying for all $\boldsymbol{v} \in H_0^1(\Omega)^n$:

–

$$\forall i, 1 \le i \le M_s, \quad \int_{\Omega_{s,i}} \operatorname{div}\big(\Theta_s^h(\boldsymbol{v}) - \boldsymbol{v}\big) = 0. \tag{3.23}$$

– For any $\Gamma_{ij}$ in $\Gamma_{sd}$,

$$\int_{\Gamma_{ij}} \big(\Theta_s^h(\boldsymbol{v}) - \boldsymbol{v}\big) \cdot \boldsymbol{n}_{ij} = 0. \tag{3.24}$$

– There exists a constant $C$ independent of $\boldsymbol{v}$, $h$, $H$, and the diameter of $\Omega_{s,i}$, $1 \le i \le M_s$, such that

$$\|\Theta_s^h(\boldsymbol{v})\|_{X_s} \le C|\boldsymbol{v}|_{H^1(\Omega)}. \tag{3.25}$$

A general construction strategy discussed in Sect. 4.1 gives an operator $\Theta_s^h$ that satisfies (3.23) and (3.24). The stability bound (3.25) is shown to hold for the specific examples presented in Sects. 7.1 and 7.2.

**Lemma 3.5** *Assume that an operator $\Theta_s^h$ satisfying (3.23)–(3.25) exists; then there exists a linear operator $\Pi_s^h : H_0^1(\Omega)^n \mapsto V_s^h$ such that for all $\boldsymbol{v} \in H_0^1(\Omega)^n$,*

$$\forall w^h \in W_s^h, \quad \sum_{i=1}^{M_s} \int_{\Omega_{s,i}} w^h \operatorname{div}(\Pi_s^h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \tag{3.26}$$

$$\forall \Gamma_{ij} \in \Gamma_{sd}, \quad \int_{\Gamma_{ij}} \big(\Pi_s^h(\boldsymbol{v}) - \boldsymbol{v}\big) \cdot \boldsymbol{n}_{ij} = 0, \tag{3.27}$$

*and there exists a constant $C$ independent of $\boldsymbol{v}$, $h$, $H$, and the diameter of $\Omega_{s,i}$, $1 \le i \le M_s$, such that*

$$\|\Pi_s^h(\boldsymbol{v})\|_{X_s} \le C|\boldsymbol{v}|_{H^1(\Omega)}. \tag{3.28}$$

*Proof* The operator $\Pi_s^h$ is constructed by correcting $\Theta_s^h$:

$$\Pi_s^h(\boldsymbol{v}) = \Theta_s^h(\boldsymbol{v}) + \mathbf{c}_s^h(\boldsymbol{v})$$

where $\mathbf{c}_s^h(\boldsymbol{v})|_{\Omega_{s,i}} \in X_{0,s,i}^h$ and

$$\forall w^h \in W_{0,s,i}^h,\ 1 \le i \le M_s,\quad \int_{\Omega_{s,i}} w^h \operatorname{div} \mathbf{c}_s^h(\boldsymbol{v}) = \int_{\Omega_{s,i}} w^h \operatorname{div}\big(\boldsymbol{v} - \Theta_s^h(\boldsymbol{v})\big). \tag{3.29}$$

Existence of $\mathbf{c}_s^h(\boldsymbol{v})$ follows directly from (3.3) and with the same constant

$$|\mathbf{c}_s^h(\boldsymbol{v})|_{H^1(\Omega_{s,i})} \le \frac{1}{\beta_s^\star} \|\operatorname{div}(\boldsymbol{v} - \Theta_s^h(\boldsymbol{v}))\|_{L^2(\Omega_{s,i})}. \tag{3.30}$$

The restriction $w^h \in W_{0,s,i}^h$ is relaxed by applying (3.23) and using the fact that $\mathbf{c}_s^h(\boldsymbol{v})$ belongs to $X_{0,s,i}^h$. Finally, (3.28) follows from the above bound and (3.25).

The idea of constructing the operator $\Pi_s^h$ via the interior inf-sup condition (3.3) and the simplified operator $\Theta_s^h$ satisfying (3.23) and (3.25) is not new. It can be found for instance in [36] and [12].

To control the bilinear form $b$ in $\Omega_d$, we make the following assumption: There exists a linear operator $\Pi_d^h : H_0^1(\Omega)^n \mapsto V_d^h$ satisfying for all $\boldsymbol{v} \in H_0^1(\Omega)^n$:

–

$$\forall w^h \in W_d^h,\quad \sum_{i=1}^{M_d} \int_{\Omega_{d,i}} w^h \operatorname{div}\big(\Pi_d^h(\boldsymbol{v}) - \boldsymbol{v}\big) = 0. \tag{3.31}$$

– For any $\Gamma_{ij}$ in $\Gamma_{sd}$,

$$\forall \mu^H \in \Lambda_{sd}^H,\quad \int_{\Gamma_{ij}} \mu^H \big(\Pi_d^h(\boldsymbol{v}) - \Pi_s^h(\boldsymbol{v})\big) \cdot \boldsymbol{n}_{ij} = 0. \tag{3.32}$$

– There exists a constant $C$ independent of $\boldsymbol{v}$, $h$, $H$, and the diameter of $\Omega_{d,i}$, $1 \le i \le M_d$, such that

$$|\!|\!| \Pi_d^h(\boldsymbol{v}) |\!|\!|_{X_d} \le C|\boldsymbol{v}|_{H^1(\Omega)}. \tag{3.33}$$

The construction of the operator $\Pi_d^h$ is presented in Sect. 4. In particular, the general construction strategy discussed in Sect. 4.1 gives an operator that satisfies (3.31) and (3.32). The stability bound (3.33) is shown to hold for various cases in Sect. 4.4.

The next lemma follows readily from the properties of $\Pi_s^h$ and $\Pi_d^h$.

**Lemma 3.6** *Under the above assumptions, there exists a linear operator* $\Pi^h \in \mathcal{L}(H_0^1(\Omega)^n; V^h)$ *such that for all* $\boldsymbol{v} \in H_0^1(\Omega)^n$

$$\forall w^h \in W^h, \quad \sum_{i=1}^{M} \int_{\Omega_i} w^h \operatorname{div}\big(\Pi^h(\boldsymbol{v}) - \boldsymbol{v}\big) = 0, \tag{3.34}$$

$$|||\Pi^h(\boldsymbol{v})|||_X \le C|\boldsymbol{v}|_{H^1(\Omega)}, \tag{3.35}$$

*with a constant* $C$ *independent of* $\boldsymbol{v}$*,* $h$*,* $H$*, and the diameter of* $\Omega_i$*,* $1 \le i \le M$*.*

*Proof* Take $\Pi^h(\boldsymbol{v})|_{\Omega_s} = \Pi_s^h(\boldsymbol{v})$ and $\Pi^h(\boldsymbol{v})|_{\Omega_d} = \Pi_d^h(\boldsymbol{v})$. Then (3.34) follows from (3.26) and (3.31). The matching condition of the functions of $V^h$ at the interfaces of $\Gamma_{sd}$ holds by virtue of (3.32). Finally, the stability bound (3.35) stems from (3.28) and (3.33).

The following inf-sup condition between $W^h$ and $V^h$ is an immediate consequence of a simple variant of Fortin's Lemma [18,34] and Lemma 3.6.

**Theorem 3.1** *Under the above assumptions, there exists a constant* $\beta^\star > 0$*, independent of* $h$*,* $H$*, and the diameter of* $\Gamma_{ij}$ *for all* $i < j$*, such that*

$$\forall w^h \in W^h, \quad \sup_{\boldsymbol{v}^h \in V^h} \frac{b(\boldsymbol{v}^h, w^h)}{|||\boldsymbol{v}^h|||_X} \ge \beta^\star \|w^h\|_{L^2(\Omega)}. \tag{3.36}$$

Finally, well-posedness of the discrete scheme (3.13) follows from Lemma 3.3 or 3.4 and Theorem 3.1.

**Corollary 3.1** *Under the above assumptions, problem* (3.13) *has a unique solution* $(\boldsymbol{u}^h, p^h) \in V^h \times W^h$ *and*

$$|||\boldsymbol{u}^h|||_X + \|p^h\|_{L^2(\Omega)} \le C\big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|q_d\|_{L^2(\Omega_d)}\big), \tag{3.37}$$

*with a constant* $C$ *independent of* $h$*,* $H$*, and the diameter of* $\Gamma_{ij}$ *for all* $i < j$*.*

*Proof* Since (3.13) is set into finite dimension, it is sufficient to prove uniqueness, and as uniqueness follows from (3.37), it suffices to prove this stability estimate. Thus let $(\boldsymbol{u}^h, p^h)$ solve (3.13), which is a typical linear problem with a non-homogeneous constraint. By virtue of the discrete inf-sup condition (3.36), there exists a function $\boldsymbol{u}_q^h \in V^h$ such that

$$\forall w^h \in W^h, \quad b(\boldsymbol{u}_q^h, w^h) = -\int_{\Omega_d} w^h\, q_d,$$

$$|||\boldsymbol{u}_q^h|||_X \le \frac{1}{\beta^\star}\|q_d\|_{L^2(\Omega_d)}. \tag{3.38}$$

Then $\boldsymbol{u}_0^h = \boldsymbol{u}^h - \boldsymbol{u}_q^h$ solves (3.13) with $q_d = 0$ and the coercivity condition (3.15) or (3.18) and the discrete inf-sup condition (3.36) imply that

$$\|\!|\boldsymbol{u}_0^h|\!\|_X + \|p^h\|_{L^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|q_d\|_{L^2(\Omega_d)}).$$

With (3.38), this gives (3.37).

## 4 Elements of construction of $\Theta_s^h$ and $\Pi_d^h$

Constructing an operator $\Theta_s^h$ with values in $V_s^h$, satisfying (3.23)–(3.25), uniformly stable with respect to the diameter of the subdomains and interfaces, is not straightforward, particularly in 3-D. On the other hand, a general construction of $\Pi_d^h$ in $\Omega_d$ can be found in [5], and we shall adapt it so that it matches suitably $\Theta_s^h$ on $\Gamma_{sd}$. Let us describe our strategy in each region.

### 4.1 General construction strategy

Let $\boldsymbol{v} \in H_0^1(\Omega)^n$. In $\Omega_s$, we propose the following three-step construction.

1. Starting step. In each $\Omega_{s,i}$, $1 \le i \le M_s$, take $\Theta_s^h(\boldsymbol{v}) = S^h(\boldsymbol{v})$, where $S^h(\boldsymbol{v})$ is a Scott and Zhang [56] approximation operator, constructed so that $S^h(\boldsymbol{v})|_{\partial\Omega_{s,i}}$ only uses values of $\boldsymbol{v}$ restricted to $\partial\Omega_{s,i}$, and in particular, $S^h(\boldsymbol{v})|_{\Gamma_s} = \boldsymbol{0}$. Thus $S^h(\boldsymbol{v})|_{\Omega_{s,i}} \in X_{s,i}^h$ and $S^h(\boldsymbol{v}) \in X_s^h$.
2. First correction step. For each $\Gamma_{ij} \in \Gamma_{ss} \cup \Gamma_{sd}$ with $i < j$, correct $\Theta_s^h(\boldsymbol{v})$ in $\Omega_{s,i}$ by setting

$$\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} := \Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} + \boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}),$$

where $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}) \in X_{s,i}^h$, $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}) = \boldsymbol{0}$ on $\partial\Omega_{s,i}\backslash\Gamma_{ij}$,

$$\int_{\Gamma_{ij}} \boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}) \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right) \cdot \boldsymbol{n}_{ij}, \tag{4.1}$$

and satisfies suitable bounds. The existence of $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ (without bounds) is guaranteed by assumption (3.7). In particular, one can define it on $\Gamma_{ij}$ to satisfy (4.1), extend it by zero on $\partial\Omega_{s,i}\backslash\Gamma_{ij}$, and extend it arbitrarily inside $\Omega_{s,i}$ to a function in $X_{s,i}^h$. Note that the correction $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ only affects values at points located in the *interior* of $\Omega_{s,i}$ and in the *interior* of $\Gamma_{ij}$. It has no influence on values at points located on the other interfaces. For this reason, the discrete term that appears in the right-hand side of (4.1) is the trace on $\Gamma_{ij}$ of $S^h(\boldsymbol{v})$ coming from $\Omega_{s,i}$; this term has not been yet corrected. Once this correction step is performed on all $\Gamma_{ij} \in \Gamma_{ss} \cup \Gamma_{sd}$ with $i < j$, the resulting function $\Theta_s^h(\boldsymbol{v})$ satisfies on all these interfaces

$$\int_{\Gamma_{ij}} \left(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - \boldsymbol{v}\right) \cdot \boldsymbol{n}_{ij} = 0. \tag{4.2}$$

Note that this step does not modify the trace on $\Gamma_{ij}$ of $S^h(\boldsymbol{v})$ coming from $\Omega_{s,j}$. This trace will be modified in the next step.

3. Second correction step. For each $\Gamma_{ij} \in \Gamma_{ss}$ with $i < j$, correct $\Theta_s^h(\boldsymbol{v})$ in $\Omega_{s,j}$ by setting

$$\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,j}} := \Theta_s^h(\boldsymbol{v})|_{\Omega_{s,j}} + \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}),$$

where $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) \in X_{s,j}^h$, $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) = \boldsymbol{0}$ on $\partial\Omega_{s,j} \backslash \Gamma_{ij}$,

$$\forall \boldsymbol{\mu}^H \in \Lambda_s^H, \quad \int_{\Gamma_{ij}} \boldsymbol{\mu}^H \cdot \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) = \int_{\Gamma_{ij}} \boldsymbol{\mu}^H \cdot \left(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - S^h(\boldsymbol{v})|_{\Omega_{s,j}}\right), \tag{4.3}$$

and satisfies suitable bounds. The existence of $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ (without bounds) is guaranteed by assumption (3.8). In particular, one can define it on $\Gamma_{ij}$ to satisfy (4.3), extend it by zero on $\partial\Omega_{s,j} \backslash \Gamma_{ij}$, and extend it arbitrarily inside $\Omega_{s,j}$ to a function in $X_{s,j}^h$. Here also, the correction $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ has no influence on values at points located on the other interfaces. Once this correction step is performed on all $\Gamma_{ij} \in \Gamma_{ss}$ with $i < j$, the resulting function $\Theta_s^h(\boldsymbol{v})$ satisfies on all these interfaces $b_s(\Theta_s^h(\boldsymbol{v}), \boldsymbol{\mu}^H) = 0$ for all $\boldsymbol{\mu}^H \in \Lambda_s^H$, i.e.

$$\forall \boldsymbol{\mu}^H \in \Lambda_s^H, \quad \int_{\Gamma_{ij}} [\Theta_s^h(\boldsymbol{v})] \cdot \boldsymbol{\mu}^H = 0. \tag{4.4}$$

Finally, if the second assumption in Hypothesis 3.2 holds, then $\boldsymbol{n} \in \Lambda_s^H$ and (4.2) and (4.4) imply that on all these interfaces,

$$\int_{\Gamma_{ij}} \left(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,j}} - \boldsymbol{v}\right) \cdot \boldsymbol{n} = 0. \tag{4.5}$$

Therefore $\Theta_s^h(\boldsymbol{v})$ satisfies (3.23) and (3.24). Specific constructions of the corrections $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ and $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ that guarantee that $\Theta_s^h(\boldsymbol{v})$ also satisfies (3.25) are presented in the forthcoming subsections.

Next, we propose the following two-step construction algorithm in $\Omega_d$.

1. Starting step. Set $P_d^h(\boldsymbol{v}) = R^h(\boldsymbol{v}) \in X_d^h$, where $R^h(\boldsymbol{v})$ is a standard mixed approximation operator associated with $W_d^h$. It preserves the normal component on the boundary:

$$\forall \Gamma_{ij} \subset \partial\Omega_{d,k}, 1 \le k \le M_d, \forall \boldsymbol{v}^h \in X_d^h, \int_{\Gamma_{ij}} \boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} \left(R^h(\boldsymbol{v})|_{\Omega_{d,k}} - \boldsymbol{v}\right) \cdot \boldsymbol{n}_{ij} = 0, \tag{4.6}$$

and satisfies

$$\forall 1 \le i \le M_d, \forall w^h \in W_d^h, \quad \int_{\Omega_{d,i}} w^h \mathrm{div}\big(R^h(\boldsymbol{v}) - \boldsymbol{v}\big) = 0. \tag{4.7}$$

2. Correction step. It remains to prescribe the jump condition. For each $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$ with $i < j$, correct $P_d^h(\boldsymbol{v})$ in $\Omega_{d,j}$ by setting:

$$P_d^h(\boldsymbol{v})|_{\Omega_{d,j}} := P_d^h(\boldsymbol{v})|_{\Omega_{d,j}} + \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}),$$

where $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) \in X_{d,j}^h$, $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) \cdot \boldsymbol{n}_j = 0$ on $\partial\Omega_{d,j} \backslash \Gamma_{ij}$, $\mathrm{div}\, \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) = 0$ in $\Omega_{d,j}$,

$$\begin{aligned} \forall \mu^H \in \Lambda_d^H, \int_{\Gamma_{ij}} \mu^H \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) \cdot \boldsymbol{n}_{ij} &= \int_{\Gamma_{ij}} \mu^H \left(R^h(\boldsymbol{v})|_{\Omega_{d,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\right) \cdot \boldsymbol{n}_{ij}, \\ \forall \mu^H \in \Lambda_{sd}^H, \int_{\Gamma_{ij}} \mu^H \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) \cdot \boldsymbol{n}_{ij} &= \int_{\Gamma_{ij}} \mu^H \left(\Pi_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\right) \cdot \boldsymbol{n}_{ij}, \end{aligned} \tag{4.8}$$

and $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ satisfies adequate bounds. Existence of a non necessarily divergence-free $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ (without bounds) follows from (3.9); it suffices to extend suitably $R^h(\boldsymbol{v})|_{\Omega_{d,j}}$ and $R^h(\boldsymbol{v})|_{\Omega_{d,i}}$ or $\Pi_s^h(\boldsymbol{v})|_{\Omega_{s,i}}$. The zero divergence will be prescribed in the examples. Note that $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ has no effect on interfaces other than $\Gamma_{ij}$ and no effect on the restriction of $P_d^h(\boldsymbol{v})$ in $\Omega_{d,i}$ or on that of $\Pi_s^h(\boldsymbol{v})$ in $\Omega_{s,i}$. Therefore these corrections can be superimposed.

When step 2 is done on all $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$ with $i < j$, the resulting function $P_d^h(\boldsymbol{v})$ has zero normal trace on $\Gamma_d$, it belongs to $V_d^h$ since, due to the first equation in (4.8), it satisfies for all $\Gamma_{ij} \in \Gamma_{dd}$ with $i < j$

$$\forall \mu^H \in \Lambda_d^H, \quad \int_{\Gamma_{ij}} \mu^H [P_d^h(\boldsymbol{v}) \cdot \boldsymbol{n}] = 0, \tag{4.9}$$

and, as the corrections are assumed to be divergence-free in each subdomain,

$$\forall w^h \in W_d^h, \forall 1 \le i \le M_d, \quad \int_{\Omega_{d,i}} w^h \mathrm{div}\big(P_d^h(\boldsymbol{v}) - \boldsymbol{v}\big) = 0. \tag{4.10}$$

Furthermore, due to the second equation in (4.8), it satisfies for all $\Gamma_{ij} \in \Gamma_{sd}$,

$$\forall \mu^H \in \Lambda_{sd}^H, \quad \int_{\Gamma_{ij}} \mu^H \left(\Pi_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - P_d^h(\boldsymbol{v})|_{\Omega_{d,j}}\right) \cdot \boldsymbol{n}_{ij} = 0. \tag{4.11}$$

Therefore, taking $\Pi_d^h(\boldsymbol{v}) = P_d^h(\boldsymbol{v})$ in $\Omega_d$, it satisfies (3.31) and (3.32).

The remainder of this section is devoted to corrections $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ and $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ where the constants in the stability bounds (3.25) and (3.33) are shown to be independent of the discretization parameters and the diameter of the subdomains. In particular, the bounds for the corrections will stem from the fact that each involves differences between $\boldsymbol{v}$ and good approximations of $\boldsymbol{v}$. Beforehand, we need to refine the assumptions on the meshes at the interfaces and refine Hypothesis 3.1 on the mesh of subdomains.

**Hypothesis 4.1** *For $i < j$, let $T$ be any element of $\mathcal{T}_i^h$ that is adjacent to $\Gamma_{ij}$, and let $\{T_\ell\}$ denote the set of elements of $\mathcal{T}_j^h$ that intersect $T$. The number of elements in this set is bounded by a fixed integer and there exists a constant $C$ such that*

$$|T_\ell|\,|T|^{-1} \le C.$$

*The same is true if the indices $i$ and $j$ of the triangulations are interchanged. These constants are independent of $i$, $j$, $h$, and the diameters of the interfaces and subdomains.*

**Hypothesis 4.2** 1. *Each $\Omega_i$, $1 \le i \le M$, is the image of a "reference" polygonal or polyhedral domain by an homothety and a rigid body motion:*

$$\Omega_i = F_i(\hat{\Omega}_i), \quad \boldsymbol{x} = F_i(\hat{\boldsymbol{x}}) = A_i R_i \hat{\boldsymbol{x}} + \boldsymbol{b}_i, \tag{4.12}$$

*where $A_i = \mathrm{diam}(\Omega_i)$, $R_i$ is an orthogonal matrix with constant coefficients and $\boldsymbol{b}_i$ a constant vector.*

2. *There exists a constant $\sigma_1$ independent of $M$ such that for any pair of adjacent subdomains $\Omega_i$ and $\Omega_j$, $1 \le i, j \le M$, we have*

$$A_i\, A_j^{-1} \le \sigma_1. \tag{4.13}$$

By (4.12) $\mathrm{diam}(\hat{\Omega}_i) = 1$. In addition, it follows from Hypothesis 3.1 that on one hand the reference domains $\hat{\Omega}_i$ can take at most $L$ geometrical shapes and on the other hand,

$$\forall i, 1 \le i \le M, \quad \mathrm{diam}(\hat{B}_i) \ge \frac{1}{\sigma}, \tag{4.14}$$

where $\hat{B}_i$ is the largest ball contained in $\hat{\Omega}_i$ and $\sigma$ is the constant of (3.1).

### 4.2 A construction of $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$

In this section we construct a correction $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ satisfying (4.1) and suitable bounds needed to establish the stability estimate (3.25). Recall the approximation properties of the Scott and Zhang operator of degree $r \geq 1$. Let $v \in H^t(\Omega_{s,i})$, for some real number $t \geq 1$ and let $T$ be an element of $\mathcal{T}_i^h$, $1 \leq i \leq M_s$. Let $\Delta_T$ denote the macro-element that is used for defining the values of $S^h(v)$ in $T$. Then (cf. [56]) there exists a constant $C$ that depends only on $r$, $t$, and the shape regularity of $\mathcal{T}_i^h$, such that the following local approximation error holds

$$\|v - S^h(v)\|_{L^2(T)} + h_T|v - S^h(v)|_{H^1(T)} \leq Ch_T^{\min(r,t)}|v|_{H^{\min(r,t)}(\Delta_T)}. \tag{4.15}$$

Owing to the regularity of $\mathcal{T}_i^h$, when (4.15) is summed over all $T$ in $\mathcal{T}_i^h$, it gives

$$\|v - S^h(v)\|_{L^2(\Omega_{s,i})} + h|v - S^h(v)|_{H^1(\Omega_{s,i})} \leq Ch^{\min(r,t)}|v|_{H^{\min(r,t)}(\Omega_{s,i})}. \tag{4.16}$$

Let $\boldsymbol{v} \in H_0^1(\Omega)^n$. Consider the case when $n = 3$, the case $n = 2$ being simpler, and also consider the case when $\Gamma_{ij}$ is a polyhedral surface, not necessarily a flat plane. Let $X^h_{i,\Gamma_{ij}}$ denote the trace of $X^h_{s,i}$ on $\Gamma_{ij}$, and $\mathcal{T}^h_{i,\Gamma_{ij}}$ the trace of $\mathcal{T}_i^h$ on $\Gamma_{ij}$, $1 \leq i \leq M_s$, $i < j$; $\mathcal{T}^h_{i,\Gamma_{ij}}$ is a triangular mesh of each flat face in $\Gamma_{ij}$. In all cases considered, the restriction to each $T$ of functions of $X^h_{s,i}$ contains at least $\mathbb{P}_1^n$. Now, choose one of the faces, say $F$, of $\Gamma_{ij}$. For each *interior* vertex $\boldsymbol{a}_k$ of $\mathcal{T}^h_{i,\Gamma_{ij}}$, $1 \leq k \leq N_F$, on $F$, let $\mathcal{O}_k$ denote the macro-element of all triangles of $\mathcal{T}^h_{i,\Gamma_{ij}}$ (i.e. faces on $F$) that share the vertex $\boldsymbol{a}_k$. Thus

$$F = \cup_{k=1}^{N_F}\mathcal{O}_k.$$

The set $\{\mathcal{O}_k\}$ is not a partition of $F$, but it can be transformed into a partition by setting

$$\Delta_1 = \mathcal{O}_1, \text{ and recursively } \Delta_k = \mathcal{O}_k \setminus \cup_{\ell=1}^{k-1} \Delta_\ell.$$

Note that some $\Delta_k$ may be empty. By construction, we have

$$F = \cup_{k=1}^{N_F}\Delta_k, \quad \Delta_k \cap \Delta_\ell = \emptyset,\ k \neq \ell,\ \Delta_k \subset \mathcal{O}_k.$$

This can be done for all flat faces of $\Gamma_{ij}$. For each $k$, let $b_k$ be the piecewise $\mathbb{P}_1$ "bubble" function such that

$$b_k(\boldsymbol{a}_\ell) = \delta_{k,\ell},$$

extended by zero to all vertices inside $\Omega_{s,i}$, and define $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ by

$$\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}) = \sum_{F\subset\Gamma_{ij}} \sum_{k=1}^{N_F} \boldsymbol{c}_k b_k, \text{ where } \boldsymbol{c}_k = \frac{1}{\int_{\mathcal{O}_k} b_k} \int_{\Delta_k} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right),\ \boldsymbol{c}_k = \boldsymbol{0}, \text{ if } \Delta_k = \emptyset. \tag{4.17}$$

**Lemma 4.1** *The correction $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ defined by (4.17) satisfies (4.1), more precisely*

$$\forall F \subset \Gamma_{ij}, \quad \int_F \boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}) = \int_F \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right), \tag{4.18}$$

*and there exists a constant $C$ independent of $h$ and the diameter of $\Omega_{s,i}$ and $\Gamma_{ij}$ such that*

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^n,\ |\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})|_{H^1(\Omega_{s,i})} \le C|\boldsymbol{v}|_{H^1(\Omega_{s,i})}, \quad \|\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{s,i})} \le Ch|\boldsymbol{v}|_{H^1(\Omega_{s,i})}. \tag{4.19}$$

*Proof* For each $k$, the support of the trace of $b_k$ on $\Gamma_{ij}$ is contained in a single flat plane, say $F$, therefore,

$$\begin{aligned}\int_F \boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}) &= \int_F \sum_{k=1}^{N_F} \boldsymbol{c}_k b_k = \sum_{k=1}^{N_F} \boldsymbol{c}_k \int_F b_k = \sum_{k=1}^{N_F} \boldsymbol{c}_k \int_{\mathcal{O}_k} b_k = \sum_{k=1}^{N_F} \int_{\Delta_k} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right) \\ &= \int_F \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right),\end{aligned}$$

since the set $(\Delta_k)$ is a partition of $\Gamma$.

To derive the estimate (4.19), we consider the faces $T'$ in $\Delta_k$, pass to the reference element $\hat{T}$, where $T$ is the element of $\mathcal{T}^h_i$ adjacent to $T'$, apply a trace theorem in $\hat{T}$, and revert to $T$. This gives

$$\begin{aligned}\left|\int_{\Delta_k} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right)\right| &\le \sum_{T'\subset\Delta_k} \|\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\|_{L^1(T')} \\ &\le \hat{C} \sum_{T'\subset\Delta_k} |T'|\,|T|^{-\frac{1}{2}} \Big(\|\boldsymbol{v} - S^h(\boldsymbol{v})\|_{L^2(T)} + h_T|\boldsymbol{v} - S^h(\boldsymbol{v})|_{H^1(T)}\Big).\end{aligned}$$

In this proof, $\hat{C}$ denotes constants that depend only on the reference element and the shape regularity of the triangulation. Then considering the regularity of $\mathcal{T}^h_i$, the local approximation formula (4.15) with $r = t = 1$ implies that

$$\left|\int_{\Delta_k} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right)\right| \leq \hat{C} \sum_{T' \subset \Delta_k} h_T |T'|\,|T|^{-\frac{1}{2}} |\boldsymbol{v}|_{H^1(\Delta_T)} \leq \hat{C} |\Delta_k|\, \rho_k^{-\frac{1}{2}} |\boldsymbol{v}|_{H^1(D_k)}, \tag{4.20}$$

where $D_k$ is the set of elements of $\mathcal{T}_i^h$ where the values of $\boldsymbol{v}$ are taken for computing $S^h(\boldsymbol{v})$, and

$$\rho_k = \min_{T \in D_k} \rho_T.$$

Therefore, considering that $|\Delta_k| \leq |\mathcal{O}_k|$, $\boldsymbol{c}_k$ defined by (4.17) satisfies

$$|\boldsymbol{c}_k| \leq \hat{C}\, \rho_k^{-\frac{1}{2}} |\boldsymbol{v}|_{H^1(D_k)}. \tag{4.21}$$

Now,

$$|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(\Omega_{s,i})}^2 = \int_{\Omega_{s,i}} \left| \sum_{F \subset \Gamma_{ij}} \sum_{k=1}^{N_F} \boldsymbol{c}_k \nabla\, b_k \right|^2 = \sum_{T \in \mathcal{T}_i^h} \int_T \left| \sum_{F \subset \Gamma_{ij}} \sum_{k=1}^{N_F} \boldsymbol{c}_k \nabla\, b_k \right|^2.$$

But given an element $T$ in $\mathcal{T}_i^h$ there is at most a fixed (and small) number of indices $k$ such that $b_k|_T \neq 0$. Therefore

$$|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(T)}^2 = \int_T \left| \sum_{F \subset \Gamma_{ij}} \sum_{k=1}^{N_F} \boldsymbol{c}_k \nabla\, b_k \right|^2 \leq \hat{C} \sum_k |\boldsymbol{c}_k|^2 |b_k|_{H^1(T)}^2,$$

where the sum runs over all indices $k$ such that $b_k|_T \neq 0$. By substituting (4.21) into this inequality and estimating the norm of $b_k$, we easily derive that

$$|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(T)}^2 \leq \hat{C} \sum_k \frac{1}{\rho_k} \frac{1}{\rho_T^2} |T|\,|\boldsymbol{v}|_{H^1(D_k)}^2. \tag{4.22}$$

Since $\rho_k \rho_T^2$ has the same order as $|T|$, then by summing (4.22) over all $T$ in $\mathcal{T}_i^h$ and considering that there is at most a fixed (and small) number of repetitions in the $D_k$, we obtain the first inequality in (4.19). The second inequality in (4.19) follows in a similar manner from the representation (4.17) and bound (4.21).

### 4.3 A construction of $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ in $\Omega_s$. The 2-D case

Constructing a suitable correction solving (4.3) is fairly complex because it amounts to satisfying many conditions per interface. In 2-D, it can be done by following the sharp approach of Crouzeix and Thomée [23]. This is a general procedure that can also be used in 3-D, but as observed in Remark 4.1, it restricts the meshes. To bypass this restriction, we present a more ad hoc construction in 3-D, which is postponed to the Appendix: Sect. 7.2.

We slightly modify the underlying Scott and Zhang operator by asking that the modified operator $\tilde{S}^h$ be continuous at the end points of all interfaces $\Gamma_{ij}$ in $\Gamma_{ss} \cup \Gamma_{sd}$, and coincide elsewhere with $S^h$. This only concerns the set of points, say $\boldsymbol{a}_k$, $1 \le k \le P$, that do not lie on $\Gamma_s$, since $S^h(v)$ is necessarily zero on $\Gamma_s$. Note that these are all interior cross points of the triangulation of subdomains $\mathcal{T}_\Omega$. To achieve continuity at any such point $\boldsymbol{a}$, we pick an element $T_{\boldsymbol{a}}$ in $\Omega_s$ with vertex $\boldsymbol{a}$ and set

$$\tilde{S}^h(v)(\boldsymbol{a}) = \frac{1}{|T_{\boldsymbol{a}}|} \int_{T_{\boldsymbol{a}}} v.$$

Both corrections $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ and $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ are defined to satisfy respectively (4.1) and (4.3) with $S^h(\boldsymbol{v})$ replaced by $\tilde{S}^h(\boldsymbol{v})$. For $v \in H^1(\Omega)$, this does not change the approximation properties (4.16) of $\tilde{S}^h$ and hence (4.19), except that the domain of definition of $v$ is possibly a little larger than $\Omega_{s,i}$ or $\Omega_{s,j}$ near the end points of $\Gamma_{ij}$.

**Notation 4.1** From now on, we denote by $\tilde{\Omega}_{s,k}$ the possibly slightly enlarged domain of definition of $v$ in case $\tilde{S}^h(v)$ is used, with the convention that it coincides with $\Omega_{s,k}$ when $\tilde{S}^h(v)$ is not used.

Owing to the continuity of $\tilde{S}^h(\boldsymbol{v})$ at the end points of $\Gamma_{ij}$ and the fact that all previously made corrections vanish at the end points of the interfaces $\Gamma_{ij}$ in $\Gamma_{ss} \cup \Gamma_{sd}$, the integrand in the right-hand side of (4.3) also vanishes at the end points of $\Gamma_{ij}$. Thus the trace of $\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - \tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}}$ on $\Gamma_{ij}$ belongs to $H^{\frac{1}{2}}_{00}(\Gamma_{ij})^2$, see Sect. 1.1. We shall see that this property is crucial for estimating uniformly the gradient of $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$.

We propose to split the construction of $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ into two parts:

1. Construct an operator $\pi^h_s \in \mathcal{L}(H^{\frac{1}{2}}_{00}(\Gamma_{ij})^2, X^h_{j,\Gamma_{ij}})$ such that for all $\boldsymbol{\ell} \in H^{\frac{1}{2}}_{00}(\Gamma_{ij})^2$,

$$\begin{aligned} \forall \boldsymbol{\mu}_H \in \Lambda^H_s, \quad & \int_{\Gamma_{ij}} (\pi^h_s(\boldsymbol{\ell}) - \boldsymbol{\ell}) \cdot \boldsymbol{\mu}_H = 0, \\ & \pi^h_s(\boldsymbol{\ell}) = \boldsymbol{0}, \text{ at the end points of } \Gamma_{ij}, \\ & |\pi^h_s(\boldsymbol{\ell})|_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})} \le \kappa_s \, |\boldsymbol{\ell}|_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})}, \end{aligned} \tag{4.23}$$

   with a constant $\kappa_s$ that is independent of $h$, $H$, and the measure of $\Gamma_{ij}$.
2. Define $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ to be a suitable extension of $\pi^h_s(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - \tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}})$ inside $\Omega_{s,j}$ so that it satisfies all uniform properties required for the stability bound (3.25).

Note that $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ satisfies (4.3) by construction. Also, the existence of the operator $\pi^h_s$ implies that condition (3.8) holds.

We now discuss the construction of an operator satisfying (4.23). Uniformity with respect to the diameters of $\Gamma_{ij}$ and $\Omega_{s,j}$ is obtained by reverting to the reference subdomains. Let $\hat{\Omega}_j$ be associated with $\Omega_{s,j}$ by Hypothesis 4.2 and let $\hat{\Gamma} = F_j^{-1}(\Gamma_{ij})$. Let the image by $F_j^{-1}$ of $\mathcal{T}^h_{j,\Gamma_{ij}}$ be denoted by $\hat{\mathcal{T}}_j$. Similarly, set

$$\hat{X}_j = \{\hat{\boldsymbol{v}} = \boldsymbol{v}^h \circ F_j^{-1};\ \boldsymbol{v}^h \in X^h_{j,\Gamma_{ij}}, \boldsymbol{v}^h = \boldsymbol{0},\ \text{at the end points of } \Gamma_{ij}\},$$
$$\hat{\Lambda} = \{\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}^H \circ F_j^{-1};\ \boldsymbol{\mu}^H \in \Lambda_s^H\}.$$

Then, given $\hat{\boldsymbol{\ell}}$ in $H_{00}^{\frac{1}{2}}(\hat{\Gamma})^2$, if we construct $\hat{\pi}_j(\hat{\boldsymbol{\ell}})$ in $\hat{X}_j$ unique solution of

$$\forall \hat{\boldsymbol{\mu}} \in \hat{\Lambda}, \quad \int_{\hat{\Gamma}} \hat{\pi}_j(\hat{\boldsymbol{\ell}}) \cdot \hat{\boldsymbol{\mu}} = \int_{\hat{\Gamma}} \hat{\boldsymbol{\ell}} \cdot \hat{\boldsymbol{\mu}}, \tag{4.24}$$

and satisfying

$$|\hat{\pi}_j(\hat{\boldsymbol{\ell}})|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})} < \hat{C}|\hat{\boldsymbol{\ell}}|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})}, \tag{4.25}$$

with a constant $\hat{C}$ independent of $j$ and $\hat{\boldsymbol{\ell}}$, then by reverting to $\Omega_{s,j}$ and defining $\pi_j^h(\boldsymbol{\ell})$ by

$$\pi_j^h(\boldsymbol{\ell}) \circ F_j = \hat{\pi}_j(\boldsymbol{\ell} \circ F_j) = \hat{\pi}_j(\hat{\boldsymbol{\ell}}), \tag{4.26}$$

we shall obtain an operator satisfying (4.23). Indeed, uniqueness of the solution of (4.24) will guarantee that the mapping $\hat{\pi}_j$ is linear, and the inequality in (4.23) with the same constant will follow from the invariance of the $H_{00}^{\frac{1}{2}}$ seminorm by a homothety and a rigid body motion. But since explicit constructions of $\hat{\pi}_j$ are fairly technical, we postpone them to the Appendix: Sect. 7.1 and discuss the extension part (2) now. First we construct a suitable function $v$ in $H^1(\Omega_{s,j})$.

**Lemma 4.2** *For any $\Omega_{s,j} \subset \Omega_s$, any interface $\Gamma \subset \partial\Omega_{s,j}$ and any $\ell \in H_{00}^{\frac{1}{2}}(\Gamma)$ there exists a function $v = E(\ell) \in H^1(\Omega_{s,j})$ satisfying*

$$v|_\Gamma = \ell, \quad v|_{\partial\Omega_{s,j}\setminus\Gamma} = 0,$$

*the mapping $E$ is linear and there exists a constant $C$ independent of $\ell$, $v$, and the diameter of $\Gamma$ and $\Omega_{s,j}$ such that*

$$|v|_{H^1(\Omega_{s,j})} \leq C|\ell|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \quad \|v\|_{L^2(\Omega_{s,j})} \leq CA_j|\ell|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \tag{4.27}$$

*where $A_j$ is the diameter of $\Omega_{s,j}$.*

*Proof* With the notation of Hypothesis 4.2, let $\hat{\ell} = \ell \circ F_j$. Then $\hat{\ell}$ belongs to $H_{00}^{\frac{1}{2}}(\hat{\Gamma})$; hence it can be extended by zero to $\partial\hat{\Omega}_j$ and the extended function, say $\hat{\ell}_0$ belongs to $H^{\frac{1}{2}}(\partial\hat{\Omega}_j)$ with

$$|\hat{\ell}_0|_{H^{\frac{1}{2}}(\partial\hat{\Omega}_j)} \leq \hat{C}|\hat{\ell}|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})}. \tag{4.28}$$

There exists an extension operator $\hat{E} \in \mathcal{L}(H^{\frac{1}{2}}(\partial\hat{\Omega}_j); H^1(\hat{\Omega}_j))$ such that

$$\|\hat{E}\|_{\mathcal{L}(H^{\frac{1}{2}}(\partial\hat{\Omega}_j); H^1(\hat{\Omega}_j))} \leq \hat{C}. \tag{4.29}$$

Take $\hat{v} = \hat{E}(\hat{\ell}_0)$ and $v = \hat{v} \circ F_j^{-1}$. The mapping $\ell \mapsto v$ is linear, $v$ belongs to $H^1(\Omega_{s,j})$, its trace satisfies the statement of the lemma, and it remains to check (4.27). The following inequalities stem from (4.29), (4.28) and a straightforward scaling argument:

$$\begin{aligned} |v|_{H^1(\Omega_{s,j})} &\leq |\hat{v}|_{H^1(\hat{\Omega}_j)} \leq \hat{C}|\hat{\ell}_0|_{H^{\frac{1}{2}}(\partial\hat{\Omega}_j)} \leq \hat{C}|\hat{\ell}|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})} \leq \hat{C}|\ell|_{H_{00}^{\frac{1}{2}}(\Gamma)}, \\ \|v\|_{L^2(\Omega_{s,j})} &\leq \hat{C}A_j\|\hat{v}\|_{L^2(\hat{\Omega}_j)} \leq \hat{C}A_j|\ell|_{H_{00}^{\frac{1}{2}}(\Gamma)}. \end{aligned} \tag{4.30}$$

Next, we take $\Gamma = \Gamma_{ij}$ and define

$$\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) = S^h(\boldsymbol{w}), \quad \boldsymbol{w} = E(\pi_s^h(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - \tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}})), \tag{4.31}$$

where the Scott and Zhang interpolant $S^h$ is constructed so that $S^h(\boldsymbol{w})$ vanishes on $\partial\Omega_{s,j}\backslash\Gamma_{ij}$. The uniform approximation properties (4.16) of $S^h$, (4.25), and (4.27) imply, with constants $C$ independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$ and $\Omega_{s,j}$:

$$\|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,j})} \leq C|\boldsymbol{w}|_{H^1(\Omega_{s,j})} \leq C\left|\{\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - \tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}}\}\right|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}, \tag{4.32}$$

and it remains to derive a uniform bound for the last norm in terms of $\boldsymbol{v}$. This is the object of the next lemma.

**Lemma 4.3** *Let $\Gamma_{ij} \in \Gamma_{ss}$, $i < j$. There exists a constant $C$ independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$ and $\Omega_{s,j}$, such that*

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^2, \quad \left|\{\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - \tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}}\}\right|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \leq C|\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,i}\cup\tilde{\Omega}_{s,j})}, \tag{4.33}$$

*where $\tilde{\Omega}_{s,k}$ is defined in Notation 4.1.*

*Proof* Recall that the trace on $\Gamma_{ij}$ of $\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}}$ is $\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,i}} + \boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$. As $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ vanishes on $\partial\Omega_{s,i}\backslash\Gamma_{ij}$, then by passing to the reference subdomain $\hat{\Omega}_i$, we obtain

$$|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} = |\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}) \circ F_i|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})} \leq \hat{C}|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v}) \circ F_i|_{H^1(\hat{\Omega}_i)} \leq \hat{C}|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(\Omega_{s,i})}.$$

Therefore it is bounded owing to (4.19), with $\tilde{\Omega}_{s,i}$ instead of $\Omega_{s,i}$. Thus it suffices to consider the jump $[\tilde{S}^h(\boldsymbol{v})]$ through $\Gamma_{ij}$.

First, by writing $[\tilde{S}^h(\boldsymbol{v})] = [\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}]$, by passing to the reference subdomains $\hat{\Omega}_k$, $k = i, j$, and by using the approximation properties (4.16) of $S^h$, we derive

$$\begin{aligned} \left|(\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v})|_{\Omega_{s,k}}\right|_{H^{\frac{1}{2}}(\Gamma_{ij})} &= \left|\{(\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}) \circ F_k|_{\hat{\Omega}_k}\}\right|_{H^{\frac{1}{2}}(\hat{\Gamma})} \leq \hat{C}\|(\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}) \circ F_k\|_{H^1(\hat{\Omega}_k)} \\ &\leq \hat{C}\left(|\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}|_{H^1(\Omega_{s,k})}^2 + A_k^{-2}\|\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}\|_{L^2(\Omega_{s,k})}^2\right)^{\frac{1}{2}} \\ &\leq \hat{C}\left(|\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,k})}^2 + (h\,A_k^{-1})^2|\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,k})}^2\right)^{\frac{1}{2}}. \end{aligned} \tag{4.34}$$

Since $h < A_k$, this bounds the first part of the norm and it remains to estimate

$$\int_{\Gamma_{ij}} \frac{1}{d(\boldsymbol{x})}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2,$$

where $d(\boldsymbol{x})$ denotes the distance between $\boldsymbol{x}$ and the end points of $\Gamma_{ij}$. This part does not have the same immediate bound as the above correction because the jump does not vanish on the other boundaries of the subdomains. Let us first consider the case when $\Gamma_{ij}$ is a straight line. There is no loss of generality in assuming that $\Gamma_{ij}$ lies on the axis $y = 0$ with end point at the origin: $\Gamma_{ij} = ]0, L[$. Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = L$ denote the nodes of $\mathcal{T}_{i,\Gamma_{ij}}^h$ and exceptionally set $h_n = x_{n+1} - x_n$. Then $d(\boldsymbol{x}) = \mathrm{Min}(x, L - x)$ and since

$$\int_{\Gamma_{ij}} \frac{1}{d(\boldsymbol{x})}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2 \leq \int_0^L \frac{1}{x}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2 + \int_0^L \frac{1}{L - x}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2,$$

it suffices to examine the first term:

$$\int_0^L \frac{1}{x}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2 = \sum_{n=0}^{N} \int_{x_n}^{x_{n+1}} \frac{1}{x}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2.$$

For any $n \geq 1$, we have $x_n \geq h_{n-1}$, and since $[\tilde{S}^h(\boldsymbol{v})]$ belongs to $\left(H^1(0, L) \cap \mathcal{C}^0(0, L)\right)^2$, we can write

$$\int_{x_n}^{x_{n+1}} \frac{1}{x}\left|[\tilde{S}^h(\boldsymbol{v})]\right|^2 \leq \frac{1}{x_n}\int_{x_n}^{x_{n+1}}\left([\tilde{S}^h(\boldsymbol{v})(x_n)] + \int_{x_n}^{x}\frac{d}{dt}[\tilde{S}^h(\boldsymbol{v})(t)]dt\right)^2 dx$$
$$\leq \frac{2h_n}{h_{n-1}}\left([\tilde{S}^h(\boldsymbol{v})(x_n)]^2 + \frac{h_n}{2}\left\|\frac{d}{dx}[\tilde{S}^h(\boldsymbol{v})]\right\|^2_{L^2(x_n,x_{n+1})}\right). \quad (4.35)$$

An equivalence of norms yields

$$\frac{2h_n}{h_{n-1}}\left|[\tilde{S}^h(\boldsymbol{v})(x_n)]\right|^2 \leq \frac{2h_n}{h_{n-1}}\|[\tilde{S}^h(\boldsymbol{v})]\|^2_{L^\infty(x_n,x_{n+1})} \leq \hat{C}\frac{1}{h_{n-1}}\|[\tilde{S}^h(\boldsymbol{v})]\|^2_{L^2(x_n,x_{n+1})}$$
$$= \hat{C}\frac{1}{h_{n-1}}\|(\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,i}} - \boldsymbol{v}) - (\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}} - \boldsymbol{v})\|^2_{L^2(x_n,x_{n+1})}.$$

By arguing as in the proof of Lemma 4.1, we obtain

$$\|\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,i}} - \boldsymbol{v}\|_{L^2(x_n,x_{n+1})} \leq \hat{C}h_T^{\frac{1}{2}}|\boldsymbol{v}|_{H^1(\tilde{\Delta}_{n,i})}, \quad (4.36)$$

where $T$ is the element of $\mathcal{T}_i^h$ adjacent to $[x_n, x_{n+1}]$ and $\tilde{\Delta}_{n,k}$ denotes the set of elements in $\Omega_{s,k}$ used in defining $\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,k}}$ on $[x_n, x_{n+1}]$.

The estimation of the second term is more technical because now $\tilde{S}^h(\boldsymbol{v})$ is constructed on $\mathcal{T}_j^h$ whereas $[x_n, x_{n+1}]$ is the side of an element of $\mathcal{T}_i^h$. Let $\{T_\ell\}$ denote the set of elements of $\mathcal{T}_j^h$ that intersect $[x_n, x_{n+1}]$. The argument of Lemma 4.1 gives:

$$\|\tilde{S}^h(\boldsymbol{v})|_{\Omega_{s,j}} - \boldsymbol{v}\|_{L^2(x_n,x_{n+1})} \leq \hat{C}\sum_\ell h_{T_\ell}^{\frac{1}{2}}|\boldsymbol{v}|_{H^1(\tilde{\Delta}_{n,\ell})}. \quad (4.37)$$

Then combining (4.36) and (4.37), using the regularity of the triangulation and Hypothesis 4.1, we obtain

$$\frac{2h_n}{h_{n-1}}\left|[\tilde{S}^h(\boldsymbol{v})(x_n)]\right|^2 \leq \hat{C}\left(|\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{n,i})} + |\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{n,j})}\right). \quad (4.38)$$

Similarly,

$$\frac{h_n^2}{h_{n-1}}\left\|\frac{d}{dx}[\tilde{S}^h(\boldsymbol{v})]\right\|^2_{L^2(x_n,x_{n+1})} \leq \hat{C}\frac{1}{h_{n-1}}\|[\tilde{S}^h(\boldsymbol{v})]\|^2_{L^2(x_n,x_{n+1})}$$
$$= \hat{C}\frac{1}{h_{n-1}}\|[\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}]\|^2_{L^2(x_n,x_{n+1})}$$
$$\leq \hat{C}\left(|\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{n,i})} + |\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{n,j})}\right). \quad (4.39)$$

When $n = 0$, as $[\tilde{S}^h(\boldsymbol{v})(0)] = \mathbf{0}$, there only remains the second term in the first line of (4.35), and we immediately derive

$$\int_0^{x_1} \frac{1}{x} \big|[\tilde{S}^h(\boldsymbol{v})]\big|^2 = \int_0^{x_1} \frac{1}{x} \Big( \int_0^{x} \frac{d}{dt} \big([\tilde{S}^h(\boldsymbol{v})]\big) dt \Big)^2 dx$$
$$\leq \int_0^{x_1} \Big\| \frac{d}{dx} \big([\tilde{S}^h(\boldsymbol{v})]\big) \Big\|^2_{L^2(0,x)} \leq h_1 \Big\| \frac{d}{dx} \big([\tilde{S}^h(\boldsymbol{v})]\big) \Big\|^2_{L^2(0,x_1)}$$
$$\leq \hat{C} \big( |\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{0,i})} + |\boldsymbol{v}|^2_{H^1(\tilde{\Delta}_{0,j})} \big).$$

Finally, when $\Gamma_{ij}$ is a polygonal line, the above argument is valid on the segments that share an end point with $\Gamma_{ij}$ and is simpler on the other segments as $d(\boldsymbol{x})$ is not small compared to $h$.

Inequalities (4.32) and (4.33) imply the following.

**Corollary 4.1** *Let $\Gamma_{ij} \in \Gamma_{ss}$, $i < j$. There exists a constant $C$ independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$ and $\Omega_{s,j}$, such that*

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^2, \quad \|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{H^1(\Omega_{s,j})} \leq C|\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,i} \cup \tilde{\Omega}_{s,j})}. \tag{4.40}$$

**Corollary 4.2** *The approximation operator $\Theta_s^h$ constructed in Sect. 4.1 with the above corrections $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ and $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ satisfies assumption* (3.25).

*Proof* Since $\Theta_s^h(\boldsymbol{v})$ is constructed by correcting the Scott-Zhang interpolant $\tilde{S}^h$ with $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ and $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$, assumption (3.25) follows from (4.16), (4.19), (4.40), and the Poincaré inequality (1.1).

*Remark 4.1* The above correction $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ has a straightforward extension in 3-D, but its use is limited because now it requires continuity of $\tilde{S}_h(\boldsymbol{v})$ on the *edges* of subdomains; thus the meshes must match on these edges. This is not required by the correction defined in Sect. 7.2.

### 4.4 A construction of $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ in $\Omega_d$

Here we construct a correction $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ in $\Omega_d$ satisfying (4.8) and suitable continuity bounds that are needed to establish the stability estimate (3.33). Recall that the existence of $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ relies on (3.9). In the construction below we directly show that (3.9) holds for a wide range of mesh configurations.

Let $\boldsymbol{v}$ be given in $H_0^1(\Omega)^n$. Recall that the mixed approximation operator $R^h$ defined in each $\Omega_{d,i}$ takes its values in $X_d^h$ and satisfies (4.6) on each $\Gamma_{ij} \subset \partial\Omega_{d,k}$, $1 \leq k \leq M_d$, and (4.7) in each $\Omega_{d,i}$, $1 \leq i \leq M_d$:

$$\forall \boldsymbol{v}^h \in X_d^h, \quad \int_{\Gamma_{ij}} \boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} \left(R^h(\boldsymbol{v})|_{\Omega_{d,k}} - \boldsymbol{v}\right) \cdot \boldsymbol{n}_{ij} = 0,$$

$$\forall w^h \in W_d^h, \quad \int_{\Omega_{d,i}} w^h \mathrm{div}\left(R^h(\boldsymbol{v}) - \boldsymbol{v}\right) = 0.$$

Furthermore there exists a constant $C$ independent of $h$ and the geometry of $\Omega_{d,i}$, such that

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^n, \quad \|R^h(\boldsymbol{v})\|_{H(\mathrm{div};\Omega_{d,i})} \le C\|\boldsymbol{v}\|_{H^1(\Omega_{d,i})}, \quad 1 \le i \le M_d. \tag{4.41}$$

This is easily established by observing that the moments defining the degrees of freedom of $R^h(\boldsymbol{v})$ are invariant by homothety and rigid-body motion; in particular the normal vector is preserved. In addition, it satisfies (3.6):

$$\forall i, 1 \le i \le M_d, \ \forall \boldsymbol{v}^h \in X_{d,i}^h, \ \mathrm{div}\, \boldsymbol{v}^h \in W_{d,i}^h.$$

The above properties also imply (3.5): for all $i$, $1 \le i \le M_d$,

$$\inf_{w^h \in W_{0,d,i}^h} \sup_{\boldsymbol{v}^h \in X_{0,d,i}^h} \frac{\int_{\Omega_{d,i}} w^h \mathrm{div}\, \boldsymbol{v}^h}{\|\boldsymbol{v}^h\|_{H(\mathrm{div};\Omega_{d,i})} \|w^h\|_{L^2(\Omega_{d,i})}} \ge \beta_d^\star,$$

with a constant $\beta_d^\star > 0$ independent of $h$ and $A_i$.

Now, let $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$; by analogy with the situation in the Stokes region, we denote by $X_{d,j,\Gamma_{ij}}^h$ the trace space of $X_{d,j}^h$ on $\Gamma_{ij}$. Following [5], we define the space of normal traces

$$X_{j,\Gamma_{ij}}^{\boldsymbol{n}} = \{\boldsymbol{w} \cdot \boldsymbol{n}_{ij};\ \boldsymbol{w} \in X_{d,j,\Gamma_{ij}}^h\},$$

and the orthogonal projection $Q_{j,\Gamma_{ij}}^h$ from $L^2(\Gamma_{ij})$ into $X_{j,\Gamma_{ij}}^{\boldsymbol{n}}$. Then we make the following assumption: There exists a constant $C$, independent of $H$, $h$, $i$, $j$, and the diameters of $\Gamma_{ij}$ and $\Omega_{d,j}$, such that

$$\forall \mu^H \in \Lambda_d^H, \forall \mu^H \in \Lambda_{sd}^H, \quad \|\mu^H\|_{L^2(\Gamma_{ij})} \le C\|Q_{j,\Gamma_{ij}}^h(\mu^H)\|_{L^2(\Gamma_{ij})}. \tag{4.42}$$

It is shown in [60] that (4.42) holds for both continuous and discontinuous mortar spaces, if the mortar grid $\mathcal{T}_{ij}^H$ is a coarsening by two of $\mathcal{T}_{j,\Gamma_{ij}}^h$. A similar inequality for more general grid configurations is shown in [52]. Formula (4.42) implies that the projection $Q_{j,\Gamma_{ij}}^h$ is an isomorphism from the restriction of $\Lambda_{sd}^H$, respectively $\Lambda_d^H$, to $\Gamma_{ij}$, say $\Lambda_{sd,ij}^H$ respectively $\Lambda_{d,ij}^H$, onto its image in $X_{j,\Gamma_{ij}}^{\boldsymbol{n}}$, and the norm of its inverse is bounded by $C$. Then a standard algebraic argument shows that its dual operator, namely the orthogonal projection from $X_{j,\Gamma_{ij}}^{\boldsymbol{n}}$ into $\Lambda_{sd,ij}^H$, respectively $\Lambda_{d,ij}^H$, is also an isomorphism from the orthogonal complement in $X_{j,\Gamma_{ij}}^{\boldsymbol{n}}$ of the projection's kernel onto $\Lambda^H_{sd,ij}$, respectively $\Lambda^H_{d,ij}$, and the norm of its inverse is also bounded by $C$. As a consequence, for each $f \in L^2(\Gamma_{ij})$, there exists $\boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} \in X^{\boldsymbol{n}}_{j,\Gamma_{ij}}$ such that

$$\forall \mu_H \in \Lambda^H_d, \forall \mu_H \in \Lambda^H_{sd}, \quad \int_{\Gamma_{ij}} \mu^H \boldsymbol{v}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} f \mu^H,$$

and there exists a constant $C$ independent of $h$, and the diameter of $\Gamma_{ij}$, such that

$$\|\boldsymbol{v}^h \cdot \boldsymbol{n}_{ij}\|_{L^2(\Gamma_{ij})} \le C\|f\|_{L^2(\Gamma_{ij})}.$$

This implies that (3.9) holds. Furthermore, the solution $\boldsymbol{v}^h \cdot \boldsymbol{n}_{ij}$ is unique in the orthogonal complement of the projection's kernel and by virtue of this uniqueness, $\boldsymbol{v}^h \cdot \boldsymbol{n}_{ij}$ depends linearly on $f$. This result permits to partially solve (4.8).

**Lemma 4.4** *Let $\boldsymbol{v} \in H^1_0(\Omega)^n$. Under assumption (4.42), for each $\Gamma_{ij} \in \Gamma_{dd} \cup \Gamma_{sd}$, there exists $\boldsymbol{w}^h \cdot \boldsymbol{n}_{ij} \in X^{\boldsymbol{n}}_{j,\Gamma_{ij}}$ such that*

$$\begin{aligned} &\forall \mu^H \in \Lambda^H_d, \quad \int_{\Gamma_{ij}} \mu^H \boldsymbol{w}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \mu^H \big[R^h(\boldsymbol{v}) \cdot \boldsymbol{n}\big], \\ &\|\boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}\|_{L^2(\Gamma_{ij})} \le C\big\|\big[R^h(\boldsymbol{v}) \cdot \boldsymbol{n}\big]\big\|_{L^2(\Gamma_{ij})}, \end{aligned} \tag{4.43}$$

$$\begin{aligned} &\forall \mu^H \in \Lambda^H_{sd}, \quad \int_{\Gamma_{ij}} \mu^H \boldsymbol{w}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \mu^H \big(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\big) \cdot \boldsymbol{n}_{ij}, \\ &\|\boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}\|_{L^2(\Gamma_{ij})} \le C\big\|\big(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\big) \cdot \boldsymbol{n}_{ij}\big\|_{L^2(\Gamma_{ij})}, \end{aligned} \tag{4.44}$$

*with the constant $C$ of (4.42). The mapping $\boldsymbol{v} \mapsto \boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}$ is linear.*

Lemma 4.4 constructs a normal trace $\boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}$ on $\Gamma_{ij}$ and we must extend it inside $\Omega_{d,j}$. To this end, let $\ell^h \in L^2(\partial\Omega_{d,j})$ be the extension of $\boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}$ by zero on $\partial\Omega_{d,j}$. Next, we solve the problem: Find $q \in H^1(\Omega_{d,j}) \cap L^2_0(\Omega_{d,j})$ such that

$$\Delta q = 0 \quad \text{in } \Omega_{d,j}, \quad \frac{\partial q}{\partial \boldsymbol{n}_j} = \ell^h \quad \text{on } \partial\Omega_{d,j}. \tag{4.45}$$

**Lemma 4.5** *Problem (4.45) has one and only one solution $q \in H^{\frac{3}{2}}(\Omega_{d,j}) \cap L^2_0(\Omega_{d,j})$ and*

$$\begin{aligned} &|q|_{H^1(\Omega_{d,j})} \le C\sqrt{A_j}\|[R^h(\boldsymbol{v}) \cdot \boldsymbol{n}]\|_{L^2(\Gamma_{ij})}, \\ &|q|_{H^{\frac{3}{2}}(\Omega_{d,j})} \le C\|[R^h(\boldsymbol{v}) \cdot \boldsymbol{n}]\|_{L^2(\Gamma_{ij})}, \quad \Gamma_{ij} \in \Gamma_{dd}, \\ &|q|_{H^1(\Omega_{d,j})} \le C\sqrt{A_j}\big\|\big(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\big) \cdot \boldsymbol{n}_{ij}\big\|_{L^2(\Gamma_{ij})}, \end{aligned} \tag{4.46}$$

$$|q|_{H^{\frac{3}{2}}(\Omega_{d,j})} \leq C \| \big(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}\big) \cdot \boldsymbol{n}_{ij} \|_{L^2(\Gamma_{ij})}, \ \Gamma_{ij} \in \Gamma_{sd}, \tag{4.47}$$

*with constants $C$ independent of $h$, $H$, $q$, $i$, $j$, and the diameters of $\Gamma_{ij}$ and $\Omega_{d,j}$.*

*Proof* Since $\mu^H$ contains the constant functions, $\ell^h$ satisfies on $\Gamma_{ij} \in \Gamma_{dd}$, using (4.43) and (4.6),

$$\int_{\partial\Omega_{d,j}} \ell^h = \int_{\Gamma_{ij}} \boldsymbol{w}^h \cdot \boldsymbol{n}_{ij} = \int_{\Gamma_{ij}} \big[(R^h(\boldsymbol{v}) - \boldsymbol{v}) \cdot \boldsymbol{n}\big] = 0,$$

and this implies the unique solvability of (4.45). A similar argument holds on $\Gamma_{ij} \in \Gamma_{sd}$ using (4.44) and (3.24). In order to check (4.46), we pass to the reference subdomain $\hat{\Omega}_j$ associated with $\Omega_{d,j}$, let $\hat{\boldsymbol{n}}_j$ be its exterior unit normal vector, $\hat{\Gamma} = F_j^{-1}(\Gamma_{ij})$, $\hat{q} = q \circ F_j$ and $\hat{\ell} = \ell^h \circ F_j = (\boldsymbol{w}^h \circ F_j) \cdot \hat{\boldsymbol{n}}_j$; then $\hat{q} \in H^1(\hat{\Omega}_j) \cap L^2_0(\hat{\Omega}_j)$ is the unique solution of

$$\hat{\Delta}\, \hat{q} = 0 \quad \text{in } \hat{\Omega}_j, \quad \frac{\hat{\partial} \hat{q}}{\hat{\partial} \hat{\boldsymbol{n}}_j} = A_j \hat{\ell} \quad \text{on } \partial\hat{\Omega}_j.$$

As $\hat{\ell}$ is in $L^2(\partial\hat{\Omega}_j)$, it follows from [43] that $\hat{q}$ belongs to $H^{\frac{3}{2}}(\hat{\Omega}_j)$ and

$$\|\hat{q}\|_{H^{\frac{3}{2}}(\hat{\Omega}_j)} \leq \hat{C} A_j \|\hat{\ell}\|_{L^2(\partial\hat{\Omega}_j)},$$

with a constant $\hat{C}$ that only depends on the geometry of $\hat{\Omega}_j$. Then (4.46) is a direct consequence of (4.43) and the following bounds:

$$\|\ell^h\|_{L^2(\Gamma_{ij})} = A_j^{\frac{n-1}{2}} \|\hat{\ell}\|_{L^2(\partial\hat{\Omega}_j)}, \quad |q|_{H^1(\Omega_{d,j})} = A_j^{\frac{n}{2}-1} |\hat{q}|_{H^1(\hat{\Omega}_j)},$$
$$|q|_{H^{\frac{3}{2}}(\Omega_{d,j})} = A_j^{\frac{n-3}{2}} |\hat{q}|_{H^{\frac{3}{2}}(\hat{\Omega}_j)}.$$

The argument for (4.47) is similar, using (4.44).

Now define $\boldsymbol{c} = \nabla q$ in $\Omega_{d,j}$. Then $\boldsymbol{c}$ belongs to $H(\text{div}; \Omega_{d,j}) \cap H^{\frac{1}{2}}(\Omega_{d,j})^n$ and $\text{div}\, \boldsymbol{c} = 0$. Therefore $R^h(\boldsymbol{c})$ is well defined [18] and satisfies the approximation property for divergence-free functions [48]

$$\|\boldsymbol{c} - R^h(\boldsymbol{c})\|_{L^2(\Omega_{d,j})} \leq C h^r |\boldsymbol{c}|_{H^r(\Omega_{d,j})}, \quad 0 < r \leq \frac{1}{2}, \tag{4.48}$$

with a constant $C$ independent of $h$, $j$, and the diameter of $\Omega_{d,j}$. We are now ready to define the correction $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$. In particular, take $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v}) = R^h(\boldsymbol{c})$ applied in $\Omega_{d,j}$. Note that $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ belongs to $X^h_{d,j}$, and (4.7) and (4.6) imply that $\text{div}\, \boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v}) = 0$

in $\Omega_{d,j}$ and $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v}) \cdot \boldsymbol{n}_j = \ell^h = \boldsymbol{w}^h \cdot \boldsymbol{n}_{ij}$ on $\Gamma_{ij}$. Therefore (4.43) and (4.44) imply that $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ satisfies (4.8). Furthermore, (4.48) yields

$$\begin{aligned}\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} &\leq \|R^h(\boldsymbol{c}) - \boldsymbol{c}\|_{L^2(\Omega_{d,j})} + \|\boldsymbol{c}\|_{L^2(\Omega_{d,j})} \\ &\leq C h_j^{\frac{1}{2}} |\boldsymbol{c}|_{H^{\frac{1}{2}}(\Omega_{d,j})} + \|\boldsymbol{c}\|_{L^2(\Omega_{d,j})},\end{aligned}$$

with a constant $C$ independent of the geometry of $\Omega_{d,j}$. Considering that $h_j \leq A_j$, Lemma 4.5 gives

$$\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} \leq C\sqrt{A_j}\|[R^h(\boldsymbol{v}) \cdot \boldsymbol{n}]\|_{L^2(\Gamma_{ij})}, \quad \Gamma_{ij} \in \Gamma_{dd}, \tag{4.49}$$

$$\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} \leq C\sqrt{A_j}\|(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}) \cdot \boldsymbol{n}_{ij}\|_{L^2(\Gamma_{ij})}, \quad \Gamma_{ij} \in \Gamma_{sd}, \tag{4.50}$$

with a constant $C$ independent of $h$, $H$, $\boldsymbol{v}$, $i$, $j$, and the diameters of $\Gamma_{ij}$ and $\Omega_{d,j}$.

**Corollary 4.3** *The approximation operator $\Pi^h_d$ constructed in Sect. 4.1 with corrections $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ described above satisfies assumption* (3.33).

*Proof* Since $\Pi^h_d$ is a correction of the mixed interpolant $R^h$, which is stable in the sense of (4.41), it remains to bound $\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{H(\mathrm{div};\Omega_{d,j})}$. By construction $\mathrm{div}\, \boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v}) = 0$. In the following we will make use of the trace inequality [38]

$$\forall \varphi \in H^1(\Omega_{d,j}), \quad \|\varphi\|_{L^2(\Gamma_{ij})} \leq C\big(A_j^{-\frac{1}{2}}\|\varphi\|_{L^2(\Omega_{d,j})} + A_j^{\frac{1}{2}}|\varphi|_{H^1(\Omega_{d,j})}\big). \tag{4.51}$$

For $\Gamma_{ij} \in \Gamma_{dd}$, using (4.49), (4.6), and (4.51), we have

$$\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} \leq C\sqrt{A_j}(\|\boldsymbol{v} \cdot \boldsymbol{n}_i\|_{L^2(\Gamma_{ij})} + \|\boldsymbol{v} \cdot \boldsymbol{n}_j\|_{L^2(\Gamma_{ij})}) \leq C\|\boldsymbol{v}\|_{H^1(\Omega_{d,i} \cup \Omega_{d,j})},$$

using also that $A_j \leq 1$. For $\Gamma_{ij} \in \Gamma_{sd}$, we employ (4.50) and (4.1) to obtain

$$\begin{aligned}\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} &\leq C\sqrt{A_j}\big(\|(\boldsymbol{v} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}) \cdot \boldsymbol{n}_j\|_{L^2(\Gamma_{ij})} \\ &\quad + \|(\boldsymbol{v} - (S^h(\boldsymbol{v}) + \boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}))|_{\Omega_{s,i}}) \cdot \boldsymbol{n}_i\|_{L^2(\Gamma_{ij})}\big),\end{aligned}$$

with $\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})$ defined in (4.1) and constructed in Sect. 4.2, and if necessary $S^h$ replaced by $\tilde{S}^h$. For the first term in the right-hand side, using (4.6), we have

$$\|(\boldsymbol{v} - R^h(\boldsymbol{v})|_{\Omega_{d,j}}) \cdot \boldsymbol{n}_j\|_{L^2(\Gamma_{ij})} \leq \|\boldsymbol{v} \cdot \boldsymbol{n}_j\|_{L^2(\Gamma_{ij})} \leq C\|\boldsymbol{v}\|_{H^1(\Omega_{d,j})}.$$

For the second term on the right, using (4.51) and the approximation property (4.16) of $S^h$ or $\tilde{S}^h$ we have

$$
\begin{aligned}
\|(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}) \cdot \boldsymbol{n}_i\|_{L^2(\Gamma_{ij})} &\le C(A_i^{-\frac{1}{2}} \|\boldsymbol{v} - S^h(\boldsymbol{v})\|_{L^2(\Omega_{s,i})} + A_i^{\frac{1}{2}} |\boldsymbol{v} - S^h(\boldsymbol{v})|_{H^1(\Omega_{s,i})}) \\
&\le C(A_i^{-\frac{1}{2}} h_i |\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,i})} + A_i^{\frac{1}{2}} |\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,i})}) \\
&\le C|\boldsymbol{v}|_{H^1(\tilde{\Omega}_{s,i})},
\end{aligned}
$$

using that $h_i < A_i \le 1$, with a similar bound for $\|\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\tilde{\Omega}_{s,i})}$, in view of (4.19). The proof is completed by combining all bounds and using the Poincaré inequality (1.1).

## 5 Error analysis

In this section we establish *a priori* error estimates for our method. Let us assume that the finite element spaces $X^h_s$ and $W^h_s$ in $\Omega_s$ contain at least polynomials of degree $r_s$ and $r_s - 1$, respectively. Let $X^h_d$ and $W^h_d$ in $\Omega_d$ contain at least polynomials of degree $r_d$ and $l_d$, respectively. In all cases under consideration, either $l_d = r_d$ or $l_d = r_d - 1$. Let $\Lambda^H_{sd}$, $\Lambda^H_d$, and $\Lambda^H_s$ contain at least polynomials of degree $r_{sd}$, $r_{dd}$, and $r_{ss}$, respectively. In the analysis we will make use of the following well known approximation properties of the mixed interpolant $R^h$ [18,34]: for all $\boldsymbol{v} \in H^t(\Omega)^n$, $t \ge 1$, there exists a constant $C$ that depends only on $r_d$, $l_d$, $t$, and the shape regularity of $\mathcal{T}^h_i$, such that for all $T$ in $\mathcal{T}^h_i$

$$\|\boldsymbol{v} - R^h(\boldsymbol{v})\|_{L^2(T)} \le Ch^r|\boldsymbol{v}|_{H^r(T)}, \quad 1 \le r \le \min(r_d + 1, t), \tag{5.1}$$

$$\|\mathrm{div}(\boldsymbol{v} - R^h(\boldsymbol{v}))\|_{L^2(T)} \le Ch^r|\mathrm{div}\, \boldsymbol{v}|_{H^r(T)}, \quad 0 \le r \le \min(l_d + 1, t - 1), \tag{5.2}$$

$$\|(\boldsymbol{v} - R^h(\boldsymbol{v})|_{\Omega_{d,i}}) \cdot \boldsymbol{n}\|_{L^2(T')} \le Ch^r|\boldsymbol{v} \cdot \boldsymbol{n}|_{H^r(T')}, \quad 0 \le r \le \min\left(r_d + 1, t - \frac{1}{2}\right), \tag{5.3}$$

where $T'$ denotes an arbitrary element of the trace $\mathcal{T}^h_{i,\Gamma_{ij}}$ on $\Gamma_{ij}$. We begin with the following approximation result for the operator $\Pi^h$ defined in Lemma 3.6.

**Lemma 5.1** *Under the assumptions of Lemma 3.6, the operator* $\Pi^h \in \mathcal{L}(H^1_0(\Omega)^n; V^h)$ *satisfies for all* $\boldsymbol{v} \in \left(H^t(\Omega) \cap H^1_0(\Omega)\right)^n$, $t \ge 1$,

$$\|\!|\boldsymbol{v} - \Pi^h(\boldsymbol{v})|\!\|_{X_s} \le Ch^r|\boldsymbol{v}|_{H^{r+1}(\Omega)}, \quad 0 \le r \le \min(r_s, t - 1), \tag{5.4}$$

$$
\begin{aligned}
&\|\!|\boldsymbol{v} - \Pi^h(\boldsymbol{v})|\!\|_{X_d} \le C\left(h^r\|\boldsymbol{v}\|_{H^{r+\frac{1}{2}}(\Omega)} + h^q\|\mathrm{div}\, \boldsymbol{v}\|_{H^q(\Omega)} + h^s\|\boldsymbol{v}\|_{H^{s+1}(\Omega)}\right), \\
&\frac{1}{2} \le r \le \min\left(r_d + 1, t - \frac{1}{2}\right),\ 0 \le q \le \min(l_d + 1, t - 1),\ 0 \le s \le \min(r_s, t - 1).
\end{aligned}
\tag{5.5}
$$

*Proof* To show (5.4), recall that $\Pi^h(\boldsymbol{v})|_{\Omega_s} = \Pi^h_s(\boldsymbol{v})$, $\Pi^h_s(\boldsymbol{v}) = \Theta^h_s(\boldsymbol{v}) + \mathbf{c}^h_s(\boldsymbol{v})$, where $\mathbf{c}^h_s(\boldsymbol{v})$ is defined in (3.29), and $\Theta^h_s$ is a correction of the Scott-Zhang operator constructed in Sect. 4. The triangle inequality and (3.30) imply that

$$\|\boldsymbol{v} - \Pi_s^h(\boldsymbol{v})\|_{H^1(\Omega_{s,i})} \le C\|\boldsymbol{v} - \Theta_s^h(\boldsymbol{v})\|_{H^1(\Omega_{s,i})}.$$

Recall that $\Theta_s^h(\boldsymbol{v})$ is constructed by correcting the Scott-Zhang interpolant $S^h$ or $\tilde{S}^h$ with $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ and $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$. Since $\|\boldsymbol{v} - S^h(\boldsymbol{v})\|_{H^1(\Omega_{s,i})}$ is bounded optimally in (4.16), it remains to bound $\|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,i})}$ and $\|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,j})}$. The bound on $\|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,i})}$ follows from a simple modification of the argument in the proof of Lemma 4.1. More precisely, by using (4.15) with $0 \le r \le \min(r_s, t-1)$, (4.20) can be modified as

$$\left| \int_{\Delta_k} \left(\boldsymbol{v} - S^h(\boldsymbol{v})|_{\Omega_{s,i}}\right) \right| \le C h_T^r |\Delta_k|\, \rho_k^{-\frac{1}{2}} |\boldsymbol{v}|_{H^{r+1}(D_k)}.$$

Propagating the above bound through the proof gives

$$\|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})\|_{L^2(\Omega_{s,i})} + h|\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(\Omega_{s,i})} \le C h^{r+1} |\boldsymbol{v}|_{H^{r+1}(\tilde{\Omega}_{s,i})}, \quad 0 \le r \le \min(r_s, t-1).$$

We proceed with the bound on $\|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,j})}$. Considering first the 2-D case presented in Sect. 4.3, inequality (4.34) in the proof of Lemma 4.3 can be modified, using the approximation property (4.16) of $\tilde{S}^h$, as

$$|\left(\tilde{S}^h(\boldsymbol{v}) - \boldsymbol{v}\right)|_{\Omega_{s,k}}|_{H^{\frac{1}{2}}(\Gamma_{ij})} \le C h^r |\boldsymbol{v}|_{H^{r+1}(\tilde{\Omega}_{s,k})}, \quad 0 \le r \le \min(r_s, t-1),$$

and (4.38) and (4.39) can be modified, using the approximation property (4.15) of $\tilde{S}^h$, as

$$\int_{x_n}^{x_{n+1}} \frac{1}{x} \left|[\tilde{S}^h(\boldsymbol{v})]\right|^2 \le \hat{C} h^{2r} |\boldsymbol{v}|^2_{H^{r+1}(\tilde{\Delta}_{n,i} \cup \tilde{\Delta}_{n,j})}, \quad 0 \le r \le \min(r_s, t-1),$$

implying

$$\|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{H^1(\Omega_{s,j})} \le C h^r |\boldsymbol{v}|_{H^{r+1}(\tilde{\Omega}_{s,i} \cup \tilde{\Omega}_{s,j})}, \quad 0 \le r \le \min(r_s, t-1).$$

We next consider the 3-D case presented in Sect. 7.2. Modifying the proof of Lemma 7.4 in a similar way gives

$$\|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{L^2(\Omega_{s,j})} + h|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(\Omega_{s,j})} \le C h^{r+1} |\boldsymbol{v}|_{H^{r+1}(\Omega_{s,i} \cup \Omega_{s,j})}, \quad 0 \le r \le \min(r_s, t-1).$$

A combination of the above inequalities gives (5.4).

We continue with the proof of (5.5). Recall that $\Pi^h(\boldsymbol{v})|_{\Omega_d} = \Pi_d^h(\boldsymbol{v})$, where $\Pi_d^h(\boldsymbol{v}) = R^h(\boldsymbol{v}) + \boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ is a corrected mixed interpolant with a correction satisfying (4.8). Since $\|\boldsymbol{v} - R^h(\boldsymbol{v})\|_{H(\mathrm{div};\Omega_{d,i})}$ is bounded optimally in (5.1)–(5.2), it remains to bound $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ constructed in Sect. 4.4. By construction $\operatorname{div} \boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v}) = 0$, so we only need to control $\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})}$. Consider first $\Gamma_{ij} \in \Gamma_{dd}$. Using (5.3), bound (4.49) gives

$$\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} \le C A_j^{\frac{1}{2}} h^r |\boldsymbol{v}|_{H^r(\Gamma_{ij})} \le C h^r \|\boldsymbol{v}\|_{H^{r+\frac{1}{2}}(\Omega_{d,i} \cup \Omega_{d,j})},$$
$$0 < r \le \min\left(r_d + 1, t - \frac{1}{2}\right),$$

where we have used the trace inequality [38]

$$|\varphi|_{H^r(\Gamma_{ij})} \le A_i^{-\frac{1}{2}} \|\varphi\|_{H^{r+\frac{1}{2}}(\Omega_{d,i})}, \quad r > 0. \tag{5.6}$$

For $\Gamma_{ij} \in \Gamma_{sd}$, we modify the argument in Corollary 4.3 to use the full approximation properties (4.16) of $S^h$ or $\tilde{S}^h$ to obtain

$$\|\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})\|_{L^2(\Omega_{d,j})} \le C\left(h^r \|\boldsymbol{v}\|_{H^{r+\frac{1}{2}}(\tilde{\Omega}_{s,i} \cup \tilde{\Omega}_{d,j})} + h^s \|\boldsymbol{v}\|_{H^{s+1}(\tilde{\Omega}_{s,i} \cup \tilde{\Omega}_{d,j})}\right),$$
$$0 < r \le \min\left(r_d + 1, t - \frac{1}{2}\right), \quad 0 \le s \le \min(r_s, t - 1).$$

A combination of the above inequalities completes the proof of (5.5).

Next, we need to approximate the functions for the pressure. For any $q \in L^2(\Omega_i)$, let $\mathcal{P}^h q$ be its $L^2(\Omega_i)$-projection onto $W_i^h := W^h|_{\Omega_i}$,

$$(q - \mathcal{P}^h q, w^h) = 0, \quad \forall w^h \in W_i^h,$$

satisfying the approximation property

$$\|q - \mathcal{P}^h q\|_{0,\Omega_i} \le C\, h^r |q|_{H^r(\Omega_i)}, \quad 0 \le r \le r_i + 1, \tag{5.7}$$

where $r_i$ is the polynomial degree in the space $W_i^h$: $r_i = r_s - 1$ in $\Omega_s$ and $r_i = l_d$ in $\Omega_d$.

## 5.1 Error estimates

From the error equation:

$$\forall \boldsymbol{v}^h \in Z^h, \forall q_h \in W_h, \quad a(\boldsymbol{u} - \boldsymbol{u}^h, \boldsymbol{v}^h) = -(b(\boldsymbol{v}^h, p - q^h) + b_{sd}(\boldsymbol{v}^h, \lambda_{sd}) + b_d(\boldsymbol{v}^h, \lambda_d) + b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s)), \tag{5.8}$$

and Lemmas 3.3 and 3.4, we immediately derive

$$\|\boldsymbol{u} - \boldsymbol{u}^h\|_X \leq C \left( \inf_{\boldsymbol{v}^h \in V^h(q_d)} \|\boldsymbol{u} - \boldsymbol{v}^h\|_X + \inf_{w^h \in W^h} \|p - w^h\|_W \right) + C\mathcal{R}_u^h, \tag{5.9}$$

where $V^h(q_d) = \{\boldsymbol{v} \in V^h;\ \forall w \in W^h, b(\boldsymbol{v}, w) = \int_{\Omega_d} w\, q_d\}$, and

$$\mathcal{R}_u^h = \sup_{\boldsymbol{v}^h \in Z^h} \frac{1}{\|\boldsymbol{v}^h\|_X} \left| b_{sd}(\boldsymbol{v}^h, \lambda_{sd}) + b_d(\boldsymbol{v}^h, \lambda_d) + b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s) \right| \tag{5.10}$$

is the consistency error at the interfaces. Similarly a simple variant of (5.8) with $\boldsymbol{v}^h \in V^h$ and Theorem 3.1 yield

$$\|p - p^h\|_W \leq C \left( \|\boldsymbol{u} - \boldsymbol{u}^h\|_X + \inf_{w^h \in W^h} \|p - w^h\|_W \right) + C\mathcal{R}_p^h, \tag{5.11}$$

where $\mathcal{R}_p^h$ is given by (5.10) with $Z^h$ replaced by $V^h$. As the bound on the approximation error of $W^h$ follows from (5.7) and Lemma 5.1 estimates the approximation error of $V^h(q_d)$, it suffices to bound $\mathcal{R}_p^h$. Note that by virtue of (3.11), we have for all $\boldsymbol{\mu}_s^H \in \Lambda_s^H$, $\mu_d^H \in \Lambda_d^H$, $\mu_{sd}^H \in \Lambda_{sd}^H$,

$$\mathcal{R}_p^h = \sup_{\boldsymbol{v}^h \in V^h} \frac{1}{\|\boldsymbol{v}^h\|_X} \left| b_{sd}(\boldsymbol{v}^h, \lambda_{sd} - \mu_{sd}^H) + b_d(\boldsymbol{v}^h, \lambda_d - \mu_d^H) + b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s - \boldsymbol{\mu}_s^H) \right|. \tag{5.12}$$

The bound for $b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s - \boldsymbol{\mu}_s^H)$ is straightforward because $\boldsymbol{v}_h$ is measured in $H^1$ in each $\Omega_{s,i}$, and therefore its trace can be considered on each $\Gamma_{i,j}$. But deriving an optimal bound for the other terms is more intricate because $\boldsymbol{v}_h$ is measured in $H(\mathrm{div})$ in each $\Omega_{d,i}$ and this only controls the trace of the normal component in $H^{-\frac{1}{2}}(\partial\Omega_{d,i})$. Consequently, $\lambda_{sd} - \mu_{sd}^H$ and $\lambda_d - \mu_d^H$ must belong to $H^{\frac{1}{2}}(\partial\Omega_{d,i})$. In 2-D, this is easily achieved by interpolating $\lambda_{sd}$ and $\lambda_d$ with a variant of the Scott and Zhang interpolant that is continuous at the vertices of the subdomains; a similar idea was used in Sect. 4.3. But in 3-D, this construction requires an additional assumption on the mortar grids in the Darcy region.

**Hypothesis 5.1** *For each $\Omega_{d,i}$ in 3-D, the mortar grids $\mathcal{T}_{ij}^H$ on all $\Gamma_{ij} \subset \partial\Omega_{d,i} \backslash \partial\Omega$ are chosen such that their traces on the boundaries of $\Gamma_{ij}$ coincide.*

This assumption implies conformity of mortar grids along interface boundaries in each connected region in $\Omega_d$ and it allows us to consider the space $\Lambda_d^{H,c}$, which is the subset of continuous functions in $\Lambda_d^H \cup \Lambda_{sd}^H$. On $\Gamma_{dd} \cup \Gamma_{sd}$, let $\mathcal{I}^H$ be the Scott–Zhang interpolant in $\Lambda_d^{H,c}$. We assume that its definition depends only on function values on $\Gamma_{dd} \cup \Gamma_{sd}$. On $\Gamma_{ss}$, let $\mathcal{I}^H$ be the $L^2(\Gamma_{ij})$-orthogonal projection operator in $\Lambda_s^H$. This operator has the following approximation properties, assuming sufficient smoothness of $\mu$ and $\boldsymbol{\mu}$ (the index $\star$ stands for $dd$ or $sd$):

$$\forall \Gamma_{ij} \in \Gamma_\star, \forall \tau \in \Gamma_{ij}, \quad \|\mu - \mathcal{I}^H(\mu)\|_{H^t(\tau)} \le C H^{r-t} |\mu|_{H^r(\Delta_\tau)}, \quad 0 \le t \le 1, \ t \le r \le r_\star + 1, \tag{5.13}$$

$$\forall \Gamma_{ij} \in \Gamma_{ss}, \forall \tau \in \Gamma_{ij}, \quad \|\boldsymbol{\mu} - \mathcal{I}^H(\boldsymbol{\mu})\|_{L^2(\tau)} \le C H^r |\boldsymbol{\mu}|_{H^r(\Delta_\tau)}, \quad 0 \le r \le r_{ss} + 1, \tag{5.14}$$

where $\Delta_\tau$ is the macroelement used in defining $\mathcal{I}^H(\mu)$ restricted to $\tau$. The union of all $\Delta_\tau$ for $\tau$ in $\partial\Omega_i$ may slightly overlap interfaces that are not part of $\partial\Omega_i$; we denote the overlap by $\mathcal{O}$.

**Lemma 5.2** *There exists a constant $C$ independent of $h$, $H$, and the diameters of the subdomains such that, for all $\boldsymbol{v}^h \in X^h$:*

$$|b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s - \mathcal{I}^H(\boldsymbol{\lambda}_s))| \le C\, H^s \sum_{i=1}^{M_s} A_i^{-\frac{1}{2}} \|\boldsymbol{v}^h\|_{H^1(\Omega_{s,i})} |\boldsymbol{\lambda}_s|_{H^s(\partial\Omega_{s,i}\cap\Gamma_{ss}\cup\mathcal{O})}, \quad 0 \le s \le r_{ss} + 1, \tag{5.15}$$

*provided $\boldsymbol{\lambda}_s$ is sufficiently smooth.*

*Proof* We have

$$\begin{aligned} b_s(\boldsymbol{v}^h, \boldsymbol{\lambda}_s - \mathcal{I}^H(\boldsymbol{\lambda}_s)) &= \sum_{\Gamma_{ij}\in\Gamma_{ss}} \int_{\Gamma_{ij}} [\boldsymbol{v}^h] \cdot (\boldsymbol{\lambda}_s - \mathcal{I}^H(\boldsymbol{\lambda}_s)) \\ &\le \sum_{i=1}^{M_s} \|\boldsymbol{v}^h\|_{L^2(\partial\Omega_{s,i}\cap\Gamma_{ss})} \|\boldsymbol{\lambda}_s - \mathcal{I}^H(\boldsymbol{\lambda}_s)\|_{L^2(\partial\Omega_{s,i}\cap\Gamma_{ss})}. \end{aligned}$$

Then (5.15) follows readily from (5.14) and the trace inequality (4.51).

**Lemma 5.3** *Under Hypothesis 5.1, there exists a constant $C$ independent of $h$, $H$, and the diameters of the subdomains such that, for all $\boldsymbol{v}^h \in X^h$:*

$$\begin{aligned} &|b_{sd}(\boldsymbol{v}^h, \lambda_{sd} - \mathcal{I}^H(\lambda_{sd})) + b_d(\boldsymbol{v}^h, \lambda_d - \mathcal{I}^H(\lambda_d))| \\ &\le C \left( \sum_{i=1}^{M_d} \|\boldsymbol{v}^h\|_{H(\mathrm{div};\Omega_{d,i})} \left( H^{q-\frac{1}{2}} |\lambda|_{H^q(\partial\Omega_{d,i}\cap\Gamma_{dd}\cup\mathcal{O})} + H^{r-\frac{1}{2}} |\lambda|_{H^r(\partial\Omega_{d,i}\cap\Gamma_{sd}\cup\mathcal{O})} \right) \right. \\ &\qquad \left. + \sum_{i=1}^{M_d} \sum_j A_j^{-\frac{1}{2}} \|\boldsymbol{v}^h\|_{H^1(\Omega_{s,j})} H^r |\lambda|_{H^r(\partial\Omega_{d,i}\cap\Gamma_{sd}\cup\mathcal{O})} \right), \\ &\qquad \frac{1}{2} \le q \le r_{dd} + 1, \ \frac{1}{2} \le r \le r_{sd} + 1, \end{aligned} \tag{5.16}$$

*provided $\lambda_{sd}$ and $\lambda_d$ are sufficiently smooth, and where the sum over $j$ runs over all $\Omega_{s,j}$ adjacent to $\Omega_{d,i}$.*

*Proof* In the following $\lambda$ denotes either $\lambda_d$ or $\lambda_{sd}$ depending on the type of interface. We can write

$$
\begin{aligned}
& b_{sd}(\boldsymbol{v}^h, \lambda - \mathcal{I}^H(\lambda)) + b_d(\boldsymbol{v}^h, \lambda - \mathcal{I}^H(\lambda)) \\
& = \sum_{\Gamma_{ij} \in \Gamma_{sd} \cup \Gamma_{dd}} \int_{\Gamma_{ij}} [\boldsymbol{v}^h \cdot \boldsymbol{n}](\lambda - \mathcal{I}^H(\lambda)) \\
& = \sum_{i=1}^{M_d} \int_{\partial\Omega_{d,i}} \boldsymbol{v}^h \cdot \boldsymbol{n}_i(\lambda - \mathcal{I}^H(\lambda)) + \sum_{\Gamma_{ij} \in \Gamma_{sd}} \int_{\Gamma_{ij}} \boldsymbol{v}^h \cdot \boldsymbol{n}_s(\lambda - \mathcal{I}^H(\lambda)),
\end{aligned}
$$

where we have used that $\boldsymbol{v}^h \cdot \boldsymbol{n} = 0$ on $\partial\Omega_{d,i} \cap \partial\Omega$ and $\lambda - \mathcal{I}^H(\lambda)$ has been extended continuously from $H^{\frac{1}{2}}(\partial\Omega_{d,i} \backslash \partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega_{d,i})$. The argument of Lemma 5.2 can be used to bound the second sum above by the last sum in (5.16). Thus it is left to bound the first sum.

Consider first one subdomain $\Omega_{d,i}$ that is not adjacent to $\Gamma_{sd}$. Let us switch to the reference domain $\hat{\Omega}_i$ and let $\hat{E}$ denote the extension operator defined in (4.29) relative to $\hat{\Omega}_i$. For any $f$ in $H^{\frac{1}{2}}(\partial\Omega_{d,i})$, define $E(f)$ by: $E(f) = \hat{E}(f \circ F_i)$; therefore $E$ maps $H^{\frac{1}{2}}(\partial\Omega_{d,i})$ into $H^1(\Omega_{d,i})$. Thus, by Green's formula

$$
\begin{aligned}
\int_{\partial\Omega_{d,i}} \boldsymbol{v}^h \cdot \boldsymbol{n}_i(\lambda - \mathcal{I}^H(\lambda)) &= \int_{\partial\Omega_{d,i}} \boldsymbol{v}^h \cdot \boldsymbol{n}_i E(\lambda - \mathcal{I}^H(\lambda)) \\
&= \int_{\Omega_{d,i}} \operatorname{div} \boldsymbol{v}^h E(\lambda - \mathcal{I}^H(\lambda)) + \int_{\Omega_{d,i}} \boldsymbol{v}^h \cdot \nabla E(\lambda - \mathcal{I}^H(\lambda)).
\end{aligned}
$$

By reverting to the reference domain and observing that $\mathcal{I}^H$ is invariant under the rigid body motion $F_i$, this reads

$$
\left| \int_{\partial\Omega_{d,i}} \boldsymbol{v}^h \cdot \boldsymbol{n}_i(\lambda - \mathcal{I}^H(\lambda)) \right| \leq A_i^{n-1} \|\hat{\boldsymbol{v}}\|_{H(\hat{\operatorname{div}};\hat{\Omega}_i)} \|\hat{E}(\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^1(\hat{\Omega}_i)}.
$$

On one hand, the bound in (4.29) gives

$$
\begin{aligned}
\|\hat{E}(\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^1(\hat{\Omega}_i)} &\leq \hat{C} \|\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^{\frac{1}{2}}(\partial\hat{\Omega}_i)} \leq \hat{C} \|\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^{\frac{1}{2}}(\partial\hat{\Omega}_i \backslash \partial\hat{\Omega})} \\
&\leq \hat{C} \Big( \sum_{\hat{\Gamma} \subset \partial\hat{\Omega}_i \backslash \partial\hat{\Omega}} \|\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|^2_{H^{\frac{1}{2}}(\hat{\Gamma})} \Big)^{\frac{1}{2}},
\end{aligned}
$$

with constants here and below independent of $h$, $H$, and the diameters of $\Omega_{d,i}$ and $\Gamma_{ij}$. By space interpolation between $L^2(\hat{\Gamma})$ and $H^1(\hat{\Gamma})$, the estimates (5.13) summed on $\hat{\Gamma}$ with $t = 0$ and $t = 1$ readily imply that

$$
\|\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^{\frac{1}{2}}(\hat{\Gamma})} \leq \hat{C} \hat{H}_i^{r-\frac{1}{2}} |\hat{\lambda}|_{H^r(\Delta_{\hat{\Gamma}})}, \quad \frac{1}{2} \leq r \leq r_{dd} + 1, \tag{5.17}
$$

where the mesh size $\hat{H}_i$ on $\partial\hat{\Omega}_i$ is related to the mesh size $H_i$ on $\partial\Omega_i$ by

$$H_i = A_i \hat{H}_i.$$

In the case when $r$ is not an integer, since (5.17) is stated in the reference region $\hat{\Gamma}$, we choose the fractional seminorm in the right-hand side to be defined by (1.8) and incorporate the equivalence constant into $\hat{C}$. The motivation for this choice is that the intrinsic norm is easily transformed by $F_i$. Therefore

$$\begin{aligned}\|\hat{\lambda} - \hat{\mathcal{I}}(\hat{\lambda}))\|_{H^{\frac{1}{2}}(\partial\hat{\Omega}_i)} &\le \hat{C}\hat{H}_i^{r-\frac{1}{2}}|\hat{\lambda}|_{H^r(\partial\hat{\Omega}_i\setminus\partial\hat{\Omega}\cup\hat{\mathcal{O}})} \\ &\le \hat{C}\hat{H}_i^{r-\frac{1}{2}}A_i^{r-\frac{n-1}{2}}|\lambda|_{H^r(\partial\Omega_{d,i}\setminus\partial\Omega\cup\mathcal{O})}.\end{aligned}$$

On the other hand,

$$\|\hat{\boldsymbol{v}}\|_{H(\hat{\text{div}};\hat{\Omega}_i)} = A_i^{-\frac{n}{2}}\Big(A_i^2\|\text{div}\,\boldsymbol{v}^h\|^2_{L^2(\Omega_{d,i})} + \|\boldsymbol{v}^h\|^2_{L^2(\Omega_{d,i})}\Big)^{\frac{1}{2}}.$$

By collecting these inequalities, we derive

$$\left|\int_{\partial\Omega_{d,i}} \boldsymbol{v}^h\cdot\boldsymbol{n}_i(\lambda - \mathcal{I}^H(\lambda))\right| \le \hat{C}\big(A_i\hat{H}_i\big)^{r-\frac{1}{2}}\|\boldsymbol{v}^h\|_{H(\text{div};\Omega_{d,i})}|\lambda|_{H^r(\partial\Omega_{d,i}\setminus\partial\Omega\cup\mathcal{O})},$$
$$\frac{1}{2} \le r \le r_{dd}+1.$$

By applying (5.17) to a portion of $\Gamma_{sd}$, using (5.13) the same argument handles the case when $\Omega_{d,i}$ is adjacent to $\Gamma_{sd}$. Then (5.16) follows easily from these estimates.

*Remark 5.1* We can skip Hypothesis 5.1, work locally on each $\Gamma_{ij}$, and use the $L^2$ norm for both factors $\boldsymbol{v}^h\cdot\boldsymbol{n}_i$ and $\lambda - \mathcal{I}^H(\lambda)$. But this gives rise to a factor of the form $\big(H_i/h\big)^{\frac{1}{2}}$ in (5.16), that is clearly not optimal when $h$ is very small compared to $H_i$.

Let $\mathcal{R}^h$ denote either $\mathcal{R}^h_u$ or $\mathcal{R}^h_p$. The next lemma estimates $\mathcal{R}^h$.

**Lemma 5.4** *Under Hypothesis 5.1, the consistency error $\mathcal{R}^h$ satisfies*

$$\begin{aligned}\mathcal{R}^h \le C\Big(&A^{-\frac{1}{2}}H^{r-\frac{1}{2}}\|p_d\|_{H^{r+\frac{1}{2}}(\Omega_d)} + A^{-\frac{1}{2}}H^{q-\frac{1}{2}}\|p_d\|_{H^{q+\frac{1}{2}}(\Omega_d)} \\ &+A^{-1}H^s(\|\boldsymbol{u}_s\|_{H^{s+\frac{3}{2}}(\Omega_s)} + \|p_s\|_{H^{s+\frac{1}{2}}(\Omega_s)})\Big), \\ &\frac{1}{2}\le q\le r_{dd}+1,\ \frac{1}{2}\le r\le r_{sd}+1,\ 0 < s\le r_{ss}+1,\end{aligned} \tag{5.18}$$

*where* $A = \min_{1\le i\le M} A_i$.

*Proof* By combining (5.15) and (5.16), we easily derive

$$\mathcal{R}^h \le C \left( H^{q-\frac{1}{2}} \left( \sum_{i=1}^{M_d} |\lambda|^2_{H^q(\partial\Omega_{d,i}\cap\Gamma_{dd}\cup\mathcal{O})} \right)^{\frac{1}{2}} + H^{r-\frac{1}{2}} \left( \sum_{i=1}^{M_d} |\lambda|^2_{H^r(\partial\Omega_{d,i}\cap\Gamma_{sd}\cup\mathcal{O})} \right)^{\frac{1}{2}} \right.$$
$$+ H^s \left( \sum_{i=1}^{M_s} A_i^{-1} |\boldsymbol{\lambda}_s|^2_{H^s(\partial\Omega_{s,i}\cap\Gamma_{ss}\cup\mathcal{O})} \right)^{\frac{1}{2}}$$
$$\left. + H^r \left( \sum_{i=1}^{M_s} A_i^{-1} |\lambda|^2_{H^r(\partial\Omega_{s,i}\cap\Gamma_{sd}\cup\mathcal{O})} \right)^{\frac{1}{2}} \right),$$
$$\frac{1}{2} \le q \le r_{dd}+1,\ \frac{1}{2} \le r \le r_{sd}+1,\ 0 \le s \le r_{ss}+1.$$

In view of the representation (2.47), local trace theorems yield:

$$|\lambda_{sd}|_{H^r(\partial\Omega_{d,i}\cap\Gamma_{sd})} \le C A_i^{-\frac{1}{2}} \|p_d\|_{H^{r+\frac{1}{2}}(\Omega_{d,i})},$$
$$|\lambda_d|_{H^q(\partial\Omega_{d,i}\cap\Gamma_{dd})} \le C A_i^{-\frac{1}{2}} \|p_d\|_{H^{q+\frac{1}{2}}(\Omega_{d,i})},$$
$$|\boldsymbol{\lambda}_s|_{H^s(\partial\Omega_{s,i}\cap\Gamma_{ss})} \le C A_i^{-\frac{1}{2}} \big(\|\boldsymbol{u}_s\|_{H^{s+\frac{3}{2}}(\Omega_{s,i})} + \|p_s\|_{H^{s+\frac{1}{2}}(\Omega_{s,i})}\big).$$

Then (5.18) follows from these bounds, that are easily extended to include $\mathcal{O}$, and the fact that $A_i^{-1} < H^{-1}$.

Now we use the abstract error bounds (5.9) and (5.11), the approximation results (5.4), (5.5), and (5.7), and the consistency error bound (5.18) to obtain the following convergence result.

**Theorem 5.1** *We have*

$$\begin{aligned}
&|||\boldsymbol{u} - \boldsymbol{u}^h|||_X + \|p - p^h\|_W \\
&\quad \le C\big(h^{r_1}(\|\boldsymbol{u}\|_{H^{r_1+1}(\Omega)} + \|p\|_{H^{r_1}(\Omega)}) + h^{r_2}\|\boldsymbol{u}\|_{H^{r_2+\frac{1}{2}}(\Omega)} \\
&\quad + h^{r_3}(\|\mathrm{div}\,\boldsymbol{u}\|_{H^{r_3}(\Omega)} + \|p\|_{H^{r_3}(\Omega)}) \\
&\quad + A^{-1}H^{r_4}(\|\boldsymbol{u}_s\|_{H^{r_4+\frac{3}{2}}(\Omega_s)} + \|p_s\|_{H^{r_4+\frac{1}{2}}(\Omega_s)}) \\
&\quad + A^{-\frac{1}{2}}H^{r_5-\frac{1}{2}}\|p_d\|_{H^{r_5+\frac{1}{2}}(\Omega_d)} + A^{-\frac{1}{2}}H^{r_6-\frac{1}{2}}\|p_d\|_{H^{r_6+\frac{1}{2}}(\Omega_d)}\big), \\
&\qquad 0 \le r_1 \le r_s,\ \frac{1}{2} \le r_2 \le r_d+1,\ 0 \le r_3 \le l_d+1, \\
&\qquad 0 < r_4 \le r_{ss}+1,\ \frac{1}{2} \le r_5 \le r_{dd}+1,\ \frac{1}{2} \le r_6 \le r_{sd}+1.
\end{aligned}$$

*Remark 5.2* In the above estimate, the fine scale subdomain approximation error terms are of optimal order with constants independent of the size of the subdomains $A$. The constants of the coarse scale mortar consistency error terms deteriorate with decrease in $A$, since in that case the number of interfaces grows. Nevertheless, higher order mortar polynomials can be employed to balance the error terms, giving optimal fine scale convergence.

## 6 Numerical tests

In this section we validate our analysis by carrying out some numerical experiments. In all tests the computational domain is taken to be $\overline{\Omega} = \overline{\Omega}_s \cup \overline{\Omega}_d$, where $\Omega_s = (0, 1) \times (\frac{1}{2}, 1)$ and $\Omega_d = (0, 1) \times (0, \frac{1}{2})$. For simplicity we set

$$\boldsymbol{\sigma}(\boldsymbol{u}_s, p_s) = -p_s \boldsymbol{I} + \nu_s \nabla \boldsymbol{u}_s$$

in the Stokes equation in $\Omega_s$, and

$$\boldsymbol{K} = K\boldsymbol{I}$$

in the Darcy equation in $\Omega_d$, where $K$ is a positive constant.

To test for convergence we construct the following analytical solution satisfying the flow equations in $\Omega_s$ and $\Omega_d$ along with the conditions on the interface $\Gamma_{sd}$:

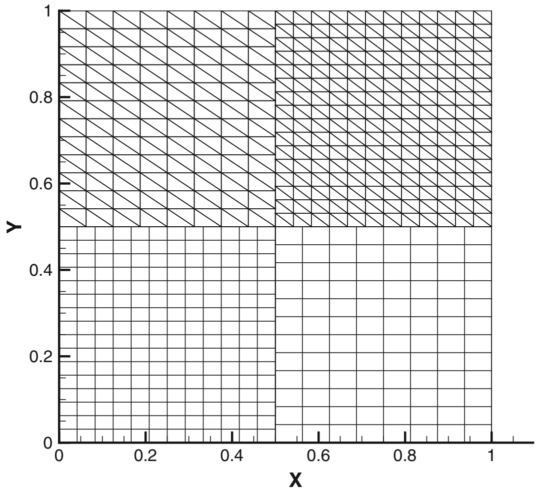
$$\begin{aligned}
\boldsymbol{u}_s &= \begin{bmatrix} (2-x)(1.5-y)(y-\xi) \\ -\frac{y^3}{3} + \frac{y^2}{2}(\xi + 1.5) - 1.5\xi y - 0.5 + \sin(\omega x) \end{bmatrix}, \\
\boldsymbol{u}_d &= \begin{bmatrix} \omega \cos(\omega x) y \\ \chi(y + 0.5) + \sin(\omega x) \end{bmatrix}, \\
p_s &= -\frac{\sin(\omega x) + \chi}{2K} + \nu_s(0.5 - \xi) + \cos(\pi y), \\
p_d &= -\frac{\chi}{K}\frac{(y+0.5)^2}{2} - \frac{\sin(\omega x) y}{K},
\end{aligned}$$

where

$$\nu_s = 0.1,\ K = 1,\ \alpha = 0.5,\ G = \frac{\sqrt{\nu_s K}}{\alpha}, \xi = \frac{1-G}{2(1+G)},\ \chi = \frac{-30\xi - 17}{48},\ \text{and } \omega = 6.0.$$

The right-hand sides $\boldsymbol{f}_s$, $\boldsymbol{f}_d$, and $q_d$ for the Stokes–Darcy flow system are obtained by substituting the analytical solution into (2.6), (2.3), and (2.4), respectively. The boundary conditions are as follows: for the Stokes region, the velocity $\boldsymbol{u}_s$ is specified on the left and top boundaries, and the normal and tangential stresses $(\boldsymbol{\sigma n}_s) \cdot \boldsymbol{n}_s$ and $(\boldsymbol{\sigma n}_s) \cdot \boldsymbol{\tau}_s$ are specified on the right boundary; for the Darcy region, the normal velocity $\boldsymbol{u}_d \cdot \boldsymbol{n}_d$ is specified on the left boundary and the pressure $p_d$ is specified on the bottom and right boundaries. Each region $\Omega_s$ and $\Omega_d$ is divided into two

**Fig. 2** Non-matching subdomain meshes $8 \times 12$–$12 \times 16$




subdomains, giving a total of four subdomains. The subdomain grids do not match across the interfaces. The Stokes subdomains are discretized by the Taylor–Hood triangular finite elements with quadratic velocities and linear pressures ($r_s = 2$). The Darcy subdomains are discretized by the lowest order Raviart–Thomas rectangular finite elements ($r_d = l_d = 0$). We use discontinuous piecewise linear mortars on all interfaces ($r_{ss} = r_{dd} = r_{sd} = 1$). To test convergence, we solve the problem on a sequence of grid refinements. On the coarsest level, the subdomain grids are $3 \times 4$ in the lower left and upper right subdomains and $2 \times 3$ in the other two subdomains. We test two cases, $H = 2h$ and $H = \sqrt{h}$. In both cases the coarsest mortar grids have a single element per interface. In the first case the mortar grids are refined by two each time the subdomain grids are refined by two. In the second case the mortar grids are refined by two each time the subdomain grids are refined by four. The non-matching grids on the middle level of refinement are shown in Fig. 2. The computed solution and the numerical error on this grid level with mortar meshes $H = 2h$ are displayed in Fig. 3. The numerical errors and convergence rates on all refinement levels are reported in Tables 1, 2, 3, 4, 5, and 6. We report separately the errors in $\Omega_s$ and $\Omega_d$ in their respective norms. In the Darcy region we also report the discrete $L^2$ errors $\|\cdot\|_{M,\Omega_d}$ for the pressure and the velocity, computed via the use of the midpoint quadrature rule on each element.

In the case $H = 2h$ we observe second order of convergence for $\|\boldsymbol{u}_s - \boldsymbol{u}_s^h\|_{H^1(\Omega_s)}$ and $\|p_s - p_s^h\|_{L^2(\Omega_s)}$, as well as first order convergence for $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{H(\mathrm{div};\Omega_d)}$ and $\|p_d - p_d^h\|_{L^2(\Omega_d)}$. These rates are consistent with the error terms $O(h^{r_s}) = O(h^2)$ and $O(h^{r_d+1}) = O(h^{l_d+1}) = O(h)$ that appear in Theorem 5.1. Note that in the case $H = 2h$ the interface consistency error terms are $O(h^{r_{ss}+1}) = O(h^2)$ and $O(h^{r_{dd}+\frac{1}{2}}) = O(h^{r_{sd}+\frac{1}{2}}) = O(h^{\frac{3}{2}})$. While the error terms in Theorem 5.1 depend on the global solution norms, the results indicate that the lower convergence rates in $\Omega_d$ and on $\Gamma_{sd}$ do not affect the second order convergence in $\Omega_s$. This suggests that it may be possible

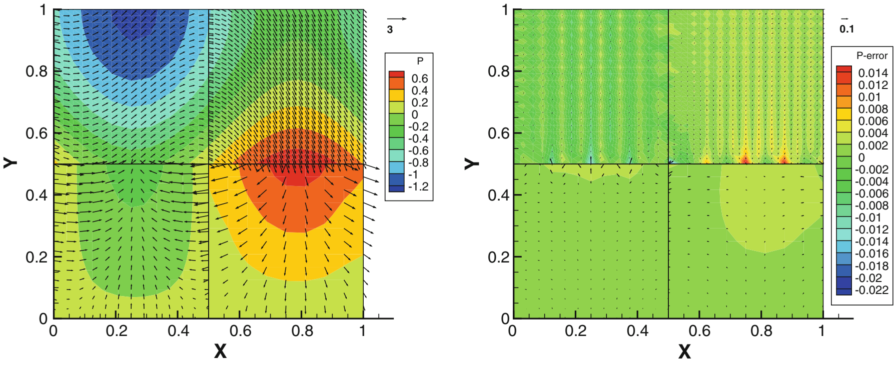



**Fig. 3** *Test 1*: computed solution (*left*) and error (*right*) on subdomain meshes $8 \times 12$–$12 \times 16$ and mortar meshes $H = 2h$

**Table 1** Test 1: $H = 2h$

| Mesh | $\|\boldsymbol{u}_s - \boldsymbol{u}_s^h\|_{H^1(\Omega_s)}$ | Rate | $\|p_s - p_s^h\|_{L^2(\Omega_s)}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 3.93e−01 | | 2.64e−02 | |
| 4×6 6×8 | 8.95e−02 | 2.13 | 6.37e−03 | 2.05 |
| 8×12 12×16 | 2.10e−02 | 2.09 | 1.53e−03 | 2.06 |
| 16×24 24×32 | 5.08e−03 | 2.05 | 3.75e−04 | 2.03 |
| 32×48 48×64 | 1.25e−03 | 2.02 | 9.29e−05 | 2.01 |

Numerical errors and convergence rates in $\Omega_s$

**Table 2** Test 1: $H = 2h$

| Mesh | $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{H(\text{div};\Omega_d)}$ | Rate | $\|p_d - p_d^h\|_{L^2(\Omega_d)}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 3.51e+00 | | 1.01e−01 | |
| 4×6 6×8 | 1.79e+00 | 0.97 | 5.05e−02 | 1.00 |
| 8×12 12×16 | 9.00e−01 | 0.99 | 2.52e−02 | 1.00 |
| 16×24 24×32 | 4.50e−01 | 1.00 | 1.26e−02 | 1.00 |
| 32×48 48×64 | 2.20e−01 | 1.00 | 6.28e−03 | 1.00 |

Numerical errors and convergence rates in $\Omega_d$

to establish interior convergence error estimates. Such estimates were derived in [53] for mortar discretizations for Darcy flow. We also observe superconvergence at the cell centers in the Darcy region, as indicated by the $O(h^2)$ convergence of $\|p_d - p_d^h\|_{M,\Omega_d}$ and the $O(h^{\frac{3}{2}})$ convergence of $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{M,\Omega_d}$. Superconvergence for the pressure on unstructured grids and for the velocity on logically rectangular grids in mixed finite element discretizations for Darcy flow has been extensively studied in the literature, see, e.g. [5] and the references therein. The reduced superconvergence order for the Darcy velocity is due to the $O(h^{\frac{3}{2}})$ mortar error term, as shown in [5].

**Table 3** Test 1: $H = 2h$

| Mesh | $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{M,\Omega_d}$ | Rate | $\|p_d - p_d^h\|_{M,\Omega_d}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 2.60e−01 | | 1.65e−02 | |
| 4×6 6×8 | 6.71e−02 | 1.95 | 4.13e−03 | 2.00 |
| 8×12 12×16 | 2.09e−02 | 1.68 | 1.01e−03 | 2.03 |
| 16×24 24×32 | 6.85e−03 | 1.61 | 2.50e−04 | 2.01 |
| 32×48 48×64 | 2.32e−03 | 1.56 | 6.22e−05 | 2.01 |

Numerical errors and convergence rates in $\Omega_d$ using the discrete $\|\cdot\|_{M,\Omega_d}$ norm

**Table 4** Test 2: $H = \sqrt{h}$

| Mesh | $\|\boldsymbol{u}_s - \boldsymbol{u}_s^h\|_{H^1(\Omega_s)}$ | Rate | $\|p_s - p_s^h\|_{L^2(\Omega_s)}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 3.93e−01 | | 2.64e−02 | |
| 8×12 12×16 | 5.59e−02 | 1.41 | 3.51e−03 | 1.46 |
| 32×48 48×64 | 1.03e−02 | 1.22 | 6.02e−04 | 1.27 |

Numerical errors and convergence rates in $\Omega_s$

**Table 5** Test 2: $H = \sqrt{h}$

| Mesh | $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{H(\mathrm{div};\Omega_d)}$ | Rate | $\|p_d - p_d^h\|_{L^2(\Omega_d)}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 3.51e+00 | | 1.01e−01 | |
| 8×12 12×16 | 9.10e−01 | 0.97 | 2.53e−02 | 1.00 |
| 32×48 48×64 | 2.30e−01 | 0.99 | 6.32e−03 | 1.00 |

Numerical errors and convergence rates in $\Omega_d$

**Table 6** Test 2: $H = \sqrt{h}$

| Mesh | $\|\boldsymbol{u}_d - \boldsymbol{u}_d^h\|_{M,\Omega_d}$ | Rate | $\|p_d - p_d^h\|_{M,\Omega_d}$ | Rate |
|---|---|---|---|---|
| 2×3 3×4 | 2.60e−01 | | 1.65e−02 | |
| 8×12 12×16 | 1.10e−01 | 0.62 | 3.65e−03 | 1.09 |
| 32×48 48×64 | 5.01e−02 | 0.57 | 7.57e−04 | 1.13 |

Numerical errors and convergence rates in $\Omega_d$ using the discrete $\|\cdot\|_{M,\Omega_d}$ norm

In the case $H = \sqrt{h}$, we observe approximately $O(h)$ convergence for all error norms. Note that in this case the interface consistency error terms are $O(h^{(r_{ss}+1)/2}) = O(h)$ and $O(h^{(r_{dd}+\frac{1}{2})/2}) = O(h^{(r_{sd}+\frac{1}{2})/2}) = O(h^{\frac{3}{4}})$, so their effect on the convergence in the Stokes and Darcy regions is more significant. In this multiscale case, one may utilize higher order mortars to recover optimal fine scale subdomain convergence, see [6] for the Darcy case.

## 7 Appendix

This Appendix is devoted to specific examples of construction of $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ in $\Omega_s$.

### 7.1 The 2-D case

Recall that the construction of the correction $\boldsymbol{c}^h_{j,\Gamma_{ij}}(\boldsymbol{v})$ in $\Omega_s$ in 2-D required an operator $\pi_s^h$ satisfying (4.23). In particular, given $\hat{\boldsymbol{\ell}}$ in $H^{\frac{1}{2}}_{00}(\hat{\Gamma})^2$, one needs to construct $\hat{\pi}_j(\hat{\boldsymbol{\ell}})$ in $\hat{X}_j$, unique solution of (4.24):

$$\forall \hat{\boldsymbol{\mu}} \in \hat{\Lambda}, \quad \int_{\hat{\Gamma}} \hat{\pi}_j(\hat{\boldsymbol{\ell}}) \cdot \hat{\boldsymbol{\mu}} = \int_{\hat{\Gamma}} \hat{\boldsymbol{\ell}} \cdot \hat{\boldsymbol{\mu}},$$

and satisfying (4.25).

Here we present two simple examples of how to construct such an operator when $\Gamma_{ij}$ is a straight line segment and the traces of the discrete spaces on $\Gamma_{ij}$ are piecewise $\mathbb{P}_1$ finite elements, Without loss of generality, we can suppose that $\hat{\Gamma}$ is the segment [0, 1], that the nodes of $\hat{\mathcal{T}}_j$ are $0 = \hat{x}_0 < \hat{x}_1 < \cdots < \hat{x}_N < \hat{x}_{N+1} = 1$, and we set $\hat{h}_n = \hat{x}_{n+1} - \hat{x}_n$, $0 \le n \le N$. We choose each component of the functions of $\hat{X}_j$ piecewise $\mathbb{P}_1$ in the subintervals $[\hat{x}_n, \hat{x}_{n+1}]$, with degrees of freedom at the nodes $\hat{x}_n$, $1 \le n \le N$.

#### *7.1.1 First example*

As a first example, we choose each component of the mortar functions of $\hat{\Lambda}$ also piecewise $\mathbb{P}_1$ in the subintervals $[\hat{x}_n, \hat{x}_{n+1}]$, with degrees of freedom at the nodes $\hat{x}_n$, $n = 0$, $2 \le n \le N-1$, and $n = N+1$. The nodes $\hat{x}_1$ and $\hat{x}_N$ are deleted so that $\hat{X}_j$ and $\hat{\Lambda}$ have the same dimension $2N$; thus the matrix of the linear system (4.24) is square. In order to express these functions in terms of a basis, it is convenient to modify slightly the standard Lagrange basis functions for the velocity. More precisely, we define

$$\begin{gathered}
\hat{\varphi}_0(\hat{x})|_{[\hat{x}_0,\hat{x}_2]} = \frac{1}{\hat{h}_0 + \hat{h}_1}(\hat{x}_2 - \hat{x}), \quad \hat{\varphi}_{N+1}(\hat{x})|_{[\hat{x}_{N-1},\hat{x}_{N+1}]} = \frac{1}{\hat{h}_N + \hat{h}_{N-1}}(\hat{x} - \hat{x}_{N-1}),\\
\hat{\varphi}_2(\hat{x})|_{[\hat{x}_0,\hat{x}_2]} = \frac{1}{\hat{h}_0 + \hat{h}_1}(\hat{x} - \hat{x}_0), \quad \hat{\varphi}_2(\hat{x})|_{[\hat{x}_2,\hat{x}_3]} = \frac{1}{\hat{h}_2}(\hat{x}_3 - \hat{x}),\\
\hat{\varphi}_{N-1}(\hat{x})|_{[\hat{x}_{N-2},\hat{x}_{N-1}]} = \frac{1}{\hat{h}_{N-1}}(\hat{x} - \hat{x}_{N-2}), \qquad (7.1)\\
\hat{\varphi}_{N-1}(\hat{x})|_{[\hat{x}_{N-1},\hat{x}_{N+1}]} = \frac{1}{\hat{h}_{N-1} + \hat{h}_N}(\hat{x}_{N+1} - \hat{x}),\\
\hat{\varphi}_n(\hat{x})|_{[\hat{x}_{n-1},\hat{x}_n]} = \frac{1}{\hat{h}_{n-1}}(\hat{x} - \hat{x}_{n-1}), \ \hat{\varphi}_n(\hat{x})|_{[\hat{x}_n,\hat{x}_{n+1}]} = \frac{1}{\hat{h}_n}(\hat{x}_{n+1} - \hat{x}),\\
n = 1, 3 \le n \le N-2, n = N,
\end{gathered}$$

extended by zero elsewhere. We take

$$\hat{X}_j = \left\{ \hat{\boldsymbol{v}} \in H_0^1(0,1)^2;\ \hat{\boldsymbol{v}}(\hat{x}) = \sum_{n=1}^{N} \hat{\boldsymbol{v}}_n \hat{\varphi}_n(\hat{x}) \right\}, \tag{7.2}$$

$$\hat{\Lambda} = \left\{ \hat{\boldsymbol{\mu}} \in H^1(0,1)^2;\ \hat{\boldsymbol{\mu}}(\hat{x}) = \hat{\boldsymbol{\mu}}(\hat{x}_0)\hat{\varphi}_0(\hat{x}) + \sum_{n=2}^{N-1} \hat{\boldsymbol{\mu}}(\hat{x}_n)\hat{\varphi}_n(\hat{x}) + \hat{\boldsymbol{\mu}}(\hat{x}_{N+1})\hat{\varphi}_{N+1}(\hat{x}) \right\}. \tag{7.3}$$

On one hand, the set $\{\hat{\varphi}_0, \hat{\varphi}_n, 2 \le n \le N-1, \hat{\varphi}_{N+1}\}$ is indeed a Lagrange basis for $\hat{\Lambda}$. On the other hand, the set $\{\hat{\varphi}_n, 1 \le n \le N\}$ is a basis but not a Lagrange basis for $\hat{X}_j$; however, it is easy to check that

$$\begin{aligned} &\hat{\boldsymbol{v}}_1 = \hat{\boldsymbol{v}}(\hat{x}_1) - \hat{\boldsymbol{v}}(\hat{x}_2)\hat{\varphi}_2(\hat{x}_1), \quad \hat{\boldsymbol{v}}_n = \hat{\boldsymbol{v}}(\hat{x}_n), \quad 2 \le n \le N-1, \\ &\hat{\boldsymbol{v}}_N = \hat{\boldsymbol{v}}(\hat{x}_N) - \hat{\boldsymbol{v}}(\hat{x}_{N-1})\hat{\varphi}_{N-1}(\hat{x}_N). \end{aligned} \tag{7.4}$$

The system (4.24) can be decoupled into two independent systems in $\mathbb{R}^N$, one for each component, and each system has the form $\boldsymbol{M}\boldsymbol{\alpha} = \boldsymbol{b}$, where $\boldsymbol{\alpha} \in \mathbb{R}^N$ is the unknown, the non zero coefficients of $\boldsymbol{M}$ per row are

$$\begin{gathered} M_{1,1} = \frac{1}{6}(\hat{h}_0 + 2\hat{h}_1), \quad M_{1,2} = \frac{1}{6}(\hat{h}_0 + \hat{h}_1), \\ M_{2,1} = \frac{1}{6}(2\hat{h}_0 + \hat{h}_1), \quad M_{2,2} = \frac{1}{3}(\hat{h}_0 + \hat{h}_1 + \hat{h}_2), \quad M_{2,3} = \frac{\hat{h}_2}{6}, \\ M_{n,n-1} = \frac{\hat{h}_{n-1}}{6}, \quad M_{n,n} = \frac{1}{3}(\hat{h}_{n-1} + \hat{h}_n), \quad M_{n,n+1} = \frac{\hat{h}_n}{6}, 3 \le n \le N-2, \\ M_{N-1,N-2} = \frac{\hat{h}_{N-2}}{6}, \quad M_{N-1,N-1} = \frac{1}{3}(\hat{h}_{N-2} + \hat{h}_{N-1} + \hat{h}_N), \\ M_{N-1,N} = \frac{1}{6}(\hat{h}_{N-1} + 2\hat{h}_N), \quad M_{N-1,N} = \frac{1}{6}(2\hat{h}_{N-1} + \hat{h}_N), \\ M_{N,N} = \frac{1}{6}(\hat{h}_{N-1} + \hat{h}_N). \end{gathered}$$

Denoting by $\hat{\ell}$ a generic component of $\hat{\boldsymbol{\ell}}$, the components of $\boldsymbol{b} \in \mathbb{R}^N$ are

$$\begin{aligned} &b_1 = \int_{\hat{x}_0}^{\hat{x}_2} \hat{\ell}\hat{\varphi}_0, \quad b_2 = \int_{\hat{x}_0}^{\hat{x}_3} \hat{\ell}\hat{\varphi}_2, \quad b_n = \int_{\hat{x}_{n-1}}^{\hat{x}_{n+1}} \hat{\ell}\hat{\varphi}_n, \quad 3 \le n \le N-2, \\ &b_{N-1} = \int_{\hat{x}_{N-2}}^{\hat{x}_{N+1}} \hat{\ell}\hat{\varphi}_{N-1}, \quad b_N = \int_{\hat{x}_{N-1}}^{\hat{x}_{N+1}} \hat{\ell}\hat{\varphi}_{N+1}. \end{aligned} \tag{7.5}$$

The matrix $\boldsymbol{M}$ is tridiagonal but not symmetric, the coefficients of its three diagonals are all strictly positive, and it is strictly diagonally dominant, hence invertible, but this is not sufficient to obtain a sharp estimate for its inverse. To this end, we use the approach of Crouzeix and Thomée [23]. More precisely, let $\boldsymbol{D}$ be the principal diagonal of $\boldsymbol{M}$ and factor $\boldsymbol{M}$ into $\boldsymbol{M} = \boldsymbol{D}(\boldsymbol{I} + \boldsymbol{K})$ where $\boldsymbol{K}$ is a tridiagonal matrix with principal diagonal zero. Denote the diagonal terms of $\boldsymbol{D}$ by $d_n > 0$ and note that $\boldsymbol{D}^{\frac{1}{2}}$ is well-defined. The next lemma relates $\hat{\pi}_j(\hat{\ell})$ and $\boldsymbol{\alpha}$. Recall that $|\cdot|$ denotes the Euclidean vector norm.

**Lemma 7.1** *For each component $\hat{\ell}$ of $\hat{\boldsymbol{\ell}}$ we have*

$$\|\hat{\pi}_j(\hat{\ell})\|_{L^2(\hat{\Gamma})} \leq 2|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\alpha}|. \tag{7.6}$$

*Proof* Considering the support of the basis functions $\hat{\varphi}_n$, we have

$$\|\hat{\pi}_j(\hat{\ell})\|^2_{L^2(\hat{\Gamma})} = \int_{\hat{x}_0}^{\hat{x}_2} (\alpha_1\hat{\varphi}_1 + \alpha_2\hat{\varphi}_2)^2 + \sum_{n=2}^{N-2} \int_{\hat{x}_n}^{\hat{x}_{n+1}} (\alpha_n\hat{\varphi}_n + \alpha_{n+1}\hat{\varphi}_{n+1})^2 + \int_{\hat{x}_{N-1}}^{\hat{x}_{N+1}} (\alpha_{N-1}\hat{\varphi}_{N-1} + \alpha_{N+1}\hat{\varphi}_{N+1})^2.$$

By substituting the expression of these integrals and rearranging terms, we derive

$$\|\hat{\pi}_j(\hat{\ell})\|^2_{L^2(\hat{\Gamma})} \leq \frac{2}{3}\Big(\alpha_1^2(\hat{h}_0 + \hat{h}_1) + \alpha_2^2(\hat{h}_0 + \hat{h}_1 + \hat{h}_2) + \sum_{n=3}^{N-2} \alpha_n^2(\hat{h}_{n-1} + \hat{h}_n) + \alpha_{N-1}^2(\hat{h}_{N-2} + \hat{h}_{N-1} + \hat{h}_N) + \alpha_N^2(\hat{h}_{N-1} + \hat{h}_N)\Big).$$

In view of the diagonal terms $d_n$ of $\boldsymbol{D}$, this can be written

$$\|\hat{\pi}_j(\hat{\ell})\|^2_{L^2(\hat{\Gamma})} \leq 2\left(\alpha_1^2\frac{\hat{h}_0 + \hat{h}_1}{3} + \sum_{n=2}^{N-1} \alpha_n^2 d_n + \alpha_N^2\frac{\hat{h}_{N-1} + \hat{h}_N}{3}\right).$$

But

$$\frac{\hat{h}_0 + \hat{h}_1}{3} = 2d_1\frac{\hat{h}_0 + \hat{h}_1}{\hat{h}_0 + 2\hat{h}_1} < 2d_1, \text{ and similarly } \frac{\hat{h}_{N-1} + \hat{h}_N}{3} < 2d_N.$$

Hence

$$\|\hat{\pi}_j(\hat{\ell})\|_{L^2(\hat{\Gamma})} \leq 2\left(\sum_{n=1}^{N} \alpha_n^2 d_n\right)^{\frac{1}{2}}, \tag{7.7}$$

whence (7.6).

This lemma shows that an $L^2$ bound for $\hat{\pi}_j(\hat{\ell})$ relies on a bound for $|\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\alpha}|$. But $\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\alpha}$ has also the following expression

$$\boldsymbol{D}^{\frac{1}{2}}\boldsymbol{\alpha} = \big(\boldsymbol{I} + \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{K}\boldsymbol{D}^{-\frac{1}{2}}\big)^{-1}\big(\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}\big).$$

Therefore Lemma 7.1 implies that

$$\|\hat{\pi}_j(\hat{\ell})\|_{L^2(\hat{\Gamma})} \leq 2\|\big(\boldsymbol{I} + \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{K}\boldsymbol{D}^{-\frac{1}{2}}\big)^{-1}\|_2|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}|, \tag{7.8}$$

where $\|\cdot\|_2$ denotes the matrix norm subordinated by the Euclidean norm, and more generally $\|\cdot\|_p$ is the matrix norm subordinated by the $l^p$ vector norm. The next lemma gives a bound for $|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}|$.

**Lemma 7.2** *We have*

$$|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}| \leq \sqrt{3}\|\hat{\ell}\|_{L^2(\hat{\Gamma})}. \tag{7.9}$$

*Proof* By integrating the basis functions $\hat{\varphi}_n$, we derive

$$\begin{aligned}
|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}|^2 &= \sum_{n=1}^{N} d_n^{-1} b_n^2 \\
&\leq \frac{1}{3}\Bigg(\frac{\hat{h}_0+\hat{h}_1}{d_1}\|\hat{\ell}\|^2_{L^2(\hat{x}_0,\hat{x}_2)} + \frac{\hat{h}_0+\hat{h}_1+\hat{h}_2}{d_2}\|\hat{\ell}\|^2_{L^2(\hat{x}_0,\hat{x}_3)} \\
&\qquad + \sum_{n=3}^{N-2}\frac{\hat{h}_{n-1}+\hat{h}_n}{d_n}\|\hat{\ell}\|^2_{L^2(\hat{x}_{n-1},\hat{x}_{n+1})} + \frac{\hat{h}_{N-2}+\hat{h}_{N-1}+\hat{h}_N}{d_{N-1}}\|\hat{\ell}\|^2_{L^2(\hat{x}_{N-2},\hat{x}_{N+1})} \\
&\qquad + \frac{\hat{h}_{N-1}+\hat{h}_N}{d_N}\|\hat{\ell}\|^2_{L^2(\hat{x}_{N-1},\hat{x}_{N+1})}\Bigg).
\end{aligned} \tag{7.10}$$

Then proceeding as in Lemma 7.1, we obtain

$$\begin{aligned}
|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{b}|^2 \leq{}& \|\hat{\ell}\|^2_{L^2(\hat{x}_0,\hat{x}_1)} + \|\hat{\ell}\|^2_{L^2(\hat{x}_0,\hat{x}_2)} + \sum_{n=1}^{N}\|\hat{\ell}\|^2_{L^2(\hat{x}_{n-1},\hat{x}_{n+1})} + \|\hat{\ell}\|^2_{L^2(\hat{x}_N,\hat{x}_{N+1})} \\
&+ \|\hat{\ell}\|^2_{L^2(\hat{x}_{N-1},\hat{x}_{N+1})},
\end{aligned}$$

whence (7.9).

It remains to evaluate $\|\big(\boldsymbol{I} + \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{K}\boldsymbol{D}^{-\frac{1}{2}}\big)^{-1}\|_2$. Following Crouzeix and Thomée, we introduce the following constraint on the mesh length $\hat{h}_n$:

**Hypothesis 7.1** *There exist two constants $c_0 > 0$ and $\gamma > 0$ independent of $N$ such that*

$$\forall n, m, 0 \le n, m \le N, \quad \frac{\hat{h}_n}{\hat{h}_m} \le c_0 \gamma^{|n-m|}. \tag{7.11}$$

This condition holds for quasi-uniform meshes, but it is also satisfied by much more general meshes.

**Proposition 7.1** *If Hypothesis 7.1 holds with suitable constants $c_0$ and $\gamma$, then there exists a constant $C$ independent of $N$ and $j$ such that*

$$\|\left(\boldsymbol{I} + \boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K} \boldsymbol{D}^{-\frac{1}{2}}\right)^{-1}\|_2 \le C. \tag{7.12}$$

*Hence with the same constant $C$,*

$$\forall \hat{\boldsymbol{\ell}} \in L^2(\hat{\Gamma})^2, \quad \|\hat{\pi}_j(\hat{\boldsymbol{\ell}})\|_{L^2(\hat{\Gamma})} \le 2\sqrt{3} C \|\hat{\boldsymbol{\ell}}\|_{L^2(\hat{\Gamma})}. \tag{7.13}$$

*Proof* The proof follows the ideas of [23] but is more complex because the functions of $\hat{X}_j$ vanish at the end points of $\hat{\Gamma}$ whereas those of $\hat{\Lambda}$ do not. Let us prove convergence of the series

$$\sum_{r=1}^{\infty} \|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_2;$$

this will yield

$$\|\left(\boldsymbol{I} + \boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K} \boldsymbol{D}^{-\frac{1}{2}}\right)^{-1}\|_2 \le 1 + \sum_{r=1}^{\infty} \|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_2. \tag{7.14}$$

As $\boldsymbol{K}$ has three diagonals, then $\boldsymbol{K}^r$ has $2r + 1$ diagonals. In addition, since the coefficients of these diagonals are non negative, this implies that

$$\|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_1 \le (2r+1) \|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_\infty,$$

and by interpolation,

$$\|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_2 \le (2r+1)^{\frac{1}{2}} \|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_\infty.$$

Therefore

$$\|\boldsymbol{D}^{\frac{1}{2}} \boldsymbol{K}^r \boldsymbol{D}^{-\frac{1}{2}}\|_2 \le \sup_{|n-m| \le 2r} \left(\frac{d_n}{d_m}\right)^{\frac{1}{2}} (2r+1)^{\frac{1}{2}} \|\boldsymbol{K}\|_\infty^r. \tag{7.15}$$

Thus a sufficient condition for the series convergence is:

$$\sup_{|n-m|\leq 2r}\Big(\frac{d_n}{d_m}\Big)^{\frac{1}{2}} \leq c\Big(\frac{\delta}{\|\boldsymbol{K}\|_\infty}\Big)^r, \tag{7.16}$$

where $c > 0$ and $0 < \delta < 1$ are two constants independent of $r$. First, assuming Hypothesis 7.1, a simple but tedious computation gives for $0 < \gamma < 2$

$$\sup_{|n-m|\leq 2r}\frac{d_n}{d_m} \leq 2c_0(1+\gamma+\gamma^2)\gamma^{2r}. \tag{7.17}$$

Hence the series converges if

$$\gamma < \mathrm{Min}\Big(2, \frac{1}{\|\boldsymbol{K}\|_\infty}\Big).$$

It is easy to check that

$$\|\boldsymbol{K}\|_\infty = \mathrm{Max}\Bigg(\frac{\hat{h}_0+\hat{h}_1}{\hat{h}_0+2\hat{h}_1}, \frac{\hat{h}_N+\hat{h}_{N-1}}{\hat{h}_N+2\hat{h}_{N-1}}, \frac{1}{2}\Big(1+\frac{\hat{h}_0}{\hat{h}_0+\hat{h}_1+\hat{h}_2}\Big),$$
$$\frac{1}{2}\Big(1+\frac{\hat{h}_N}{\hat{h}_{N-2}+\hat{h}_{N-1}+\hat{h}_N}\Big)\Bigg).$$

To simplify the notation, set $\theta = c_0\gamma$. Owing to Hypothesis 7.1, for $\theta \geq 1$, we have

$$\|\boldsymbol{K}\|_\infty \leq \frac{1}{2}\Big(1+\mathrm{Max}\big(\frac{\theta}{2+\theta}, \frac{\theta^2}{1+\theta+\theta^2}\big)\Big) = \frac{1}{2}\Big(1+\frac{\theta^2}{1+\theta+\theta^2}\Big). \tag{7.18}$$

Take for instance $\theta = \frac{3}{2}$; then (7.18) yields $\|\boldsymbol{K}\|_\infty < \frac{3}{4}$. Therefore the series converges for $\gamma = \frac{4}{3}$ and in turn $c_0 = \frac{9}{8}$. Then (7.13) is an immediate consequence of Lemmas 7.1 and 7.2.

Now we estimate $\hat{\pi}_j(\hat{\boldsymbol{\ell}})$ in $H^1(\hat{\Gamma})^2$ for $\hat{\boldsymbol{\ell}} \in H_0^1(\hat{\Gamma})^2$. First, we have the analogue of Lemma 7.1.

**Lemma 7.3** *We keep the notation of Lemma 7.1. Under Hypothesis 7.1 with $\gamma > 1$, we have*

$$|\hat{\pi}_j(\hat{\ell})|_{H^1(\hat{\Gamma})} \leq \sqrt{\frac{2}{3}}\Big(c_0\gamma(1+\gamma)+2+\frac{c_0\gamma^2}{1+\gamma}\Big)^{\frac{1}{2}}|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{\alpha}|. \tag{7.19}$$

*Proof* Let us denote the derivation on $\hat{\Gamma}$ with a prime. Then

$$\hat{\pi}_j(\hat{\ell})' = \sum_{n=1}^{N}\alpha_n\hat{\varphi}_n',$$

and a straightforward computation gives

$$\|\hat{\pi}_j(\hat{\ell})'\|^2_{L^2(\hat{\Gamma})} \leq 2\left(\alpha_1^2\left(\frac{1}{\hat{h}_0}+\frac{1}{\hat{h}_1}\right)+\alpha_2^2\left(\frac{1}{\hat{h}_0+\hat{h}_1}+\frac{1}{\hat{h}_2}\right)+\sum_{n=3}^{N-2}\alpha_n^2\left(\frac{1}{\hat{h}_{n-1}}+\frac{1}{\hat{h}_n}\right)\right.$$
$$\left.+\alpha_{N-1}^2\left(\frac{1}{\hat{h}_{N-2}}+\frac{1}{\hat{h}_{N-1}+\hat{h}_N}\right)+\alpha_N^2\left(\frac{1}{\hat{h}_{N-1}}+\frac{1}{\hat{h}_N}\right)\right).$$

Hypothesis 7.1 yields

$$\frac{1}{\hat{h}_0}+\frac{1}{\hat{h}_1} \leq \frac{1}{2d_1}(1+c_0\gamma),\ \ \frac{1}{\hat{h}_0+\hat{h}_1}+\frac{1}{\hat{h}_2} \leq \frac{1}{3d_2}\left(c_0\gamma(1+\gamma)+2+\frac{c_0\gamma^2}{1+\gamma}\right),$$
$$\frac{1}{\hat{h}_{n-1}}+\frac{1}{\hat{h}_n} \leq \frac{2}{3d_n}(1+c_0\gamma),$$
$$\frac{1}{\hat{h}_{N-2}}+\frac{1}{\hat{h}_{N-1}+\hat{h}_N} \leq \frac{1}{3d_{N-1}}\left(c_0\gamma(1+\gamma)+2+\frac{c_0\gamma^2}{1+\gamma}\right),$$
$$\frac{1}{\hat{h}_{N-1}}+\frac{1}{\hat{h}_N} \leq \frac{1}{2d_N}(1+c_0\gamma).$$

Then (7.19) follows readily from the fact that for $\gamma > 1$,

$$c_0\gamma(1+\gamma)+2+\frac{c_0\gamma^2}{1+\gamma} > 2(1+c_0\gamma).$$

Comparing with (7.8) and the proof of Lemma 7.2, we see that a direct estimate of $\hat{\pi}_j(\hat{\boldsymbol{\ell}})'$ relies on a bound for $|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|$, which we can hardly expect to be sharp. This difficulty can be bypassed by using the fact that $\hat{\boldsymbol{\ell}}$ belongs to $H_0^1(\hat{\Gamma})^2$ and arguing as in [23]. Uniqueness of the solution of (4.24) in $\hat{X}_j$ implies that $\hat{\pi}_j(\hat{I}(\hat{\boldsymbol{\ell}})) = \hat{I}(\hat{\boldsymbol{\ell}})$ where $\hat{I}$ denotes the standard Lagrange interpolant in $\hat{X}_j$, that is well-defined because $\hat{\Gamma}$ is a line segment. This permits to write

$$\hat{\pi}_j(\hat{\boldsymbol{\ell}})' = \hat{\pi}_j(\hat{\boldsymbol{\ell}} - \hat{I}(\hat{\boldsymbol{\ell}}))' + (\hat{I}(\hat{\boldsymbol{\ell}}) - \hat{\boldsymbol{\ell}})' + \hat{\boldsymbol{\ell}}'. \tag{7.20}$$

First, we easily derive that

$$|\hat{I}(\hat{\boldsymbol{\ell}}) - \hat{\boldsymbol{\ell}}|_{H^1(\hat{\Gamma})} \leq 2|\hat{\boldsymbol{\ell}}|_{H^1(\hat{\Gamma})}.$$

Therefore

$$|\hat{\pi}_j(\hat{\boldsymbol{\ell}})|_{H^1(\hat{\Gamma})} \leq 3|\hat{\boldsymbol{\ell}}|_{H^1(\hat{\Gamma})} + |\hat{\pi}_j(\hat{\boldsymbol{\ell}} - \hat{I}(\hat{\boldsymbol{\ell}}))|_{H^1(\hat{\Gamma})}, \tag{7.21}$$

and it suffices to derive a bound for this last term. Let $\boldsymbol{M\alpha} = \boldsymbol{b}$ be the system (4.24) for a generic component of $\hat{\boldsymbol{\ell}} - \hat{I}(\hat{\boldsymbol{\ell}})$. Applying (7.12) and the analogue of (7.8), we obtain

$$|\hat{\pi}_j(\hat{\ell} - \hat{I}(\hat{\ell}))|_{H^1(\hat{\Gamma})} \leq \|\big(\boldsymbol{I} + \boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{K}\boldsymbol{D}^{\frac{1}{2}}\big)^{-1}\|_2 |\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}| \leq C|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|, \quad (7.22)$$

with the constant $C$ of (7.12), provided $c_0$ and $\gamma$ are chosen as in the proof of Proposition 7.1. Here we use the fact that (7.15) is also valid for $\|\boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{K}^r\boldsymbol{D}^{\frac{1}{2}}\|_2$. It remains to estimate $|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|$.

**Proposition 7.2** *Let $\hat{\boldsymbol{\ell}}$ belong to $H_0^1(\hat{\Gamma})^2$. If Hypothesis 7.1 holds with the constants $c_0$ and $\gamma$ chosen as in the proof of Proposition 7.1, then each component $\hat{\ell}$ of $\hat{\boldsymbol{\ell}}$ satisfies*

$$|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}| < \sqrt{30}c|\hat{\ell}|_{H^1(\hat{\Gamma})}, \quad (7.23)$$

*where $c$ depends only on the interpolant $\hat{I}$. Therefore,*

$$\forall \hat{\boldsymbol{\ell}} \in H_0^1(\hat{\Gamma})^2, \quad |\hat{\pi}_j(\hat{\boldsymbol{\ell}})|_{H^1(\hat{\Gamma})} < \big(3 + \sqrt{30}cC\big)|\hat{\boldsymbol{\ell}}|_{H^1(\hat{\Gamma})}, \quad (7.24)$$

*with the constants $c$ and $C$ of* (7.23) *and* (7.12).

*Proof* We sketch the proof. To simplify, set $w = \hat{\ell} - \hat{I}(\hat{\ell})$. Arguing as in Lemma 7.2, we recover a bound for $|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|$ by replacing $d_n$ by $d_n^3$ and $\hat{\ell}$ by $w$ in (7.10). The key point here is that the properties of the interpolant $\hat{I}$ imply

$$\|w\|^2_{L^2(\hat{x}_0,\hat{x}_1)} \leq c\big(\hat{h}_0^2|\hat{\ell}|^2_{H^1(\hat{x}_0,\hat{x}_1)} + \hat{h}_1^2|\hat{\ell}|^2_{H^1(\hat{x}_1,\hat{x}_2)}\big),$$

for a constant $c$ independent of $N$, with analogous expressions for the norms in the other subintervals. The factors involving $\hat{h}_n$ in $|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|$ can be bounded by applying (7.11) with $c_0\gamma = \frac{3}{2}$, as in the proof of Proposition 7.1. This gives

$$|\boldsymbol{D}^{-\frac{3}{2}}\boldsymbol{b}|^2 < 10c\Bigg(|\hat{\ell}|^2_{H^1(\hat{x}_0,\hat{x}_1)} + |\hat{\ell}|^2_{H^1(\hat{x}_0,\hat{x}_2)} + \sum_{n=1}^{N}|\hat{\ell}|^2_{H^1(\hat{x}_{n-1},\hat{x}_{n+1})} + |\hat{\ell}|^2_{H^1(\hat{x}_N,\hat{x}_{N+1})} + |\hat{\ell}|^2_{H^1(\hat{x}_{N-1},\hat{x}_{N+1})}\Bigg),$$

and (7.23) follows.

Finally, space interpolation between (7.13) and (7.24) gives

$$\forall \hat{\boldsymbol{\ell}} \in H_{00}^{\frac{1}{2}}(\hat{\Gamma})^2, \quad |\hat{\pi}_j(\hat{\boldsymbol{\ell}})|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})} < \hat{C}|\hat{\boldsymbol{\ell}}|_{H_{00}^{\frac{1}{2}}(\hat{\Gamma})}, \quad (7.25)$$

with a constant $\hat{C}$ independent of $N$.

*7.1.2 Second example*

In the first example, $\hat{\Lambda}$ and $\hat{\boldsymbol{X}}_j$ have the same dimension, but of course this is not necessary. For the unique solvability of (3.8), it is sufficient that the set of degrees of freedom of $\hat{\Lambda}$ be an adequate subset of those of $\hat{\boldsymbol{X}}_j$, sufficiently rich to guarantee a good approximation property of the Lagrange interpolant $\hat{I}$. Let us briefly give another simple choice of $\hat{\Lambda}$ with roughly half as many degrees of freedom as in the first example. Let $N$ be an odd integer, keep the same $\hat{\varphi}_0$ and $\hat{\varphi}_{N+1}$ as in (7.1), and choose

$$\begin{aligned}
\hat{\varphi}_1(\hat{x})|_{[\hat{x}_0,\hat{x}_1]} &= \frac{1}{\hat{h}_0}(\hat{x}-\hat{x}_0), \quad \hat{\varphi}_1(\hat{x})|_{[\hat{x}_1,\hat{x}_2]} = \frac{1}{\hat{h}_1}(\hat{x}_2-\hat{x}),\\
\hat{\varphi}_N(\hat{x})|_{[\hat{x}_{N-1},\hat{x}_N]} &= \frac{1}{\hat{h}_{N-1}}, \quad (\hat{x}-\hat{x}_{N-1}) \ \hat{\varphi}_N(\hat{x})|_{[\hat{x}_N,\hat{x}_{N+1}]} = \frac{1}{\hat{h}_N}(\hat{x}_{N+1}-\hat{x}),\\
\hat{\varphi}_{2n}(\hat{x})|_{[\hat{x}_{2n-2},\hat{x}_{2n}]} &= \frac{1}{\hat{h}_{2n-2}+\hat{h}_{2n-1}}(\hat{x}-\hat{x}_{2n-2}),\\
\hat{\varphi}_{2n}(\hat{x})|_{[\hat{x}_{2n},\hat{x}_{2n+2}]} &= \frac{1}{\hat{h}_{2n}+\hat{h}_{2n+1}}(\hat{x}_{2n+2}-\hat{x}), \quad 1 \le n \le \frac{N}{2},
\end{aligned} \tag{7.26}$$

extended by zero elsewhere. We take

$$\hat{X}_j = \left\{ \hat{\boldsymbol{v}} \in H_0^1(0,1)^2;\ \hat{\boldsymbol{v}}(\hat{x}) = \hat{\boldsymbol{v}}_1\hat{\varphi}_1(\hat{x}) + \sum_{n=1}^{N/2} \hat{\boldsymbol{v}}(\hat{x}_{2n})\hat{\varphi}_{2n}(\hat{x}) + \hat{\boldsymbol{v}}_N\hat{\varphi}_N(\hat{x}) \right\}, \tag{7.27}$$

$$\hat{\Lambda} = \left\{ \hat{\boldsymbol{\mu}} \in H^1(0,1)^2;\ \hat{\boldsymbol{\mu}}(\hat{x}) = \sum_{n=0}^{(N+1)/2} \hat{\boldsymbol{\mu}}(\hat{x}_{2n})\hat{\varphi}_{2n}(\hat{x}) \right\}. \tag{7.28}$$

With this choice, (4.24) is uniquely solvable and it can be checked that all the results established for the first example carry over to this second example, with different constants.

## 7.2 The 3-D case

In addition to restricting the mesh in 3-D (see Remark 4.1), the above proofs are complex because they apply to a situation where the matrix of the system is irreducible. The proofs are much easier, when (3.8) represents local and independent conditions, but in general, this can only be achieved by suitably modifying the discrete velocity spaces. In this section we present an example for which (3.8) holds by explicitly constructing the solution of (4.3). The correction satisfies suitable continuity bounds that are needed to establish the stability estimate (3.25). We follow the approach of BenBelgacem [12] who presents a local construction in 3-D by adding to the space $X_{s,j}^h$ in $\Omega_{s,j}$ a stabilizing bubble function in each face $T'$ of the trace $\mathcal{T}_{j,\Gamma_{ij}}^h$ of the triangulation $\mathcal{T}_j^h$ on $\Gamma_{ij}$ and choosing for $\Lambda^H$ constant vectors on each face $T'$. More precisely, for each $j$, $1 \le j \le M_s$, and for each $T$ in $\mathcal{T}_j^h$, let $\mathcal{P}(T)$ denote a polynomial space such as the mini-element or Bernardi–Raugel element, used in approximating the velocity of the Stokes problem with order one. For each face $T'$ of $\mathcal{T}_{j,\Gamma_{ij}}^h$, let $T$ be the element of $\mathcal{T}_j^h$ with face $T'$, and define

$$b_{T'}|_T = \lambda_1\lambda_2\lambda_3,$$

where $\lambda_k$, $k = 1, 2, 3$, denote the barycentric coordinates of the three vertices of $T'$; it is extended by zero outside $T$. Then set

$$\begin{aligned} X_{s,j}^h = \{\boldsymbol{v} \in \mathcal{C}^0(\overline{\Omega}_{s,j})^3;\ \forall T \in \mathcal{T}_j^h, T \text{ not adjacent to } \Gamma_{ij}, \boldsymbol{v}|_T \in \mathcal{P}(T), \\ \forall T \in \mathcal{T}_j^h, T \text{ adjacent to } \Gamma_{ij}, \boldsymbol{v}|_T \in \mathcal{P}(T) + b_{T'}\mathbb{R}^3\}, \end{aligned} \tag{7.29}$$

and choose

$$\Lambda_H = \{\boldsymbol{\mu}_H \in L^2(\Gamma_{ij})^3;\ \forall T' \in \mathcal{T}_{j,\Gamma_{ij}}^h, \boldsymbol{\mu}_H|_{T'} \in \mathbb{P}_0^3\}. \tag{7.30}$$

As $b_{T'}$ vanishes at all vertices of $T$, it does not change the approximating properties of $\mathcal{P}(T)$. Note that $\Lambda_H$ does not satisfy Hypothesis 3.2 (3). Nevertheless a discrete Korn inequality holds in $V_s^h$, see Propositions 7.3 and 7.4 below. With this choice, we can show that (3.8) holds. Indeed, for $\boldsymbol{v} \in \tilde{X}$, it is easy to see that

$$\boldsymbol{v}^h = \sum_{T' \in \mathcal{T}_{j,\Gamma_{ij}}^h} \boldsymbol{v}_{T'} b_{T'}, \quad \text{where } \boldsymbol{v}_{T'} = \frac{1}{\int_{T'} b_{T'}} \int_{T'} \boldsymbol{v}$$

satisfies (3.8). We now define

$$\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v}) = \sum_{T' \in \mathcal{T}_{j,\Gamma_{ij}}^h} \boldsymbol{c}_{T'} b_{T'}, \quad \text{where } \boldsymbol{c}_{T'} = \frac{1}{\int_{T'} b_{T'}} \int_{T'} \left(\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}} - S^h(\boldsymbol{v})|_{\Omega_{s,j}}\right), \tag{7.31}$$

where $\Theta_s^h(\boldsymbol{v})|_{\Omega_{s,i}}$ is computed in the preceding subsection by correcting $S^h(\boldsymbol{v})|_{\Omega_{s,i}}$ with $\boldsymbol{c}_{i,\Gamma_{ij}}^h(\boldsymbol{v})$ defined in (4.17).

**Lemma 7.4** *The correction $\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})$ defined by (7.31) satisfies (4.3) and there exists a constant $C$ independent of $h$ and the diameter of $\Omega_{s,i}$, $\Omega_{s,j}$, and $\Gamma_{ij}$ such that for all $\boldsymbol{v}$ in $H_0^1(\Omega)^n$*

$$|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})|_{H^1(\Omega_{s,j})} \le C|\boldsymbol{v}|_{H^1(\Omega_{s,i}\cup\Omega_{s,j})}, \quad \|\boldsymbol{c}_{j,\Gamma_{ij}}^h(\boldsymbol{v})\|_{L^2(\Omega_{s,j})} \le Ch|\boldsymbol{v}|_{H^1(\Omega_{s,i}\cup\Omega_{s,j})}. \tag{7.32}$$

*Proof* For any face $T'$ of $\mathcal{T}^h_{j,\Gamma_{ij}}$, we can write

$$\boldsymbol{c}_{T'} = \frac{1}{\int_{T'} b_{T'}} \int\limits_{T'} \big(\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} - \boldsymbol{v} - (S^h(\boldsymbol{v})|_{\Omega_{s,j}} - \boldsymbol{v})\big).$$

But on $\Gamma_{ij}$,

$$\Theta^h_s(\boldsymbol{v})|_{\Omega_{s,i}} = S^h(\boldsymbol{v})|_{\Omega_{s,i}} + \boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}),$$

owing to the support of each correction. Therefore

$$\boldsymbol{c}_{T'} = \frac{1}{\int_{T'} b_{T'}} \int\limits_{T'} \big(\boldsymbol{c}^h_{i,\Gamma_{ij}}(\boldsymbol{v}) + (S^h(\boldsymbol{v})|_{\Omega_{s,i}} - \boldsymbol{v}) - (S^h(\boldsymbol{v})|_{\Omega_{s,j}} - \boldsymbol{v})\big) = \boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{A}_3.$$

Let $T$ be the element of $\mathcal{T}^h_j$ adjacent to $T'$. By arguing as in the proof of Lemma 4.1, we easily derive

$$|\boldsymbol{A}_3 b_{T'}|_{H^1(T)} \leq \hat{C} |\boldsymbol{v}|_{H^1(\Delta_T)}, \tag{7.33}$$

where $\Delta_T$ is the macro-element of $\mathcal{T}^h_j$ required to define $S^h(\boldsymbol{v})$ in $T$. The estimation of $\boldsymbol{A}_2$ is similar, but more technical because $T'$ belongs to $\mathcal{T}^h_j$ whereas $S^h(\boldsymbol{v})|_{\Omega_{s,i}}$ is constructed on $\mathcal{T}^h_i$. Following the argument used in the proof of Lemma 4.3, we derive

$$|\boldsymbol{A}_2 b_{T'}|_{H^1(T)} \leq \hat{C} \sum_{\ell} |\boldsymbol{v}|_{H^1(\Delta_{T_\ell})}, \tag{7.34}$$

where $\{T_\ell\}$ denote the set of elements of $\mathcal{T}^h_i$ that intersect $T$.

The bound for $\boldsymbol{A}_1$ follows the same lines. With the notation of Lemma 4.1,

$$\boldsymbol{A}_1 = \frac{1}{\int_{T'} b_{T'}} \int\limits_{T'} \sum_k \boldsymbol{c}_k b_k,$$

where the sum runs over the indices $k$ for which $\mathcal{O}_k \cap T' \neq \emptyset$. According to Hypothesis 4.1, the number of terms in this sum is bounded by a fixed integer. Thus

$$|\boldsymbol{A}_1| \leq \sum_k \hat{C} \rho_k^{-\frac{1}{2}} |\boldsymbol{v}|_{H^1(D_k)},$$

and Hypothesis 4.1 implies that

$$|\boldsymbol{A}_1 b_{T'}|_{H^1(T)} \leq \hat{C} |\boldsymbol{v}|_{H^1(\tilde{\Delta}_{T_\ell})}, \tag{7.35}$$

where $\tilde{\Delta}_{T_\ell}$ is a macro-element in $\Omega_{s,i}$ and again the number of elements in $\tilde{\Delta}_{T_\ell}$ is bounded by a fixed integer. All constants are independent of $h$ and the diameter of $\Omega_{s,i}$, $\Omega_{s,j}$, and $\Gamma_{ij}$. Finally, the first inequality in (7.32) follows readily by summing (7.33)–(7.35) and applying Hypothesis 4.1. The argument for the second inequality in (7.32) is similar.

### *7.2.1 A discrete Korn inequality in $V_s^h$*

Since all assumptions except Hypothesis 3.2 (3) hold, it suffices to examine the jump of functions of $V_s^h$ through the interfaces $\Gamma_{ij} \in \Gamma_{ss}$. The next two lemmas show that the projection of this jump on polynomials of $\mathbb{P}_1$ is small.

**Lemma 7.5** *Let $\Gamma_{ij} \in \Gamma_{ss}$, $i < j$. There exists a constant $C$, independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$, and $\Omega_{s,j}$ such that*

$$\forall \boldsymbol{v}^h \in V_s^h, \forall \boldsymbol{p} \in \mathbb{P}_1^3, \left| \int\limits_{\Gamma_{ij}} [\boldsymbol{v}^h] \cdot \boldsymbol{p} \right| \le C h^{\frac{3}{2}} |\Gamma_{ij}|^{\frac{1}{2}} \big( |\boldsymbol{v}^h|_{H^1(D_{s,i})} + |\boldsymbol{v}^h|_{H^1(D_{s,j})} \big), \tag{7.36}$$

*where $D_{s,k}$ denotes the layer of elements in $\Omega_{s,k}$ adjacent to $\Gamma_{ij}$.*

*Proof* All constants in this proof are independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$, and $\Omega_{s,j}$. Let $T'$ be an element (i.e. a face) of $\mathcal{T}^h_{j,\Gamma_{ij}}$, which is the mesh of $\Lambda^H$. There is no loss of generality in assuming that $T'$ lies on the $x_1 - x_2$ plane. Consider a generic component $v^h$ of $\boldsymbol{v}^h$. Since the jump already satisfies

$$\int\limits_{T'} [v^h] = 0,$$

it suffices to bound the product of the jump by $x_k$, $k = 1, 2$, say $x_1$. Then for any constant $c$,

$$\int\limits_{T'} [v^h] x_1 = \int\limits_{T'} \big( [v^h] - c \big) \left( x_1 - \frac{1}{|T'|} \int\limits_{T'} x_1 \right).$$

A scaling argument gives

$$\left\| x_1 - \frac{1}{|T'|} \int\limits_{T'} x_1 \right\|_{L^2(T')} \le \hat{C} |T'|^{\frac{1}{2}} h_{T'}.$$

Similarly,

$$\inf_{c \in \mathbb{R}} \|[v^h] - c\|_{L^2(T')} \le \hat{C} h_{T'} |[v^h]|_{H^1(T')} \le \hat{C} h_{T'} \big( |v^h|_{\Omega_{s,i}}|_{H^1(T')} + |v^h|_{\Omega_{s,j}}|_{H^1(T')} \big).$$

On one hand, equivalence of norms gives

$$|v^h|_{\Omega_{s,j}}|_{H^1(T')} \leq \hat{C}\,\rho_T^{-\frac{1}{2}}|v^h|_{\Omega_{s,j}}|_{H^1(T)},$$

where $T$ is the element of $\mathcal{T}_j^h$ adjacent to $T'$. On the other hand, arguing as in the proof of Lemma 7.4, we prove that

$$|v^h|_{\Omega_{s,i}}|_{H^1(T')} \leq \hat{C}\,\rho_T^{-\frac{1}{2}}|v^h|_{\Omega_{s,i}}|_{H^1(\Delta_T)},$$

where $\Delta_T$ is the union of all elements of $\mathcal{T}_i^h$ that intersect $T'$. Then taking the product and summing over all faces $T'$ of $\mathcal{T}_{j,\Gamma_{ij}}^h$, we obtain

$$\left|\int_{\Gamma_{ij}} [v^h]x_1\right| \leq \hat{C}h^{\frac{3}{2}}\left(\sum_{T'\in\mathcal{T}_{j,\Gamma_{ij}}^h} |T'|\right)^{\frac{1}{2}}\big(|v^h|_{H^1(D_{s,i})}+|v^h|_{H^1(D_{s,j})}\big),$$

whence (7.36).

**Lemma 7.6** *Let $\Gamma_{ij}\in\Gamma_{ss}$, $i<j$, and let $\Pi([\boldsymbol{v}^h])\in\mathbb{P}_1^3$ denote the projection of $[\boldsymbol{v}^h]$ onto $\mathbb{P}_1^3$ on $\Gamma_{ij}$. There exists a constant $C$, independent of $h$ and the diameters of $\Gamma_{ij}$, $\Omega_{s,i}$, and $\Omega_{s,j}$ such that*

$$\forall \boldsymbol{v}^h\in V_s^h,\quad \|\Pi([\boldsymbol{v}^h])\|_{L^2(\Gamma_{ij})} \leq Ch^{\frac{3}{2}}\big(|\boldsymbol{v}^h|_{H^1(D_{s,i})}+|\boldsymbol{v}^h|_{H^1(D_{s,j})}\big). \quad (7.37)$$

*Proof* By definition $\Pi([\boldsymbol{v}^h])\in\mathbb{P}_1^3$ solves

$$\forall \boldsymbol{p}\in\mathbb{P}_1^3,\quad \int_{\Gamma_{ij}}\Pi([\boldsymbol{v}^h])\cdot\boldsymbol{p} = \int_{\Gamma_{ij}}[\boldsymbol{v}^h]\cdot\boldsymbol{p}.$$

By passing to the reference subdomain $\hat{\Omega}_j$, this reads

$$\forall \hat{\boldsymbol{p}}\in\mathbb{P}_1^3,\quad \int_{\hat{\Gamma}}\Pi([\boldsymbol{v}^h])\circ F_j\cdot\hat{\boldsymbol{p}} = |\hat{\Gamma}|\,|\Gamma_{ij}|^{-1}\int_{\Gamma_{ij}}[\boldsymbol{v}^h]\cdot\boldsymbol{p}.$$

The coefficients of the polynomial of degree one $\Pi([\boldsymbol{v}^h])\circ F_j$ solve a linear system whose matrix only depends on $\hat{\Gamma}$. Therefore, in view of (7.36),

$$\|\Pi([\boldsymbol{v}^h])\|_{L^2(\Gamma_{ij})} \leq |\Gamma_{ij}|^{\frac{1}{2}}\|\Pi([\boldsymbol{v}^h])\circ F_j\|_{L^\infty(\hat{\Gamma})} \leq \hat{C}h^{\frac{3}{2}}\big(|\boldsymbol{v}^h|_{H^1(D_{s,i})}+|\boldsymbol{v}^h|_{H^1(D_{s,j})}\big).$$

This last result enables us to prove Korn's inequality in $V_s^h$.

**Proposition 7.3** *Let $|\Gamma_s| > 0$ and $\Omega_s$ be connected. Then there exists $h_0 > 0$ such that for all $h \le h_0$,*

$$\forall \boldsymbol{v}^h \in V_s^h, \quad \sum_{i=1}^{M_s} |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,i})} \le C \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}^h)\|^2_{L^2(\Omega_{s,i})}; \tag{7.38}$$

*the constants $C$ and $h_0$ are independent of $h$ and the diameters of the interfaces $\Gamma_{ij}$ and the subdomains $\Omega_{s,k}$.*

*Proof* Let $\boldsymbol{v}^h \in V_s^h$. Formula (1.12) in [16] gives, with a constant $C_1$ that depends only on the shape regularity of the mesh of subdomains $\mathcal{T}_\Omega$,

$$\begin{aligned}
&\sum_{i=1}^{M_s} |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,i})} \\
&\quad \le C_1 \left( \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}^h)\|^2_{L^2(\Omega_{s,i})} + \sum_{i<j} \frac{1}{\operatorname{diam}(\Gamma_{ij})} \|\Pi([\boldsymbol{v}^h])\|^2_{L^2(\Gamma_{ij})} \right) \\
&\quad \le C_1 \left( \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}^h)\|^2_{L^2(\Omega_{s,i})} + \hat{C} \sum_{i<j} \frac{1}{\operatorname{diam}(\Gamma_{ij})} h^3 \big( |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,i})} + |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,j})} \big) \right),
\end{aligned}$$

owing to (7.37). But $h < \operatorname{diam}(\Gamma_{ij})$ and there exists an $h_0 > 0$ such that

$$\forall h \le h_0, \quad h^2 \, (C_1 \hat{C})^{-1} \le \frac{1}{2}.$$

Since $C_1$ and $\hat{C}$ are independent of $h$ and the diameters of the interfaces $\Gamma_{ij}$ and the subdomains, then so is $h_0$. Hence for all $h \le h_0$,

$$\sum_{i=1}^{M_s} |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,i})} \le 2C \sum_{i=1}^{M_s} \|\boldsymbol{D}(\boldsymbol{v}^h)\|^2_{L^2(\Omega_{s,i})}.$$

By arguing as in the proof of Lemma 3.4, we easily derive Korn's inequality when $|\Gamma_s| = 0$ and $\Omega_s$ is connected.

**Proposition 7.4** *Let $|\Gamma_s| = 0$ and $\Omega_s$ be connected, i.e. $\Gamma_{sd} = \partial\Omega_s$. Then there exists $h_0 > 0$ such that for all $h \le h_0$,*

$$\begin{aligned}
&\forall \boldsymbol{v}^h \in V_s^h, \\
&\sum_{i=1}^{M_s} |\boldsymbol{v}^h|^2_{H^1(\Omega_{s,i})} \le C \sum_{i=1}^{M_s} \left( \|\boldsymbol{D}(\boldsymbol{v}^h)\|^2_{L^2(\Omega_{s,i})} + \left( \sum_{l=1}^{2} \int_{\Gamma_{sd}} |\boldsymbol{v}_s^h \cdot \boldsymbol{\tau}_l| \right)^2 \right);
\end{aligned} \tag{7.39}$$

*the constants $C$ and $h_0$ are independent of $h$ and the diameters of the interfaces $\Gamma_{ij}$ and the subdomains $\Omega_{s,k}$.*

The case when $\Omega_s$ is not connected follows from these two propositions applied to each connected component of $\Omega_s$ according that it is or not adjacent to $\Gamma_s$.

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