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A multipoint stress-flux mixed finite element method for the Stokes-Biot model

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Abstract

In this paper we present and analyze a fully-mixed formulation for the coupled problem arising in the interaction between a free fluid and a poroelastic medium. The flows in the free fluid and poroelastic regions are governed by the Stokes and Biot equations, respectively, and the transmission conditions are given by mass conservation, balance of stresses, and the Beavers-Joseph-Saffman law. We apply dual-mixed formulations in both domains, where the symmetry of the Stokes and poroelastic stress tensors is imposed by setting the vorticity and structure rotation tensors as auxiliary unknowns. In turn, since the transmission conditions become essential, they are imposed weakly by introducing the traces of the fluid velocity, structure velocity, and the poroelastic media pressure on the interface as the associated Lagrange multipliers. The existence and uniqueness of a solution are established for the continuous weak formulation, as well as a semidiscrete continuous-in-time formulation with non-matching grids, together with the corresponding stability bounds. In addition, we develop a new multipoint stress-flux mixed finite element method by involving the vertex quadrature rule, which allows for local elimination of the stresses, rotations, and Darcy fluxes. Well-posedness and

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error analysis with corresponding rates of convergence for the fully-discrete scheme are complemented by several numerical experiments.

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1 Introduction

The interaction of a free fluid with a deformable porous medium, referred to as fluid-poroelastic structure interaction (FPSI), is a challenging multiphysics problem. It has applications to predicting and controlling processes arising in gas and oil extraction from naturally or hydraulically fractured reservoirs, modeling arterial flows, and designing industrial filters, to name a few. For this physical phenomenon, the free fluid region can be modeled by the Stokes (or Navier–Stokes) equations, while the flow through the deformable porous medium is modeled by the Biot system of poroelasticity. In the latter, the volumetric deformation of the elastic porous matrix is complemented with the Darcy equation that describes the average velocity of the fluid in the pores. The two regions are coupled via dynamic and kinematic interface conditions, including balance of forces, continuity of normal velocity, and a no slip or slip with friction tangential velocity condition. The model exhibits features of both coupled Stokes-Darcy flows and fluid-structure interaction (FSI).

To the authors' knowledge, one of the first works in analyzing the Stokes-Biot coupled problem is [56], where well-posedness for the fully dynamic problem is established by developing an appropriate variational formulation and using semigroup methods. One of the first numerical studies is presented in [13], where monolithic and iterative partitioned methods are developed for the solution of the coupled system. A non-iterative operator splitting scheme with a non-mixed Darcy formulation is developed in [22]. Finite element methods for mixed Darcy formulations, where the continuity of normal flux condition becomes essential, are considered in [21] using Nitsche's coupling and in [9] using a pressure Lagrange multiplier. More recently, a nonlinear quasi-static Stokes–Biot model for non-Newtonian fluids is studied in [3]. The authors establish well-posedness of the weak formulation in Banach space setting, along with stability and convergence of the finite element approximation. In [26], the fully dynamic coupled Navier-Stokes/Biot system with a pressure-based Darcy formulation is analyzed. Additional works include optimization-based decoupling method [25], a second order in time split scheme [46], various discretization methods [14, 24, 59], dimensionally reduced model for flow through fractures [23], and coupling with transport [5]. All of the above mentioned works are based on displacement formulations for the elasticity equation. In a recent work [49], the first mathematical and numerical study of a stress-displacement mixed elasticity formulation for the Stokes-Biot model is presented.

The goal of the present paper is to develop a new fully mixed formulation of the quasi-static Stokes-Biot model, which is based on dual mixed formulations for all three components - Darcy, elasticity, and Stokes. In particular, we use a velocity-pressure Darcy formulation, a weakly symmetric stress-displacement-rotation elasticity formulation, and a weakly symmetric stress-velocity-vorticity Stokes formulation. This formulation exhibits multiple advantages, including local conservation of mass for the Darcy fluid, local poroelastic and Stokes momentum conservation, and accurate approximations with continuous normal components across element edges or faces for the Darcy velocity, the poroelastic stress, and the free fluid stress. In addition, dual mixed formulations are known for their locking-free properties and robustness with respect to the physical parameters, including the regimes of almost incompressible materials, low poroelastic storativity, and low permeability [47, 63]. We note that our analysis also applies to the dual mixed elasticity and Stokes formulations with strong stress symmetry, i.e., stress-displacement for elasticity and stress-velocity for Stokes. However, we focus on the weakly-symmetric formulations, since they allow for finite element approximations with fewer degrees of freedom, see, e.g., [11, 12]. Moreover, in certain low-order cases they are suitable for multipoint stress mixed finite element approximations [6–8], which are discussed below.

Our five-field dual mixed Biot formulation is based on the model developed in [47] and studied further in [8]. It is also considered in [49] for the Stokes-Biot problem. Our analysis also extends to the strongly symmetric mixed four-field Biot formulation developed in [62]. Our three-field dual mixed Stokes formulation is based on the models developed in [36, 37]. In particular, we introduce the stress tensor and subsequently eliminate the pressure unknown, by utilizing the deviatoric stress. In order to impose the symmetry of the Stokes stress and poroelastic stress tensors, the vorticity and structure rotation, respectively, are introduced as additional unknowns. The transmission conditions consisting of mass conservation, conservation of momentum, and the Beavers–Joseph–Saffman slip with friction condition are imposed weakly via the incorporation of additional Lagrange multipliers: the traces of the fluid velocity, structure velocity and the poroelastic media pressure on the interface. The resulting variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness and uniqueness of both the continuous and semidiscrete continuous-in-time formulations are proved by employing some classical results for parabolic problems [55, 57] and monotone operators, and an abstract theory for twofold saddle point problems [1, 35]. In the discrete problem, for the three components of the model we consider suitable stable mixed finite element spaces on non-matching grids across the interface, coupled through either conforming or non-conforming Lagrange multiplier discretizations. We develop stability and error analysis, establishing rates of convergence to the true solution. The estimates we establish are uniform in the limit of the storativity coefficient going to zero.

Another main contribution of this paper is the development of a new mixed finite element method for the Stokes-Biot model that can be reduced to a positive definite cell-centered pressure-velocities-traces system. We recall the multipoint flux mixed finite element (MFMFE) method for Darcy flow developed in [20, 42, 60, 61], where the lowest order Brezzi-Douglas-Marini $\mathbb{BDM}_1$ velocity spaces [18, 19, 50] and piecewise constant pressure space are utilized. An alternative formulation based on a broken Raviart-Thomas velocity space is developed in [45]. The use of the vertex quadrature rule for the velocity bilinear form localizes the interaction between velocity degrees of freedom around mesh vertices and leads to a block-diagonal mass matrix. Consequently, the velocity can be locally eliminated, resulting in a cell-centered pressure system. In turn, the multipoint stress mixed finite element (MSMFE) method for elasticity is developed in [6, 7]. It utilizes stable weakly symmetric elasticity finite element triples with $\mathbb{BDM}_1$ stress spaces [7, 11, 12, 17, 32, 48]. Similarly to the MFMFE method, an application of the vertex quadrature rule for the stress and rotation bilinear forms allows for local stress and rotation elimination, resulting in a cell-centered displacement system. We also refer the reader to the related finite volume multipoint stress approximation (MPSA) method for elasticity [43, 51, 52]. Recently, combining the MSMFE and MFMFE methods, a multipoint stress-flux mixed finite element (MSFMFE) method for the Biot poroelasticity model is developed in [8]. There, the dual mixed finite element system is reduced to a cell-centered displacement-pressure system. The reduced system is comparable in cost to the finite volume method developed in [53].

In this paper we note for the first time that the MSMFE method for elasticity can be applied to the weakly symmetric stress-velocity-vorticity Stokes formulation from [36, 37] when $\mathbb{BDM}_1$-based stable finite element triples are utilized. With the application of the vertex quadrature rule, the fluid stress and vorticity can be locally eliminated, resulting in a positive definite cell-centered velocity system. To the best of our knowledge, this is the first such scheme for Stokes in the literature.

Finally, we combine the MFMFE method for Darcy flow with the MSMFE methods for the elasticity and Stokes equations to develop a multipoint stress-flux mixed finite element for the Stokes-Biot system. We analyze the stability and convergence of the semidiscrete formulation. We further consider the fully discrete system with backward Euler time discretization and show that the algebraic system on each time step can be reduced to a positive definite cell-centered pressure-velocities-traces system.

The rest of this work is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. In Sect. 2 we introduce the model problem and in Sect. 3 we derive a fully-mixed variational formulation, which is written as a degenerate evolution problem with a twofold saddle point structure. Next, existence, uniqueness and stability of the solution of the weak formulation are obtained in Sect. 4. The corresponding semidiscrete continuous-in-time approximation is introduced and analyzed in Sect. 5, where the discrete analogue of the theory used in the continuous case is employed to prove its well-posedness. Error estimates and rates of convergence are also derived there. In Sect. 6, the multipoint stress-flux mixed finite element method is presented and the corresponding rates of convergence are provided, along with the analysis of the reduced cell-centered system. Finally, numerical experiments illustrating the accuracy of our mixed finite element method and its applications to coupling surface and subsurface flows and flow through poroelastic medium with a cavity are reported in Sect. 7.

We end this section by introducing some definitions and fixing some notations. Let $\mathcal{O} \subset \mathrm{R}^n$, $n \in \{2, 3\}$, denote a domain with Lipschitz boundary. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1, +\infty]$, we denote by $\mathrm{L}^\mathrm{p}(\mathcal{O})$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^\mathrm{p}(\mathcal{O})}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\mathcal{O})}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\mathcal{O}) = \mathrm{L}^\mathrm{p}(\mathcal{O})$. If $\mathrm{p} = 2$ we write $\mathrm{H}^\mathrm{s}(\mathcal{O})$ in place of $\mathrm{W}^{\mathrm{s},2}(\mathcal{O})$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^\mathrm{s}(\mathcal{O})}$. Similar notation is used for a section Γ of the boundary of $\mathcal{O}$. By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space M. The $\mathrm{L}^2(\mathcal{O})$ inner product for scalar, vector, or tensor valued functions is denoted by $(\cdot,\cdot)_{\mathcal{O}}$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. For any vector field $\mathbf{v} = (v_i)_{i=1,\dots,n}$, we set the gradient and divergence operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,\dots,n} \quad \text{and} \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}.$$

For any tensor fields $\boldsymbol{\tau} := (\tau_{ij})_{i,j=1,\dots,n}$ and $\boldsymbol{\zeta} := (\zeta_{ij})_{i,j=1,\dots,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,\dots,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n}\tau_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n}\tau_{ij}\zeta_{ij},$$
$$\text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I},$$

where $\mathbf{I}$ is the identity matrix in $\mathrm{R}^{n\times n}$. In addition, we recall the Hilbert space

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \Big\{\mathbf{v}\in\mathbf{L}^2(\mathcal{O}): \quad \operatorname{div}(\mathbf{v})\in\mathrm{L}^2(\mathcal{O})\Big\},$$

equipped with the norm $\|\mathbf{v}\|^2_{\mathbf{H}(\operatorname{div};\mathcal{O})} := \|\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{O})} + \|\operatorname{div}(\mathbf{v})\|^2_{\mathrm{L}^2(\mathcal{O})}$. The space of matrix valued functions whose rows belong to $\mathbf{H}(\operatorname{div};\mathcal{O})$ will be denoted by $\mathbb{H}(\mathbf{div};\mathcal{O})$ and endowed with the norm $\|\boldsymbol{\tau}\|^2_{\mathbb{H}(\mathbf{div};\mathcal{O})} := \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\mathcal{O})} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{\mathbf{L}^2(\mathcal{O})}$. Finally, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, we let $\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})$ be the space of classes of functions $f:(0,T)\to\mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} < \infty$, with

$$\|f\|^{\mathrm{p}}_{\mathrm{L}^{\mathrm{p}}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^{\mathrm{p}}_{\mathrm{V}}\,dt, \quad \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]}\|f(t)\|_{\mathrm{V}}.$$

2 The model problem

Let $\Omega\subset\mathrm{R}^n$, $n\in\{2,3\}$, be a Lipschitz domain, which is subdivided into two non-overlapping and possibly non-connected regions: fluid region Ω_f and poroelastic region Ω_p. Let $\Gamma_{fp} = \partial\Omega_f\cap\partial\Omega_p$ denote the (nonempty) interface between these regions and let $\Gamma_f = \partial\Omega_f\setminus\Gamma_{fp}$ and $\Gamma_p = \partial\Omega_p\setminus\Gamma_{fp}$ denote the external parts on the boundary $\partial\Omega$. We denote by $\mathbf{n}_f$ and $\mathbf{n}_p$ the unit normal vectors that point outward from $\partial\Omega_f$ and $\partial\Omega_p$, respectively, noting that $\mathbf{n}_f = -\mathbf{n}_p$ on Γ_{fp}. Let $(\mathbf{u}_\star, p_\star)$ be the velocity-pressure pair in $\Omega_\star$ with $\star\in\{f,p\}$, and let $\boldsymbol{\eta}_p$ be the displacement in Ω_p. Let $\mu>0$ be the fluid viscosity, let $\mathbf{f}_\star$ be the body force terms, and let $q_\star$ be the external source or sink terms.

We assume that the flow in Ω_f is governed by the Stokes equations, which are written in the following stress-velocity-pressure formulation:

$$\begin{aligned} &\boldsymbol{\sigma}_f = -p_f\,\mathbf{I} + 2\,\mu\,\mathbf{e}(\mathbf{u}_f), \quad -\mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f, \quad \operatorname{div}(\mathbf{u}_f) = q_f \quad \text{in} \quad \Omega_f \times (0,T], \\ &\boldsymbol{\sigma}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}} \times (0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}} \times (0,T], \end{aligned} \tag{2.1}$$

where $\boldsymbol{\sigma}_f$ is the stress tensor, $\mathbf{e}(\mathbf{u}_f) := \frac{1}{2}\left(\nabla\mathbf{u}_f + (\nabla\mathbf{u}_f)^{\mathrm{t}}\right)$ stands for the deformation rate tensor, $\Gamma_f = \Gamma_f^{\mathrm{N}} \cup \Gamma_f^{\mathrm{D}}$, and $T > 0$ is the final time. Next, we adopt the approach from [1, 36], and include as a new variable the vorticity tensor $\boldsymbol{\gamma}_f$,

$$\boldsymbol{\gamma}_f := \frac{1}{2}\left(\nabla\mathbf{u}_f - (\nabla\mathbf{u}_f)^{\mathrm{t}}\right).$$

In this way, owing to the fact that $\operatorname{tr}(\mathbf{e}(\mathbf{u}_f)) = \operatorname{div}(\mathbf{u}_f) = q_f$, we find that (2.1) can be rewritten, equivalently, as the set of equations with unknowns $\boldsymbol{\sigma}_f$, $\boldsymbol{\gamma}_f$ and $\mathbf{u}_f$, given by

$$\begin{aligned} &\frac{1}{2\,\mu}\,\boldsymbol{\sigma}_f^{\mathrm{d}} = \nabla\mathbf{u}_f - \boldsymbol{\gamma}_f - \frac{1}{n}\,q_f\,\mathbf{I}, \quad -\mathbf{div}(\boldsymbol{\sigma}_f) = \mathbf{f}_f \quad \text{in} \quad \Omega_f \times (0,T], \\ &\boldsymbol{\sigma}_f = \boldsymbol{\sigma}_f^{\mathrm{t}}, \quad p_f = -\frac{1}{n}\left(\operatorname{tr}(\boldsymbol{\sigma}_f) - 2\,\mu\,q_f\right) \quad \text{in} \quad \Omega_f \times (0,T], \\ &\boldsymbol{\sigma}_f\mathbf{n}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{N}} \times (0,T], \quad \mathbf{u}_f = \mathbf{0} \quad \text{on} \quad \Gamma_f^{\mathrm{D}} \times (0,T]. \end{aligned} \tag{2.2}$$

Notice that the fourth equation in (2.2) has allowed us to eliminate the pressure p_f from the system and provides a formula for its approximation through a post-processing procedure. For simplicity we assume that $|\Gamma_f^{\mathrm{N}}| > 0$, which will allow us to control $\boldsymbol{\sigma}_f$ by $\boldsymbol{\sigma}_f^{\mathrm{d}}$. The case $|\Gamma_f^{\mathrm{N}}| = 0$ can be handled as in [36–38] by introducing an additional variable corresponding to the mean value of $\operatorname{tr}(\boldsymbol{\sigma}_f)$.

In turn, let $\boldsymbol{\sigma}_e$ and $\boldsymbol{\sigma}_p$ be the elastic and poroelastic stress tensors, respectively, satisfying

$$A\,\boldsymbol{\sigma}_e = \mathbf{e}(\boldsymbol{\eta}_p) \quad \text{and} \quad \boldsymbol{\sigma}_p := \boldsymbol{\sigma}_e - \alpha_p\,p_p\,\mathbf{I} \quad \text{in} \quad \Omega_p \times (0,T], \tag{2.3}$$

where $0 < \alpha_p \leq 1$ is the Biot–Willis constant, and A is the symmetric and positive definite compliance tensor, which in the isotropic case has the form, for all tensors $\boldsymbol{\tau}$,

$$A(\boldsymbol{\tau}) := \frac{1}{2\,\mu_p}\left(\boldsymbol{\tau} - \frac{\lambda_p}{2\,\mu_p + n\,\lambda_p}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}\right), \quad \text{with} \quad A^{-1}(\boldsymbol{\tau}) = 2\,\mu_p\,\boldsymbol{\tau} + \lambda_p\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}, \tag{2.4}$$

satisfying

$$\forall\,\boldsymbol{\tau} \in \mathrm{R}^{n\times n}, \quad \frac{1}{2\mu_{\max} + n\,\lambda_{\max}}\,\boldsymbol{\tau} : \boldsymbol{\tau} \;\leq\; A(\boldsymbol{\tau}) : \boldsymbol{\tau} \;\leq\; \frac{1}{2\mu_{\min}}\,\boldsymbol{\tau} : \boldsymbol{\tau} \quad \forall\,\mathbf{x} \in \Omega_p. \tag{2.5}$$

In this case, $\boldsymbol{\sigma}_e := \lambda_p \operatorname{div}(\boldsymbol{\eta}_p)\,\mathbf{I} + 2\,\mu_p\,\mathbf{e}(\boldsymbol{\eta}_p)$, and $0 < \lambda_{\min} \leq \lambda_p(\mathbf{x}) \leq \lambda_{\max}$ and $0 < \mu_{\min} \leq \mu_p(\mathbf{x}) \leq \mu_{\max}$ are the Lamé parameters. The poroelasticity region Ω_p is governed by the quasi-static Biot system [15]:

$$\begin{aligned}
&-\mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p, \quad \mu\,\mathbf{K}^{-1}\mathbf{u}_p + \nabla\,p_p = \mathbf{0},\\
&\frac{\partial}{\partial t}\left(s_0\,p_p + \alpha_p \operatorname{div}(\boldsymbol{\eta}_p)\right) + \operatorname{div}(\mathbf{u}_p) = q_p \quad \text{in} \quad \Omega_p \times (0,T],\\
&\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}} \times (0,T], \quad p_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{D}} \times (0,T],\\
&\boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{N}} \times (0,T], \quad \boldsymbol{\eta}_p = \mathbf{0} \quad \text{on} \quad \tilde{\Gamma}_p^{\mathrm{D}} \times (0,T],
\end{aligned} \tag{2.6}$$

where $\Gamma_p = \Gamma_p^{\mathrm{N}} \cup \Gamma_p^{\mathrm{D}} = \tilde{\Gamma}_p^{\mathrm{N}} \cup \tilde{\Gamma}_p^{\mathrm{D}}$, $s_0 > 0$ is a storativity coefficient and $\mathbf{K}(\mathbf{x})$ is the symmetric and uniformly positive definite rock permeability tensor, satisfying, for some constants $0 < k_{\min} \leq k_{\max}$,

$$\forall\,\mathbf{w} \in \mathrm{R}^n, \quad k_{\min}\,\mathbf{w}\cdot\mathbf{w} \;\leq\; (\mathbf{K}\mathbf{w})\cdot\mathbf{w} \;\leq\; k_{\max}\,\mathbf{w}\cdot\mathbf{w} \quad \forall\,\mathbf{x} \in \Omega_p. \tag{2.7}$$

To avoid the issue with restricting the mean value of the pressure, we assume that $|\Gamma_p^{\mathrm{D}}| > 0$. We also assume that Γ_f^{D}, Γ_p^{D}, and $\tilde{\Gamma}_p^{\mathrm{D}}$ are not adjacent to the interface Γ_{fp}, i.e., $\exists\,s > 0$ such that $\operatorname{dist}(\Gamma_f^{\mathrm{D}}, \Gamma_{fp}) \geq s$, $\operatorname{dist}(\Gamma_p^{\mathrm{D}}, \Gamma_{fp}) \geq s$, and $\operatorname{dist}(\tilde{\Gamma}_p^{\mathrm{D}}, \Gamma_{fp}) \geq s$. This assumption is used to simplify the characterization of the normal trace spaces on Γ_{fp}.

Next, we introduce the following transmission conditions on the interface Γ_{fp} [9, 13, 21, 56]:

$$\begin{aligned}
&\mathbf{u}_f \cdot \mathbf{n}_f + \left(\frac{\partial\,\boldsymbol{\eta}_p}{\partial t} + \mathbf{u}_p\right)\cdot \mathbf{n}_p = 0, \quad \boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp} \times (0,T],\\
&\boldsymbol{\sigma}_f \mathbf{n}_f + \mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \sqrt{\mathbf{K}_j^{-1}} \left\{\left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot \mathbf{t}_{f,j}\right\} \mathbf{t}_{f,j} = -p_p \mathbf{n}_f \quad \text{on} \quad \Gamma_{fp} \times (0,T],
\end{aligned} \tag{2.8}$$

where $\mathbf{t}_{f,j}$, $1 \leq j \leq n-1$, is an orthogonal system of unit tangent vectors on Γ_{fp}, $\mathbf{K}_j = (\mathbf{K}\,\mathbf{t}_{f,j})\cdot \mathbf{t}_{f,j}$, and $\alpha_{\mathrm{BJS}} \geq 0$ is an experimentally determined friction coefficient. The first and second equations in (2.8) correspond to mass conservation and conservation of momentum on Γ_{fp}, respectively, whereas the third one can be decomposed into its normal and tangential components, as follows:

$$\begin{aligned}
&(\boldsymbol{\sigma}_f \mathbf{n}_f)\cdot \mathbf{n}_f = -p_p,\\
&(\boldsymbol{\sigma}_f \mathbf{n}_f)\cdot \mathbf{t}_{f,j} = -\mu\,\alpha_{\mathrm{BJS}} \sqrt{\mathbf{K}_j^{-1}} \left(\mathbf{u}_f - \frac{\partial\,\boldsymbol{\eta}_p}{\partial t}\right)\cdot \mathbf{t}_{f,j} \quad \text{on} \quad \Gamma_{fp} \times (0,T],
\end{aligned}$$

representing balance of normal stress and the Beaver–Joseph–Saffman (BJS) slip with friction condition, respectively.

Finally, the above system of equations is complemented by the initial condition $p_p(\mathbf{x},0) = p_{p,0}(\mathbf{x})$ in Ω_p. In Lemma 4.9 below we will construct compatible initial data for the rest of the variables from $p_{p,0}$ in a way that all equations in the system (2.2)–(2.8), except for the unsteady conservation of mass equation in the first row of (2.6), hold at $t = 0$.

3 The weak formulation

In this section we proceed analogously to [3, Section 3] (see also [36]) and derive a weak formulation of the coupled problem given by (2.2), (2.3)–(2.6), and (2.8).

3.1 Preliminaries

For the stress tensor, velocity, and vorticity in the Stokes region, we use the following Hilbert spaces, respectively,

$$\begin{aligned}\mathbb{X}_f &:= \Big\{\boldsymbol{\tau}_f \in \mathbb{H}(\mathbf{div};\Omega_f) : \boldsymbol{\tau}_f\mathbf{n}_f = \mathbf{0} \ \text{ on } \ \Gamma_f^{\mathrm{N}}\Big\}, \quad \mathbf{V}_f := \mathbf{L}^2(\Omega_f),\\ \mathbb{Q}_f &:= \Big\{\boldsymbol{\chi}_f \in \mathbb{L}^2(\Omega_f) : \boldsymbol{\chi}_f^{\mathrm{t}} = -\boldsymbol{\chi}_f\Big\},\end{aligned}$$

endowed with the corresponding norms

$$\|\boldsymbol{\tau}_f\|_{\mathbb{X}_f} := \|\boldsymbol{\tau}_f\|_{\mathbb{H}(\mathbf{div};\Omega_f)}, \quad \|\mathbf{v}_f\|_{\mathbf{V}_f} := \|\mathbf{v}_f\|_{\mathbf{L}^2(\Omega_f)}, \quad \|\boldsymbol{\chi}_f\|_{\mathbb{Q}_f} := \|\boldsymbol{\chi}_f\|_{\mathbb{L}^2(\Omega_f)}.$$

For the unknowns in the Biot region we introduce the following Hilbert spaces:

$$\begin{aligned}\mathbb{X}_p &:= \Big\{\boldsymbol{\tau}_p \in \mathbb{H}(\mathbf{div};\Omega_p) : \boldsymbol{\tau}_p\mathbf{n}_p = \mathbf{0} \ \text{ on } \ \tilde{\Gamma}_p^{\mathrm{N}}\Big\}, \quad \mathbf{V}_s := \mathbf{L}^2(\Omega_p),\\ \mathbb{Q}_p &:= \Big\{\boldsymbol{\chi}_p \in \mathbb{L}^2(\Omega_p) : \boldsymbol{\chi}_p^{\mathrm{t}} = -\boldsymbol{\chi}_p\Big\},\\ \mathbf{V}_p &:= \Big\{\mathbf{v}_p \in \mathbf{H}(\mathrm{div};\Omega_p) : \ \mathbf{v}_p\cdot\mathbf{n}_p = 0 \ \text{ on } \ \Gamma_p^{\mathrm{N}}\Big\}, \quad \mathrm{W}_p := \mathrm{L}^2(\Omega_p),\end{aligned}$$

endowed with the standard norms

$$\begin{aligned}&\|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} := \|\boldsymbol{\tau}_p\|_{\mathbb{H}(\mathbf{div};\Omega_p)}, \quad \|\mathbf{v}_s\|_{\mathbf{V}_s} := \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega_p)}, \quad \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p} := \|\boldsymbol{\chi}_p\|_{\mathbb{L}^2(\Omega_p)},\\ &\|\mathbf{v}_p\|_{\mathbf{V}_p} := \|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)}, \quad \|w_p\|_{\mathrm{W}_p} := \|w_p\|_{\mathrm{L}^2(\Omega_p)}.\end{aligned}$$

Finally, analogously to [3, 9, 33, 36, 49] we need to introduce the Lagrange multiplier spaces $\Lambda_p := (\mathbf{V}_p\cdot\mathbf{n}_p|_{\Gamma_{fp}})'$, $\boldsymbol{\Lambda}_f := (\mathbb{X}_f\,\mathbf{n}_f|_{\Gamma_{fp}})'$, and $\boldsymbol{\Lambda}_s := (\mathbb{X}_p\,\mathbf{n}_p|_{\Gamma_{fp}})'$. According to the normal trace theorem, since $\mathbf{v}_p \in \mathbf{V}_p \subset \mathbf{H}(\mathrm{div};\Omega_p)$, then $\mathbf{v}_p\cdot\mathbf{n}_p \in \mathrm{H}^{-1/2}(\partial\Omega_p)$. It is shown in [33] that, if $\mathbf{v}_p\cdot\mathbf{n}_p = 0$ on $\partial\Omega_p\setminus\Gamma_{fp}$, then $\mathbf{v}_p\cdot\mathbf{n}_p \in \mathrm{H}^{-1/2}(\Gamma_{fp})$. This argument has been modified in [9] for the case $\mathbf{v}_p\cdot\mathbf{n}_p = 0$ on Γ_p^{N} and $\mathrm{dist}\,(\Gamma_p^{\mathrm{D}},\Gamma_{fp}) \geq s > 0$. In particular, it holds that

$$\langle\mathbf{v}_p\cdot\mathbf{n}_p,\xi\rangle_{\Gamma_{fp}} \leq C\|\mathbf{v}_p\|_{\mathbf{H}(\mathrm{div};\Omega_p)}\|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \forall\,\mathbf{v}_p\in\mathbf{V}_p,\ \xi\in\mathrm{H}^{1/2}(\Gamma_{fp}). \quad (3.1)$$

Similarly,

$$\langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \boldsymbol{\psi} \rangle_{\Gamma_{fp}} \leq C \|\boldsymbol{\tau}_\star\|_{\mathbb{H}(\mathrm{div};\Omega_\star)} \|\boldsymbol{\psi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \quad \forall\, \boldsymbol{\tau}_\star \in \mathbb{X}_\star,\ \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma_{fp}),\ \star \in \{f, p\}. \tag{3.2}$$

Therefore we can take $\Lambda_p := \mathrm{H}^{1/2}(\Gamma_{fp})$, $\boldsymbol{\Lambda}_f := \mathbf{H}^{1/2}(\Gamma_{fp})$, and $\boldsymbol{\Lambda}_s := \mathbf{H}^{1/2}(\Gamma_{fp})$, endowed with the norms

$$\|\xi\|_{\Lambda_p} := \|\xi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, \quad \|\boldsymbol{\psi}\|_{\boldsymbol{\Lambda}_f} := \|\boldsymbol{\psi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}, \quad \text{and} \quad \|\boldsymbol{\phi}\|_{\boldsymbol{\Lambda}_s} := \|\boldsymbol{\phi}\|_{\mathbf{H}^{1/2}(\Gamma_{fp})}. \tag{3.3}$$

3.2 Lagrange multiplier formulation

We now proceed with the derivation of our Lagrange multiplier variational formulation for the coupling of the Stokes and Biot problems. To this end, and inspired by [3, 37], we begin by introducing the structure velocity $\mathbf{u}_s := \partial_t \boldsymbol{\eta}_p \in \mathbf{V}_s$ satisfying $\mathbf{u}_s = \mathbf{0}$ on $\tilde{\Gamma}_p^{\mathrm{D}} \times (0, T]$ (cf. the last equation in (2.6)), and three Lagrange multipliers modeling the Stokes velocity, structure velocity and Darcy pressure on the interface, respectively,

$$\boldsymbol{\varphi} := \mathbf{u}_f|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_f, \quad \boldsymbol{\theta} := \mathbf{u}_s|_{\Gamma_{fp}} \in \boldsymbol{\Lambda}_s, \quad \text{and} \quad \lambda := p_p|_{\Gamma_{fp}} \in \Lambda_p.$$

The reason for introducing these Lagrange multipliers is twofold. First, $\mathbf{u}_f$, $\mathbf{u}_s$, and p_p are all modeled in the L^2 space, thus they do not have sufficient regularity for their traces on Γ_{fp} to be well defined. Second, the Lagrange multipliers are utilized to impose weakly the transmission conditions (2.8).

To impose the symmetry condition of $\boldsymbol{\sigma}_p$ in a weak sense we introduce the rotation operator $\boldsymbol{\rho}_p := \dfrac{1}{2}(\nabla \boldsymbol{\eta}_p - \nabla \boldsymbol{\eta}_p^{\mathrm{t}})$. Notice that in the weak formulation we will use its time derivative, that is, the structure rotation velocity

$$\boldsymbol{\gamma}_p := \partial_t \boldsymbol{\rho}_p = \frac{1}{2} \left(\nabla \mathbf{u}_s - (\nabla \mathbf{u}_s)^{\mathrm{t}} \right) \in \mathbb{Q}_p.$$

From the definition of the elastic and poroelastic stress tensors $\boldsymbol{\sigma}_e, \boldsymbol{\sigma}_p$ (cf. (2.3)) and recalling that $\boldsymbol{\sigma}_e$ is connected to the displacement $\boldsymbol{\eta}_p$ through the relation $A(\boldsymbol{\sigma}_e) = \mathbf{e}(\boldsymbol{\eta}_p)$, we deduce the identities

$$\mathrm{div}(\boldsymbol{\eta}_p) = \mathrm{tr}(\mathbf{e}(\boldsymbol{\eta}_p)) = \mathrm{tr}(A\boldsymbol{\sigma}_e) = \mathrm{tr} A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}) \tag{3.4}$$

and

$$\partial_t\, A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}) = \nabla\, \mathbf{u}_s - \boldsymbol{\gamma}_p\,. \tag{3.5}$$

Then, similarly to [3, 9, 36, 37], we test the first equation of (2.2), the second equation of (2.6), and (3.5) with arbitrary $\boldsymbol{\tau}_f \in \mathbb{X}_f$, $\mathbf{v}_p \in \mathbf{V}_p$, and $\boldsymbol{\tau}_p \in \mathbb{X}_p$, respectively, integrate by parts, utilize the fact that $\boldsymbol{\sigma}_f^{\mathrm{d}} : \boldsymbol{\tau}_f = \boldsymbol{\sigma}_f^{\mathrm{d}} : \boldsymbol{\tau}_f^{\mathrm{d}}$, test the third equation of (2.6) with $w_p \in \mathrm{W}_p$ employing (3.4), impose the remaining equations weakly, and utilize the transmission conditions in (2.8) to obtain the following variational problem. Given

$$\mathbf{f}_f : [0, T] \to \mathbf{V}'_f, \quad q_f : [0, T] \to \mathbb{X}'_f, \quad \mathbf{f}_p : [0, T] \to \mathbf{V}'_s, \quad q_p : [0, T] \to \mathrm{W}'_p,$$

find $(\boldsymbol{\sigma}_f, \mathbf{u}_f, \boldsymbol{\gamma}_f, \boldsymbol{\sigma}_p, \mathbf{u}_s, \boldsymbol{\gamma}_p, \mathbf{u}_p, p_p, \boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda) : [0, T] \to \mathbb{X}_f \times \mathbf{V}_f \times \mathbb{Q}_f \times \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p \times \mathbf{V}_p \times \mathrm{W}_p \times \boldsymbol{\Lambda}_f \times \boldsymbol{\Lambda}_s \times \Lambda_p$, such that $\forall\, \boldsymbol{\tau}_f \in \mathbb{X}_f$, $\mathbf{v}_f \in \mathbf{V}_f$, $\boldsymbol{\chi}_f \in \mathbb{Q}_f$, $\boldsymbol{\tau}_p \in \mathbb{X}_p$, $\mathbf{v}_s \in \mathbf{V}_s$, $\boldsymbol{\chi}_p \in \mathbb{Q}_p$, $\mathbf{v}_p \in \mathbf{V}_p$, $w_p \in \mathrm{W}_p$, $\boldsymbol{\psi} \in \boldsymbol{\Lambda}_f$, $\boldsymbol{\phi} \in \boldsymbol{\Lambda}_s$, $\xi \in \Lambda_p$, and for a.e. $t \in (0, T)$,

$$\frac{1}{2\mu}\,(\boldsymbol{\sigma}^{\mathrm{d}}_f, \boldsymbol{\tau}^{\mathrm{d}}_f)_{\Omega_f} + (\mathbf{u}_f, \mathbf{div}(\boldsymbol{\tau}_f))_{\Omega_f} + (\boldsymbol{\gamma}_f, \boldsymbol{\tau}_f)_{\Omega_f} - \left\langle \boldsymbol{\tau}_f \mathbf{n}_f, \boldsymbol{\varphi} \right\rangle_{\Gamma_{fp}} = -\frac{1}{n}\,(q_f\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f}, \tag{3.6a}$$

$$-\,(\mathbf{v}_f, \mathbf{div}(\boldsymbol{\sigma}_f))_{\Omega_f} = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f}, \tag{3.6b}$$

$$-\,(\boldsymbol{\sigma}_f, \boldsymbol{\chi}_f)_{\Omega_f} = 0, \tag{3.6c}$$

$$(\partial_t\, A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p} + (\mathbf{u}_s, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p} + (\boldsymbol{\gamma}_p, \boldsymbol{\tau}_p)_{\Omega_p} - \left\langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\theta} \right\rangle_{\Gamma_{fp}} = 0, \tag{3.6d}$$

$$-\,(\mathbf{v}_s, \mathbf{div}(\boldsymbol{\sigma}_p))_{\Omega_p} = (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}, \tag{3.6e}$$

$$-\,(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p)_{\Omega_p} = 0, \tag{3.6f}$$

$$\mu\,(\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p} - (p_p, \mathrm{div}(\mathbf{v}_p))_{\Omega_p} + \left\langle \mathbf{v}_p \cdot \mathbf{n}_p, \lambda \right\rangle_{\Gamma_{fp}} = 0, \tag{3.6g}$$

$$(s_0\, \partial_t\, p_p, w_p)_{\Omega_p} + \alpha_p\,(\partial_t\, A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), w_p\, \mathbf{I})_{\Omega_p} + (w_p, \mathrm{div}(\mathbf{u}_p))_{\Omega_p} = (q_p, w_p)_{\Omega_p}, \tag{3.6h}$$

$$-\left\langle \boldsymbol{\varphi} \cdot \mathbf{n}_f + (\boldsymbol{\theta} + \mathbf{u}_p) \cdot \mathbf{n}_p, \xi \right\rangle_{\Gamma_{fp}} = 0, \tag{3.6i}$$

$$\left\langle \boldsymbol{\sigma}_f \mathbf{n}_f, \boldsymbol{\psi} \right\rangle_{\Gamma_{fp}} + \mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\psi} \cdot \mathbf{n}_f, \lambda \right\rangle_{\Gamma_{fp}} = 0, \tag{3.6j}$$

$$\left\langle \boldsymbol{\sigma}_p \mathbf{n}_p, \boldsymbol{\phi} \right\rangle_{\Gamma_{fp}} - \mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\,(\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi} \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \lambda \right\rangle_{\Gamma_{fp}} = 0. \tag{3.6k}$$

Equations (3.6i)–(3.6k) impose weakly the transmission conditions (2.8). In particular, Eq. (3.6i) imposes the mass conservation, Eq. (3.6j) imposes the last equation in (2.8), which is a combination of balance of normal stress and the BJS condition, while Eq. (3.6k) imposes the conservation of momentum. We emphasize that this is a new formulation. To our knowledge, this is the first fully dual-mixed formulation for the Stokes-Biot problem.

We will discuss the construction of initial conditions for the problem (3.6) later on in Lemma 4.9.

Remark 3.1 The time differentiated Eq. (3.6d) allows us to eliminate the displacement variable $\boldsymbol{\eta}_p$ and obtain a formulation that uses only $\mathbf{u}_s$. By integrating in time the

Eq. (3.6d) and using the initial data constructed in Lemma 4.9, we can recover the original equation

$$(A(\boldsymbol{\sigma}_p+\alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p}+(\boldsymbol{\eta}_p, \mathbf{div}(\boldsymbol{\tau}_p))_{\Omega_p}+(\boldsymbol{\rho}_p, \boldsymbol{\tau}_p)_{\Omega_p}-\left\langle \boldsymbol{\tau}_p \mathbf{n}_p, \boldsymbol{\omega}\right\rangle_{\Gamma_{fp}} = 0, \quad (3.7)$$

where $\boldsymbol{\omega} := \boldsymbol{\eta}_p|_{\Gamma_{fp}}$.

To simplify the notation, we set the following bilinear forms:

$$\begin{aligned}
& a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) := \frac{1}{2\,\mu}\,(\boldsymbol{\sigma}_f^{\mathrm{d}}, \boldsymbol{\tau}_f^{\mathrm{d}})_{\Omega_f}, \quad a_p(\mathbf{u}_p, \mathbf{v}_p) := \mu\,(\mathbf{K}^{-1}\mathbf{u}_p, \mathbf{v}_p)_{\Omega_p}, \\
& a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) := (A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I}), \boldsymbol{\tau}_p + \alpha_p\, w_p\, \mathbf{I})_{\Omega_p}, \\
& b_f(\boldsymbol{\tau}_f, \mathbf{v}_f) := (\mathbf{div}(\boldsymbol{\tau}_f), \mathbf{v}_f)_{\Omega_f}, \quad b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) := (\mathbf{div}(\boldsymbol{\tau}_p), \mathbf{v}_s)_{\Omega_p}, \\
& b_p(\mathbf{v}_p, w_p) := -(\mathrm{div}(\mathbf{v}_p), w_p)_{\Omega_p}, \quad b_\Gamma(\mathbf{v}_p, \xi) := \left\langle \mathbf{v}_p \cdot \mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}}, \\
& b_{\mathrm{sk},\star}(\boldsymbol{\tau}_\star, \boldsymbol{\chi}_\star) := (\boldsymbol{\tau}_\star, \boldsymbol{\chi}_\star)_{\Omega_\star}, \quad b_{\mathbf{n}_\star}(\boldsymbol{\tau}_\star, \boldsymbol{\psi}) := -\left\langle \boldsymbol{\tau}_\star \mathbf{n}_\star, \boldsymbol{\psi}\right\rangle_{\Gamma_{fp}}, \text{ with } \star \in \{f, p\},
\end{aligned} \quad (3.8)$$

and

$$\begin{aligned}
& c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) := \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi} - \boldsymbol{\theta}) \cdot \mathbf{t}_{f,j}, (\boldsymbol{\psi} - \boldsymbol{\phi}) \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}}, \\
& c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \xi) := \left\langle \boldsymbol{\psi} \cdot \mathbf{n}_f, \xi\right\rangle_{\Gamma_{fp}} + \left\langle \boldsymbol{\phi} \cdot \mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}}.
\end{aligned} \quad (3.9)$$

There are many different ways of ordering the variables in (3.6). For the sake of the subsequent analysis, we proceed as in [36] and [3], and adopt one leading to an evolution problem in a doubly-mixed form. In particular, we combine the equations for the variables associated with the coercive bilinear forms a_f, a_p, and a_e, namely $\boldsymbol{\sigma}_f$, $\boldsymbol{\sigma}_p$, $\mathbf{u}_p$, and p_p. We further combine the interface Eqs. (3.6i)–(3.6k), and also combine the remaining equations. Hence, (3.6) results in

$$\begin{aligned}
& a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + a_e(\partial_t\, \boldsymbol{\sigma}_p, \partial_t\, p_p; \boldsymbol{\tau}_p, w_p) + (s_0\, \partial_t\, p_p, w_p)_{\Omega_p} \\
& \quad + b_p(\mathbf{v}_p, p_p) - b_p(\mathbf{u}_p, w_p) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\varphi}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}) + b_\Gamma(\mathbf{v}_p, \lambda) \\
& \quad + b_f(\boldsymbol{\tau}_f, \mathbf{u}_f) + b_s(\boldsymbol{\tau}_p, \mathbf{u}_s) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\gamma}_f) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_p) \\
& \quad = -\frac{1}{n}\,(q_f\, \mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p}, \\
& - b_{\mathbf{n}_f}(\boldsymbol{\sigma}_f, \boldsymbol{\psi}) - b_{\mathbf{n}_p}(\boldsymbol{\sigma}_p, \boldsymbol{\phi}) - b_\Gamma(\mathbf{u}_p, \xi) + c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) \\
& \quad + c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \lambda) - c_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\theta}; \xi) = 0, \\
& - b_f(\boldsymbol{\sigma}_f, \mathbf{v}_f) - b_s(\boldsymbol{\sigma}_p, \mathbf{v}_s) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_f, \boldsymbol{\chi}_f) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_p, \boldsymbol{\chi}_p) \\
& \quad = (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p},
\end{aligned} \quad (3.10)$$

Now, we group the spaces and test functions as follows:

$$\mathbf{X} := \mathbb{X}_f \times \mathbf{V}_p \times \mathbb{X}_p \times \mathrm{W}_p, \quad \mathbf{Y} := \mathbf{\Lambda}_f \times \mathbf{\Lambda}_s \times \Lambda_p, \quad \mathbf{Z} := \mathbf{V}_f \times \mathbf{V}_s \times \mathbb{Q}_f \times \mathbb{Q}_p,$$
$$\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda) \in \mathbf{Y}, \quad \underline{\mathbf{u}} := (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z},$$
$$\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_f, \mathbf{v}_p, \boldsymbol{\tau}_p, w_p) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \boldsymbol{\phi}, \xi) \in \mathbf{Y}, \quad \underline{\mathbf{v}} := (\mathbf{v}_f, \mathbf{v}_s, \boldsymbol{\chi}_f, \boldsymbol{\chi}_p) \in \mathbf{Z},$$

where the spaces $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are endowed, respectively, with the following norms:

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}} := \|\boldsymbol{\tau}_f\|_{\mathbb{X}_f} + \|\mathbf{v}_p\|_{\mathbf{V}_p} + \|\boldsymbol{\tau}_p\|_{\mathbb{X}_p} + \|w_p\|_{\mathrm{W}_p},$$
$$\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} := \|\boldsymbol{\psi}\|_{\mathbf{\Lambda}_f} + \|\boldsymbol{\phi}\|_{\mathbf{\Lambda}_s} + \|\xi\|_{\Lambda_p},$$
$$\|\underline{\mathbf{v}}\|_{\mathbf{Z}} := \|\mathbf{v}_f\|_{\mathbf{V}_f} + \|\mathbf{v}_s\|_{\mathbf{V}_s} + \|\boldsymbol{\chi}_f\|_{\mathbb{Q}_f} + \|\boldsymbol{\chi}_p\|_{\mathbb{Q}_p}.$$

Hence, we can write (3.10) in an operator notation as a degenerate evolution problem in a doubly-mixed form:

$$\begin{aligned} \frac{\partial}{\partial t}\,\mathcal{E}(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{A}(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}(t)) + \mathcal{B}'(\underline{\mathbf{u}}(t)) &= \mathbf{F}(t) \ \text{ in } \mathbf{X}', \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}(t)) + \mathcal{C}(\underline{\boldsymbol{\varphi}}(t)) &= \mathbf{0} \quad \text{ in } \mathbf{Y}', \\ -\mathcal{B}\,(\underline{\boldsymbol{\sigma}}(t)) &= \mathbf{G}(t) \ \text{ in } \mathbf{Z}', \end{aligned} \tag{3.11}$$

where, according to (3.8)–(3.9), the operators $\mathcal{A} : \mathbf{X} \to \mathbf{X}'$, $\mathcal{B}_1 : \mathbf{X} \to \mathbf{Y}'$, $\mathcal{C} : \mathbf{Y} \to \mathbf{Y}'$, and $\mathcal{B} : \mathbf{X} \to \mathbf{Z}'$, are defined by

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) &:= a_f(\boldsymbol{\sigma}_f, \boldsymbol{\tau}_f) + a_p(\mathbf{u}_p, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_p) - b_p(\mathbf{u}_p, w_p), \\ \mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\psi}}) &:= b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\psi}) + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi}) + b_\Gamma(\mathbf{v}_p, \xi), \\ \mathcal{C}(\underline{\boldsymbol{\varphi}})(\underline{\boldsymbol{\psi}}) &:= c_{\mathrm{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\psi}, \boldsymbol{\phi}) + c_\Gamma(\boldsymbol{\psi}, \boldsymbol{\phi}; \lambda) - c_\Gamma(\boldsymbol{\varphi}, \boldsymbol{\theta}; \xi), \end{aligned} \tag{3.12}$$

and

$$\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}}) := b_f(\boldsymbol{\tau}_f, \mathbf{v}_f) + b_s(\boldsymbol{\tau}_p, \mathbf{v}_s) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\chi}_f) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\chi}_p), \tag{3.13}$$

whereas the operator $\mathcal{E} : \mathbf{X} \to \mathbf{X}'$ is given by

$$\mathcal{E}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) := a_e(\boldsymbol{\sigma}_p, p_p; \boldsymbol{\tau}_p, w_p) + (s_0\, p_p, w_p)_{\Omega_p}, \tag{3.14}$$

and the functionals $\mathbf{F} \in \mathbf{X}'$, $\mathbf{G} \in \mathbf{Z}'$ are defined as

$$\mathbf{F}(\underline{\boldsymbol{\tau}}) := -\frac{1}{n}\,(q_f\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f} + (q_p, w_p)_{\Omega_p} \quad \text{and} \quad \mathbf{G}(\underline{\mathbf{v}}) := (\mathbf{f}_f, \mathbf{v}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{v}_s)_{\Omega_p}. \tag{3.15}$$

4 Well-posedness of the model

In this section we establish the solvability of (3.11) (equivalently (3.10)) given suitable initial data. To that end we first collect some previous results that will be used in the forthcoming analysis.

4.1 Preliminaries

We begin by recalling the following key result given in [55, Theorem IV.6.1(b)] that will be used to establish the existence of a solution to (3.11). In what follows, $Rg(\mathcal{A})$ denotes the range of $\mathcal{A}$.

Theorem 4.1 *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual E^*, and let E'_b be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \big(\mathcal{N}(x)(x)\big)^{1/2}, \quad x \in E.$$

Let $\mathcal{M} \subset E \times E'_b$ be a relation with domain $\mathcal{D} = \Big\{x \in E \,:\, \mathcal{M}(x) \neq \emptyset\Big\}$.

Assume $\mathcal{M}$ is monotone and $Rg(\mathcal{N} + \mathcal{M}) = E'_b$. Then, for each $u_0 \in \mathcal{D}$ and for each $f \in \mathrm{W}^{1,1}(0, T; E'_b)$, there is a solution u of

$$\frac{d}{dt}\big(\mathcal{N}(u(t))\big) + \mathcal{M}\big(u(t)\big) \ni f(t) \quad a.e.\ 0 < t < T, \tag{4.1}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0, T; E'_b),\ u(t) \in \mathcal{D},\ \textit{for all}\ 0 \le t \le T, \textit{and}\ \mathcal{N}(u(0)) = \mathcal{N}(u_0).$$

In addition, in order to show the range condition of Theorem 4.1 in our context, we will require the following theorem whose proof can be derived similarly to [35, Theorem 2.2] (see also [1, Theorem 3.13] for a generalized nonlinear Banach version).

Theorem 4.2 *Let X, Y, and Z be Hilbert spaces, and let X', Y', Z' be their respective duals. Let $A : X \to X'$, $S : Y \to Y'$, $B_1 : X \to Y'$, and $B : X \to Z'$ be linear bounded operators. We also let $B'_1 : Y \to X'$ and $B' : Z \to X'$ be the corresponding adjoints. Finally, we let V be the kernel of B, that is*

$$V := \Big\{\boldsymbol{\tau} \in X : \quad B(\boldsymbol{\tau})(\mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in Z\Big\}.$$

Assume that

(i) $A|_V : V \to V'$ *is elliptic, that is, there exists a constant $\alpha > 0$ such that*

$$A(\boldsymbol{\tau})(\boldsymbol{\tau}) \ \ge\ \alpha\, \|\boldsymbol{\tau}\|_X^2 \quad \forall\, \boldsymbol{\tau} \in V.$$

(ii) *S is positive semi-definite on Y, that is,*

$$S(\boldsymbol{\psi})(\boldsymbol{\psi}) \,\geq\, 0 \quad \forall\, \boldsymbol{\psi} \in Y.$$

(iii) *B_1 satisfies an inf-sup condition on $V \times Y$, that is, there exists $\beta_1 > 0$ such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in V} \frac{B_1(\boldsymbol{\tau})(\boldsymbol{\psi})}{\|\boldsymbol{\tau}\|_X} \,\geq\, \beta_1\,\|\boldsymbol{\psi}\|_Y \quad \forall\, \boldsymbol{\psi} \in Y.$$

(iv) *B satisfies an inf-sup condition on $X \times Z$, that is, there exists $\beta > 0$ such that*

$$\sup_{\mathbf{0}\neq\boldsymbol{\tau}\in X} \frac{B(\boldsymbol{\tau})(\mathbf{v})}{\|\boldsymbol{\tau}\|_X} \,\geq\, \beta\,\|\mathbf{v}\|_Z \quad \forall\, \mathbf{v} \in Z.$$

Then, for each $(F_1, F_2, G) \in X' \times Y' \times Z'$ there exists a unique $(\boldsymbol{\sigma}, \boldsymbol{\varphi}, \mathbf{u}) \in X \times Y \times Z$, such that

$$\begin{aligned} A(\boldsymbol{\sigma})(\boldsymbol{\tau}) + B_1'(\boldsymbol{\varphi})(\boldsymbol{\tau}) + B'(\mathbf{u})(\boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in X,\\ B_1(\boldsymbol{\sigma})(\boldsymbol{\psi}) - S(\boldsymbol{\varphi})(\boldsymbol{\psi}) &= F_2(\boldsymbol{\psi}) \quad \forall\, \boldsymbol{\psi} \in Y,\\ B(\boldsymbol{\sigma})(\mathbf{v}) &= G(\mathbf{v}) \quad\ \ \forall\, \mathbf{v} \in Z. \end{aligned}$$

Moreover, there exists $C > 0$, depending only on α, β_1, β, $\|A\|$, $\|S\|$, and $\|B_1\|$ such that

$$\|(\boldsymbol{\sigma}, \boldsymbol{\varphi}, \mathbf{u})\|_{X\times Y\times Z} \,\leq\, C\left\{\|F_1\|_{X'} + \|F_2\|_{Y'} + \|G\|_{Z'}\right\}.$$

At this point we recall, for later use, that there exist positive constants $c_1(\Omega_f)$ and $c_2(\Omega_f)$, such that (see, [19, Proposition IV.3.1] and [34, Lemma 2.5], respectively)

$$c_1(\Omega_f)\,\|\boldsymbol{\tau}_{f,0}\|^2_{\mathbb{L}^2(\Omega_f)} \,\leq\, \|\boldsymbol{\tau}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{div}(\boldsymbol{\tau}_f)\|^2_{\mathbf{L}^2(\Omega_f)} \quad \forall\, \boldsymbol{\tau}_f = \boldsymbol{\tau}_{f,0} + \ell\,\mathbf{I} \in \mathbb{H}(\mathbf{div};\Omega_f) \tag{4.2}$$

and

$$c_2(\Omega_f)\,\|\boldsymbol{\tau}_f\|^2_{\mathbb{X}_f} \,\leq\, \|\boldsymbol{\tau}_{f,0}\|^2_{\mathbb{X}_f} \quad \forall\, \boldsymbol{\tau}_f = \boldsymbol{\tau}_{f,0} + \ell\,\mathbf{I} \in \mathbb{X}_f, \tag{4.3}$$

where $\boldsymbol{\tau}_{f,0} \in \mathbb{H}_0(\mathbf{div};\Omega_f) := \left\{\boldsymbol{\tau}_f \in \mathbb{H}(\mathbf{div};\Omega_f) : \quad (\mathrm{tr}(\boldsymbol{\tau}_f), 1)_{\Omega_f} = 0\right\}$ and $\ell \in \mathrm{R}$. We emphasize that (4.3) holds since each $\boldsymbol{\tau}_f \in \mathbb{X}_f$ satisfies the boundary condition $\boldsymbol{\tau}_f\mathbf{n}_f = \mathbf{0}$ on Γ^{N}_f with $|\Gamma^{\mathrm{N}}_f| > 0$.

4.2 A reduced problem

Now, we proceed to analyze the solvability of (3.11) (equivalently (3.10)). First, recalling the definition of the operators $\mathcal{A}, \mathcal{B}_1, \mathcal{B}, \mathcal{C}$, and $\mathcal{E}$ (cf. (3.12), (3.13) and (3.14)), we note that problem (3.11) can be written in the form of (4.1) with

$$E = \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}, \quad u = \begin{pmatrix} \underline{\boldsymbol{\sigma}} \\ \underline{\boldsymbol{\varphi}} \\ \underline{\mathbf{u}} \end{pmatrix}, \mathcal{N} = \begin{pmatrix} \mathcal{E} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathcal{M} = \begin{pmatrix} \mathcal{A} & \mathcal{B}_1' & \mathcal{B}' \\ -\mathcal{B}_1 & \mathcal{C} & \mathbf{0} \\ -\mathcal{B} & \mathbf{0} & \mathbf{0} \end{pmatrix}, f = \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \\ \mathbf{G} \end{pmatrix}. \tag{4.4}$$

In addition, the seminorm induced by the operator $\mathcal{E}$ is $|\underline{\boldsymbol{\tau}}|_{\mathcal{E}}^2 := s_0 \|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$, which is equivalent to $\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{L}^2(\Omega_p)}$ since $s_0 > 0$. We denote by $\mathbb{X}_{p,2}$ and $\mathrm{W}_{p,2}$ the closures of the spaces $\mathbb{X}_p$ and W_p, respectively, with respect to the norms $\|\boldsymbol{\tau}_p\|_{\mathbb{X}_{p,2}} := \|\boldsymbol{\tau}_p\|_{\mathbb{L}^2(\Omega_p)}$ and $\|w_p\|_{\mathrm{W}_{p,2}} := \|w_p\|_{\mathrm{L}^2(\Omega_p)}$. Note that $\mathbb{X}_{p,2} = \mathbb{L}^2(\Omega_p)$ and $\mathrm{W}_{p,2} = \mathrm{W}_p = \mathrm{L}^2(\Omega_p)$, therefore $\mathbb{X}'_{p,2} = \mathbb{L}^2(\Omega_p)$ and $\mathrm{W}'_{p,2} = \mathrm{W}'_p = \mathrm{L}^2(\Omega_p)$. Next, denoting $\mathbf{X}'_{2,0} := \mathbf{0} \times \mathbf{0} \times \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, $\mathbf{Y}'_{2,0} := \mathbf{0} \times \mathbf{0} \times \mathbf{0}$, and $\mathbf{Z}'_{2,0} := \mathbf{0} \times \mathbf{0} \times \mathbf{0} \times \mathbf{0}$, the Hilbert space E'_b and domain $\mathcal{D}$ in Theorem 4.1 for our context are

$$E'_b := \mathbf{X}'_{2,0} \times \mathbf{Y}'_{2,0} \times \mathbf{Z}'_{2,0}, \quad \mathcal{D} := \left\{ (\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z} : \quad \mathcal{M}(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in E'_b \right\}. \tag{4.5}$$

Remark 4.1 The above definition of the space E'_b and the corresponding domain $\mathcal{D}$ implies that, in order to apply Theorem 4.1 for our problem (3.11), we need to restrict $\mathbf{f}_f = \mathbf{0}, q_f = 0$, and $\mathbf{f}_p = \mathbf{0}$. To avoid this restriction we will employ a translation argument [57] to reduce the existence for (3.11) to existence for the following initial-value problem: Given initial data $(\widehat{\underline{\boldsymbol{\sigma}}}_0, \widehat{\underline{\boldsymbol{\varphi}}}_0, \widehat{\underline{\mathbf{u}}}_0) \in \mathcal{D}$ and source terms $(\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p}, \widehat{f}_{p_p}) : [0, T] \to \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, find $(\widehat{\underline{\boldsymbol{\sigma}}}, \widehat{\underline{\boldsymbol{\varphi}}}, \widehat{\underline{\mathbf{u}}}) \in [0, T] \to \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ such that $(\widehat{\boldsymbol{\sigma}}_p(0), \widehat{p}_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0}, \widehat{p}_{p,0})$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{E}(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{A}(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{B}_1'(\widehat{\underline{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\widehat{\underline{\mathbf{u}}}(t)) = \widehat{\mathbf{F}}(t) \quad & \text{in } \mathbf{X}'_{2,0}, \\ -\mathcal{B}_1(\widehat{\underline{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\widehat{\underline{\boldsymbol{\varphi}}}(t)) = \mathbf{0} \quad & \text{in } \mathbf{Y}'_{2,0}, \\ -\mathcal{B}(\widehat{\underline{\boldsymbol{\sigma}}}(t)) = \mathbf{0} \quad & \text{in } \mathbf{Z}'_{2,0}, \end{aligned} \tag{4.6}$$

where $\widehat{\mathbf{F}} = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p}, \widehat{f}_{p_p})^{\mathrm{t}}$.

In order to apply Theorem 4.1 for problem (4.6), we need to: (1) establish the required properties of the operators $\mathcal{N}$ and $\mathcal{M}$, (2) prove the range condition $Rg(\mathcal{N} + \mathcal{M}) = E'_b$, and (3) construct compatible initial data $(\widehat{\underline{\boldsymbol{\sigma}}}_0, \widehat{\underline{\boldsymbol{\varphi}}}_0, \widehat{\underline{\mathbf{u}}}_0) \in \mathcal{D}$. We proceed with a sequence of lemmas establishing these results.

4.2.1 Operator properties

Lemma 4.3 *The linear operators $\mathcal{N}$ and $\mathcal{M}$ defined in (4.4) are continuous and monotone. In addition, $\mathcal{N}$ is symmetric.*

Proof First, from the definition of the operators $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}$ and $\mathcal{B}$ (cf. (3.12), (3.13), (3.14)) it is clear that both $\mathcal{N}$ and $\mathcal{M}$ (cf. (4.4)) are linear and continuous, using the trace inequalities (3.1)–(3.2) for the continuity of $\mathcal{B}_1$. In turn, $\mathcal{N}$ is symmetric since $\mathcal{E}$ is. Finally, using (2.7), we have

$$\mathcal{E}(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) = s_0\|w_p\|^2_{\mathrm{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p w_p \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)},$$
$$\mathcal{A}(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \geq \frac{1}{2\,\mu}\,\|\boldsymbol{\tau}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \mu\, k^{-1}_{\max}\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} \quad \forall\, \underline{\boldsymbol{\tau}} \in \mathbf{X}, \tag{4.7}$$

and recalling the definition of the operator $\mathcal{C}$ (cf. (3.9), (3.12)), we obtain

$$\mathcal{C}(\underline{\boldsymbol{\psi}})(\underline{\boldsymbol{\psi}}) = \mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\psi} - \boldsymbol{\phi})\cdot \mathbf{t}_{f,j}, (\boldsymbol{\psi} - \boldsymbol{\phi})\cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \geq \frac{\mu\,\alpha_{\mathrm{BJS}}}{\sqrt{k_{\max}}}\,|\boldsymbol{\psi} - \boldsymbol{\phi}|^2_{\mathrm{BJS}}, \tag{4.8}$$

for all $\underline{\boldsymbol{\psi}} = (\boldsymbol{\psi}, \boldsymbol{\phi}, \xi) \in \mathbf{Y}$, where $|\boldsymbol{\psi} - \boldsymbol{\phi}|^2_{\mathrm{BJS}} := \sum_{j=1}^{n-1} \|(\boldsymbol{\psi} - \boldsymbol{\phi})\cdot \mathbf{t}_{f,j}\|^2_{\mathrm{L}^2(\Gamma_{fp})}$. Thus, combining (4.7) and (4.8), and the fact that the operators $\mathcal{E}$, $\mathcal{A}$, $\mathcal{C}$ are linear, we deduce the monotonicity of the operators $\mathcal{N}$ and $\mathcal{M}$ completing the proof. □

4.2.2 The range condition

Next, we establish the range condition $Rg(\mathcal{N} + \mathcal{M}) = E'_b$, which is done by solving the related resolvent system $(\mathcal{N} + \mathcal{M}(v)) = f$ in E'_b for $v \in \mathcal{D}$. In fact, we will show a stronger result by considering a resolvent system where all source terms in $\mathbf{F}$ and $\mathbf{G}$ may be non-zero. This stronger result will be used in the translation argument for proving existence of the original problem (3.11). More precisely, let

$$\mathbf{X}_2 := \mathbb{X}_f \times \mathbf{V}_p \times \mathbb{X}_{p,2} \times \mathrm{W}_{p,2} \supset \mathbf{X}$$

and note that $\mathbf{X}'_2 = \mathbb{X}'_f \times \mathbf{V}'_p \times \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2} \subset \mathbf{X}'$. We consider the following resolvent system:

$$\begin{aligned}
(\mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}}) + \mathcal{B}'_1(\underline{\boldsymbol{\varphi}}) + \mathcal{B}'(\underline{\mathbf{u}}) &= \widehat{\mathbf{F}} \quad \text{in } \mathbf{X}'_2,\\
-\mathcal{B}_1(\underline{\boldsymbol{\sigma}}) + \mathcal{C}(\underline{\boldsymbol{\varphi}}) &= \mathbf{0} \quad \text{in } \mathbf{Y}',\\
-\mathcal{B}(\underline{\boldsymbol{\sigma}}) &= \widehat{\mathbf{G}} \quad \text{in } \mathbf{Z}',
\end{aligned} \tag{4.9}$$

where $\widehat{\mathbf{F}} \in \mathbf{X}'_2$ and $\widehat{\mathbf{G}} \in \mathbf{Z}'$ are such that

$$\widehat{\mathbf{F}}(\underline{\boldsymbol{\tau}}) := (\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_f}, \boldsymbol{\tau}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\mathbf{u}_p}, \mathbf{v}_p)_{\Omega_p} + (\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p}, \boldsymbol{\tau}_p)_{\Omega_p} + (\widehat{f}_{p_p}, w_p)_{\Omega_p},$$
$$\widehat{\mathbf{G}}(\underline{\mathbf{v}}) := (\widehat{\mathbf{f}}_{\mathbf{u}_f}, \mathbf{v}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\mathbf{u}_s}, \mathbf{v}_s)_{\Omega_p} + (\widehat{\mathbf{f}}_{\boldsymbol{\gamma}_f}, \boldsymbol{\chi}_f)_{\Omega_f} + (\widehat{\mathbf{f}}_{\boldsymbol{\gamma}_p}, \boldsymbol{\chi}_p)_{\Omega_p}.$$

We next focus on proving that the resolvent system (4.9) is well-posed. We start with the following preliminary lemma.

Lemma 4.4 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ *be a solution to* (4.9). *Then, for any positive constant* κ*, it satisfies*

$$\begin{aligned} (\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\sigma}}) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}) + \mathcal{B}'(\underline{\mathbf{u}}) &= \ \widetilde{\mathbf{F}} \ \text{ in } \ \mathbf{X}_2', \\ \mathcal{B}_1(\underline{\boldsymbol{\sigma}}) - \mathcal{C}(\underline{\boldsymbol{\varphi}}) &= \ \mathbf{0} \ \text{ in } \ \mathbf{Y}', \\ \mathcal{B}(\underline{\boldsymbol{\sigma}}) &= -\widehat{\mathbf{G}} \ \text{ in } \ \mathbf{Z}', \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} &\widetilde{\mathcal{A}}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) := \mathcal{A}(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) \\ &\quad + \kappa \left\{ (\mathrm{div}(\mathbf{u}_p), \mathrm{div}(\mathbf{v}_p))_{\Omega_p} + \big(s_0\, p_p + \alpha_p \operatorname{tr}\big(A(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})\big), \mathrm{div}(\mathbf{v}_p)\big)_{\Omega_p} \right\}, \end{aligned} \tag{4.11}$$

and

$$\widetilde{\mathbf{F}}(\underline{\boldsymbol{\tau}}) := \widehat{\mathbf{F}}(\underline{\boldsymbol{\tau}}) + \kappa \left(\widehat{f}_{p_p}, \mathrm{div}(\mathbf{v}_p) \right)_{\Omega_p}.$$

Conversely, if $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ *is a solution to* (4.10)*, then it is also a solution to* (4.9).

Proof Let $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ be a solution to (4.9). Using that $\mathrm{div}\, \mathbf{V}_p = \mathrm{W}_p$, we take $\underline{\boldsymbol{\tau}} = (\mathbf{0}, w_p) = (\mathbf{0}, \mathrm{div}(\mathbf{v}_p)) \in \mathbf{X}$ in the first row of (4.9), multiply by a positive constant κ and add that term to (4.9), to obtain (4.10). Conversely, if $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ satisfies (4.10) we employ similar arguments, but now subtracting, to recover (4.9). □

Problem (4.10) has the same structure as the one in Theorem 4.2. Therefore, in what follows we apply this result to establish the well-posedness of (4.10). To that end, we first observe that the kernel of the operator $\mathcal{B}$, cf. (3.13), can be written as

$$\mathbf{V} := \left\{ \underline{\boldsymbol{\tau}} \in \mathbf{X} : \quad \mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}}) = 0 \quad \forall\, \mathbf{v} \in \mathbf{Z} \right\} = \widetilde{\mathbb{X}}_f \times \mathbf{V}_p \times \widetilde{\mathbb{X}}_p \times \mathrm{W}_p, \tag{4.12}$$

where

$$\widetilde{\mathbb{X}}_\star := \left\{ \boldsymbol{\tau}_\star \in \mathbb{X}_\star : \quad \boldsymbol{\tau}_\star = \boldsymbol{\tau}_\star^{\mathrm{t}} \ \text{ and } \ \mathbf{div}(\boldsymbol{\tau}_\star) = \mathbf{0} \ \text{ in } \ \Omega_\star \right\}, \quad \star \in \{f, p\}.$$

We next verify the hypotheses of Theorem 4.2. We begin by noting that the operators $\widetilde{\mathcal{A}}, \mathcal{B}_1, \mathcal{C}, \mathcal{B}$, and $\mathcal{E}$ are linear and continuous. Next, we proceed with the ellipticity of the operator $\mathcal{E} + \widetilde{\mathcal{A}}$ on $\mathbf{V}$.

Lemma 4.5 *Assume that*

$$\kappa \in \left(0, 2\, \min\left\{\delta_1, \frac{\delta_2}{\alpha_p}\right\}\right) \ \ \textit{with} \ \ \delta_1 \in \left(0, \frac{2}{s_0}\right) \ \ \textit{and} \ \ \delta_2 \in \left(0, \frac{4\mu_{min}}{n\,\alpha_p}\left(1 - \frac{s_0}{2}\,\delta_1\right)\right).$$

Then, the operator $\mathcal{E} + \widetilde{\mathcal{A}}$ *is elliptic on* $\mathbf{V}$.

Proof From the definition of $\widetilde{\mathcal{A}}$, cf. (4.11), and considering $\underline{\boldsymbol{\tau}} \in \mathbf{V}$ we get

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) &= \frac{1}{2\,\mu}\,\|\boldsymbol{\tau}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \mu\|\mathbf{K}^{-1/2}\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)} + s_0\,\|w_p\|^2_{\mathrm{W}_p}\\ &+\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \kappa\,\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)} + s_0\,\kappa\,(w_p, \mathrm{div}(\mathbf{v}_p))_{\Omega_p}\\ &+\alpha_p\,\kappa\,(A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I}), A^{1/2}(\mathrm{div}(\mathbf{v}_p)\,\mathbf{I}))_{\Omega_p}.\end{aligned}$$

Hence, using the Cauchy–Schwarz and Young's inequalities, (2.7), (2.5), and (4.2)–(4.3), we obtain

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) &\geq \frac{C_{\mathrm{d}}}{2\,\mu}\|\boldsymbol{\tau}_f\|^2_{\mathbb{X}_f} + \mu\,k_{\max}^{-1}\|\mathbf{v}_p\|^2_{\mathbf{L}^2(\Omega_p)}\\ &+\kappa\left(\left(1-\frac{s_0}{2}\delta_1\right) - \frac{n\,\alpha_p}{4\mu_{\min}}\delta_2\right)\|\mathrm{div}(\mathbf{v}_p)\|^2_{\mathrm{L}^2(\Omega_p)}\\ &+\left(1 - \frac{\alpha_p}{2\,\delta_2}\,\kappa\right)\|A^{1/2}\,(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\left(1 - \frac{\kappa}{2\,\delta_1}\right)\|w_p\|^2_{\mathrm{W}_p},\end{aligned}$$

where $C_{\mathrm{d}} := C_1(\Omega_f)\,C_2(\Omega_f)$. Then, using the stipulated hypotheses on δ_1, δ_2 and κ, we can define the positive constants

$$\begin{aligned}&\alpha_1(\Omega_f) := \frac{C_{\mathrm{d}}}{2\,\mu}, \quad \alpha_2(\Omega_p) := \min\left\{\mu\,k_{\max}^{-1}, \kappa\,\left(\left(1 - \frac{s_0}{2}\,\delta_1\right) - \frac{n\,\alpha_p}{4\mu_{\min}}\,\delta_2\right)\right\},\\ &\alpha_3(\Omega_p) := \frac{s_0}{2}\,\left(1 - \frac{\kappa}{2\,\delta_1}\right), \quad \alpha_4(\Omega_p) := \min\left\{\left(1 - \frac{\alpha_p}{2\,\delta_2}\,\kappa\right), \alpha_3(\Omega_p)\right\}\end{aligned}$$

which allow us to obtain

$$\begin{aligned}(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) &\geq \alpha_1(\Omega_f)\,\|\boldsymbol{\tau}_f\|^2_{\mathbb{X}_f} + \alpha_2(\Omega_p)\,\|\mathbf{v}_p\|^2_{\mathbf{V}_p} + \alpha_3(\Omega_p)\,\|w_p\|^2_{\mathrm{W}_p}\\ &+\alpha_4(\Omega_p)\,\left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p}\right). \quad (4.13)\end{aligned}$$

In turn, from (2.5) and using the triangle inequality, we deduce

$$\begin{aligned}\|\boldsymbol{\tau}_p\|^2_{\mathbb{L}^2(\Omega_p)} &\leq (2\,\mu_{\max} + n\,\lambda_{\max})\,\left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}(\alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\right)\\ &\leq C_p\,\left(\|A^{1/2}(\boldsymbol{\tau}_p + \alpha_p\,w_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|w_p\|^2_{\mathrm{W}_p}\right), \quad (4.14)\end{aligned}$$

where $C_p := (2\,\mu_{\max} + n\,\lambda_{\max})\max\left\{1, \frac{n\,\alpha_p^2}{2\mu_{\min}}\right\}$. A combination of (4.13) and (4.14), and the fact that $\mathbf{div}(\boldsymbol{\tau}_p) = \mathbf{0}$ in Ω_p, implies

$$(\mathcal{E} + \widetilde{\mathcal{A}})(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\tau}}) \;\geq\; \alpha(\Omega_f, \Omega_p)\,\|\underline{\boldsymbol{\tau}}\|^2_{\mathbf{X}} \quad \forall\,\underline{\boldsymbol{\tau}} \in \mathbf{V},$$

with $\alpha(\Omega_f, \Omega_p) := \min\left\{\alpha_1(\Omega_f), \alpha_2(\Omega_p), \alpha_3(\Omega_p), \alpha_4(\Omega_p)/C_p\right\}$, hence $\mathcal{E} + \widetilde{\mathcal{A}}$ is elliptic on $\mathbf{V}$. □

Remark 4.2 To maximize the ellipticity constant $\alpha(\Omega_f, \Omega_p)$, we can choose explicitly the parameter κ by taking the parameters δ_1 and δ_2 as the middle points of their feasible ranges. More precisely, we can simply take

$$\delta_1 = \frac{1}{s_0}, \quad \delta_2 = \frac{\mu_{min}}{n\,\alpha_p}, \quad \kappa = \min\left\{\frac{1}{s_0}, \frac{\mu_{\min}}{n\,\alpha_p^2}\right\}.$$

We continue with the verification of the hypotheses of Theorem 4.2.

Lemma 4.6 *There exist positive constants β_1 and β, such that*

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{V}} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\psi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \geq \beta_1\,\|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \quad \forall\,\underline{\boldsymbol{\psi}} \in \mathbf{Y}, \tag{4.15}$$

and

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{v}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \geq \beta\,\|\underline{\mathbf{v}}\|_{\mathbf{Z}} \quad \forall\,\underline{\mathbf{v}} \in \mathbf{Z}. \tag{4.16}$$

Proof We begin with the proof of (4.15). Due the diagonal character of operator $\mathcal{B}_1$, cf. (3.12), we need to show individual inf-sup conditions for $b_{\mathbf{n}_f}$, $b_{\mathbf{n}_p}$, and b_Γ. The inf-sup condition for b_Γ follows from a slight adaptation of the argument in [31, Lemma 3.2] to account for the presence of Dirichlet boundary Γ_p^{D}, using that dist $(\Gamma_p^{\mathrm{D}}, \Gamma_{fp}) \geq s > 0$. The inf-sup conditions for $b_{\mathbf{n}_f}$ and $b_{\mathbf{n}_p}$ follow in a similar way. Since the kernel space $\mathbf{V}$ consists of symmetric and divergence-free tensors, the argument in [31, Lemma 3.2] must be modified to account for that. For example, in Ω_f we solve a problem

$$\mathbf{div}(\mathbf{e}(\mathbf{v}_f)) = \mathbf{0} \;\text{ in }\; \Omega_f, \quad \mathbf{e}(\mathbf{v}_f)\,\mathbf{n}_f = \boldsymbol{\xi} \;\text{ on }\; \Gamma_{fp}\cup\Gamma_f^{\mathrm{N}}, \quad \mathbf{v}_f = \mathbf{0} \;\text{ on }\; \Gamma_f^{\mathrm{D}}, \tag{4.17}$$

for given datum $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_{fp}\cup\Gamma_f^{\mathrm{N}})$ such that $\boldsymbol{\xi} = \mathbf{0}$ on Γ_f^{N}. We recall that Γ_f^{N} is adjacent to Γ_{fp}. Furthermore, $|\Gamma_f^{\mathrm{D}}| > 0$, which guarantees the solvability of the problem. We refer to [31, Lemma 3.2] for further details.

Finally, proceeding as above, using the diagonal character of operator $\mathcal{B}$, cf. (3.13), and employing the theory developed in [34, Section 2.4.3] to our context, we can deduce (4.16). □

Now, we are in a position to establish that the resolvent system associated to (4.6) is well-posed.

Lemma 4.7 *For $\mathcal{N}$, $\mathcal{M}$ and E_b' defined in (4.4)–(4.5), it holds that $Rg(\mathcal{N}+\mathcal{M}) = E_b'$, that is, given $f \in E_b'$, there exists $v \in \mathcal{D}$ such that $(\mathcal{N}+\mathcal{M})(v) = f$.*

Proof Let us consider $\widehat{\mathbf{F}} = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p}, \widehat{f}_{p_p})^{\mathrm{t}}$ and $\widehat{\mathbf{G}} = \mathbf{0}$ in (4.9)–(4.10) and κ as in Lemma 4.5. The well-posedness of (4.10) follows from (4.8), Lemmas 4.5 and 4.6, and a straightforward application of Theorem 4.2 with $A = \mathcal{E} + \widetilde{\mathcal{A}}$, $B_1 = \mathcal{B}_1$, $S = \mathcal{C}$, and $B = \mathcal{B}$. Then, employing Lemma 4.4 we conclude that there exists a unique solution of the resolvent system of (4.6), implying the range condition. □

4.2.3 Existence of a solution of the reduced problem

We are now ready to establish existence for the auxiliary initial value problem (4.6), assuming compatible initial data.

Lemma 4.8 *For each compatible initial data* $(\underline{\widehat{\boldsymbol{\sigma}}}_0, \underline{\widehat{\boldsymbol{\varphi}}}_0, \underline{\widehat{\mathbf{u}}}_0) \in \mathcal{D}$ *and each* $(\widehat{\mathbf{f}}_p, \widehat{q}_p) \in \mathrm{W}^{1,1}(0,T;\mathbb{X}'_{p,2}) \times \mathrm{W}^{1,1}(0,T;\mathrm{W}'_{p,2})$*, the problem* (4.6) *has a solution* $(\underline{\widehat{\boldsymbol{\sigma}}}, \underline{\widehat{\boldsymbol{\varphi}}}, \underline{\widehat{\mathbf{u}}}) : [0,T] \to \mathbf{X}\times\mathbf{Y}\times\mathbf{Z}$ *such that* $(\widehat{\boldsymbol{\sigma}}_p, \widehat{p}_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\widehat{\boldsymbol{\sigma}}_p(0), \widehat{p}_p(0)) = (\widehat{\boldsymbol{\sigma}}_{p,0}, \widehat{p}_{p,0})$.

Proof The assertion of the lemma follows by applying Theorem 4.1 with $E, \mathcal{N}, \mathcal{M}$ defined in (4.4), using Lemmas 4.3 and 4.7. □

We will employ Lemma 4.8 to obtain existence of a solution to our problem (3.11). To that end, we first construct compatible initial data $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$.

4.3 Compatible initial data

Lemma 4.9 *Assume that the initial condition* $p_{p,0} \in \mathrm{H}_p$*, where*

$$\mathrm{H}_p := \Big\{ w_p \in \mathrm{H}^1(\Omega_p) : \quad \mathbf{K}\nabla w_p \in \mathbf{H}^1(\Omega_p),\ \mathbf{K}\nabla w_p \cdot \mathbf{n}_p = 0 \text{ on } \Gamma_p^{\mathrm{N}}, \quad w_p = 0 \text{ on } \Gamma_p^{\mathrm{D}} \Big\}. \tag{4.18}$$

Then, there exist $\underline{\boldsymbol{\sigma}}_0 := (\boldsymbol{\sigma}_{f,0}, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{p,0}, p_{p,0}) \in \mathbf{X}$, $\underline{\boldsymbol{\varphi}}_0 := (\boldsymbol{\varphi}_0, \boldsymbol{\theta}_0, \lambda_0) \in \mathbf{Y}$*, and* $\underline{\mathbf{u}}_0 := (\mathbf{u}_{f,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{f,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbf{Z}$ *such that*

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}_0) + \mathcal{B}_1'(\underline{\boldsymbol{\varphi}}_0) + \mathcal{B}'(\underline{\mathbf{u}}_0) &= \widehat{\mathbf{F}}_0 \quad \text{in } \mathbf{X}_2', \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_0) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_0) &= \mathbf{0} \quad \text{in } \mathbf{Y}', \\ -\mathcal{B}(\underline{\boldsymbol{\sigma}}_0) &= \mathbf{G}(0) \ \text{ in } \mathbf{Z}', \end{aligned} \tag{4.19}$$

where $\widehat{\mathbf{F}}_0 = \big(\frac{1}{n} q_f(0)\mathbf{I}, \mathbf{0}, \widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p,0}, \widehat{f}_{p_p,0}\big)^{\mathrm{t}} \in \mathbf{X}_2'$*, with some* $(\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p,0}, \widehat{f}_{p_p,0}) \in \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$.

Proof Following the approach from [3, Lemma 4.15], the initial data are constructed by solving a sequence of well-defined subproblems. We take the following steps.

1. Define $\mathbf{u}_{p,0} := -\frac{1}{\mu}\mathbf{K}\nabla p_{p,0}$, with $p_{p,0} \in \mathrm{H}_p$, cf. (4.18). It follows that $\mathbf{u}_{p,0} \in \mathbf{H}(\mathrm{div};\Omega_p)$ and

$$\mu\,\mathbf{K}^{-1}\mathbf{u}_{p,0} = -\nabla p_{p,0}, \quad \mathrm{div}(\mathbf{u}_{p,0}) = -\frac{1}{\mu}\mathrm{div}(\mathbf{K}\nabla p_{p,0}) \quad \text{in} \quad \Omega_p, \quad \mathbf{u}_{p,0}\cdot\mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_p^{\mathrm{N}}. \tag{4.20}$$

Next, defining $\lambda_0 := p_{p,0}|_{\Gamma_{fp}} \in \Lambda_p$, (4.20) implies

$$a_p(\mathbf{u}_{p,0}, \mathbf{v}_p) + b_p(\mathbf{v}_p, p_{p,0}) + b_\Gamma(\mathbf{v}_p, \lambda_0) = 0 \quad \forall\, \mathbf{v}_p \in \mathbf{V}_p. \tag{4.21}$$

2. Define $(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\varphi}_0, \mathbf{u}_{f,0}, \boldsymbol{\gamma}_{f,0}) \in \mathbb{X}_f \times \boldsymbol{\Lambda}_f \times \mathbf{V}_f \times \mathbb{Q}_f$ as the unique solution of the problem

$$
\begin{aligned}
&a_f(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\tau}_f) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\varphi}_0) + b_f(\boldsymbol{\tau}_f, \mathbf{u}_{f,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_f, \boldsymbol{\gamma}_{f,0}) = -\frac{1}{n}\,(q_f(0)\,\mathbf{I}, \boldsymbol{\tau}_f)_{\Omega_f},\\
&-b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\psi}) = -\mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot\mathbf{t}_{f,j}, \boldsymbol{\psi}\cdot\mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\psi}\cdot\mathbf{n}_f, \lambda_0\rangle_{\Gamma_{fp}},\\
&-b_f(\boldsymbol{\sigma}_{f,0}, \mathbf{v}_f) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\chi}_f) = (\mathbf{f}_f(0), \mathbf{v}_f)_{\Omega_f},
\end{aligned}
\tag{4.22}
$$

for all $(\boldsymbol{\tau}_f, \boldsymbol{\psi}, \mathbf{v}_f, \boldsymbol{\chi}_f) \in \mathbb{X}_f \times \boldsymbol{\Lambda}_f \times \mathbf{V}_f \times \mathbb{Q}_f$. Note that (4.22) is well-posed, since it corresponds to the weak solution of the Stokes problem in a mixed formulation and its solvability can be shown using classical Babuška-Brezzi theory. Note also that $\mathbf{u}_{p,0}$ and λ_0 are data for this problem.

3. Define $(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\omega}_0, \boldsymbol{\eta}_{p,0}, \boldsymbol{\rho}_{p,0}) \in \mathbb{X}_p \times \boldsymbol{\Lambda}_s \times \mathbf{V}_s \times \mathbb{Q}_p$, as the unique solution of the problem

$$
\begin{aligned}
&(A(\boldsymbol{\sigma}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\omega}_0) + b_s(\boldsymbol{\tau}_p, \boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\rho}_{p,0})\\
&\quad = -(A(\alpha_p\, p_{p,0}\,\mathbf{I}), \boldsymbol{\tau}_p)_{\Omega_p}\\
&-b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}) = \mu\,\alpha_{\mathrm{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}\mathbf{u}_{p,0}\cdot\mathbf{t}_{f,j}, \boldsymbol{\phi}\cdot\mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} - \langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \lambda_0\rangle_{\Gamma_{fp}}\\
&-b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_s) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_p) = (\mathbf{f}_p(0), \mathbf{v}_s)_{\Omega_p},
\end{aligned}
\tag{4.23}
$$

for all $(\boldsymbol{\tau}_p, \boldsymbol{\phi}, \mathbf{v}_s, \boldsymbol{\chi}_p) \in \mathbb{X}_p \times \boldsymbol{\Lambda}_s \times \mathbf{V}_s \times \mathbb{Q}_p$. Problem (4.23) corresponds to the weak solution of the elasticity problem in a mixed formulation and its solvability can be shown using classical Babuška-Brezzi theory. Note that $p_{p,0}$, $\mathbf{u}_{p,0}$, and λ_0 are data for this problem. Here $\boldsymbol{\eta}_{p,0}$, $\boldsymbol{\rho}_{p,0}$, and $\boldsymbol{\omega}_0$ are auxiliary variables that are not part of the constructed initial data. However, they can be used to recover the variables $\boldsymbol{\eta}_p$, $\boldsymbol{\rho}_p$, and $\boldsymbol{\omega}$ that satisfy the non-differentiated Eq. (3.7).

4. Define $\boldsymbol{\theta}_0 \in \boldsymbol{\Lambda}_s$ as

$$
\boldsymbol{\theta}_0 := \boldsymbol{\varphi}_0 - \mathbf{u}_{p,0} \quad \text{on} \quad \Gamma_{fp}, \tag{4.24}
$$

where $\boldsymbol{\varphi}_0$ and $\mathbf{u}_{p,0}$ are data obtained in the previous steps. Note that (4.24) implies that the BJS terms in (4.22) and (4.23) can be rewritten with $\mathbf{u}_{p,0}\cdot\mathbf{t}_{f,j} = (\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0)\cdot\mathbf{t}_{f,j}$ and that the Eq. (3.6i) holds for the initial data, that is,

$$
-\left\langle \boldsymbol{\varphi}_0\cdot\mathbf{n}_f + (\boldsymbol{\theta}_0 + \mathbf{u}_{p,0})\cdot\mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}} = 0 \quad \forall\, \xi \in \Lambda_p. \tag{4.25}
$$

5. Finally, define $(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{u}_{s,0}, \boldsymbol{\gamma}_{p,0}) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$, as the unique solution of the problem

$$
\begin{aligned}
&(A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} + b_s(\boldsymbol{\tau}_p, \mathbf{u}_{s,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_p, \boldsymbol{\gamma}_{p,0}) = -b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\theta}_0)\\
&-b_s(\widehat{\boldsymbol{\sigma}}_{p,0}, \mathbf{v}_s) - b_{\mathrm{sk},p}(\widehat{\boldsymbol{\sigma}}_{p,0}, \boldsymbol{\chi}_p) = 0,
\end{aligned}
\tag{4.26}
$$

for all $(\boldsymbol{\tau}_p, \mathbf{v}_s, \boldsymbol{\chi}_p) \in \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p$. Problem (4.26) corresponds to the weak solution of the elasticity problem in Ω_p with Dirichlet datum $\boldsymbol{\theta}_0$ on Γ_{fp}.

Combining (4.21), (4.22), the second and third equations in (4.23), (4.25), and the first equation in (4.26), we obtain $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0) \in \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ satisfying (4.19) with

$$(\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p,0}, \boldsymbol{\tau}_p)_{\Omega_p} = -(A(\widehat{\boldsymbol{\sigma}}_{p,0}), \boldsymbol{\tau}_p)_{\Omega_p} \quad \text{and} \quad (\widehat{f}_{p_p,0}, w_p)_{\Omega_p} = -b_p(\mathbf{u}_{p,0}, w_p). \tag{4.27}$$

The above equations imply

$$\|\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\widehat{f}_{p_p,0}\|_{\mathrm{L}^2(\Omega_p)} \leq C\left(\|\widehat{\boldsymbol{\sigma}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} + \|\mathrm{div}(\mathbf{u}_{p,0})\|_{\mathrm{L}^2(\Omega_p)}\right).$$

Standard stability arguments for (4.22) and (4.26), together with the definition (4.24) of $\boldsymbol{\theta}_0$, imply that $\|\widehat{\boldsymbol{\sigma}}_{p,0}\|_{\mathbb{L}^2(\Omega_p)} \leq C(\|q_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{f}_f(0)\|_{\mathrm{L}^2(\Omega_f)} + \|\mathbf{u}_{p,0}\|_{\mathbf{H}^1(\Omega_p)} + \|\lambda_0\|_{\mathrm{H}^{1/2}(\Gamma_{fp})})$. Hence $(\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_p,0}, \widehat{f}_{p_p,0}) \in \mathbb{X}'_{p,2} \times \mathrm{W}'_{p,2}$, completing the proof. □

4.4 The main result

We are now ready to prove existence and uniqueness of a solution of the problem (3.11).

Theorem 4.10 *For each* $p_{p,0} \in \mathrm{H}_p$ *and compatible initial data* $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$ *constructed in* Lemma 4.9 *and each*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_f), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}'_s), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}'_f),$$
$$q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}'_p),$$

there exists a unique solution of (3.11), $(\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}}) : [0,T] \to \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$, *such that* $(\boldsymbol{\sigma}_p, p_p) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega_p)) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_p)$ *and* $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

Proof For each fixed time $t \in [0,T]$, Lemma 4.7 implies that there exists a solution to the resolvent system (4.9) with $\widehat{\mathbf{F}} = \mathbf{F}(t)$ and $\widehat{\mathbf{G}} = \mathbf{G}(t)$ defined in (3.15). More precisely, there exist $(\underline{\widetilde{\boldsymbol{\sigma}}}(t), \underline{\widetilde{\boldsymbol{\varphi}}}(t), \underline{\widetilde{\mathbf{u}}}(t))$ such that

$$\begin{aligned}
(\mathcal{E}+\mathcal{A})(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) + \mathcal{B}'_1(\underline{\widetilde{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\underline{\widetilde{\mathbf{u}}}(t)) &= \mathbf{F}(t) \ \text{ in } \mathbf{X}'_2, \\
-\mathcal{B}_1(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\underline{\widetilde{\boldsymbol{\varphi}}}(t)) &= \mathbf{0} \quad\ \text{ in } \mathbf{Y}', \\
-\mathcal{B}(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) &= \mathbf{G}(t) \ \text{ in } \mathbf{Z}'.
\end{aligned} \tag{4.28}$$

We look for a solution to (3.11) in the form $\underline{\boldsymbol{\sigma}}(t) = \underline{\widetilde{\boldsymbol{\sigma}}}(t) + \underline{\widehat{\boldsymbol{\sigma}}}(t)$, $\underline{\boldsymbol{\varphi}}(t) = \underline{\widetilde{\boldsymbol{\varphi}}}(t) + \underline{\widehat{\boldsymbol{\varphi}}}(t)$, and $\underline{\mathbf{u}}(t) = \underline{\widetilde{\mathbf{u}}}(t) + \underline{\widehat{\mathbf{u}}}(t)$. Subtracting (4.28) from (3.11) leads to the reduced evolution problem for $(\underline{\widehat{\boldsymbol{\sigma}}}(t), \underline{\widehat{\boldsymbol{\varphi}}}(t), \underline{\widehat{\mathbf{u}}}(t))$:

$$\begin{aligned}
\partial_t\mathcal{E}(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{A}(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{B}'_1(\underline{\widehat{\boldsymbol{\varphi}}}(t)) + \mathcal{B}'(\underline{\widehat{\mathbf{u}}}(t)) &= \mathcal{E}(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) - \partial_t\mathcal{E}(\underline{\widetilde{\boldsymbol{\sigma}}}(t)) \ \text{ in } \mathbf{X}'_{2,0}, \\
-\mathcal{B}_1(\underline{\widehat{\boldsymbol{\sigma}}}(t)) + \mathcal{C}(\underline{\widehat{\boldsymbol{\varphi}}}(t)) &= \mathbf{0} \quad \text{ in } \mathbf{Y}'_{2,0}, \\
-\mathcal{B}(\underline{\widehat{\boldsymbol{\sigma}}}(t)) &= \mathbf{0} \quad \text{ in } \mathbf{Z}'_{2,0},
\end{aligned} \tag{4.29}$$

with initial condition $\widehat{\underline{\sigma}}(0) = \underline{\sigma}_0 - \widetilde{\underline{\sigma}}(0)$, $\widehat{\underline{\varphi}}(0) = \underline{\varphi}_0 - \widetilde{\underline{\varphi}}(0)$, and $\widehat{\underline{\mathbf{u}}}(0) = \underline{\mathbf{u}}_0 - \widetilde{\underline{\mathbf{u}}}(0)$. Subtracting (4.28) at $t = 0$ from (4.19) gives

$$\begin{aligned} \mathcal{A}(\widehat{\underline{\sigma}}(0)) + \mathcal{B}_1'(\widehat{\underline{\varphi}}(0)) + \mathcal{B}'(\widehat{\underline{\mathbf{u}}}(0)) &= \mathcal{E}(\widetilde{\underline{\sigma}}(0)) + \widehat{\mathbf{F}}_0 - \mathbf{F}(0) \ \text{ in } \ \mathbf{X}_{2,0}', \\ -\mathcal{B}_1(\widehat{\underline{\sigma}}(0)) + \mathcal{C}(\widehat{\underline{\varphi}}(0)) &= \mathbf{0} \ \text{ in } \ \mathbf{Y}_{2,0}', \\ -\mathcal{B}(\widehat{\underline{\sigma}}(0)) &= \mathbf{0} \ \text{ in } \ \mathbf{Z}_{2,0}'. \end{aligned} \tag{4.30}$$

We emphasize that in (4.30), $\widehat{\mathbf{F}}_0 - \mathbf{F}(0) = (\mathbf{0}, \mathbf{0}, \widehat{\mathbf{f}}_{\sigma_p,0}, \widehat{f}_{p_p,0} - q_p(0))^{\mathrm{t}} \in \mathbf{X}_{2,0}'$. Thus, $\mathcal{M}(\widehat{\underline{\sigma}}(0), \widehat{\underline{\varphi}}(0), \widehat{\underline{\mathbf{u}}}(0)) \in E_b'$, i.e., $(\widehat{\underline{\sigma}}(0), \widehat{\underline{\varphi}}(0), \widehat{\underline{\mathbf{u}}}(0)) \in \mathcal{D}$ (cf. (4.5)). Thus, the reduced evolution problem (4.29) is in the form of (4.6). According to Lemma 4.8, it has a solution, which establishes the existence of a solution to (3.11) with the stated regularity satisfying $(\boldsymbol{\sigma}_p(0), p_p(0)) = (\boldsymbol{\sigma}_{p,0}, p_{p,0})$.

We next show that the solution of (3.11) is unique. Since the problem is linear, it is sufficient to prove that the problem with zero data has only the zero solution. Taking $\mathbf{F} = \mathbf{G} = \mathbf{0}$ in (3.11) and testing it with the solution $(\underline{\sigma}, \underline{\varphi}, \underline{\mathbf{u}})$ yields

$$\begin{aligned} &\frac{1}{2}\,\partial_t \left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\, \|p_p\|^2_{\mathrm{W}_p} \right) \\ &\quad + \frac{1}{2\,\mu}\, \|\boldsymbol{\sigma}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{u}_p, \mathbf{u}_p) + \mathcal{C}(\underline{\varphi})(\underline{\varphi}) = 0, \end{aligned}$$

which together with (4.14), (2.7) to bound a_p (cf. (3.8)), the semi-definite positive property of $\mathcal{C}$ (cf. (4.8)), integrating in time from 0 to $t \in (0, T]$, and using that the initial data are zero, implies

$$\|\boldsymbol{\sigma}_p\|^2_{\mathbb{L}^2(\Omega_p)} + \|p_p\|^2_{\mathrm{W}_p} + \int_0^t \left(\|\boldsymbol{\sigma}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} \right) ds \le 0. \tag{4.31}$$

It follows from (4.31) that $\boldsymbol{\sigma}_f^{\mathrm{d}}(t) = \mathbf{0}$, $\mathbf{u}_p(t) = \mathbf{0}$, $\boldsymbol{\sigma}_p(t) = \mathbf{0}$, and $p_p(t) = 0$ for all $t \in (0, T]$.

Now, taking $\underline{\tau} \in \mathbf{V}$ (cf. (4.12)) in the first equation of (3.11) and employing the inf-sup condition of $\mathcal{B}_1$ (cf. (4.15)), with $\underline{\psi} = \underline{\varphi} = (\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda) \in \mathbf{Y}$, yields

$$\widetilde{\beta}\,\|\underline{\varphi}\|_{\mathbf{Y}} \le \sup_{\mathbf{0} \neq \underline{\tau} \in \mathbf{V}} \frac{\mathcal{B}_1(\underline{\tau})(\underline{\varphi})}{\|\underline{\tau}\|_{\mathbf{X}}} = - \sup_{\mathbf{0} \neq \underline{\tau} \in \mathbf{V}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\sigma})(\underline{\tau})}{\|\underline{\tau}\|_{\mathbf{X}}} = 0.$$

Thus, $\boldsymbol{\varphi}(t) = \mathbf{0}$, $\boldsymbol{\theta}(t) = \mathbf{0}$, and $\lambda(t) = 0$ for all $t \in (0, T]$. In turn, from the inf-sup condition of $\mathcal{B}$ (cf. (4.16)), with $\underline{\mathbf{v}} = \underline{\mathbf{u}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$, we get

$$\beta\,\|\underline{\mathbf{u}}\|_{\mathbf{Z}} \le \sup_{\mathbf{0} \neq \underline{\tau} \in \mathbf{X}} \frac{\mathcal{B}(\underline{\tau})(\underline{\mathbf{u}})}{\|\underline{\tau}\|_{\mathbf{X}}} = - \sup_{\mathbf{0} \neq \underline{\tau} \in \mathbf{X}} \frac{(\partial_t\,\mathcal{E} + \mathcal{A})(\underline{\sigma})(\underline{\tau}) + \mathcal{B}_1(\underline{\tau})(\underline{\varphi})}{\|\underline{\tau}\|_{\mathbf{X}}} = 0.$$

Therefore, $\mathbf{u}_f(t) = \mathbf{0}$, $\mathbf{u}_s(t) = \mathbf{0}$, $\boldsymbol{\gamma}_f(t) = \mathbf{0}$, and $\boldsymbol{\gamma}_p(t) = \mathbf{0}$ for all $t \in (0, T]$. Finally, from the third row in (3.10), we have the identity

$$b_f(\boldsymbol{\sigma}_f, \mathbf{v}_f) = 0 \quad \forall\, \mathbf{v}_f \in \mathbf{V}_f.$$

Taking $\mathbf{v}_f = \mathbf{div}(\boldsymbol{\sigma}_f) \in \mathbf{V}_f$, we deduce that $\mathbf{div}(\boldsymbol{\sigma}_f(t)) = \mathbf{0}$ for all $t \in (0, T]$, which combined with the fact that $\boldsymbol{\sigma}_f^{\mathrm{d}}(t) = \mathbf{0}$ for all $t \in (0, T]$, and estimates (4.2)–(4.3) yields $\boldsymbol{\sigma}_f(t) = \mathbf{0}$ for all $t \in (0, T]$. Then, (3.11) has a unique solution. □

Corollary 4.11 *The solution of* (3.11) *established in* Theorem 4.10 *satisfies* $\boldsymbol{\sigma}_f(0) = \boldsymbol{\sigma}_{f,0}$, $\mathbf{u}_f(0) = \mathbf{u}_{f,0}$, $\boldsymbol{\gamma}_f(0) = \boldsymbol{\gamma}_{f,0}$, $\mathbf{u}_p(0) = \mathbf{u}_{p,0}$, $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0$, $\lambda(0) = \lambda_0$, *and* $\boldsymbol{\theta}(0) = \boldsymbol{\theta}_0$.

Proof Let $\overline{\boldsymbol{\sigma}}_f := \boldsymbol{\sigma}_f(0) - \boldsymbol{\sigma}_{f,0}$, with a similar definition and notation for the rest of the variables. Since Theorem 4.1 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0, T; E_b')$, we can take $t \to 0$ in all equations without time derivatives in (4.29), and therefore also in (3.11). Using that the initial data $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$ satisfy the same equations at $t = 0$ (cf. (4.19)), and that $\overline{\boldsymbol{\sigma}}_p = \mathbf{0}$ and $\overline{p}_p = 0$, we obtain

$$\frac{1}{2\mu}\,(\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}}, \boldsymbol{\tau}_f^{\mathrm{d}})_{\Omega_f} + (\overline{\mathbf{u}}_f, \mathbf{div}(\boldsymbol{\tau}_f))_{\Omega_f} + (\overline{\boldsymbol{\gamma}}_f, \boldsymbol{\tau}_f)_{\Omega_f} - \langle \boldsymbol{\tau}_f\mathbf{n}_f, \overline{\boldsymbol{\varphi}}\rangle_{\Gamma_{fp}} = 0, \tag{4.32a}$$

$$\mu\,(\mathbf{K}^{-1}\overline{\mathbf{u}}_p, \mathbf{v}_p)_{\Omega_p} + \langle \mathbf{v}_p\cdot\mathbf{n}_p, \overline{\lambda}\rangle_{\Gamma_{fp}} = 0, \tag{4.32b}$$

$$-\,(\mathbf{v}_f, \mathbf{div}(\overline{\boldsymbol{\sigma}}_f))_{\Omega_f} = 0, \tag{4.32c}$$

$$-\,(\overline{\boldsymbol{\sigma}}_f, \boldsymbol{\chi}_f)_{\Omega_f} = 0, \tag{4.32d}$$

$$-\left\langle \overline{\boldsymbol{\varphi}}\cdot\mathbf{n}_f + \left(\overline{\boldsymbol{\theta}} + \overline{\mathbf{u}}_p\right)\cdot\mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}} = 0, \tag{4.32e}$$

$$\langle \overline{\boldsymbol{\sigma}}_f\mathbf{n}_f, \boldsymbol{\psi}\rangle_{\Gamma_{fp}} + \mu\,\alpha_{\mathrm{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}\left(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}\right)\cdot\mathbf{t}_{f,j}, \boldsymbol{\psi}\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle \boldsymbol{\psi}\cdot\mathbf{n}_f, \overline{\lambda}\rangle_{\Gamma_{fp}} = 0, \tag{4.32f}$$

$$-\,\mu\,\alpha_{\mathrm{BJS}}\sum_{j=1}^{n-1}\left\langle\sqrt{\mathbf{K}_j^{-1}}\left(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}\right)\cdot\mathbf{t}_{f,j}, \boldsymbol{\phi}\cdot\mathbf{t}_{f,j}\right\rangle_{\Gamma_{fp}} + \langle \boldsymbol{\phi}\cdot\mathbf{n}_p, \overline{\lambda}\rangle_{\Gamma_{fp}} = 0. \tag{4.32g}$$

Taking $(\boldsymbol{\tau}_f, \mathbf{v}_p, \mathbf{v}_f, \boldsymbol{\chi}_f, \xi, \boldsymbol{\psi}, \boldsymbol{\phi}) = (\overline{\boldsymbol{\sigma}}_f, \overline{\mathbf{u}}_p, \overline{\mathbf{u}}_f, \overline{\boldsymbol{\gamma}}_f, \overline{\lambda}, \overline{\boldsymbol{\varphi}}, \overline{\boldsymbol{\theta}})$ and combining the equations results in

$$\|\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + \|\overline{\mathbf{u}}_p\|^2_{\mathbf{L}^2(\Omega_p)} + |\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}}|^2_{\mathrm{BJS}} \le 0\,, \tag{4.33}$$

implying $\overline{\boldsymbol{\sigma}}_f^{\mathrm{d}} = \mathbf{0}$, $\overline{\mathbf{u}}_p = \mathbf{0}$, and $(\overline{\boldsymbol{\varphi}} - \overline{\boldsymbol{\theta}})\cdot\mathbf{t}_{f,j} = 0$. The inf-sup conditions (4.15)–(4.16), together with (4.32), imply that $\overline{\mathbf{u}}_f = \mathbf{0}$, $\overline{\boldsymbol{\gamma}}_f = \mathbf{0}$, $\overline{\boldsymbol{\varphi}} = \mathbf{0}$, and $\overline{\lambda} = 0$. Then (4.33) yields $\overline{\boldsymbol{\theta}}\cdot\mathbf{t}_{f,j} = 0$. In turn, Eq. (4.32e) implies that $\left\langle\overline{\boldsymbol{\theta}}\cdot\mathbf{n}_p, \xi\right\rangle_{\Gamma_{fp}} = 0$ for all $\xi \in \mathrm{H}^{1/2}(\Gamma_{fp})$. Note that $\mathbf{n}_p$ may be discontinuous on Γ_{fp}, thus $\overline{\boldsymbol{\theta}}\cdot\mathbf{n}_p \in \mathrm{L}^2(\Gamma_{fp})$. Since $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, then $\overline{\boldsymbol{\theta}}\cdot\mathbf{n}_p = 0$, and we conclude that $\overline{\boldsymbol{\theta}} = \mathbf{0}$. In addition, taking $\mathbf{v}_f = \mathbf{div}(\overline{\boldsymbol{\sigma}}_f) \in \mathbf{V}_f$ in (4.32c) we deduce that $\mathbf{div}(\overline{\boldsymbol{\sigma}}_f) = \mathbf{0}$, which, combined with (4.2)–(4.3), yields $\overline{\boldsymbol{\sigma}}_f = \mathbf{0}$, completing the proof. □

Remark 4.3 As we noted in Remark 3.1, the Eq. (3.6d) can be used to recover the non-differentiated Eq. (3.7). In particular, recalling the initial data construction (4.23), let

$$\forall\, t \in [0, T], \quad \boldsymbol{\eta}_p(t) = \boldsymbol{\eta}_{p,0} + \int_0^t \mathbf{u}_s(s)\, ds, \quad \boldsymbol{\rho}_p(t) = \boldsymbol{\rho}_{p,0} + \int_0^t \boldsymbol{\gamma}_p(s)\, ds,$$
$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_0 + \int_0^t \boldsymbol{\theta}(s)\, ds.$$

Then (3.7) follows from integrating (3.6d) from 0 to $t \in (0, T]$ and using the first equation in (4.23).

We end this section with a stability bound for the solution of (3.11). We will use the inf-sup condition

$$\|p_p\|_{\mathrm{W}_p} + \|\lambda\|_{\Lambda_p} \le c \sup_{\mathbf{0}\neq\mathbf{v}_p\in\mathbf{V}_p} \frac{b_p(\mathbf{v}_p, p_p) + b_\Gamma(\mathbf{v}_p, \lambda)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}}, \tag{4.34}$$

which follows from a slight adaptation of [38, Lemma 3.3]. In addition, recalling the definition of the seminorm $|\boldsymbol{\psi} - \boldsymbol{\phi}|_{\mathrm{BJS}}$ for $\boldsymbol{\psi} \in \boldsymbol{\Lambda}_f, \boldsymbol{\phi} \in \boldsymbol{\Lambda}_s$, cf. (4.8), we define

$$|\boldsymbol{\psi}-\boldsymbol{\phi}|^2_{\mathrm{L}^2(0,T;\mathrm{BJS})} := \int_0^T |(\boldsymbol{\psi}-\boldsymbol{\phi})(t)|^2_{\mathrm{BJS}}\, dt, \quad |\boldsymbol{\psi}-\boldsymbol{\phi}|_{\mathrm{L}^\infty(0,T;\mathrm{BJS})} := \operatorname*{ess\,sup}_{t\in[0,T]} |(\boldsymbol{\psi}-\boldsymbol{\phi})(t)|_{\mathrm{BJS}}.$$

Theorem 4.12 *For the solution of* (3.11) *established in* Theorem 4.10*, assuming sufficient regularity of the data, there exists a positive constant C independent of s_0 such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma}_f\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\boldsymbol{\sigma}_f\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_p\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&+ |\boldsymbol{\varphi} - \boldsymbol{\theta}|_{\mathrm{L}^\infty(0,T;BJS)} + |\boldsymbol{\varphi} - \boldsymbol{\theta}|_{\mathrm{L}^2(0,T;BJS)} + \|\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_p)} \\
&+ \|\underline{\boldsymbol{\varphi}}\|_{\mathrm{L}^2(0,T;\mathbf{Y})} + \|\underline{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} \\
&+ \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} \\
&+ \|p_p\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_p\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&+ \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t\, p_p\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
\le\, & C\Big(\|\mathbf{f}_f\|_{\mathrm{H}^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{\mathrm{H}^1(0,T;\mathbf{V}_s')} + \|q_f\|_{\mathrm{H}^1(0,T;\mathbb{X}_f')} + \|q_p\|_{\mathrm{H}^1(0,T;\mathrm{W}_p')} \\
&+ (1 + \sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|_{\mathrm{H}^1(\Omega_p)}\Big).
\end{aligned} \tag{4.35}$$

Proof We begin by choosing $(\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\psi}}, \underline{\mathbf{v}}) = (\underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\varphi}}, \underline{\mathbf{u}})$ in (3.10) to get

$$\begin{aligned} &\frac{1}{2}\,\partial_t \left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\, \|p_p\|^2_{\mathrm{W}_p} \right) \\ &\quad + \frac{1}{2\,\mu}\, \|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{u}_p, \mathbf{u}_p) + c_{\mathtt{BJS}}(\boldsymbol{\varphi}, \boldsymbol{\theta}; \boldsymbol{\varphi}, \boldsymbol{\theta}) \\ &= -\frac{1}{n}\,(q_f\, \mathbf{I}, \boldsymbol{\sigma}_f)_{\Omega_f} + (q_p, p_p)_{\Omega_p} + (\mathbf{f}_f, \mathbf{u}_f)_{\Omega_f} + (\mathbf{f}_p, \mathbf{u}_s)_{\Omega_p}. \end{aligned} \tag{4.36}$$

Next, we integrate (4.36) from 0 to $t \in (0, T]$, use the coercivity bounds (4.7)–(4.8), and apply the Cauchy–Schwarz and Young's inequalities, to find

$$\begin{aligned} &\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0 \|p_p(t)\|^2_{\mathrm{W}_p} \\ &\quad + \int_0^t \left(\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi} - \boldsymbol{\theta}|^2_{\mathtt{BJS}} \right) ds \\ &\leq C \left(\int_0^t \left(\|\mathbf{f}_f\|^2_{\mathbf{V}'_f} + \|\mathbf{f}_p\|^2_{\mathbf{V}'_s} + \|q_f\|^2_{\mathbb{X}'_f} + \|q_p\|^2_{\mathrm{W}'_p} \right) ds \right. \\ &\quad \left. + \|A^{1/2}(\boldsymbol{\sigma}_p(0) + \alpha_p\, p_p(0)\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\, \|p_p(0)\|^2_{\mathrm{W}_p} \right) \\ &\quad + \delta \int_0^t \left(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s} \right) ds, \end{aligned} \tag{4.37}$$

where $\delta > 0$ will be suitably chosen. In addition, (4.34) and the first equation in (3.10), yields

$$\begin{aligned} \|p_p\|_{\mathrm{W}_p} + \|\lambda\|_{\Lambda_p} &\leq c \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{V}_p} \frac{b_p(\mathbf{v}_p, p_p) + b_\Gamma(\mathbf{v}_p, \lambda)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}} \\ &= -c \sup_{\mathbf{0} \neq \mathbf{v}_p \in \mathbf{V}_p} \frac{a_p(\mathbf{u}_p, \mathbf{v}_p)}{\|\mathbf{v}_p\|_{\mathbf{V}_p}} \leq C\, \|\mathbf{u}_p\|_{\mathbf{L}^2(\Omega_p)}. \end{aligned} \tag{4.38}$$

Taking $\underline{\boldsymbol{\tau}} \in \mathbf{V}$ (cf. (4.12)) in the first equation of (3.11), using the continuity of the operators $\mathcal{E}$ and $\mathcal{A}$ in Lemma 4.3, and the inf-sup condition of $\mathcal{B}_1$ for $\underline{\boldsymbol{\varphi}} \in \mathbf{Y}$ (cf. (4.15)), we deduce

$$\begin{aligned} \beta_1\, \|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}} &\leq \sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = -\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbf{V}} \frac{(\partial_t\, \mathcal{E} + \mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) - \mathbf{F}(\underline{\boldsymbol{\tau}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} \\ &\leq C \left(\|\boldsymbol{\sigma}_f\|_{\mathbb{X}_f} + \|\mathbf{u}_p\|_{\mathbf{V}_p} + \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p \mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} \right. \\ &\quad \left. + \sqrt{s_0}\|\partial_t\, p_p\|_{\mathrm{W}_p} + \|q_f\|_{\mathbb{X}'_f} + \|q_p\|_{\mathrm{W}'_p} \right). \end{aligned} \tag{4.39}$$

In turn, from the first equation in (3.11), applying the inf-sup condition of $\mathcal{B}$ (cf. (4.16)) for $\underline{\mathbf{u}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$, and (4.39), we obtain

$$\begin{aligned}\beta\,\|\underline{\mathbf{u}}\|_{\mathbf{Z}} &\le \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}})(\underline{\mathbf{u}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}} = -\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbf{X}} \frac{(\partial_t\,\mathcal{E}+\mathcal{A})(\underline{\boldsymbol{\sigma}})(\underline{\boldsymbol{\tau}}) + \mathcal{B}_1(\underline{\boldsymbol{\tau}})(\underline{\boldsymbol{\varphi}}) - \mathbf{F}(\underline{\boldsymbol{\tau}})}{\|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}}}\\
&\le C\Big(\|\boldsymbol{\sigma}_f\|_{\mathbb{X}_f} + \|\mathbf{u}_p\|_{\mathbf{V}_p} + \|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\\
&\qquad + \sqrt{s_0}\|\partial_t\,p_p\|_{\mathrm{W}_p} + \|q_f\|_{\mathbb{X}'_f} + \|q_p\|_{\mathrm{W}'_p}\Big). \qquad (4.40)\end{aligned}$$

In addition, taking $w_p = \mathrm{div}(\mathbf{u}_p)$, $\mathbf{v}_f = \mathbf{div}(\boldsymbol{\sigma}_f)$, and $\mathbf{v}_s = \mathbf{div}(\boldsymbol{\sigma}_p)$ in the first and third equations of (3.10), we get

$$\begin{aligned}&\|\mathbf{div}(\boldsymbol{\sigma}_f)\|_{\mathbf{L}^2(\Omega_f)} \le \|\mathbf{f}_f\|_{\mathbf{V}'_f}, \quad \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbf{L}^2(\Omega_p)} \le \|\mathbf{f}_p\|_{\mathbf{V}'_s},\\
&\|\mathrm{div}(\mathbf{u}_p)\|_{\mathrm{L}^2(\Omega_p)} \le C\Big(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\\
&\qquad + \sqrt{s_0}\|\partial_t\,p_p\|_{\mathrm{W}_p} + \|q_p\|_{\mathrm{W}'_p}\Big). \qquad (4.41)\end{aligned}$$

Then, combining (4.37)–(4.41), using (4.2)–(4.3), and choosing δ small enough, we obtain

$$\begin{aligned}&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|p_p(t)\|^2_{\mathrm{W}_p} + \int_0^t \Big(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p\|^2_{\mathbf{V}_p}\\
&\quad + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_p\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}-\boldsymbol{\theta}|^2_{\mathtt{BJS}} + \|\underline{\boldsymbol{\varphi}}\|^2_{\mathbf{Y}} + \|\underline{\mathbf{u}}\|^2_{\mathbf{Z}}\Big)\,ds\\
&\le C\Bigg(\int_0^t \Big(\|\mathbf{f}_f\|^2_{\mathbf{V}'_f} + \|\mathbf{f}_p\|^2_{\mathbf{V}'_s} + \|q_f\|^2_{\mathbb{X}'_f} + \|q_p\|^2_{\mathrm{W}'_p}\Big)\,ds\\
&\quad + \|A^{1/2}(\boldsymbol{\sigma}_p(0) + \alpha_p\,p_p(0)\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|p_p(0)\|^2_{\mathrm{W}_p}\\
&\quad + \int_0^t \Big(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p p_p\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\,p_p\|^2_{\mathrm{W}_p}\Big)\,ds\Bigg).\end{aligned} \qquad (4.42)$$

Finally, in order to bound the last two terms in (4.42), we test (3.10) with $\underline{\boldsymbol{\tau}} = (\partial_t\,\boldsymbol{\sigma}_f, \mathbf{u}_p, \partial_t\,\boldsymbol{\sigma}_p, \partial_t\,p_p) \in \mathbf{X}$, $\underline{\boldsymbol{\psi}} = (\boldsymbol{\varphi}, \boldsymbol{\theta}, \partial_t\,\lambda) \in \mathbf{Y}$, $\underline{\mathbf{v}} = (\mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p) \in \mathbf{Z}$ and differentiate in time the rows in (3.10) associated to $\mathbf{v}_p$, $\boldsymbol{\psi}$, $\boldsymbol{\phi}$, $\mathbf{v}_f$, $\mathbf{v}_s$, $\boldsymbol{\chi}_f$ and $\boldsymbol{\chi}_p$, to deduce

$$\begin{aligned}&\frac{1}{2}\,\partial_t\left(\frac{1}{2\,\mu}\,\|\boldsymbol{\sigma}^{\mathrm{d}}_f\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{u}_p,\mathbf{u}_p) + c_{\mathtt{BJS}}(\boldsymbol{\varphi},\boldsymbol{\theta};\boldsymbol{\varphi},\boldsymbol{\theta})\right)\\
&\quad + \|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\,p_p\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|\partial_t\,p_p\|^2_{\mathrm{W}_p}\\
&= \frac{1}{n}\,(q_f\,\mathbf{I}, \partial_t\,\boldsymbol{\sigma}_f)_{\Omega_f} + (q_p, \partial_t\,p_p)_{\Omega_p} + (\partial_t\,\mathbf{f}_f, \mathbf{u}_f)_{\Omega_f} + (\partial_t\,\mathbf{f}_p, \mathbf{u}_s)_{\Omega_p},\end{aligned}$$

which together with the identities

$$\int_0^t (q_f\,\mathbf{I},\partial_t\,\boldsymbol{\sigma}_f)_{\Omega_f} = (q_f\,\mathbf{I},\boldsymbol{\sigma}_f)_{\Omega_f}\Big|_0^t - \int_0^t (\partial_t\,q_f\,\mathbf{I},\boldsymbol{\sigma}_f)_{\Omega_f},$$
$$\int_0^t (q_p,\partial_t\,p_p)_{\Omega_p} = (q_p,p_p)_{\Omega_p}\Big|_0^t - \int_0^t (\partial_t\,q_p,p_p)_{\Omega_p},$$

and the positive semi-definite property of $\mathcal{C}$ (cf. (4.8)), yields

$$\begin{aligned}
&\|\boldsymbol{\sigma}^{\mathrm{d}}_f(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{u}_p(t)\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi}(t)-\boldsymbol{\theta}(t)|^2_{\mathtt{BJS}}\\
&\quad + \int_0^t \Big(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p+\alpha_p p_p\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t p_p\|^2_{\mathrm{W}_p}\Big)\,ds\\
&\quad \le C\Bigg(\int_0^t \Big(\|\partial_t\,\mathbf{f}_f\|^2_{\mathbf{V}'_f} + \|\partial_t\,\mathbf{f}_p\|^2_{\mathbf{V}'_s} + \|\partial_t\,q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t\,q_p\|^2_{\mathrm{W}'_p}\Big)\,ds\\
&\quad + \|q_f(t)\|^2_{\mathbb{X}'_f} + \|q_p(t)\|^2_{\mathrm{W}'_p} + \|q_f(0)\|^2_{\mathbb{X}'_f} + \|q_p(0)\|^2_{\mathrm{W}'_p} + \|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f}\\
&\quad + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0)-\boldsymbol{\theta}(0)|^2_{\mathtt{BJS}}\Bigg)\\
&\quad + \delta_1\Big(\|\boldsymbol{\sigma}_f(t)\|^2_{\mathbb{X}_f} + \|p_p(t)\|^2_{\mathrm{W}_p}\Big)\\
&\quad + \delta_2\int_0^t \Big(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{L}^2(\Omega_f)} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s}\Big)\,ds. \qquad (4.43)
\end{aligned}$$

Using (4.38) and the first two inequalities in (4.41), and choosing δ_1 small enough, we derive from (4.43) and (4.2)–(4.3) that

$$\begin{aligned}
&\|\boldsymbol{\sigma}_f(t)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{div}(\boldsymbol{\sigma}_p(t))\|^2_{\mathbf{L}^2(\Omega_p)} + |\boldsymbol{\varphi}(t)-\boldsymbol{\theta}(t)|^2_{\mathtt{BJS}}\\
&\quad + \|p_p(t)\|^2_{\mathrm{W}_p} + \|\lambda(t)\|^2_{\Lambda_p}\\
&\quad + \int_0^t \Big(\|\partial_t\,A^{1/2}(\boldsymbol{\sigma}_p+\alpha_p p_p\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\,p_p\|^2_{\mathrm{W}_p}\Big)\,ds\\
&\le C\Bigg(\int_0^t \Big(\|\partial_t\,\mathbf{f}_f\|^2_{\mathbf{V}'_f} + \|\partial_t\,\mathbf{f}_p\|^2_{\mathbf{V}'_s} + \|\partial_t\,q_f\|^2_{\mathrm{L}^2(\Omega_f)} + \|\partial_t q_p\|^2_{\mathrm{W}'_p}\Big)\,ds\\
&\quad + \|\mathbf{f}_f(t)\|^2_{\mathbf{V}'_f} + \|\mathbf{f}_p(t)\|^2_{\mathbf{V}'_s} + \|q_f(t)\|^2_{\mathbb{X}'_f} + \|q_p(t)\|^2_{\mathrm{W}'_p}\\
&\quad + \|q_f(0)\|^2_{\mathbb{X}'_f} + \|q_p(0)\|^2_{\mathrm{W}'_p} + \|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f}\\
&\quad + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0)-\boldsymbol{\theta}(0)|^2_{\mathtt{BJS}}\Bigg) +\\
&\delta_2\int_0^t \Big(\|\boldsymbol{\sigma}_f\|^2_{\mathbb{X}_f} + \|p_p\|^2_{\mathrm{W}_p} + \|\mathbf{u}_f\|^2_{\mathbf{V}_f} + \|\mathbf{u}_s\|^2_{\mathbf{V}_s}\Big)\,ds. \qquad (4.44)
\end{aligned}$$

We next bound the initial data terms in (4.42) and (4.44). Recalling from Corollary 4.11 that $(\underline{\boldsymbol{\sigma}}(0),\boldsymbol{\varphi}(0),\boldsymbol{\theta}(0)) = (\underline{\boldsymbol{\sigma}}_0,\boldsymbol{\varphi}_0,\boldsymbol{\theta}_0)$, using the stability of the continuous initial data problems (4.20)–(4.23) and the steady-state version of the arguments leading to (4.42), we obtain

$$
\begin{aligned}
&\|\boldsymbol{\sigma}_f(0)\|^2_{\mathbb{X}_f} + \|\mathbf{u}_p(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|A^{1/2}(\boldsymbol{\sigma}_p(0))\|^2_{\mathbb{L}^2(\Omega_p)} + \|p_p(0)\|^2_{\mathrm{W}_p} + |\boldsymbol{\varphi}(0) - \boldsymbol{\theta}(0)|^2_{\mathrm{BJS}} \\
&\leq C\left(\|p_{p,0}\|^2_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|^2_{\mathrm{H}^1(\Omega_p)} + \|\mathbf{f}_f(0)\|^2_{\mathbf{V}_f'} + \|\mathbf{f}_p(0)\|^2_{\mathbf{V}_s'} + \|q_f(0)\|^2_{\mathbb{X}_f'}\right),
\end{aligned}
\tag{4.45}
$$

Therefore, combining (4.42) with (4.44) and (4.45), choosing δ_2 small enough, and using the estimate (cf. (4.14)):

$$
\|A^{1/2}(\boldsymbol{\sigma}_p(t))\|_{\mathbb{L}^2(\Omega_p)} \leq C\left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha_p\, p_p\, \mathbf{I})(t)\|_{\mathbb{L}^2(\Omega_p)} + \|p_p(t)\|_{\mathrm{W}_p}\right),
\tag{4.46}
$$

and the Sobolev embedding of $\mathrm{H}^1(0, T)$ into $\mathrm{L}^\infty(0, T)$, we conclude (4.35). □

5 Semidiscrete continuous-in-time approximation

In this section we introduce and analyze the semidiscrete continuous-in-time approximation of (3.11). We analyze its solvability by employing the strategy developed in Sect. 4. In addition, we derive error estimates with rates of convergence.

Let $\mathcal{T}_h^f$ and $\mathcal{T}_h^p$ be shape-regular and quasi-uniform affine finite element partitions of Ω_f and Ω_p, respectively. The two partitions may be non-matching along the interface Γ_{fp}. For the discretization, we consider the following conforming finite element spaces:

$$
\begin{aligned}
&\mathbb{X}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh} \subset \mathbb{X}_f \times \mathbf{V}_f \times \mathbb{Q}_f, \quad \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph} \subset \mathbb{X}_p \times \mathbf{V}_s \times \mathbb{Q}_p, \\
&\quad \mathbf{V}_{ph} \times \mathrm{W}_{ph} \subset \mathbf{V}_p \times \mathrm{W}_p.
\end{aligned}
$$

We take $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ and $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$ to be any stable finite element spaces for mixed elasticity with weakly imposed stress symmetry, such as the Amara–Thomas [2], PEERS [10], Stenberg [58], Arnold–Falk–Winther [11, 12], or Cockburn–Gopalakrishnan–Guzman [28] families of spaces. We choose $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$ to be any stable mixed finite element Darcy spaces, such as the Raviart–Thomas or Brezzi-Douglas-Marini spaces [19]. For the Lagrange multipliers $(\boldsymbol{\Lambda}_{fh}, \boldsymbol{\Lambda}_{sh}, \Lambda_{ph})$ we consider the following two options of discrete spaces.

(S1) Conforming spaces:

$$
\boldsymbol{\Lambda}_{fh} \subset \boldsymbol{\Lambda}_f, \quad \boldsymbol{\Lambda}_{sh} \subset \boldsymbol{\Lambda}_s, \quad \Lambda_{ph} \subset \Lambda_p\,,
\tag{5.1}
$$

equipped with $\mathrm{H}^{1/2}$-norms as in (3.3). If the normal traces of the spaces $\mathbb{X}_{fh}$, $\mathbb{X}_{ph}$, or $\mathbf{V}_{ph}$ contain piecewise polynomials in P_k on simplices or Q_k on cubes with $k \geq 1$, where P_k denotes polynomials of total degree k and Q_k stands for polynomials of degree k in each variable, we take the Lagrange multiplier spaces to be continuous piecewise polynomials in P_k or Q_k on the traces of the corresponding subdomain grids. In the case of $k = 0$, we take the Lagrange multiplier spaces to be continuous piecewise polynomials in P_1 or Q_1 on grids obtained by coarsening by two the traces of the subdomain grids. Note that these choices guarantee the inf-sup conditions given below in Lemma 5.1.

(S2) Non-conforming spaces:

$$\boldsymbol{\Lambda}_{fh} := \mathbb{X}_{fh}\mathbf{n}_f|_{\Gamma_{fp}}, \quad \boldsymbol{\Lambda}_{sh} := \mathbb{X}_{ph}\mathbf{n}_p|_{\Gamma_{fp}}, \quad \Lambda_{ph} := \mathbf{V}_{ph}\cdot\mathbf{n}_p|_{\Gamma_{fp}}, \tag{5.2}$$

which consist of discontinuous piecewise polynomials and are equipped with L^2-norms.

It is also possible to mix conforming and non-conforming choices, but we will focus on **(S1)** and **(S2)** for simplicity of the presentation.

Remark 5.1 The choices **(S1)** and **(S2)** result in similar convergence rates, cf. Theorem 5.4. The conforming case **(S1)** has fewer degrees of freedom, while the non-conforming case **(S2)** provides local continuity of flux across Γ_{fp} on each element of the mesh for Λ_{ph}, which is the trace of $\mathcal{T}_h^p$ on Γ_{fp}.

Remark 5.2 We note that, since $\mathrm{H}^{1/2}(\Gamma_{fp})$ is dense in $\mathrm{L}^2(\Gamma_{fp})$, (3.6i)–(3.6k) in the continuous weak formulation hold for test functions in $\mathrm{L}^2(\Gamma_{fp})$, assuming that the solution is smooth enough so that the traces are well-defined in $\mathrm{L}^2(\Gamma_{fp})$; e.g., $\mathbf{u}_p \in \mathbf{H}^{1/2+\epsilon}(\Omega_p)$ for some $\epsilon > 0$. In particular, these equations hold for $\xi_h \in \Lambda_{ph}$, $\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_{fh}$, and $\boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}$ in both the conforming case **(S1)** and the non-conforming case **(S2)**.

Now, we group the spaces similarly to the continuous case:

$$\begin{aligned}
&\mathbf{X}_h := \mathbb{X}_{fh}\times\mathbf{V}_{ph}\times\mathbb{X}_{ph}\times\mathrm{W}_{ph}, \quad \mathbf{Y}_h := \boldsymbol{\Lambda}_{fh}\times\boldsymbol{\Lambda}_{sh}\times\Lambda_{ph},\\
&\mathbf{Z}_h := \mathbf{V}_{fh}\times\mathbf{V}_{sh}\times\mathbb{Q}_{fh}\times\mathbb{Q}_{ph},\\
&\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\varphi}}_h := (\boldsymbol{\varphi}_h, \boldsymbol{\theta}_h, \lambda_h) \in \mathbf{Y}_h,\\
&\underline{\mathbf{u}}_h := (\mathbf{u}_{fh}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{fh}, \boldsymbol{\gamma}_{ph}) \in \mathbf{Z}_h,\\
&\underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, w_{ph}) \in \mathbf{X}_h, \quad \underline{\boldsymbol{\psi}}_h := (\boldsymbol{\psi}_h, \boldsymbol{\phi}_h, \xi_h) \in \mathbf{Y}_h,\\
&\underline{\mathbf{v}}_h := (\mathbf{v}_{fh}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{fh}, \boldsymbol{\chi}_{ph}) \in \mathbf{Z}_h.
\end{aligned}$$

The spaces $\mathbf{X}_h$ and $\mathbf{Z}_h$ are endowed with the same norms as their continuous counterparts. For $\mathbf{Y}_h$ we consider the norm $\|\underline{\boldsymbol{\psi}}_h\|^2_{\mathbf{Y}_h} := \|\boldsymbol{\psi}_h\|^2_{\boldsymbol{\Lambda}_{fh}} + \|\boldsymbol{\phi}_h\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\xi_h\|^2_{\Lambda_{ph}}$, where

$$\|\xi_h\|_{\Lambda_{ph}} := \begin{cases} \|\xi_h\|_{\Lambda_p} \text{ for conforming subspaces } \textbf{(S1)} \text{ (cf. (3.3))},\\ \|\xi_h\|_{\mathrm{L}^2(\Gamma_{fp})} \text{ for non-conforming subspaces } \textbf{(S2)}. \end{cases} \tag{5.3}$$

Analogous notation is used for $\|\boldsymbol{\psi}_h\|_{\boldsymbol{\Lambda}_{fh}}$ and $\|\boldsymbol{\phi}_h\|_{\boldsymbol{\Lambda}_{sh}}$.

The continuity of all operators in the discrete case follows from their continuity in the continuous case (cf. Lemma 4.3), with the exception of $\mathcal{B}_1$ (cf. (3.12)) in the case of non-conforming Lagrange multipliers (**S2**). In this case it follows for each fixed h from the discrete trace-inverse inequality for piecewise polynomial functions, $\|\varphi\|_{\mathrm{L}^2(\Gamma)} \leq Ch^{-1/2}\|\varphi\|_{\mathrm{L}^2(\mathcal{O})}$, where $\Gamma \subset \partial\mathcal{O}$. In particular,

$$b_{\mathbf{n}_f}(\boldsymbol{\tau}_f, \boldsymbol{\psi}) \leq C\|\boldsymbol{\tau}_f\|_{\mathbb{L}^2(\Gamma_{fp})}\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Gamma_{fp})} \leq Ch^{-1/2}\|\boldsymbol{\tau}_f\|_{\mathbb{L}^2(\Omega_f)}\|\boldsymbol{\psi}\|_{\mathbf{L}^2(\Gamma_{fp})}, \quad (5.4)$$

with similar bounds for $b_{\mathbf{n}_p}(\boldsymbol{\tau}_p, \boldsymbol{\phi})$ and $b_\Gamma(\mathbf{v}_p, \xi)$.

We next discuss the discrete inf-sup conditions that are satisfied by the finite element spaces. Let

$$\widetilde{\mathbf{X}}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \quad \boldsymbol{\tau}_{fh}\mathbf{n}_f = \mathbf{0} \quad \text{and} \quad \boldsymbol{\tau}_{ph}\mathbf{n}_p = \mathbf{0} \quad \text{on} \quad \Gamma_{fp} \Big\}. \quad (5.5)$$

In addition, define the discrete kernel of the operator $\mathcal{B}$ as

$$\mathbf{V}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \quad \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Z}_h \Big\} = \widetilde{\mathbb{X}}_{fh} \times \mathbf{V}_{ph} \times \widetilde{\mathbb{X}}_{ph} \times \mathrm{W}_{ph}, \quad (5.6)$$

where

$$\widetilde{\mathbb{X}}_{\star h} := \Big\{ \boldsymbol{\tau}_{\star h} \in \mathbb{X}_{\star h} : (\boldsymbol{\tau}_{\star h}, \boldsymbol{\xi}_{\star h})_{\Omega_\star} = 0 \;\; \forall\, \boldsymbol{\xi}_{\star h} \in \mathbb{Q}_{\star h} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0} \quad \text{in} \quad \Omega_\star \Big\},$$
$$\star \in \{f, p\}.$$

In the above, $\mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0}$ follows from $\mathbf{div}(\mathbb{X}_{fh}) = \mathbf{V}_{fh}$ and $\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}$, which is true for all stable elasticity spaces.

Lemma 5.1 *There exist positive constants $\widetilde{\beta}$ and $\widetilde{\beta}_1$ such that*

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \widetilde{\mathbf{X}}_h} \frac{\mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widetilde{\beta}\, \|\underline{\mathbf{v}}_h\|_{\mathbf{Z}} \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h, \quad (5.7)$$

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \mathbf{V}_h} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widetilde{\beta}_1\, \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}_h} \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h. \quad (5.8)$$

Proof We begin with the proof of (5.7). We recall that the space $\mathbf{X}_h$ consists of stresses and velocities with zero normal traces on the Neumann boundaries, while the space $\widetilde{\mathbf{X}}_h$ involves further restriction on Γ_{fp}. The inf-sup condition (5.7) without restricting the normal stress or velocity on the subdomain boundary follows from the stability of the elasticity and Darcy finite element spaces. The restricted inf-sup condition (5.7) can be shown using the argument in [6, Theorem 4.2].

We continue with the proof of (5.8). Similarly to the continuous case, due the diagonal character of operator $\mathcal{B}_1$ (cf. (3.12)), we need to show individual inf-sup conditions for $b_{\mathbf{n}_f}$, $b_{\mathbf{n}_p}$, and b_Γ. We first focus on b_Γ. For the conforming case **(S1)** (cf. (5.1)), the proof of (5.8) can be derived from a slight adaptation of [31, Lemma 4.4] (see also [36, Section 5.3] for the case $k = 0$), whereas from [3, Section 5.1] we obtain the proof for the non-conforming version **(S2)** (cf. (5.2)). We next consider the inf-sup condition (5.8) for $b_{\mathbf{n}_f}$, with argument for $b_{\mathbf{n}_p}$ being similar. The proof utilizes a suitable interpolant of $\boldsymbol{\tau}_f := \mathbf{e}(\mathbf{v}_f)$, the solution to the auxiliary problem (4.17). Due to the stability of the spaces $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ (cf. (5.7)), there exists an interpolant $\tilde{\Pi}_h^f : \mathbb{H}^1(\Omega_f) \to \mathbb{X}_{fh}$ satisfying

$$\begin{aligned} &b_f(\tilde{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \mathbf{v}_{fh}) = 0 \quad \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh}, \\ &b_{\mathrm{sk},f}(\tilde{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \boldsymbol{\chi}_{fh}) = 0 \quad \forall\, \boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}, \\ &\langle (\tilde{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f)\mathbf{n}_f, \boldsymbol{\tau}_{fh}\mathbf{n}_f \rangle_{\Gamma_{fp}\cup\Gamma_f^N} = 0 \quad \forall\, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}. \end{aligned} \tag{5.9}$$

The interpolant $\tilde{\Pi}_h^f \boldsymbol{\tau}_f$ is defined as the elliptic projection of $\boldsymbol{\tau}_f$ satisfying Neumann boundary condition on $\Gamma_{fp}\cup\Gamma_f^N$ [44, (3.11)–(3.15)]. Due to (5.9), it holds that $\tilde{\Pi}_h^f \boldsymbol{\tau}_f \in \widetilde{\mathbb{X}}_{fh}$. With this interpolant, the proof of (5.8) for b_Γ discussed above can be easily modified for $b_{\mathbf{n}_f}$, see [31, Lemma 4.4] and [36, Section 5.3] for **(S1)** and [3, Section 5.1] for **(S2)**. □

Remark 5.3 The stability analysis requires only a discrete inf-sup condition for $\mathcal{B}$ in $\mathbf{X}_h \times \mathbf{Z}_h$. The more restrictive inf-sup condition (5.7) is used in the error analysis in order to simplify the proof.

Finally, we will utilize the following inf-sup condition: there exists a constant $c > 0$ such that

$$\|p_{ph}\|_{\mathrm{W}_p} + \|\lambda_h\|_{\Lambda_{ph}} \leq c \sup_{\mathbf{0}\neq\mathbf{v}_{ph}\in\mathbf{V}_{ph}} \frac{b_p(\mathbf{v}_{ph}, p_{ph}) + b_\Gamma(\mathbf{v}_{ph}, \lambda_h)}{\|\mathbf{v}_{ph}\|_{\mathbf{V}_p}}, \tag{5.10}$$

whose proof for the conforming case (5.1) follows from a slight adaptation of [38, Lemma 5.1], whereas the non-conforming case (5.2) can be found in [3, Section 5.1].

The semidiscrete continuous-in-time approximation to (3.11) reads: find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0, T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, and for a.e. $t \in (0, T)$,

$$\begin{aligned} \frac{\partial}{\partial t}\,\mathcal{E}(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h) &= \mathbf{F}(\underline{\boldsymbol{\tau}}_h), \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h)(\underline{\boldsymbol{\psi}}_h) &= 0, \\ -\mathcal{B}\,(\underline{\boldsymbol{\sigma}}_h)(\underline{\mathbf{v}}_h) &= \mathbf{G}(\underline{\mathbf{v}}_h). \end{aligned} \tag{5.11}$$

We next discuss the construction of compatible discrete initial data $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$.

Lemma 5.2 *Assume that* $p_{p,0} \in \mathrm{H}_p$. *Then, there exist* $\underline{\boldsymbol{\sigma}}_{h,0} = (\boldsymbol{\sigma}_{fh,0}, \mathbf{u}_{ph,0}, \boldsymbol{\sigma}_{ph,0}, p_{ph,0}) \in \mathbf{X}_h$, $\underline{\boldsymbol{\varphi}}_{h,0} = (\boldsymbol{\varphi}_{h,0}, \boldsymbol{\theta}_{h,0}, \lambda_{h,0}) \in \mathbf{Y}_h$, *and* $\underline{\mathbf{u}}_{h,0} = (\mathbf{u}_{fh,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{fh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbf{Z}_h$ *satisfying*

$$\begin{aligned} \mathcal{A}(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_{h,0}) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_{h,0}) &= \widehat{\mathbf{F}}_{h,0}(\underline{\boldsymbol{\tau}}_h) \quad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h, \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_{h,0})(\underline{\boldsymbol{\psi}}_h) &= 0 \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h, \\ -\mathcal{B}\,(\underline{\boldsymbol{\sigma}}_{h,0})(\underline{\mathbf{v}}_h) &= \mathbf{G}_0(\underline{\mathbf{v}}_h) \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h, \end{aligned} \tag{5.12}$$

where $\widehat{\mathbf{F}}_{h,0} = (\frac{1}{n} q_f(0)\mathbf{I}, \mathbf{0}, \widehat{\mathbf{f}}_{\boldsymbol{\sigma}_{ph},0}, \widehat{f}_{p_{ph},0})^{\mathrm{t}} \in \mathbf{X}_2'$ *and* $\mathbf{G}_0 = \mathbf{G}(0) \in \mathbf{Z}'$ *for some* $\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_{ph},0} \in \mathbb{X}_{p,2}'$ *and* $\widehat{f}_{p_{ph},0} \in \mathrm{W}_{p,2}'$.

Proof The construction is based on a modification of the step-by-step procedure for the continuous initial data $(\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0, \underline{\mathbf{u}}_0)$ presented in Lemma 4.9. In each step the discrete initial data is defined as a suitable projection of the continuous initial data.

1. Define $\boldsymbol{\theta}_{h,0} := P_h^{\boldsymbol{\Lambda}_s}(\boldsymbol{\theta}_0)$, where $P_h^{\boldsymbol{\Lambda}_s} : \boldsymbol{\Lambda}_s \to \boldsymbol{\Lambda}_{sh}$ is the classical L^2-projection operator, satisfying, for all $\boldsymbol{\phi} \in \mathbf{L}^2(\Gamma_{fp})$,

$$\left\langle \boldsymbol{\phi} - P_h^{\boldsymbol{\Lambda}_s}(\boldsymbol{\phi}), \boldsymbol{\phi}_h \right\rangle_{\Gamma_{fp}} = 0 \quad \forall\, \boldsymbol{\phi}_h \in \boldsymbol{\Lambda}_{sh}\,.$$

2. Define $(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\varphi}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0}) \in \mathbb{X}_{fh} \times \boldsymbol{\Lambda}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh}$ and $(\mathbf{u}_{ph,0}, p_{ph,0}, \lambda_{h,0}) \in \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph}$ by solving a coupled Stokes-Darcy problem:

$$\begin{aligned}
& a_f(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\tau}_{fh}) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\varphi}_{h,0}) + b_f(\boldsymbol{\tau}_{fh}, \mathbf{u}_{fh,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\gamma}_{fh,0}) \\
&\quad = a_f(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\tau}_{fh}) + b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\varphi}_0) + b_f(\boldsymbol{\tau}_{fh}, \mathbf{u}_{f,0}) + b_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\gamma}_{f,0}) \\
&\quad = -\frac{1}{n}\,(q_f(0)\,\mathbf{I}, \boldsymbol{\tau}_{fh})_{\Omega_f}, \\
& - b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\psi}_h) + \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_{h,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi}_h \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \\
&\qquad + \left\langle \boldsymbol{\psi}_h \cdot \mathbf{n}_f, \lambda_{h,0} \right\rangle_{\Gamma_{fp}} \\
&\quad = -b_{\mathbf{n}_f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\psi}_h) + \mu\,\alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}}(\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}, \boldsymbol{\psi}_h \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \\
&\qquad + \left\langle \boldsymbol{\psi}_h \cdot \mathbf{n}_f, \lambda_0 \right\rangle_{\Gamma_{fp}} = 0, \\
& - b_f(\boldsymbol{\sigma}_{fh,0}, \mathbf{v}_{fh}) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{fh,0}, \boldsymbol{\chi}_{fh}) = -b_f(\boldsymbol{\sigma}_{f,0}, \mathbf{v}_{fh}) - b_{\mathrm{sk},f}(\boldsymbol{\sigma}_{f,0}, \boldsymbol{\chi}_{fh}) \\
&\quad = (\mathbf{f}_f(0), \mathbf{v}_{fh})_{\Omega_f}, \\
& a_p(\mathbf{u}_{ph,0}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{ph,0}) + b_\Gamma(\mathbf{v}_{ph}, \lambda_{h,0}) \\
&\quad = a_p(\mathbf{u}_{p,0}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{p,0}) + b_\Gamma(\mathbf{v}_{ph}, \lambda_0) = 0 \\
& - b_p(\mathbf{u}_{ph,0}, w_{ph}) = -b_p(\mathbf{u}_{p,0}, w_{ph}) = -\mu^{-1}(\mathrm{div}(\mathbf{K}\nabla p_{p,0}), w_{ph})_{\Omega_p}, \\
& - \left\langle \boldsymbol{\varphi}_{h,0} \cdot \mathbf{n}_f + (\boldsymbol{\theta}_{h,0} + \mathbf{u}_{ph,0}) \cdot \mathbf{n}_p, \xi_h \right\rangle_{\Gamma_{fp}} = - \left\langle \boldsymbol{\varphi}_0 \cdot \mathbf{n}_f + (\boldsymbol{\theta}_0 + \mathbf{u}_{p,0}) \cdot \mathbf{n}_p, \xi_h \right\rangle_{\Gamma_{fp}} = 0,
\end{aligned} \tag{5.13}$$

for all $(\boldsymbol{\tau}_{fh}, \boldsymbol{\psi}_h, \mathbf{v}_{fh}, \boldsymbol{\chi}_{fh}) \in \mathbb{X}_{fh} \times \boldsymbol{\Lambda}_{fh} \times \mathbf{V}_{fh} \times \mathbb{Q}_{fh}$ and $(\mathbf{v}_{ph}, w_{ph}, \xi_h) \in \mathbf{V}_{ph} \times \mathrm{W}_{ph} \times \Lambda_{ph}$. Note that (5.13) is well-posed as a direct application of Theorem 4.2. Note also that $\boldsymbol{\theta}_{h,0}$ is datum for this problem.

3. Define $(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\omega}_{h,0}, \boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}) \in \mathbb{X}_{ph} \times \boldsymbol{\Lambda}_{sh} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$, as the unique solution of the problem

$$
\begin{aligned}
&(A(\boldsymbol{\sigma}_{ph,0}), \boldsymbol{\tau}_{ph})_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_{h,0}) + b_s(\boldsymbol{\tau}_{ph}, \boldsymbol{\eta}_{ph,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\rho}_{ph,0}) \\
&\quad + (A(\alpha_p\, p_{ph,0}\, \mathbf{I}), \boldsymbol{\tau}_{ph})_{\Omega_p} \\
&= (A(\boldsymbol{\sigma}_{p,0}), \boldsymbol{\tau}_{ph})_{\Omega_p} + b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\omega}_0) + b_s(\boldsymbol{\tau}_{ph}, \boldsymbol{\eta}_{p,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\rho}_{p,0}) \\
&\quad + (A(\alpha_p\, p_{p,0}\, \mathbf{I}), \boldsymbol{\tau}_{ph})_{\Omega_p} = 0, \\
&- b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\phi}_h) + \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} (\boldsymbol{\varphi}_{h,0} - \boldsymbol{\theta}_{h,0}) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \\
&\quad + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_{h,0} \rangle_{\Gamma_{fp}} \\
&= -b_{\mathbf{n}_p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\phi}_h) + \mu\, \alpha_{\mathtt{BJS}} \sum_{j=1}^{n-1} \left\langle \sqrt{\mathbf{K}_j^{-1}} (\boldsymbol{\varphi}_0 - \boldsymbol{\theta}_0) \cdot \mathbf{t}_{f,j}, \boldsymbol{\phi}_h \cdot \mathbf{t}_{f,j} \right\rangle_{\Gamma_{fp}} \\
&\quad + \langle \boldsymbol{\phi}_h \cdot \mathbf{n}_p, \lambda_0 \rangle_{\Gamma_{fp}} = 0, \\
&- b_s(\boldsymbol{\sigma}_{ph,0}, \mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{ph,0}, \boldsymbol{\chi}_{ph}) = -b_s(\boldsymbol{\sigma}_{p,0}, \mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi}_{ph}) \\
&\quad = (\mathbf{f}_p(0), \mathbf{v}_{sh})_{\Omega_p},
\end{aligned}
\tag{5.14}
$$

for all $(\boldsymbol{\tau}_{ph}, \boldsymbol{\phi}_h, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}) \in \mathbb{X}_{ph} \times \boldsymbol{\Lambda}_{sh} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$. Note that the well-posedness of (5.14) follows from the classical Babuška-Brezzi theory. Note also that $p_{ph,0}, \boldsymbol{\varphi}_{h,0}, \boldsymbol{\theta}_{h,0}$, and $\lambda_{h,0}$ are data for this problem.

4. Finally, define $(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$, as the unique solution of the problem

$$
\begin{aligned}
&(A(\widehat{\boldsymbol{\sigma}}_{ph,0}), \boldsymbol{\tau}_{ph})_{\Omega_p} + b_s(\boldsymbol{\tau}_{ph}, \mathbf{u}_{sh,0}) + b_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\gamma}_{ph,0}) = -b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\theta}_{h,0}), \\
&-b_s(\widehat{\boldsymbol{\sigma}}_{ph,0}, \mathbf{v}_{sh}) - b_{\mathrm{sk},p}(\widehat{\boldsymbol{\sigma}}_{ph,0}, \boldsymbol{\chi}_{ph}) = 0,
\end{aligned}
\tag{5.15}
$$

for all $(\boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{ph}) \in \mathbb{X}_{ph} \times \mathbf{V}_{sh} \times \mathbb{Q}_{ph}$. Problem (5.15) is well-posed as a direct application of the classical Babuška-Brezzi theory. Note that $\boldsymbol{\theta}_{h,0}$ is datum for this problem.

We then define $\underline{\boldsymbol{\sigma}}_{h,0} = (\boldsymbol{\sigma}_{fh,0}, \mathbf{u}_{ph,0}, \boldsymbol{\sigma}_{ph,0}, p_{ph,0}) \in \mathbf{X}_h$, $\underline{\boldsymbol{\varphi}}_{h,0} = (\boldsymbol{\varphi}_{h,0}, \boldsymbol{\theta}_{h,0}, \lambda_{h,0}) \in \mathbf{Y}_h$, and $\underline{\mathbf{u}}_{h,0} = (\mathbf{u}_{fh,0}, \mathbf{u}_{sh,0}, \boldsymbol{\gamma}_{fh,0}, \boldsymbol{\gamma}_{ph,0}) \in \mathbf{Z}_h$. The above construction implies that $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ satisfy (5.12) with $\widehat{\mathbf{F}}_{h,0} = (\frac{1}{n} q_f(0)\mathbf{I}, \mathbf{0}, \widehat{\mathbf{f}}_{\boldsymbol{\sigma}_{ph},0}, \widehat{f}_{p_{ph},0})^{\mathrm{t}} \in \mathbf{X}_2'$ and $\mathbf{G}_0 = \mathbf{G}(0) \in \mathbf{Z}'$, where $\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_{ph},0} \in \mathbb{X}_{p,2}'$ is defined by $(\widehat{\mathbf{f}}_{\boldsymbol{\sigma}_{ph},0}, \boldsymbol{\tau}_p)_{\Omega_p} = -(A(\widehat{\boldsymbol{\sigma}}_{ph,0}), \boldsymbol{\tau}_p)_{\Omega_p}\ \forall \boldsymbol{\tau}_p \in \mathbb{L}^2(\Omega_p)$ and $\widehat{f}_{p_{ph},0} \in \mathrm{W}_{p,2}'$ is defined by $(\widehat{f}_{p_{ph},0}, w_p)_{\Omega_p} = -b_p(\mathbf{u}_{ph,0}, w_p)\ \forall w_p \in \mathrm{L}^2(\Omega_p)$. □

5.1 Existence and uniqueness of a solution

Now, we establish the well-posedness of problem (5.11) and the corresponding stability bound.

Theorem 5.3 *For each* $p_{p,0} \in \mathrm{H}_p$ *and compatible discrete initial data* $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ *constructed in Lemma 5.2 and each*

$$\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'),$$
$$q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p'),$$

there exists a unique solution of (5.11), $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ *such that* $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{X}_{ph}) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_{ph})$*, and* $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0), \mathbf{u}_{fh}(0), \boldsymbol{\gamma}_{fh}(0)) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0})$*. Moreover, assuming sufficient regularity of the data, there exists a positive constant C independent of h and* s_0*, such that*

$$\begin{aligned}
&\|\boldsymbol{\sigma}_{fh}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\boldsymbol{\sigma}_{fh}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{u}_{ph}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&\quad + |\boldsymbol{\varphi}_h - \boldsymbol{\theta}_h|_{\mathrm{L}^\infty(0,T;BJS)} + |\boldsymbol{\varphi}_h - \boldsymbol{\theta}_h|_{\mathrm{L}^2(0,T;BJS)} + \|\lambda_h\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} \\
&\quad + \|\underline{\boldsymbol{\varphi}}_h\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} + \|\underline{\mathbf{u}}_h\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} \\
&\quad + \|p_{ph}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\quad + \|\partial_t\, A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha_p p_{ph}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t\, p_{ph}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\le C\Big(\|\mathbf{f}_f\|_{\mathrm{H}^1(0,T;\mathbf{V}_f')} + \|\mathbf{f}_p\|_{\mathrm{H}^1(0,T;\mathbf{V}_s')} + \|q_f\|_{\mathrm{H}^1(0,T;\mathbb{X}_f')} \\
&\quad + \|q_p\|_{\mathrm{H}^1(0,T;\mathrm{W}_p')} + (1+\sqrt{s_0})\|p_{p,0}\|_{\mathrm{W}_p} + \|\mathbf{K}\nabla p_{p,0}\|_{\mathrm{H}^1(\Omega_p)}\Big).
\end{aligned} \tag{5.16}$$

Proof From the fact that $\mathbf{X}_h \subset \mathbf{X}$, $\mathbf{Z}_h \subset \mathbf{Z}$, and $\mathbf{div}(\mathbb{X}_{fh}) = \mathbf{V}_{fh}$, $\mathbf{div}(\mathbb{X}_{ph}) = \mathbf{V}_{sh}$, $\mathrm{div}(\mathbf{V}_{ph}) = \mathrm{W}_{ph}$, considering $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ satisfying (5.12), and employing the continuity and monotonicity properties of the operators $\mathcal{N}$ and $\mathcal{M}$ (cf. Lemma 4.3 and (5.4)), as well as the discrete inf-sup conditions (5.7), (5.8), and (5.10), the proof is identical to the proofs of Theorems 4.10 and 4.12, and Corollary 4.11. We note that the proof of Corollary 4.11 works in the discrete case due to the choice of the discrete initial data as the elliptic projection of the continuous initial data (cf. (5.13)–(5.15)). □

Remark 5.4 The construction of the initial data in Lemma 5.2 provides compatible initial data $(\boldsymbol{\eta}_{ph,0}, \boldsymbol{\rho}_{ph,0}, \boldsymbol{\omega}_{h,0})$ for the non-differentiated elasticity variables in the sense of the first equation in (4.23) (cf. (5.14)). As in the continuous case, we can recover them as follows:

$$\boldsymbol{\eta}_{ph}(t) = \boldsymbol{\eta}_{ph,0} + \int_0^t \mathbf{u}_{sh}(s)\,ds, \quad \boldsymbol{\rho}_{ph}(t) = \boldsymbol{\rho}_{ph,0} + \int_0^t \boldsymbol{\gamma}_{ph}(s)\,ds,$$
$$\boldsymbol{\omega}_h(t) = \boldsymbol{\omega}_{h,0} + \int_0^t \boldsymbol{\theta}_h(s)\,ds\,,$$

for each $t \in [0,T]$. Then (3.7) holds discretely, which follows from integrating the equation associated to $\boldsymbol{\tau}_{ph}$ in (5.11) from 0 to $t \in (0,T]$ and using the first equation in (5.14).

5.2 Error analysis

We proceed with establishing rates of convergence. To that end, let us set $\mathrm{V} \in \left\{\mathrm{W}_p, \mathbf{V}_f, \mathbf{V}_s, \mathbb{Q}_f, \mathbb{Q}_p\right\}$, $\Lambda \in \left\{\mathbf{\Lambda}_f, \mathbf{\Lambda}_s, \Lambda_p\right\}$ and let V_h, Λ_h be the discrete counterparts. Let $P_h^{\mathrm{V}} : \mathrm{V} \to \mathrm{V}_h$ and $P_h^{\Lambda} : \Lambda \to \Lambda_h$ be the L^2-projection operators, satisfying

$$\begin{aligned} (u - P_h^{\mathrm{V}}(u), v_h)_{\Omega_\star} &= 0 \quad \forall\, v_h \in \mathrm{V}_h, \\ \langle \varphi - P_h^{\Lambda}(\varphi), \psi_h\rangle_{\Gamma_{fp}} &= 0 \quad \forall\, \psi_h \in \Lambda_h, \end{aligned} \tag{5.17}$$

where $\star \in \{f, p\}$, $u \in \left\{p_p, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p\right\}$, $\varphi \in \left\{\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda\right\}$, and v_h, ψ_h are the corresponding discrete test functions. We have the approximation properties [27]:

$$\begin{aligned} \|u - P_h^{\mathrm{V}}(u)\|_{\mathrm{L}^2(\Omega_\star)} &\le C h^{s_u+1}\, \|u\|_{\mathrm{H}^{s_u+1}(\Omega_\star)}, \\ \|\varphi - P_h^{\Lambda}(\varphi)\|_{\Lambda_h} &\le C h^{s_\varphi + r}\, \|\varphi\|_{\mathrm{H}^{s_\varphi+1}(\Gamma_{fp})}, \end{aligned} \tag{5.18}$$

where $s_u \in \left\{s_{p_p}, s_{\mathbf{u}_f}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_f}, s_{\boldsymbol{\gamma}_p}\right\}$ and $s_\varphi \in \left\{s_{\boldsymbol{\varphi}}, s_{\boldsymbol{\theta}}, s_\lambda\right\}$ are the degrees of polynomials in the spaces V_h and Λ_h, respectively, and (cf. (5.3)),

$$\|\varphi\|_{\Lambda_h} := \begin{cases} \|\varphi\|_{\mathrm{H}^{1/2}(\Gamma_{fp})}, & \text{with } r = 1/2 \text{ in (5.18) for conforming spaces } \mathbf{(S1)}, \\ \|\varphi\|_{\mathrm{L}^2(\Gamma_{fp})}, & \text{with } r = 1 \text{ in (5.18) for non-conforming spaces } \mathbf{(S2)}. \end{cases}$$

Next, denote $\mathrm{X} \in \left\{\mathbb{X}_f, \mathbb{X}_p, \mathbf{V}_p\right\}$, $\sigma \in \left\{\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_p, \mathbf{u}_p\right\} \in \mathrm{X}$ and let X_h and τ_h be their discrete counterparts. For the case **(S2)** when the discrete Lagrange multiplier spaces are chosen as in (5.2), (5.17) implies

$$\langle \varphi - P_h^{\Lambda}(\varphi), \tau_h \mathbf{n}_\star\rangle_{\Gamma_{fp}} = 0 \quad \forall\, \tau_h \in \mathrm{X}_h, \tag{5.19}$$

where $\star \in \{f, p\}$. We note that (5.19) does not hold for the case **(S1)**.

Let $I_h^{\mathrm{X}} : \mathrm{X} \cap \mathrm{H}^1(\Omega_\star) \to \mathrm{X}_h$ be the mixed finite element projection operator [19] satisfying

$$\begin{aligned} (\mathrm{div}(I_h^{\mathrm{X}}(\sigma)), w_h)_{\Omega_\star} &= (\mathrm{div}(\sigma), w_h)_{\Omega_\star} \qquad \forall\, w_h \in \mathrm{W}_h, \\ \left\langle I_h^{\mathrm{X}}(\sigma)\mathbf{n}_\star, \tau_h \mathbf{n}_\star\right\rangle_{\Gamma_{fp}} &= \langle \sigma \mathbf{n}_\star, \tau_h \mathbf{n}_\star\rangle_{\Gamma_{fp}} \qquad \forall\, \tau_h \in \mathrm{X}_h, \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} \|\sigma - I_h^{\mathrm{X}}(\sigma)\|_{\mathrm{L}^2(\Omega_\star)} &\le C\, h^{s_\sigma+1} \|\sigma\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)}, \\ \|\mathrm{div}(\sigma - I_h^{\mathrm{X}}(\sigma))\|_{\mathrm{L}^2(\Omega_\star)} &\le C\, h^{s_\sigma+1} \|\mathrm{div}(\sigma)\|_{\mathrm{H}^{s_\sigma+1}(\Omega_\star)}, \end{aligned} \tag{5.21}$$

where $w_h \in \left\{\mathbf{v}_{fh}, \mathbf{v}_{sh}, w_{ph}\right\}$, $\mathrm{W}_h \in \left\{\mathbf{V}_f, \mathbf{V}_s, \mathrm{W}_p\right\}$, and $s_\sigma \in \left\{s_{\boldsymbol{\sigma}_f}, s_{\boldsymbol{\sigma}_p}, s_{\mathbf{u}_p}\right\}$ – the degrees of polynomials in the spaces X_h.

Now, let $(\boldsymbol{\sigma}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p, p_p, \boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p)$ and $(\boldsymbol{\sigma}_{fh}, \mathbf{u}_{ph}, \boldsymbol{\sigma}_{ph}, p_{ph}, \boldsymbol{\varphi}_h, \boldsymbol{\theta}_h, \lambda_h, \mathbf{u}_{fh}, \mathbf{u}_{sh}, \boldsymbol{\gamma}_{fh}, \boldsymbol{\gamma}_{ph})$ be the solutions of (3.11) and (5.11), respectively. We introduce the error terms as the differences of these two solutions and decompose them into approximation and discretization errors using the interpolation operators:

$$
\begin{aligned}
\mathbf{e}_\sigma &:= \sigma - \sigma_h = (\sigma - I_h^{\mathrm{X}}(\sigma)) + (I_h^{\mathrm{X}}(\sigma) - \sigma_h) := \mathbf{e}_\sigma^I + \mathbf{e}_\sigma^h, \quad \sigma \in \{\boldsymbol{\sigma}_f, \boldsymbol{\sigma}_p, \mathbf{u}_p\},\\
\mathbf{e}_\varphi &:= \varphi - \varphi_h = (\varphi - P_h^{\Lambda}(\varphi)) + (P_h^{\Lambda}(\varphi) - \varphi_h) := \mathbf{e}_\varphi^I + \mathbf{e}_\varphi^h, \quad \varphi \in \{\boldsymbol{\varphi}, \boldsymbol{\theta}, \lambda\},\\
\mathbf{e}_u &:= u - u_h = (u - P_h^{\mathrm{V}}(u)) + (P_h^{\mathrm{V}}(u) - u_h) := \mathbf{e}_u^I + \mathbf{e}_u^h, \quad u \in \{p_p, \mathbf{u}_f, \mathbf{u}_s, \boldsymbol{\gamma}_f, \boldsymbol{\gamma}_p\}.
\end{aligned}
\tag{5.22}
$$

Then, we set the errors

$$
\mathbf{e}_{\underline{\boldsymbol{\sigma}}} := (\mathbf{e}_{\boldsymbol{\sigma}_f}, \mathbf{e}_{\mathbf{u}_p}, \mathbf{e}_{\boldsymbol{\sigma}_p}, \mathbf{e}_{p_p}), \quad \mathbf{e}_{\underline{\boldsymbol{\varphi}}} := (\mathbf{e}_{\boldsymbol{\varphi}}, \mathbf{e}_{\boldsymbol{\theta}}, \mathbf{e}_\lambda), \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{u}}} := (\mathbf{e}_{\mathbf{u}_f}, \mathbf{e}_{\mathbf{u}_s}, \mathbf{e}_{\boldsymbol{\gamma}_f}, \mathbf{e}_{\boldsymbol{\gamma}_p}).
$$

We next form the error system by subtracting the discrete problem (5.11) from the continuous one (3.11). Using that $\mathbf{X}_h \subset \mathbf{X}$ and $\mathbf{Z}_h \subset \mathbf{Z}$, as well as Remark 5.2, we obtain

$$
\begin{aligned}
(\partial_t \mathcal{E} + \mathcal{A})(\mathbf{e}_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}_{\underline{\boldsymbol{\varphi}}}) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}_{\underline{\mathbf{u}}}) &= 0 \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h,\\
-\mathcal{B}_1(\mathbf{e}_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\mathbf{e}_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h) &= 0 \qquad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h,\\
-\mathcal{B}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) &= 0 \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h.
\end{aligned}
\tag{5.23}
$$

We now establish the main result of this section.

Theorem 5.4 *For the solutions of the continuous and discrete problems* (3.11) *and* (5.11) *established in* Theorem 4.10 *and* Theorem 5.3*, respectively, assuming sufficient regularity of the true solution according to* (5.18) *and* (5.21)*, there exists a positive constant C independent of h and s_0, such that*

$$
\begin{aligned}
&\|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{e}_{\mathbf{u}_p}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{e}_{\mathbf{u}_p}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)}\\
&\quad + |\mathbf{e}_{\boldsymbol{\varphi}} - \mathbf{e}_{\boldsymbol{\theta}}|_{\mathrm{L}^\infty(0,T;BJS)} + |\mathbf{e}_{\boldsymbol{\varphi}} - \mathbf{e}_{\boldsymbol{\theta}}|_{\mathrm{L}^2(0,T;BJS)} + \|\mathbf{e}_\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})}\\
&\quad + \|\mathbf{e}_{\underline{\boldsymbol{\varphi}}}\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} + \|\mathbf{e}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))}\\
&\quad + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))}\\
&\quad + \|\mathbf{e}_{p_p}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} + \|\mathbf{e}_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)}\\
&\quad + \|\partial_t\, A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p} + \alpha_p \mathbf{e}_{p_p}\mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t\, \mathbf{e}_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)}\\
&\leq C\sqrt{\exp(T)}\left(h^{s_{\underline{\boldsymbol{\sigma}}}+1} + h^{s_{\underline{\boldsymbol{\varphi}}}+r} + h^{s_{\underline{\mathbf{u}}}+1}\right),
\end{aligned}
\tag{5.24}
$$

where $s_{\underline{\boldsymbol{\sigma}}} = \min\{s_{\boldsymbol{\sigma}_f}, s_{\mathbf{u}_p}, s_{\boldsymbol{\sigma}_p}, s_{p_p}\}$, $s_{\underline{\boldsymbol{\varphi}}} = \min\{s_{\boldsymbol{\varphi}}, s_{\boldsymbol{\theta}}, s_\lambda\}$, $s_{\underline{\mathbf{u}}} = \min\{s_{\mathbf{u}_f}, s_{\mathbf{u}_s}, s_{\boldsymbol{\gamma}_f}, s_{\boldsymbol{\gamma}_p}\}$, *and r is defined in* (5.18).

Proof We present in detail the proof for the conforming case **(S1)**. The proof in the non-conforming case **(S2)** is simpler, since several error terms are zero. We explain the differences at the end of the proof.

We proceed as in Theorem 4.12. Taking $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) = (\mathbf{e}^h_{\underline{\boldsymbol{\sigma}}}, \mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}, \mathbf{e}^h_{\underline{\mathbf{u}}})$ in (5.23), we obtain

$$\begin{aligned}
&\frac{1}{2}\,\partial_t\left(a_e(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{p_p}; \mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{p_p}) + s_0\,(\mathbf{e}^h_{p_p}, \mathbf{e}^h_{p_p})_{\Omega_p}\right) + a_f(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\sigma}_f}) + a_p(\mathbf{e}^h_{\mathbf{u}_p}, \mathbf{e}^h_{\mathbf{u}_p})\\
&\quad + c_{\mathtt{BJS}}(\mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}; \mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}})\\
&\quad = -a_f(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\sigma}_f}) - a_p(\mathbf{e}^I_{\mathbf{u}_p}, \mathbf{e}^h_{\mathbf{u}_p}) - a_e(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \partial_t\,\mathbf{e}^I_{p_p}; \mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{p_p}) - \mathcal{C}(\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}})(\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}})\\
&\quad - b_{\mathbf{n}_f}(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\boldsymbol{\varphi}}) - b_{\mathbf{n}_p}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\theta}}) - b_\Gamma(\mathbf{e}^h_{\mathbf{u}_p}, \mathbf{e}^I_\lambda) + b_{\mathbf{n}_f}(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\varphi}}) + b_{\mathbf{n}_p}(\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\boldsymbol{\theta}}) + b_\Gamma(\mathbf{e}^I_{\mathbf{u}_p}, \mathbf{e}^h_\lambda)\\
&\quad - b_{\mathrm{sk},f}(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\boldsymbol{\gamma}_f}) - b_{\mathrm{sk},p}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\gamma}_p}) + b_{\mathrm{sk},f}(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\gamma}_f}) + b_{\mathrm{sk},p}(\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\boldsymbol{\gamma}_p}),
\end{aligned} \tag{5.25}$$

where, the right-hand side of (5.25) has been simplified, since the projection properties (5.17) and (5.20), and the fact that $\operatorname{div}(\mathbf{e}^h_{\mathbf{u}_p}) \in \mathrm{W}_{ph}$, $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_f}) \in \mathbf{V}_{fh}$, and $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}) \in \mathbf{V}_{sh}$, imply that the following terms are zero:

$$\begin{aligned}
&s_0(\partial_t\,\mathbf{e}^I_{p_p}, \mathbf{e}^h_{p_p}),\ b_p(\mathbf{e}^h_{\mathbf{u}_p}, \mathbf{e}^I_{p_p}),\ b_p(\mathbf{e}^I_{\mathbf{u}_p}, \mathbf{e}^h_{p_p}),\ b_f(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\mathbf{u}_f}),\ b_f(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\mathbf{u}_f}),\\
&b_s(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\mathbf{u}_s}),\ b_s(\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\mathbf{u}_s}).
\end{aligned} \tag{5.26}$$

In turn, from the equations in (5.23) corresponding to test functions $\mathbf{v}_{fh}$, $\mathbf{v}_{sh}$, and w_{ph}, using the projection properties (5.20), we find that

$$\begin{aligned}
&b_f(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{v}_{fh}) = 0 \quad \forall\,\mathbf{v}_{fh} \in \mathbf{V}_{fh}, \quad b_s(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{v}_{sh}) = 0 \quad \forall\,\mathbf{v}_{sh} \in \mathbf{V}_{sh},\\
&b_p(\mathbf{e}^h_{\mathbf{u}_p}, w_{ph}) = a_e(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \partial_t\,\mathbf{e}^h_{p_p}; \mathbf{0}, w_{ph}) + a_e(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \partial_t\,\mathbf{e}^I_{p_p}; \mathbf{0}, w_{ph})\\
&\qquad\qquad + (s_0\,\partial_t\,\mathbf{e}^h_{p_p}, w_{ph})_{\Omega_p} \quad \forall\,w_{ph} \in \mathrm{W}_{ph}.
\end{aligned}$$

Therefore $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}) = \mathbf{0}$ in $\Omega_\star$, with $\star \in \{f, p\}$, and using (4.2)–(4.3) we deduce

$$\begin{aligned}
&\|(\mathbf{e}^h_{\boldsymbol{\sigma}_f})^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} \geq C\,\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f}, \quad \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega_p)} = 0\,,\\
&\|\operatorname{div}(\mathbf{e}^h_{\mathbf{u}_p})\|_{\mathrm{L}^2(\Omega_p)} \leq C\left(\|\partial_t\,A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\right.\\
&\qquad \left. + \|\partial_t\,A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \sqrt{s_0}\,\|\partial_t\,\mathbf{e}^h_{p_p}\|_{\mathrm{W}_p}\right).
\end{aligned} \tag{5.27}$$

Then, applying the ellipticity and continuity bounds of the bilinear forms involved in (5.25) (cf. Lemma 4.3) and the Cauchy–Schwarz and Young's inequalities, in combination with (5.27), we get

$$\partial_t \left(\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \right) + \|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p}$$
$$+ \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathrm{BJS}}$$
$$\leq C \Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} + |\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}}|^2_{\mathrm{BJS}} + \|\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f}$$
$$+ \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}$$
$$+ \|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, \mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \Big)$$
$$+ \delta_1 \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathrm{BJS}} \Big)$$
$$+ \delta_2 \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \Big),$$

where for the bound on $b_{\mathbf{n}_p}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\theta}})$ we used the trace inequality (3.2) and the fact that $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}) = \mathbf{0}$. Next, integrating from 0 to $t \in (0, T]$, using (4.14) to control the term $\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega_p)}$, and choosing δ_1 small enough, we find that

$$\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p \mathbf{e}^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\mathbf{e}^h_{p_p}(t)\|^2_{\mathrm{W}_p}$$
$$+ \int_0^t \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathrm{BJS}} \Big)\, ds$$
$$\leq C \Bigg(\int_0^t \Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}}|^2_{\mathrm{BJS}} + \|\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f}$$
$$+ \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} \Big)\, ds$$
$$+ \int_0^t \Big(\|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \Big)\, ds$$
$$+ \int_0^t \Big(\|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, \mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \Big)\, ds$$
$$+ \|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\,\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\mathbf{e}^h_{p_p}(0)\|^2_{\mathrm{W}_p} \Bigg)$$
$$+ \delta_2 \int_0^t \Big(\|\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} + \|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \Big)\, ds\,. \tag{5.28}$$

On the other hand, taking $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, 0) \in \mathbf{V}_h$ (cf. (5.6)) in the first equation of (5.23), we obtain

$$\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}) = -(\partial_t\, \mathcal{E} + \mathcal{A})(\mathbf{e}_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) - \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}})\,,$$

In the above, thanks to the projection properties (5.17), the following terms are zero: $b_p(\mathbf{v}_{ph}, \mathbf{e}^I_{p_p})$, $b_f(\boldsymbol{\tau}_{fh}, \mathbf{e}^I_{\mathbf{u}_f})$, and $b_s(\boldsymbol{\tau}_{ph}, \mathbf{e}^I_{\mathbf{u}_s})$. Then the discrete inf-sup condition of $\mathcal{B}_1$ (cf. (5.8)) for $\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}} = (\mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}, \mathbf{e}^h_{\lambda}) \in \mathbf{Y}_h$ gives

$$\begin{aligned}\|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}_h} \le C\Big(&\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\mathbf{e}^I_{\mathbf{u}_p}\|_{\mathbf{V}_p} + \|\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p}\\
&+ \|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\\
&+ \|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|_{\mathbf{V}_p} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^h_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p}\\
&+ \|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^h_{p_p}\|_{\mathrm{W}_p}\Big). \qquad (5.29)\end{aligned}$$

In turn, to bound $\|\mathbf{e}^h_{\underline{\mathbf{u}}}\|_{\mathbf{Z}}$, we test (5.23) with $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{0}, \boldsymbol{\tau}_{ph}, 0) \in \widetilde{\mathbf{X}}_h$ (cf. (5.5)), to find that

$$\mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^h_{\underline{\mathbf{u}}}) = -\Big(a_f(\mathbf{e}_{\boldsymbol{\sigma}_f}, \boldsymbol{\tau}_{fh}) + a_e(\partial_t\, \mathbf{e}_{\boldsymbol{\sigma}_p}, \partial_t\, \mathbf{e}_{p_p}; \boldsymbol{\tau}_{ph}, 0) + \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^I_{\underline{\mathbf{u}}})\Big).$$

In the above, the terms $b_f(\boldsymbol{\tau}_{fh}, \mathbf{e}^I_{\mathbf{u}_f})$ and $b_s(\boldsymbol{\tau}_{ph}, \mathbf{e}^I_{\mathbf{u}_s})$ are zero, due to the projection property (5.17). Then, the discrete inf-sup condition of $\mathcal{B}$ (cf. (5.7)) for $\mathbf{e}^h_{\underline{\mathbf{u}}} \in \mathbf{Z}_h$, yields

$$\begin{aligned}\|\mathbf{e}^h_{\underline{\mathbf{u}}}\|_{\mathbf{Z}} \le C\,\Big(&\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^I_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}\|_{\mathbb{Q}_f} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}\|_{\mathbb{Q}_p}\\
&+\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|_{\mathbb{X}_f} + \|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|_{\mathbb{L}^2(\Omega_p)}\Big). \qquad (5.30)\end{aligned}$$

Finally, to bound $\|\mathbf{e}^h_{p_p}\|_{\mathrm{W}_p}$, we test (5.23) with $\underline{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_{fh}, \mathbf{v}_{ph}, \boldsymbol{\tau}_{ph}, 0) \in \mathbf{X}_h$ to get

$$b_p(\mathbf{v}_{ph}, \mathbf{e}^h_{p_p}) + b_\Gamma(\mathbf{v}_{ph}, \mathbf{e}^h_\lambda) = -\Big(a_p(\mathbf{e}_{\mathbf{u}_p}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, \mathbf{e}^I_{p_p}) + b_\Gamma(\mathbf{v}_{ph}, \mathbf{e}^I_\lambda)\Big).$$

Note that $b_p(\mathbf{v}_{ph}, \mathbf{e}^I_{p_p}) = 0$ due to the projection property (5.17), thus the discrete inf-sup condition (5.10) gives

$$\|\mathbf{e}^h_{p_p}\|_{\mathrm{W}_p} + \|\mathbf{e}^h_\lambda\|_{\Lambda_{ph}} \le C\Big(\|\mathbf{e}^I_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{e}^I_\lambda\|_{\Lambda_{ph}} + \|\mathbf{e}^h_{\mathbf{u}_p}\|_{\mathbf{L}^2(\Omega_p)}\Big). \qquad (5.31)$$

Combining (5.28) with (5.29), (5.30), and (5.31), choosing δ_2 small enough, and employing the Grönwall's inequality to deal with the term $\int_0^t \|A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\, ds$, we obtain

$$\begin{aligned}
&\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\,\|\mathbf{e}^h_{p_p}(t)\|^2_{\mathrm{W}_p} \\
&\quad + \int_0^t \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \\
&\quad + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} \Big)\, ds \\
&\le C\,\exp(T)\,\bigg(\int_0^t \Big(\|\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} + \|\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + |\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \\
&\quad + \|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^I_{p_p}\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} \Big)\, ds \\
&\quad + \int_0^t \Big(\|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\,\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \Big)\, ds \\
&\quad + \|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^h_{p_p}\mathbf{I})(0)\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\mathbf{e}^h_{p_p}(0)\|^2_{\mathrm{W}_p} \bigg).
\end{aligned} \tag{5.32}$$

Now, in order to bound $\int_0^t \Big(\|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\,\mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\,\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \Big)\, ds$ on the right-hand side of (5.32), we test (5.23) with $\underline{\boldsymbol{\tau}}_h = (\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\mathbf{u}_p}, \partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}, \partial_t \mathbf{e}^h_{p_p})$, $\underline{\boldsymbol{\psi}}_h = (\mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}, \partial_t \mathbf{e}^h_{\lambda})$, and $\underline{\mathbf{v}}_h = (\mathbf{e}^h_{\mathbf{u}_f}, \mathbf{e}^h_{\mathbf{u}_s}, \mathbf{e}^h_{\boldsymbol{\gamma}_f}, \mathbf{e}^h_{\boldsymbol{\gamma}_p})$, differentiate in time the rows in (5.23) associated to $\mathbf{v}_{ph}, \boldsymbol{\psi}_h, \boldsymbol{\phi}_h, \mathbf{v}_{fh}, \mathbf{v}_{sh}, \boldsymbol{\chi}_{fh}, \boldsymbol{\chi}_{ph}$, and employ the projections properties (5.17)–(5.20) to eliminate some of the terms (cf. (5.26)), obtaining

$$\begin{aligned}
&\frac{1}{2}\partial_t \Big(\frac{1}{2\mu}\|(\mathbf{e}^h_{\boldsymbol{\sigma}_f})^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega_f)} + a_p(\mathbf{e}^h_{\mathbf{u}_p}, \mathbf{e}^h_{\mathbf{u}_p}) + c_{\mathtt{BJS}}(\mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}; \mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}) \Big) \\
&\quad + \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t \mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \\
&= -a_f(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_f}) - a_p(\partial_t\,\mathbf{e}^I_{\mathbf{u}_p}, \mathbf{e}^h_{\mathbf{u}_p}) - a_e(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \partial_t\,\mathbf{e}^I_{p_p}; \partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \partial_t\,\mathbf{e}^h_{p_p}) \\
&\quad - c_{\mathtt{BJS}}(\partial_t\,\mathbf{e}^I_{\boldsymbol{\varphi}}, \partial_t\,\mathbf{e}^I_{\boldsymbol{\theta}}; \mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}) + c_\Gamma(\mathbf{e}^h_{\boldsymbol{\varphi}}, \mathbf{e}^h_{\boldsymbol{\theta}}; \partial_t\,\mathbf{e}^I_{\lambda}) - c_\Gamma(\mathbf{e}^I_{\boldsymbol{\varphi}}, \mathbf{e}^I_{\boldsymbol{\theta}}; \partial_t\,\mathbf{e}^h_{\lambda}) \\
&\quad - b_{\mathbf{n}_f}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\boldsymbol{\varphi}}) - b_{\mathbf{n}_p}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\theta}}) - b_\Gamma(\mathbf{e}^h_{\mathbf{u}_p}, \partial_t\,\mathbf{e}^I_{\lambda}) \\
&\quad + b_{\mathbf{n}_f}(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\varphi}}) + b_{\mathbf{n}_p}(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\boldsymbol{\theta}}) + b_\Gamma(\mathbf{e}^I_{\mathbf{u}_p}, \partial_t\,\mathbf{e}^h_{\lambda}) - b_{\mathrm{sk},f}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\boldsymbol{\gamma}_f}) \\
&\quad - b_{\mathrm{sk},p}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\gamma}_p}) + b_{\mathrm{sk},f}(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\gamma}_f}) + b_{\mathrm{sk},p}(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\boldsymbol{\gamma}_p}).
\end{aligned} \tag{5.33}$$

Then, integrating (5.33) from 0 to $t \in (0, T]$, using the identities

$$\begin{aligned}
&\int_0^t a_f(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_f})\, ds = a_f(\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\sigma}_f})\Big|_0^t - \int_0^t a_f(\partial_t\,\mathbf{e}^I_{\boldsymbol{\sigma}_f}, \mathbf{e}^h_{\boldsymbol{\sigma}_f})\, ds\,, \\
&\int_0^t b_{\mathbf{n}_\star}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \mathbf{e}^I_{\circ})\, ds = b_{\mathbf{n}_\star}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \mathbf{e}^I_{\circ})\Big|_0^t - \int_0^t b_{\mathbf{n}_\star}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \partial_t\,\mathbf{e}^I_{\circ})\, ds\,,\ \star \in \{f, p\},\ \circ \in \{\boldsymbol{\varphi}, \boldsymbol{\theta}\}\,, \\
&\int_0^t b_{\mathrm{sk},\star}(\partial_t\,\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \mathbf{e}^I_{\boldsymbol{\gamma}_\star})\, ds = b_{\mathrm{sk},\star}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \mathbf{e}^I_{\boldsymbol{\gamma}_\star})\Big|_0^t - \int_0^t b_{\mathrm{sk},\star}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}, \partial_t\,\mathbf{e}^I_{\boldsymbol{\gamma}_\star})\, ds\,, \\
&\int_0^t \Big\langle \mathbf{e}^I_{\diamond}\cdot\mathbf{n}_f, \partial_t\,\mathbf{e}^h_{\lambda} \Big\rangle_{\Gamma_{fp}} ds = \Big\langle \mathbf{e}^I_{\diamond}\cdot\mathbf{n}_f, \mathbf{e}^h_{\lambda} \Big\rangle_{\Gamma_{fp}}\Big|_0^t - \int_0^t \Big\langle \partial_t\,\mathbf{e}^I_{\diamond}\cdot\mathbf{n}_f, \mathbf{e}^h_{\lambda} \Big\rangle_{\Gamma_{fp}} ds\,,\ \diamond \in \{\boldsymbol{\varphi}, \boldsymbol{\theta}, \mathbf{u}_p\},
\end{aligned} \tag{5.34}$$

and applying the ellipticity and continuity bounds of the bilinear forms involved (cf. Lemma 4.3), the Cauchy-Schwarz and Young's inequalities, and the fact that $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_\star}) = \mathbf{0}$ in $\Omega_\star$ with $\star \in \{f, p\}$ (cf. (5.27)), we obtain

$$
\begin{aligned}
&\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(t))\|^2_{\mathbf{L}^2(\Omega_p)} + |(\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}})(t)|^2_{\mathtt{BJS}} \\
&\quad + \int_0^t \left(\|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, \mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \right) ds \\
&\leq C \Bigg(\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{e}^I_{\mathbf{u}_p}(t)\|^2_{\mathbf{V}_p} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^I_{\boldsymbol{\varphi}}(t)\|^2_{\boldsymbol{\Lambda}_{fh}} \\
&\quad + \|\mathbf{e}^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}(t)\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} + \int_0^t \Big(\|\partial_t\, \mathbf{e}^I_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\partial_t\, \mathbf{e}^I_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} \\
&\quad + |\partial_t\,(\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}})|^2_{\mathtt{BJS}} + \|\mathbf{e}^I_{\boldsymbol{\theta}}\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\partial_t\, \mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\partial_t\, \mathbf{e}^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\partial_t\, \mathbf{e}^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \\
&\quad + \|\partial_t\, A^{1/2}\,(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^I_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + \|\partial_t\, \mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{X}_p} \Big) ds \\
&\quad + \|\mathbf{e}^I_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{e}^I_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|\mathbf{e}^I_{\boldsymbol{\varphi}}(0)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|\mathbf{e}^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}(0)\|^2_{\mathbb{Q}_f} \Bigg) \\
&\quad + \delta_3 \Bigg(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^h_{\lambda}(t)\|^2_{\Lambda_{ph}} \\
&\quad + \int_0^t \left(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} \right) ds + \int_0^t \left(\|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} \right) ds \Bigg) \\
&+ \frac{1}{2} \int_0^t \|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)}\, ds + C \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad + \|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{X}_p} + |(\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \Big). \qquad (5.35)
\end{aligned}
$$

We note that $\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{e}^h_{\lambda}(t)\|^2_{\Lambda_{ph}}$ can be bounded by using (4.14) and (5.31), whereas all the other terms with δ_3 can be bounded by the left hand side of (5.32). Thus, combining (5.32) with (5.31) and (5.35), using algebraic manipulations, and choosing δ_3 small enough, we get

$$
\begin{aligned}
&\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}(t)\|^2_{\mathbf{L}^2(\Omega_p)} + |(\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}})(t)|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\lambda}(t)\|^2_{\Lambda_{ph}} \\
&\quad + \|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\,\mathbf{I})(t)\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(t))\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{e}^h_{p_p}(t)\|^2_{\mathrm{W}_p} \\
&\quad + \int_0^t \Big(\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}\|^2_{\mathbf{V}_p} + |\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} \\
&\quad + \|\mathbf{e}^h_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \\
&\quad + \|\partial_t\, A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha_p\, \mathbf{e}^h_{p_p}\,\mathbf{I})\|^2_{\mathbb{L}^2(\Omega_p)} + s_0\|\partial_t\, \mathbf{e}^h_{p_p}\|^2_{\mathrm{W}_p} \Big) ds \\
&\leq C\, \exp(T) \Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}(t)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{e}^I_{\mathbf{u}_p}(t)\|^2_{\mathbf{V}_p} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega_p)}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\mathbf{e}^I_{\boldsymbol{\varphi}}(t)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|\mathbf{e}^I_{\boldsymbol{\theta}}(t)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}(t)\|^2_{\mathbb{Q}_f} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_p}(t)\|^2_{\mathbb{Q}_p} \\
&+ \int_0^t \left(\|\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} + \|\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\underline{\mathbf{u}}}\|^2_{\mathbf{Z}} + |\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}}|^2_{\mathtt{BJS}} + \|\partial_t\, \mathbf{e}^I_{\underline{\boldsymbol{\sigma}}}\|^2_{\mathbf{X}} \right) ds \\
&+ \int_0^t \left(\|\partial_t\, \mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}\|^2_{\mathbf{Y}_h} + |\partial_t\, (\mathbf{e}^I_{\boldsymbol{\varphi}} - \mathbf{e}^I_{\boldsymbol{\theta}})|^2_{\mathtt{BJS}} + \|\partial_t\, \mathbf{e}^I_{\boldsymbol{\gamma}_f}\|^2_{\mathbb{Q}_f} + \|\partial_t\, \mathbf{e}^I_{\boldsymbol{\gamma}_p}\|^2_{\mathbb{Q}_p} \right) ds \\
&+\|\mathbf{e}^I_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{L}^2(\Omega_f)} + \|\mathbf{e}^I_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|\mathbf{e}^I_{\boldsymbol{\varphi}}(0)\|^2_{\boldsymbol{\Lambda}_{fh}} + \|\mathbf{e}^I_{\boldsymbol{\theta}}(0)\|^2_{\boldsymbol{\Lambda}_{sh}} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_f}(0)\|^2_{\mathbb{Q}_f} \\
&+\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{L}^2(\Omega_p)} + \|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{X}_p} \\
&+(1+s_0)\|\mathbf{e}^h_{p_p}(0)\|^2_{\mathrm{W}_p} + |(\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \Bigg).
\end{aligned} \tag{5.36}
$$

Finally, we establish a bound on the initial data terms above. In fact, proceeding as in (4.45), recalling from Corollary 4.11 and Theorem 5.3 that $(\underline{\boldsymbol{\sigma}}(0), \underline{\boldsymbol{\varphi}}(0)) = (\underline{\boldsymbol{\sigma}}_0, \underline{\boldsymbol{\varphi}}_0)$ and $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0)) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0})$, using similar arguments to (5.32) in combination with the error system derived from (5.13)–(5.14), we deduce

$$
\begin{aligned}
&\|\mathbf{e}^h_{\boldsymbol{\sigma}_f}(0)\|^2_{\mathbb{X}_f} + \|\mathbf{e}^h_{\mathbf{u}_p}(0)\|^2_{\mathbf{V}_p} + \|A^{1/2}\,(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0))\|^2_{\mathbb{L}^2(\Omega_p)} + \|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0))\|^2_{\mathbf{L}^2(\Omega_p)} \\
&\quad +\|\mathbf{e}^h_{p_p}(0)\|^2_{\mathrm{W}_p} + |(\mathbf{e}^h_{\boldsymbol{\varphi}} - \mathbf{e}^h_{\boldsymbol{\theta}})(0)|^2_{\mathtt{BJS}} + \|\mathbf{e}^h_{\lambda}(0)\|^2_{\Lambda_{ph}} \\
&\quad \leq C \left(\|\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}_0}\|^2_{\mathbf{X}} + \|\mathbf{e}^I_{\underline{\widetilde{\boldsymbol{\varphi}}}_0}\|^2_{\mathbf{Y}_h} + \|\mathbf{e}^I_{\underline{\widetilde{\mathbf{u}}}_0}\|^2_{\mathbf{Z}} \right),
\end{aligned} \tag{5.37}
$$

where $\underline{\boldsymbol{\sigma}}_0 = (\boldsymbol{\sigma}_{f,0}, \mathbf{u}_{p,0}, \boldsymbol{\sigma}_{p,0}, p_{p,0})$, $\underline{\widetilde{\boldsymbol{\varphi}}}_0 = (\boldsymbol{\varphi}_0, \boldsymbol{\omega}_0, \lambda_0)$ and $\underline{\widetilde{\mathbf{u}}}_0 = (\mathbf{u}_{f,0}, \boldsymbol{\eta}_{p,0}, \boldsymbol{\gamma}_{f,0}, \boldsymbol{\rho}_{p,0})$, and $\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}_0}$, $\mathbf{e}^I_{\underline{\widetilde{\boldsymbol{\varphi}}}_0}$, $\mathbf{e}^I_{\underline{\widetilde{\mathbf{u}}}_0}$ denote their corresponding approximation errors. Thus, using the error decomposition (5.22) in combination with (5.36)–(5.37), the triangle inequality, (4.14) and the approximation properties (5.18) and (5.21), we obtain (5.24) with a positive constant C depending on parameters $\mu, \lambda_p, \mu_p, \alpha_p, k_{\min}, k_{\max}, \alpha_{\mathtt{BJS}}$, and the extra regularity assumptions for $\underline{\boldsymbol{\sigma}}$, $\underline{\boldsymbol{\varphi}}$, and $\underline{\mathbf{u}}$ whose expressions are obtained from the right-hand sides of (5.18) and (5.21). This completes the proof in the conforming case **(S1)**.

The proof in the non-conforming case **(S2)** follows by using similar arguments. We exploit the projection property (5.19) to conclude that some terms in (5.25) are zero, namely $b_{\mathbf{n}_f}(\mathbf{e}^h_{\boldsymbol{\sigma}_f}, \mathbf{e}^I_{\boldsymbol{\varphi}})$, $b_{\mathbf{n}_p}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^I_{\boldsymbol{\theta}})$, and $b_\Gamma(\mathbf{e}^h_{\mathbf{u}_p}, \mathbf{e}^I_{\lambda})$, as well as terms appearing in the operator $\mathcal{C}$ (cf. (3.9)): $\left\langle \mathbf{e}^h_{\boldsymbol{\varphi}} \cdot \mathbf{n}_f, \mathbf{e}^I_{\lambda} \right\rangle_{\Gamma_{fp}}$, $\left\langle \mathbf{e}^I_{\boldsymbol{\varphi}} \cdot \mathbf{n}_f, \mathbf{e}^h_{\lambda} \right\rangle_{\Gamma_{fp}}$, $\langle \mathbf{e}^h_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, \mathbf{e}^I_{\lambda} \rangle_{\Gamma_{fp}}$, and $\langle \mathbf{e}^I_{\boldsymbol{\theta}} \cdot \mathbf{n}_p, \mathbf{e}^h_{\lambda} \rangle_{\Gamma_{fp}}$. In addition, in the non-conforming version of (5.29) the terms $\|\mathbf{e}^I_{\lambda}\|_{\Lambda_{ph}}$, $\|\mathbf{e}^I_{\boldsymbol{\varphi}}\|_{\boldsymbol{\Lambda}_{fh}}$, and $\|\mathbf{e}^I_{\boldsymbol{\theta}}\|_{\boldsymbol{\Lambda}_{sh}}$ do not appear, since the bilinear forms $b_\Gamma(\mathbf{v}_{ph}, \mathbf{e}^I_{\lambda})$, $b_{\mathbf{n}_f}(\boldsymbol{\tau}_{fh}, \mathbf{e}^I_{\boldsymbol{\varphi}})$, and $b_{\mathbf{n}_p}(\boldsymbol{\tau}_{ph}, \mathbf{e}^I_{\boldsymbol{\theta}})$ are zero by a direct application of the projection property (5.19). □

6 A multipoint stress-flux mixed finite element method

In this section, inspired by previous works on the multipoint flux mixed finite element method for Darcy flow [20, 42, 60, 61] and the multipoint stress mixed finite element method for elasticity [6–8], we present a vertex quadrature rule that allows for local elimination of the stresses, rotations, and Darcy flux, leading to a positive-definite cell-centered pressure-velocities-traces system. We emphasize that, to the best of our knowledge, this is the first time such method is developed for the Stokes equations. To that end, the finite element spaces we consider for both $(\mathbb{X}_{fh}, \mathbf{V}_{fh}, \mathbb{Q}_{fh})$ and $(\mathbb{X}_{ph}, \mathbf{V}_{sh}, \mathbb{Q}_{ph})$ are the triple $\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1$, which have been shown to be stable for mixed elasticity with weak stress symmetry in [16, 17, 32], whereas $(\mathbf{V}_{ph}, \mathrm{W}_{ph})$ is chosen to be $\mathbf{BDM}_1 - \mathrm{P}_0$ [18], and the Lagrange multiplier spaces $(\mathbf{\Lambda}_{fh}, \mathbf{\Lambda}_{sh}, \Lambda_{ph})$ are either $\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1$ or $\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$ satisfying **(S1)** or **(S2)** (cf. (5.1), (5.2)), respectively, where $\mathrm{P}_1^{\mathrm{dc}}$ denotes the piecewise linear discontinuous finite element space and $\mathbf{P}_1^{\mathrm{dc}}$ is its corresponding vector version. We remark that the chosen finite element spaces for the stresses, rotations, and Darcy flux have degrees of freedom that can be associated with the element vertices, which, in combination with the vertex quadrature rule, allows for local elimination of these variables.

6.1 A quadrature rule setting

Let $S_\star$ denote the space of elementwise continuous functions on $\mathcal{T}_h^\star$. For any pair of tensor or vector valued functions φ and ψ with elements in $S_\star$, we define the vertex quadrature rule as in [61] (see also [6, 8]):

$$(\varphi, \psi)_{Q,\Omega_\star} := \sum_{E \in \mathcal{T}_h^\star} (\varphi, \psi)_{Q,E} = \sum_{E \in \mathcal{T}_h^\star} \frac{|E|}{s} \sum_{i=1}^{s} \varphi(\mathbf{r}_i) \cdot \psi(\mathbf{r}_i), \tag{6.1}$$

where $\star \in \{f, p\}$, $s = 3$ on triangles and $s = 4$ on tetrahedra, $\mathbf{r}_i$, $i = 1, \ldots, s$, are the vertices of the element E, and $\cdot$ denotes the inner product for both vectors and tensors.

We will apply the quadrature rule for the bilinear forms a_f, a_p, a_e and $b_{\mathrm{sk},\star}$, which will be denoted by a_f^h, a_p^h, a_e^h and $b_{\mathrm{sk},\star}^h$, respectively. These bilinear forms involve the stress spaces $\mathbb{X}_{fh}$ and $\mathbb{X}_{ph}$, the vorticity space $\mathbb{Q}_{fh}$ and rotation space $\mathbb{Q}_{ph}$, and the Darcy velocity space $\mathbf{V}_{ph}$. The $\mathbf{BDM}_1$ spaces have for degrees of freedom $s - 1$ normal components on each element edge (face), which can be associated with the vertices of the edge (face). At any element vertex $\mathbf{r}_i$, the value of a tensor or vector function is uniquely determined by its normal components at the associated two edges or three faces. Also, the vorticity space $\mathbb{Q}_{fh}$ and the rotation space $\mathbb{Q}_{ph}$ are vertex-based. Therefore the application of the vertex quadrature rule (6.1) for the bilinear forms involving the above spaces results in coupling only the degrees of freedom associated with a mesh vertex, which allows for local elimination of these variables. Next, we state a preliminary lemma to be used later on, which has been proved in [8, Lemma 3.1] and [6, Lemma 2.2].

Lemma 6.1 *There exist positive constants C_0 and C_1 independent of h, such that for any linear uniformly bounded and positive-definite operator L, there hold*

$$(L(\varphi), \varphi)_{Q,\Omega_\star} \geq C_0 \|\varphi\|^2_{\Omega_\star}, \quad (L(\varphi), \psi)_{Q,\Omega_\star} \leq C_1 \|\varphi\|_{\Omega_\star} \|\psi\|_{\Omega_\star},$$
$$\forall \varphi, \psi \in S_\star, \quad \star \in \{f, p\}.$$

Consequently, the bilinear form $(L(\varphi), \varphi)_{Q,\Omega_\star}$ is an inner product in $\mathrm{L}^2(\Omega_\star)$ and $(L(\varphi), \varphi)^{1/2}_{Q,\Omega_\star}$ is a norm equivalent to $\|\varphi\|_{\Omega_\star}$.

The semidiscrete coupled multipoint stress-flux mixed finite element method for (3.11) reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0, T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, and for a.e. $t \in (0, T)$,

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h) &= \mathbf{F}(\underline{\boldsymbol{\tau}}_h), \\ -\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h)(\underline{\boldsymbol{\psi}}_h) &= 0, \\ -\mathcal{B}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\mathbf{v}}_h) &= \mathbf{G}(\underline{\mathbf{v}}_h), \end{aligned} \tag{6.2}$$

where

$$\begin{aligned} \mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) &:= a^h_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) + a^h_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) + b_p(\mathbf{v}_{ph}, p_{ph}) - b_p(\mathbf{u}_{ph}, w_{ph}), \\ \mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h)(\underline{\boldsymbol{\tau}}_h) &:= a^h_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) + (s_0\, p_{ph}, w_{ph})_{\Omega_p}, \\ \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) &:= b_f(\boldsymbol{\tau}_{fh}, \mathbf{v}_{fh}) + b_s(\boldsymbol{\tau}_{ph}, \mathbf{v}_{sh}) + b^h_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, \boldsymbol{\chi}_{fh}) + b^h_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, \boldsymbol{\chi}_{ph}). \end{aligned}$$

We next discuss the discrete inf-sup conditions. We recall the space $\widetilde{\mathbf{X}}_h$ defined in (5.5). We also define the discrete kernel of the operator $\mathcal{B}_h$ as

$$\widehat{\mathbf{V}}_h := \Big\{ \underline{\boldsymbol{\tau}}_h \in \mathbf{X}_h : \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Z}_h \Big\} = \widehat{\mathbb{X}}_{fh} \times \mathbf{V}_{ph} \times \widehat{\mathbb{X}}_{ph} \times \mathrm{W}_{ph}, \tag{6.3}$$

where

$$\widehat{\mathbb{X}}_{\star h} := \Big\{ \boldsymbol{\tau}_{\star h} \in \mathbb{X}_{\star h} : (\boldsymbol{\tau}_{\star h}, \boldsymbol{\xi}_{\star h})_{Q,\Omega_\star} = 0\ \forall\, \boldsymbol{\xi}_{\star h} \in \mathbb{Q}_{\star h} \ \text{ and } \ \mathbf{div}(\boldsymbol{\tau}_{\star h}) = \mathbf{0} \ \text{ in } \ \Omega_\star \Big\},$$
$$\star \in \{f, p\},$$

emphasizing the difference from the discrete kernel of $\mathcal{B}$ defined in (5.6).

Lemma 6.2 *There exist positive constants $\widehat{\beta}$ and $\widehat{\beta}_1$, such that*

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \widetilde{\mathbf{X}}_h} \frac{\mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widehat{\beta}\, \|\underline{\mathbf{v}}_h\|_{\mathbf{Z}} \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{Z}_h, \tag{6.4}$$

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}}_h \in \widehat{\mathbf{V}}_h} \frac{\mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}}} \geq \widehat{\beta}_1\, \|\underline{\boldsymbol{\psi}}_h\|_{\mathbf{Y}_h} \quad \forall\, \underline{\boldsymbol{\psi}}_h \in \mathbf{Y}_h. \tag{6.5}$$

Proof The proof of (6.4) follows from a slight adaptation of the argument in [6, Theorem 4.2]. The proof of (6.5) is similar to the proof of (5.8). The main difference is replacing the interpolant satisfying (5.9) by an interpolant $\hat{\Pi}_h^f : \mathbb{H}^1(\Omega_f) \to \mathbb{X}_{fh}$ satisfying

$$
\begin{aligned}
& b_f(\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \mathbf{v}_{fh}) = 0 \quad \forall\, \mathbf{v}_{fh} \in \mathbf{V}_{fh}, \\
& b_{\mathrm{sk},f}^h(\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f, \boldsymbol{\chi}_{fh}) = 0 \quad \forall\, \boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}, \\
& \langle (\hat{\Pi}_h^f \boldsymbol{\tau}_f - \boldsymbol{\tau}_f)\mathbf{n}_f, \boldsymbol{\tau}_{fh}\mathbf{n}_f \rangle_{\Gamma_{fp} \cup \Gamma_f^N} = 0 \quad \forall\, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh},
\end{aligned}
$$

whose existence follows from the inf-sup condition for $\mathcal{B}_h$ (6.4). □

We can establish the following well-posedness result.

Theorem 6.3 *For each $p_{p,0} \in \mathrm{H}_p$ and compatible discrete initial data $(\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ constructed in* Lemma 5.2 *and each*

$$
\mathbf{f}_f \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_f'), \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{V}_s'), \quad q_f \in \mathrm{W}^{1,1}(0,T;\mathbb{X}_f'), \\
q_p \in \mathrm{W}^{1,1}(0,T;\mathrm{W}_p'),
$$

there exists a unique solution of (6.2)*, $(\underline{\boldsymbol{\sigma}}_h, \underline{\boldsymbol{\varphi}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$ such that $(\boldsymbol{\sigma}_{ph}, p_{ph}) \in \mathrm{W}^{1,\infty}(0,T;\mathbb{X}_{ph}) \times \mathrm{W}^{1,\infty}(0,T;\mathrm{W}_{ph})$, and $(\underline{\boldsymbol{\sigma}}_h(0), \underline{\boldsymbol{\varphi}}_h(0), \mathbf{u}_{fh}(0), \boldsymbol{\gamma}_{fh}(0)) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \mathbf{u}_{fh,0}, \boldsymbol{\gamma}_{fh,0})$. Moreover, assuming sufficient regularity of the data, a stability bound as in* (5.16) *also holds.*

Proof The theorem follows from similar arguments to the proof of Theorem 5.3, in conjunction with Lemmas 6.1 and 6.2. □

6.2 Error analysis

Now, we obtain the error estimates and theoretical rates of convergence for the multipoint stress-flux mixed scheme (6.2). To that end, for each $\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}$, $\mathbf{u}_{ph}$, $\mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $\boldsymbol{\sigma}_{ph}, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $p_{ph}, w_{ph} \in \mathrm{W}_{ph}$, $\boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}$, and $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$, we denote the quadrature errors by

$$
\begin{aligned}
& \delta_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) = a_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}) - a_f^h(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh}), \\
& \delta_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) = a_p(\mathbf{u}_{ph}, \mathbf{v}_{ph}) - a_p^h(\mathbf{u}_{ph}, \mathbf{v}_{ph}), \\
& \delta_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) = a_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}) - a_e^h(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph}), \\
& \delta_{\mathrm{sk},\star}(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}) = b_{\mathrm{sk},\star}(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}) - b_{\mathrm{sk},\star}^h(\boldsymbol{\chi}_{\star h}, \boldsymbol{\tau}_{\star h}), \quad \star \in \{f, p\}.
\end{aligned} \tag{6.6}
$$

Next, for the operator A (cf. (2.4)) we will say that $A \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$ if $A \in \mathbb{W}^{1,\infty}(E)$ for all $E \in \mathcal{T}_h^p$ and $\|A\|_{\mathbb{W}^{1,\infty}(E)}$ is uniformly bounded independently of h. Similar notation holds for $\mathbf{K}^{-1}$. In the next lemma we establish bounds on the quadrature errors. The proof follows from a slight adaptation of [6, Lemma 5.2] to our context (see also [8, 61]).

Lemma 6.4 *If* $\mathbf{K}^{-1} \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$ *and* $A \in \mathbb{W}^{1,\infty}_{\mathcal{T}_h^p}$, *then there is a constant* $C > 0$ *independent of* h *such that*

$$|\delta_f(\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh})| \leq C \sum_{E \in \mathcal{T}_h^f} h \, \|\boldsymbol{\sigma}_{fh}\|_{\mathbb{H}^1(E)} \, \|\boldsymbol{\tau}_{fh}\|_{\mathbb{L}^2(E)},$$

$$|\delta_p(\mathbf{u}_{ph}, \mathbf{v}_{ph})| \leq C \sum_{E \in \mathcal{T}_h^p} h \, \|\mathbf{K}^{-1}\|_{\mathbb{W}^{1,\infty}(E)} \, \|\mathbf{u}_{ph}\|_{\mathbf{H}^1(E)} \, \|\mathbf{v}_{ph}\|_{\mathbf{L}^2(E)},$$

$$|\delta_e(\boldsymbol{\sigma}_{ph}, p_{ph}; \boldsymbol{\tau}_{ph}, w_{ph})| \leq C \sum_{E \in \mathcal{T}_h^p} h \, \|A\|_{\mathbb{W}^{1,\infty}(E)} \|(\boldsymbol{\sigma}_{ph}, p_{ph})\|_{\mathbb{H}^1(E) \times \mathrm{L}^2(E)} \|(\boldsymbol{\tau}_{ph}, w_{ph})\|_{\mathbb{L}^2(E) \times \mathrm{L}^2(E)},$$

$$|\delta_{\mathrm{sk},\star}(\boldsymbol{\tau}_{\star h}, \boldsymbol{\chi}_{\star h})| \leq C \sum_{E \in \mathcal{T}_h^\star} h \, \|\boldsymbol{\tau}_{\star h}\|_{\mathbb{L}^2(E)} \, \|\boldsymbol{\chi}_{\star h}\|_{\mathbb{H}^1(E)}, \quad \star \in \{f, p\},$$

$$|\delta_{\mathrm{sk},\star}(\boldsymbol{\tau}_{\star h}, \boldsymbol{\chi}_{\star h})| \leq C \sum_{E \in \mathcal{T}_h^\star} h \, \|\boldsymbol{\tau}_{\star h}\|_{\mathbb{H}^1(E)} \, \|\boldsymbol{\chi}_{\star h}\|_{\mathbb{L}^2(E)}, \quad \star \in \{f, p\},$$

for all $\boldsymbol{\sigma}_{fh}, \boldsymbol{\tau}_{fh} \in \mathbb{X}_{fh}$, $\mathbf{u}_{ph}, \mathbf{v}_{ph} \in \mathbf{V}_{ph}$, $\boldsymbol{\sigma}_{ph}, \boldsymbol{\tau}_{ph} \in \mathbb{X}_{ph}$, $p_{ph}, w_{ph} \in \mathrm{W}_{ph}$, $\boldsymbol{\chi}_{fh} \in \mathbb{Q}_{fh}$, $\boldsymbol{\chi}_{ph} \in \mathbb{Q}_{ph}$.

We are ready to establish the convergence of the multipoint stress-flux mixed finite element method.

Theorem 6.5 *For the solutions of the continuous and semidiscrete problems* (3.11) *and* (6.2) *established in* Theorem 4.10 *and* Theorem 6.3, *respectively, assuming sufficient regularity of the true solution according to* (5.18) *and* (5.21), *there exists a positive constant* C *independent of* h *and* s_0, *such that*

$$\begin{aligned}
&\|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^\infty(0,T;\mathbb{X}_f)} + \|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\mathrm{L}^2(0,T;\mathbb{X}_f)} + \|\mathbf{e}_{\mathbf{u}_p}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{e}_{\mathbf{u}_p}\|_{\mathrm{L}^2(0,T;\mathbf{V}_p)} \\
&\quad + |\mathbf{e}_{\boldsymbol{\varphi}} - \mathbf{e}_{\boldsymbol{\theta}}|_{\mathrm{L}^\infty(0,T;BJS)} + |\mathbf{e}_{\boldsymbol{\varphi}} - \mathbf{e}_{\boldsymbol{\theta}}|_{\mathrm{L}^2(0,T;BJS)} + \|\mathbf{e}_\lambda\|_{\mathrm{L}^\infty(0,T;\Lambda_{ph})} \\
&\quad + \|\mathbf{e}_{\underline{\boldsymbol{\varphi}}}\|_{\mathrm{L}^2(0,T;\mathbf{Y}_h)} + \|\mathbf{e}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{Z})} + \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega_p))} \\
&\quad + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{e}_{p_p}\|_{\mathrm{L}^\infty(0,T;\mathrm{W}_p)} \\
&\quad + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega_p))} + \|\mathbf{e}_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\quad + \|\partial_t \, A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p} + \alpha_p \mathbf{e}_{p_p} \mathbf{I})\|_{\mathrm{L}^2(0,T;\mathbb{L}^2(\Omega_p))} + \sqrt{s_0}\|\partial_t \, \mathbf{e}_{p_p}\|_{\mathrm{L}^2(0,T;\mathrm{W}_p)} \\
&\leq C \sqrt{\exp(T)} \left(h + h^{1+r}\right),
\end{aligned} \tag{6.7}$$

where r *is defined in* (5.18).

Proof To obtain the error equations, we subtract the multipoint stress-flux mixed finite element formulation (6.2) from the continuous one (3.11). Using the error decomposition (5.22) and applying some algebraic manipulations, we obtain the error system:

$$\begin{aligned}
&\big(\partial_t\,\mathcal{E}_h + \mathcal{A}_h\big)(\mathbf{e}^h_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}}) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^h_{\underline{\mathbf{u}}})\\
&\quad = -\big(\partial_t\,\mathcal{E} + \mathcal{A}\big)(\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\tau}}_h) - \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}}) - \mathcal{B}(\underline{\boldsymbol{\tau}}_h)(\mathbf{e}^I_{\underline{\mathbf{u}}}) - \boldsymbol{\delta}_{fep}(I_h(\underline{\boldsymbol{\sigma}}), P_h(\underline{\mathbf{u}}))(\underline{\boldsymbol{\tau}}_h),\\
&-\mathcal{B}_1(\mathbf{e}^h_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\mathbf{e}^h_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h) = \mathcal{B}_1(\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}})(\underline{\boldsymbol{\psi}}_h) - \mathcal{C}(\mathbf{e}^I_{\underline{\boldsymbol{\varphi}}})(\underline{\boldsymbol{\psi}}_h)\\
&-\mathcal{B}_h(\mathbf{e}^h_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) = \mathcal{B}(\mathbf{e}^I_{\underline{\boldsymbol{\sigma}}})(\underline{\mathbf{v}}_h) + \boldsymbol{\delta}_{fp}(I_h(\underline{\boldsymbol{\sigma}}))(\underline{\mathbf{v}}_h)\,,
\end{aligned} \tag{6.8}$$

for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, where

$$\begin{aligned}
\boldsymbol{\delta}_{fep}(I_h(\underline{\boldsymbol{\sigma}}), P_h(\underline{\mathbf{u}}))(\underline{\boldsymbol{\tau}}_h) &:= -\delta_f(I_h^{\mathbb{X}_f}(\boldsymbol{\sigma}_f), \boldsymbol{\tau}_{fh}) - \delta_e(I_h^{\mathbb{X}_p}(\boldsymbol{\sigma}_p), p_p; \boldsymbol{\tau}_{ph}, w_{ph})\\
&\quad -\delta_p(I_h^{\mathbf{V}_p}(\mathbf{u}_p), \mathbf{v}_{ph}) - \delta_{\mathrm{sk},f}(\boldsymbol{\tau}_{fh}, P_h^{\mathbb{Q}_f}(\boldsymbol{\gamma}_f)) - \delta_{\mathrm{sk},p}(\boldsymbol{\tau}_{ph}, P_h^{\mathbb{Q}_p}(\boldsymbol{\gamma}_p))
\end{aligned}$$

and

$$\boldsymbol{\delta}_{fp}(I_h(\underline{\boldsymbol{\sigma}}))(\underline{\mathbf{v}}_h) := \delta_{\mathrm{sk},f}(I_h^{\mathbb{X}_f}(\boldsymbol{\sigma}_f), \boldsymbol{\chi}_{fh}) + \delta_{\mathrm{sk},p}(I_h^{\mathbb{X}_p}(\boldsymbol{\sigma}_p), \boldsymbol{\chi}_{ph})\,.$$

Notice that the error system (6.8) is similar to (5.23), except for the additional quadrature error terms. The rest of the proof follows from the arguments in the proof of (5.24), using Lemmas 6.1, 6.2 and 6.4, and utilizing the continuity bounds of the interpolation operators $I_h^{\mathbf{X}_\star}$, $I_h^{\mathbf{V}_p}$, $P_h^{\mathbb{Q}_\star}$ [6, Lemma 5.1]:

$$\begin{aligned}
\|I_h^{\mathbb{X}_\star}(\boldsymbol{\tau}_{\star h})\|_{\mathbb{H}^1(E)} &\le C\,\|\boldsymbol{\tau}_{\star h}\|_{\mathbb{H}^1(E)} \quad \forall\,\boldsymbol{\tau}_{\star h} \in \mathbb{H}^1(E)\,, \quad \star \in \{f, p\}\,,\\
\|P_h^{\mathbb{Q}_\star}(\boldsymbol{\chi}_{\star h})\|_{\mathbb{H}^1(E)} &\le C\,\|\boldsymbol{\chi}_{\star h}\|_{\mathbb{H}^1(E)} \quad \forall\,\boldsymbol{\chi}_{\star h} \in \mathbb{H}^1(E)\,,\\
\|I_h^{\mathbf{V}_p}(\mathbf{v}_{ph})\|_{\mathbf{H}^1(E)} &\le C\,\|\mathbf{v}_{ph}\|_{\mathbf{H}^1(E)} \quad \forall\,\mathbf{v}_{ph} \in \mathbf{H}^1(E)\,.
\end{aligned}$$

We omit further details, and refer to [6, 8, 61] for more details on the error analysis of the multipoint flux and multipoint stress mixed finite element methods on simplicial grids. □

6.3 Reduction to a cell-centered pressure-velocities-traces system

In this section we focus on the fully discrete problem associated to (6.2) (cf. (3.11), (5.11)), and describe how to obtain a reduced cell-centered system for the algebraic problem at each time step. For the time discretization we employ the backward Euler method. Let Δt be the time step, $T = M\,\Delta t$, $t_m = m\,\Delta t$, $m = 0, \ldots, M$. Let $d_t\,u^m := (\Delta t)^{-1}(u^m - u^{m-1})$ be the first order (backward) discrete time derivative, where $u^m := u(t_m)$. Then the fully discrete model reads: given $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\boldsymbol{\varphi}}_h^0, \underline{\mathbf{u}}_h^0) = (\underline{\boldsymbol{\sigma}}_{h,0}, \underline{\boldsymbol{\varphi}}_{h,0}, \underline{\mathbf{u}}_{h,0})$ satisfying (5.12), find $(\underline{\boldsymbol{\sigma}}_h^m, \underline{\boldsymbol{\varphi}}_h^m, \underline{\mathbf{u}}_h^m) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$, $m = 1, \ldots, M$, such that for all $(\underline{\boldsymbol{\tau}}_h, \underline{\boldsymbol{\psi}}_h, \underline{\mathbf{v}}_h) \in \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h$,

$$d_t\,\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{A}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h^m) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h^m) = \mathbf{F}(\underline{\boldsymbol{\tau}}_h)\,,$$

$$-\mathcal{B}_1(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\psi}}_h) + \mathcal{C}(\underline{\boldsymbol{\varphi}}_h^m)(\underline{\boldsymbol{\psi}}_h) = 0\,,$$
$$-\mathcal{B}_h(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\mathbf{v}}_h) = \mathbf{G}(\underline{\mathbf{v}}_h)\,. \quad (6.9)$$

Remark 6.1 The well-posedness and error estimate associated to the fully discrete problem (6.9) can be derived employing similar arguments to Theorems 6.3 and 6.5 in combination with the theory developed in [9, Sections 6 and 9]. In particular, we note that at each time step the well-posedness of the fully discrete problem (6.9), with $m = 1, \ldots, M$, follows from similar arguments to the proof of Lemma 4.7.

Notice that the first row in (6.9) can be rewritten equivalently as

$$\Big((\Delta t)^{-1}\mathcal{E}_h + \mathcal{A}_h\Big)(\underline{\boldsymbol{\sigma}}_h^m)(\underline{\boldsymbol{\tau}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\varphi}}_h^m) + \mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{u}}_h^m)$$
$$= \mathbf{F}(\underline{\boldsymbol{\tau}}_h) + (\Delta t)^{-1}\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^{m-1})(\underline{\boldsymbol{\tau}}_h)\,. \quad (6.10)$$

Let us associate with the operators in (6.9)–(6.10) matrices denoted in the same way. We then have

$$\Big((\Delta t)^{-1}\,\mathcal{E}_h + \mathcal{A}_h\Big) = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f} & 0 & 0 & 0 \\ 0 & A_{\mathbf{u}_p\mathbf{u}_p} & 0 & A^{\mathrm{t}}_{\mathbf{u}_p p_p} \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} & A^{\mathrm{t}}_{\boldsymbol{\sigma}_p p_p} \\ 0 & -A_{\mathbf{u}_p p_p} & A_{\boldsymbol{\sigma}_p p_p} & A_{p_p p_p} \end{pmatrix}, \quad \mathcal{B}_h = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 \\ A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} & 0 \end{pmatrix},$$

$$\mathcal{B}_1 = \begin{pmatrix} A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 & 0 \\ 0 & 0 & A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 \\ 0 & A_{\mathbf{u}_p\lambda} & 0 & 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} \\ A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} \\ -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & 0 \end{pmatrix},$$

with

$$\begin{aligned}
&A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f} \sim a_f^h(\cdot,\cdot),\;\; A_{\mathbf{u}_p\mathbf{u}_p} \sim a_p^h(\cdot,\cdot),\;\; A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} \sim (\Delta t)^{-1}\, a_e^h(\cdot,0;\cdot,0),\\
&A_{\boldsymbol{\sigma}_p p_p} \sim (\Delta t)^{-1} a_e^h(\cdot,0;\mathbf{0},\cdot),\\
&A_{p_p p_p} \sim (\Delta t)^{-1} a_e^h(\mathbf{0},\cdot;\mathbf{0},\cdot) + (\Delta t)^{-1}(s_0\,\cdot,\cdot)_{\Omega_p},\, A_{\mathbf{u}_p p_p} \sim b_p(\cdot,\cdot),\\
&A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \sim b_{\mathbf{n}_f}(\cdot,\cdot),\;\; A_{\mathbf{u}_p\lambda} \sim b_\Gamma(\cdot,\cdot),\\
&A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} \sim b_{\mathbf{n}_p}(\cdot,\cdot),\;\; A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} \sim c_{\mathrm{BJS}}(\cdot,\mathbf{0};\cdot,\mathbf{0}),\;\; A_{\boldsymbol{\varphi}\boldsymbol{\theta}} \sim c_{\mathrm{BJS}}(\cdot,\mathbf{0};\mathbf{0},\cdot),\\
&A_{\boldsymbol{\theta}\boldsymbol{\theta}} \sim c_{\mathrm{BJS}}(\mathbf{0},\cdot;\mathbf{0},\cdot),\;\; A_{\boldsymbol{\varphi}\lambda} \sim c_\Gamma(\cdot,\mathbf{0};\cdot),\\
&A_{\boldsymbol{\theta}\lambda} \sim c_\Gamma(\mathbf{0},\cdot;\cdot),\;\; A_{\boldsymbol{\sigma}_f\mathbf{u}_f} \sim b_f(\cdot,\cdot),\;\; A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} \sim b^h_{\mathrm{sk},f}(\cdot,\cdot),\\
&A_{\boldsymbol{\sigma}_p\mathbf{u}_s} \sim b_s(\cdot,\cdot),\;\; A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \sim b^h_{\mathrm{sk},p}(\cdot,\cdot),
\end{aligned}$$

where the notation $A \sim a$ means that the matrix A is associated with the bilinear form a. Denoting the algebraic vectors corresponding to the variables $\underline{\boldsymbol{\sigma}}_h^m$, $\underline{\boldsymbol{\varphi}}_h^m$, and $\underline{\mathbf{u}}_h^m$ in the same way, we can then write the system (6.9) in a matrix-vector form as

$$\begin{pmatrix} (\Delta t)^{-1}\,\mathcal{E}_h + \mathcal{A}_h & \mathcal{B}_1^{\mathrm{t}} & \mathcal{B}_h^{\mathrm{t}} \\ -\mathcal{B}_1 & \mathcal{C} & 0 \\ -\mathcal{B}_h & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\sigma}}_h^m \\ \underline{\boldsymbol{\varphi}}_h^m \\ \underline{\mathbf{u}}_h^m \end{pmatrix} = \begin{pmatrix} \mathbf{F} + (\Delta t)^{-1}\mathcal{E}_h(\underline{\boldsymbol{\sigma}}_h^{m-1}) \\ 0 \\ \mathbf{G} \end{pmatrix}. \quad (6.11)$$

As we noted in Sect. 6.1, due to the the use of the vertex quadrature rule, the degrees of freedom (DOFs) of the Stokes stress $\boldsymbol{\sigma}_{fh}^m$, Darcy velocity $\mathbf{u}_{ph}^m$ and poroelastic stress tensor $\boldsymbol{\sigma}_{ph}^m$ associated with a mesh vertex become decoupled from the rest of the DOFs. As a result, the assembled mass matrices have a block-diagonal structure with one block per mesh vertex. The dimension of each block equals the number of DOFs associated with the vertex. These matrices can then be easily inverted with local computations. Inverting each local block in $A_{\mathbf{u}_p\mathbf{u}_p}$ allows for expressing the Darcy velocity DOFs associated with a vertex in terms of the Darcy pressure p_{ph}^m at the centers of the elements that share the vertex, as well as the trace unknown λ_h^m on neighboring edges (faces) for vertices on Γ_{fp}. Similarly, inverting each local block in $A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ allows for expressing the Stokes stress DOFs associated with a vertex in terms of neighboring Stokes velocity $\mathbf{u}_{fh}^m$, vorticity $\boldsymbol{\gamma}_{fh}^m$, and trace $\boldsymbol{\varphi}_h^m$. Finally, inverting each local block in $A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$ allows for expressing the poroelastic stress DOFs associated with a vertex in terms of neighboring Darcy pressure p_{ph}^m, structure velocity $\mathbf{u}_{sh}^m$, structure rotation $\boldsymbol{\gamma}_{ph}^m$, and trace $\boldsymbol{\theta}_h^m$. Then we have

$$
\begin{aligned}
\mathbf{u}_{ph}^m &= -A_{\mathbf{u}_p\mathbf{u}_p}^{-1} A_{\mathbf{u}_p p_p}^{\mathrm{t}}\, p_{ph}^m - A_{\mathbf{u}_p\mathbf{u}_p}^{-1} A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\, \lambda_h^m, \\
\boldsymbol{\sigma}_{fh}^m &= -A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}\, \boldsymbol{\varphi}_h^m - A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}\, \mathbf{u}_{fh}^m - A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}\, \boldsymbol{\gamma}_{fh}^m, \\
\boldsymbol{\sigma}_{ph}^m &= -A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}\, p_{ph}^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}\, \boldsymbol{\theta}_h^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}\, \mathbf{u}_{sh}^m - A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}\, \boldsymbol{\gamma}_{ph}^m.
\end{aligned}
\tag{6.12}
$$

The reduced matrix associated to (6.11) in terms of $(p_{ph}^m, \boldsymbol{\varphi}_h^m, \boldsymbol{\theta}_h^m, \lambda_h^m, \mathbf{u}_{fh}^m, \mathbf{u}_{sh}^m, \boldsymbol{\gamma}_{fh}^m, \boldsymbol{\gamma}_{ph}^m)$ is given by

$$
\begin{pmatrix}
A_{p_p\boldsymbol{\sigma}_p p_p} + A_{p_p\mathbf{u}_p p_p} & 0 & -A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{p_p\mathbf{u}_p\lambda} & 0 & -A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & -A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \\
0 & A_{\boldsymbol{\varphi}\boldsymbol{\varphi}}+A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}}^{\mathrm{t}} & A_{\boldsymbol{\varphi}\lambda}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 \\
A_{p_p\boldsymbol{\sigma}_p\theta}^{\mathrm{t}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}}+A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\lambda}^{\mathrm{t}} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} \\
A_{p_p\mathbf{u}_p\lambda}^{\mathrm{t}} & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & A_{\lambda\mathbf{u}_p\lambda} & 0 & 0 & 0 & 0 \\
0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0 \\
A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \\
0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}} & 0 & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} & 0 \\
A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} & 0 & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & 0 & 0 & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} & 0 & A_{\gamma_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}
\end{pmatrix}
\tag{6.13}
$$

where

$$
\begin{aligned}
& A_{p_p\boldsymbol{\sigma}_p p_p} = A_{p_p p_p} - A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}, \quad A_{p_p\mathbf{u}_p p_p} = A_{\mathbf{u}_p p_p} A_{\mathbf{u}_p\mathbf{u}_p}^{-1} A_{\mathbf{u}_p p_p}^{\mathrm{t}}, \\
& A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}, \\
& A_{p_p\mathbf{u}_p\lambda} = A_{\mathbf{u}_p p_p} A_{\mathbf{u}_p\mathbf{u}_p}^{-1} A_{\mathbf{u}_p\lambda}^{\mathrm{t}}, \quad A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}, \\
& A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}, \ A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}, \\
& A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}, \ A_{\lambda\mathbf{u}_p\lambda} = A_{\mathbf{u}_p\lambda} A_{\mathbf{u}_p\mathbf{u}_p}^{-1} A_{\mathbf{u}_p\lambda}^{\mathrm{t}}, \\
& A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}, \quad A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} = A_{\boldsymbol{\sigma}_f\mathbf{u}_f} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}, \\
& A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} = A_{\boldsymbol{\sigma}_f\mathbf{u}_f} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}, \quad A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}},
\end{aligned}
$$

$$
\begin{aligned}
&A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} = A_{\boldsymbol{\sigma}_p\mathbf{u}_s}A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\mathbf{u}_s},\ A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p\mathbf{u}_s}A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\\
&A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} = A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\ A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} = A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}A^{\mathrm{t}}_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\\
&A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} = A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\ A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} = A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}A^{\mathrm{t}}_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}.
\end{aligned} \tag{6.14}
$$

Furthermore, due to the vertex quadrature rule, the vorticity and structure rotation DOFs corresponding to each vertex of the grid become decoupled from the rest of the DOFs, leading to block-diagonal matrices $A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}$ and $A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}$. Recalling the matrix definitions in (6.14), each block is symmetric and positive definite and thus locally invertible, due the positive definiteness of $A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ and $A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$ and the inf-sup condition (5.7). We then have

$$
\begin{aligned}
\boldsymbol{\gamma}^m_{fh} &= -A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}\,\boldsymbol{\varphi}^m_h - A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}\,\mathbf{u}^m_{fh},\\
\boldsymbol{\gamma}^m_{ph} &= -A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\,p^m_{ph} - A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}\,\boldsymbol{\theta}^m_h - A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}\,\mathbf{u}^m_{sh},
\end{aligned} \tag{6.15}
$$

and using some algebraic manipulation, we obtain the reduced problem $\mathbf{A}\mathbf{p}^m_h = \mathbf{F}$, with vector solution $\mathbf{p}^m_h := (p^m_{ph}, \boldsymbol{\varphi}^m_h, \boldsymbol{\theta}^m_h, \lambda^m_h, \mathbf{u}^m_{fh}, \mathbf{u}^m_{sh})$ and matrix

$$
\mathbf{A} = \begin{pmatrix}
\widetilde{A}_{p_p\boldsymbol{\sigma}_p p_p} + A_{p_p\mathbf{u}_p p_p} & 0 & -\widetilde{A}_{p_p\boldsymbol{\sigma}_p\theta} & A_{p_p\mathbf{u}_p\lambda} & 0 & -\widetilde{A}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s}\\
0 & \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} + A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\varphi}\lambda} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0\\
\widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} + A_{\boldsymbol{\theta}\boldsymbol{\theta}} & A^{\mathrm{t}}_{\boldsymbol{\theta}\lambda} & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}\\
A^{\mathrm{t}}_{p_p\mathbf{u}_p\lambda} & -A_{\boldsymbol{\varphi}\lambda} & -A_{\boldsymbol{\theta}\lambda} & A_{\lambda\mathbf{u}_p\lambda} & 0 & 0\\
0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & 0 & 0 & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & 0\\
\widetilde{A}^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} & 0 & \widetilde{A}^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & 0 & 0 & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s}
\end{pmatrix} \tag{6.16}
$$

where

$$
\begin{aligned}
\widetilde{A}_{p_p\boldsymbol{\sigma}_p p_p} &= A_{p_p\boldsymbol{\sigma}_p p_p} + A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\\
\widetilde{A}_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} &= A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}},\\
\widetilde{A}_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} &= A_{p_p\boldsymbol{\sigma}_p\mathbf{u}_s} - A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\\
\widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} &= A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} - A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}},\\
\widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} &= A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} - A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\\
\widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} &= A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}},\\
\widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} &= A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} - A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},\\
\widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} &= A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} - A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{-1}_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}A^{\mathrm{t}}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f},\\
\widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} &= A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} - A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{-1}_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}A^{\mathrm{t}}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p},
\end{aligned} \tag{6.17}
$$

and the right hand side vector $\mathbf{F}$ has been obtained by transforming the right-hand side in (6.9) accordingly to the procedure above. Note that, after solving the problem with matrix (6.16), we can recover $\mathbf{u}_{ph}^m$, $\boldsymbol{\sigma}_{fh}^m$, $\boldsymbol{\sigma}_{ph}^m$ and $\boldsymbol{\gamma}_{fh}^m$, $\boldsymbol{\gamma}_{ph}^m$ through the formulae (6.12) and (6.15), respectively, thus obtaining the full solution to (6.9).

Lemma 6.6 *The cell-centered finite difference system for the pressure-velocities-traces problem* (6.16) *is positive definite.*

Proof Consider a vector $\mathbf{q}^{\mathrm{t}} = (w_{ph}^{\mathrm{t}}\ \boldsymbol{\psi}_h^{\mathrm{t}}\ \boldsymbol{\phi}_h^{\mathrm{t}}\ \xi_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}) \neq \mathbf{0}$. Employing the matrices in (6.14) and (6.17) and some algebraic manipulations, we obtain

$$\begin{aligned}
\mathbf{q}^{\mathrm{t}}\,\mathbf{A}\,\mathbf{q} &= w_{ph}^{\mathrm{t}}\big(A_{p_p p_p} - A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}\big) w_{ph} + w_{ph}^{\mathrm{t}} A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{-1} A_{p_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} w_{ph} \\
&\quad + \big(A_{\mathbf{u}_p p_p}^{\mathrm{t}}\, w_{ph} + A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\, \xi_h\big)^{\mathrm{t}} A_{\mathbf{u}_p\mathbf{u}_p}^{-1} \big(A_{\mathbf{u}_p p_p}^{\mathrm{t}}\, w_{ph} + A_{\mathbf{u}_p\lambda}^{\mathrm{t}}\, \xi_h\big) + (\boldsymbol{\psi}_h^{\mathrm{t}}\ \boldsymbol{\phi}_h^{\mathrm{t}}) \begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\varphi}} & A_{\boldsymbol{\varphi}\boldsymbol{\theta}}^{\mathrm{t}} \\ A_{\boldsymbol{\varphi}\boldsymbol{\theta}} & A_{\boldsymbol{\theta}\boldsymbol{\theta}} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_h \\ \boldsymbol{\phi}_h \end{pmatrix} \\
&\quad + (\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}) \begin{pmatrix} \widetilde{A}_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \\ \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & \widetilde{A}_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_h \\ \mathbf{v}_{fh} \end{pmatrix} + (\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}) \begin{pmatrix} \widetilde{A}_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} \\ \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & \widetilde{A}_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_h \\ \mathbf{v}_{sh} \end{pmatrix}.
\end{aligned} \tag{6.18}$$

Now, we focus on analyzing the six terms in the right-hand side of (6.18). The first term is non-negative due to [41, Theorem 7.7.6] and the fact that the matrix $A_{p_p p_p} - A_{\boldsymbol{\sigma}_p p_p} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}}$ is a Schur complement of the matrix

$$\begin{pmatrix} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p} & A_{\boldsymbol{\sigma}_p p_p}^{\mathrm{t}} \\ A_{\boldsymbol{\sigma}_p p_p} & A_{p_p p_p} \end{pmatrix},$$

which is positive semi-definite as a consequence of the ellipticity property of the operator a_e (cf. (3.8) and (4.7)). The second term is nonnegative, since the matrix $A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}$ is positive definite, as noted in (6.15). The third term is positive for $(w_{ph}^{\mathrm{t}}\ \xi_h^{\mathrm{t}}) \neq \mathbf{0}$, due to the positive-definiteness of $A_{\mathbf{u}_p\mathbf{u}_p}^{-1}$ and the inf-sup condition (5.10). The fourth term is non-negative since the operator $\mathcal{C}$ (cf. (4.8)) is positive semi-definite. The matrices in the last two terms are Schur complements of the matrices

$$A_f := \begin{pmatrix} A_{\boldsymbol{\varphi}\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}} \\ A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\mathbf{u}_f} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} \\ A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}} & A_{\mathbf{u}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}} & A_{\boldsymbol{\gamma}_f\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f} \end{pmatrix} \text{and } A_p := \begin{pmatrix} A_{\boldsymbol{\theta}\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}} & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}} \\ A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\mathbf{u}_s} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \\ A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}} & A_{\mathbf{u}_s\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}} & A_{\boldsymbol{\gamma}_p\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p} \end{pmatrix},$$

respectively, which are positive definite. In particular, for $\mathbf{v}_f^{\mathrm{t}} = (\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}\ \boldsymbol{\chi}_{fh}^{\mathrm{t}}) \neq \mathbf{0}$ and $\mathbf{v}_p^{\mathrm{t}} = (\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}\ \boldsymbol{\chi}_{ph}^{\mathrm{t}}) \neq \mathbf{0}$, we have

$$\begin{aligned}
\mathbf{v}_f^{\mathrm{t}} A_f \mathbf{v}_f &= \big(A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}\, \boldsymbol{\psi}_h + A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}\, \mathbf{v}_{fh} \\
&\qquad + A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}\, \boldsymbol{\chi}_{fh}\big)^{\mathrm{t}} A_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}^{-1} \big(A_{\boldsymbol{\sigma}_f\boldsymbol{\varphi}}^{\mathrm{t}}\, \boldsymbol{\psi}_h + A_{\boldsymbol{\sigma}_f\mathbf{u}_f}^{\mathrm{t}}\, \mathbf{v}_{fh} + A_{\boldsymbol{\sigma}_f\boldsymbol{\gamma}_f}^{\mathrm{t}}\, \boldsymbol{\chi}_{fh}\big) > 0, \\
\mathbf{v}_p^{\mathrm{t}} A_p \mathbf{v}_p &= \big(A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}\, \boldsymbol{\phi}_h + A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}\, \mathbf{v}_{sh} \\
&\qquad + A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}\, \boldsymbol{\chi}_{ph}\big)^{\mathrm{t}} A_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}^{-1} \big(A_{\boldsymbol{\sigma}_p\boldsymbol{\theta}}^{\mathrm{t}}\, \boldsymbol{\phi}_h + A_{\boldsymbol{\sigma}_p\mathbf{u}_s}^{\mathrm{t}}\, \mathbf{v}_{sh} + A_{\boldsymbol{\sigma}_p\boldsymbol{\gamma}_p}^{\mathrm{t}}\, \boldsymbol{\chi}_{ph}\big) > 0,
\end{aligned}$$

due to the positive-definiteness of $A^{-1}_{\boldsymbol{\sigma}_f\boldsymbol{\sigma}_f}$ and $A^{-1}_{\boldsymbol{\sigma}_p\boldsymbol{\sigma}_p}$, along with the combined inf-sup condition for $\mathcal{B}_h(\underline{\boldsymbol{\tau}}_h)(\underline{\mathbf{v}}_h) + \mathcal{B}_1(\underline{\boldsymbol{\tau}}_h)(\underline{\boldsymbol{\psi}}_h)$. The latter follows from the inf-sup conditions (6.4) and (6.5), using that (6.5) holds in the kernel of $\mathcal{B}_h$. Then, applying again [41, Theorem 7.7.6], we conclude that the last two terms in (6.18) are positive for $(\boldsymbol{\psi}_h^{\mathrm{t}}\ \mathbf{v}_{fh}^{\mathrm{t}}) \neq \mathbf{0}$ and $(\boldsymbol{\phi}_h^{\mathrm{t}}\ \mathbf{v}_{sh}^{\mathrm{t}}) \neq \mathbf{0}$. Therefore $\mathbf{q}^{\mathrm{t}}\,\mathbf{A}\,\mathbf{q} > 0$ for all $\mathbf{q} \neq \mathbf{0}$, implying that the matrix $\mathbf{A}$ from (6.16) is positive definite. □

Remark 6.2 The solution of the reduced system with the matrix $\mathbf{A}$ from (6.16) results in significant computational savings compared to the original system (6.11). In particular, five of the eleven variables have been eliminated. Three of the remaining variables are Lagrange multipliers that appear only on the interface Γ_{fp}. The other three are the cell-centered velocities and Darcy pressure, with only n DOFs per element in the Stokes region and $n+1$ DOFs per element in the Biot region, which are the smallest possible number of DOFs for the sub-problems compared to any other formulation. For example, consider the Stokes-Biot formulation in [3, eq. (5.1)-(5.3)], which uses a velocity-pressure formulation for Stokes, displacement formulation for elasticity, and a velocity-pressure formulation for Darcy. In the lowest order case it uses the $\mathbf{P}_1\, b \times \mathrm{P}_1$ MINI elements for Stokes, the lowest order Raviart–Thomas spaces $\mathbf{RT}_0 \times \mathrm{P}_0$ for Darcy, and continuous piecewise linear $\mathbf{P}_1$ for the displacement. This results in $n+1$ DOFs per vertex and one DOF per element for Stokes, one DOF per edge or face and one DOF per element for Darcy, and n DOFs per vertex for elasticity. Therefore the number of DOFs is significantly larger compared to our reduced system (6.16). We further remark that the cost of eliminating the stresses, vorticity, rotation, and velocity, as well as their recovery, is asymptotically negligible compared to the cost of solving the reduced system. In particular this involves, for each vertex, solving local systems for the elimination and matrix-vector multiplications for the recovery, with small matrices of size independent of the number of elements. Finally, since the reduced system is positive definite, efficient iterative solvers such as GMRES can be utilized for its solution.

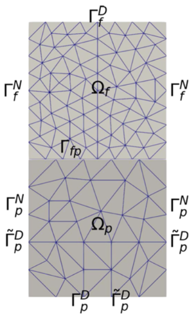
7 Numerical results

In this section we present numerical results that illustrate the behavior of the fully discrete multipoint stress-flux mixed finite element method (6.9). Our implementation is in two dimensions and it is based on `FreeFem++` [40], in conjunction with the direct linear solver `UMFPACK` [29]. For spatial discretization, we use the $(\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1)$ spaces for Stokes, the $(\mathbb{BDM}_1 - \mathbf{P}_0 - \mathbb{P}_1) - (\mathbf{BDM}_1 - \mathrm{P}_0)$ spaces for Biot, and either $(\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1)$ or $\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$ for the Lagrange multipliers. We present three examples. Example 1 is used to corroborate the rates of convergence. Example 2 is a simulation of the coupling of surface and subsurface hydrological systems, focusing on the qualitative behavior of the solution. Example 3 illustrates an application to flow in a poroelastic medium with an irregularly shaped cavity, using physically realistic parameters. We note that in all the three examples the prescribed initial conditions $p_{p,0}$ and $\boldsymbol{\sigma}_{p,0}$ are part of compatible initial data for all variables that satisfy at $t = 0$ the equations in the numerical scheme (6.9) without time derivatives, cf. (5.12).

7.1 Example 1: convergence test

In this test we study the convergence rates for the space discretization using an analytical solution. The domain is $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_p$, where $\Omega_f = (0,1) \times (0,1)$ and $\Omega_p = (0,1) \times (-1,0)$. In particular, the upper half is associated with the Stokes flow, while the lower half represents the flow in the poroelastic structure governed by the Biot system, see Fig. 1 (left). The figure also shows the types of boundary conditions imposed on the different sections of the boundary, recalling that $\Gamma_f^{\mathrm{D}}, \Gamma_f^{\mathrm{N}}, \Gamma_p^{\mathrm{D}}, \Gamma_p^{\mathrm{N}}, \tilde{\Gamma}_p^{\mathrm{D}}$, and $\tilde{\Gamma}_p^{\mathrm{N}}$ have been defined in the model problem Eqs. (2.2) and (2.6). The parameters and the true solution are given in Fig. 1 (right). The solution is designed to satisfy the interface conditions (2.8) on Γ_{fp}. The right hand side functions $\mathbf{f}_f, q_f, \mathbf{f}_p$ and q_p, the boundary conditions, and the initial condition $p_{p,0}$ are determined from (2.2)–(2.6) using the true solution. Note that the boundary conditions for $\boldsymbol{\sigma}_f, \mathbf{u}_f, \mathbf{u}_p, \boldsymbol{\sigma}_p$, and $\boldsymbol{\eta}_p$ are not homogeneous and therefore the right-hand side of the resulting system is modified accordingly. The total simulation time for this example is $T = 0.01$ and the time step is $\Delta t = 10^{-3}$. The time step is sufficiently small, so that the time discretization error does not affect the convergence rates.

Tables 1 and 2 show the convergence history for a sequence of quasi-uniform mesh refinements with non-matching grids along the interface employing conforming and non-conforming spaces for the Lagrange multipliers (cf. (5.1)–(5.2)), respectively. The grids on the coarsest level are shown in Fig. 1 (left). In the tables, h_f and h_p denote the mesh sizes in Ω_f and Ω_p, respectively, while the mesh sizes for their traces on Γ_{fp} are h_{tf} and h_{tp}. We note that the Stokes pressure and the displacement at time t_m are recovered by the post-processing formulae $p_f^m = -\frac{1}{n}(\mathrm{tr}(\boldsymbol{\sigma}_f^m) - 2\,\mu\, q_f^m)$ (cf. (2.2)) and $\boldsymbol{\eta}_p^m = \boldsymbol{\eta}_p^{m-1} + \Delta t\, \mathbf{u}_s^m$ (cf. Remark 5.4), respectively. The results illustrate that spatial rates of convergence $\mathcal{O}(h)$, as provided by Theorem 6.5, are attained for all subdomain variables in their natural norms. The Lagrange multiplier variables, which are approximated in $\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1$ and $\mathbf{P}_1^{\mathrm{dc}} - \mathbf{P}_1^{\mathrm{dc}} - \mathrm{P}_1^{\mathrm{dc}}$, exhibit rates of convergence $\mathcal{O}(h^{3/2})$ and $\mathcal{O}(h^2)$ in the $\mathrm{H}^{1/2}$ and L^2-norms on Γ_{fp}, respectively, which is consistent with the order of approximation.



$$\mu = 1, \quad \alpha_p = 1, \quad \lambda_p = 1, \quad \mu_p = 1, \quad s_0 = 1, \quad \mathbf{K} = \mathbf{I}, \quad \alpha_{\mathtt{BJS}} = 1,$$

$$\mathbf{u}_f = \pi\, \cos(\pi\, t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix},$$

$$p_f = \exp(t)\, \sin(\pi\, x)\, \cos\left(\frac{\pi\, y}{2}\right) + 2\,\pi\, \cos(\pi\, t),$$

$$p_p = \exp(t)\, \sin(\pi\, x)\, \cos\left(\frac{\pi\, y}{2}\right),$$

$$\mathbf{u}_p = -\frac{1}{\mu}\, \mathbf{K}\, \nabla p_p, \quad \boldsymbol{\eta}_p = \sin(\pi\, t) \begin{pmatrix} -3x + \cos(y) \\ y + 1 \end{pmatrix}.$$

Fig. 1 Example 1, domain and coarsest mesh level (left), parameters and analytical solution (right)

Table 1 Example 1, errors and convergence rates with $\mathbf{P}_1 - \mathbf{P}_1 - \mathrm{P}_1$ Lagrange multipliers

h_f	$\|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$		$\|\mathbf{e}_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$		$\|\mathbf{e}_{\boldsymbol{\gamma}_f}\|_{\ell^2(0,T;\mathbb{Q}_f)}$		$\|\mathbf{e}_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.1964	2.2E-02	–	2.7E-02	–	2.4E-03	–	6.3E-03	–
0.0997	1.2E-02	0.95	1.4E-02	1.00	9.3E-04	1.41	3.1E-03	1.05
0.0487	5.7E-03	0.99	6.8E-03	0.99	4.2E-04	1.11	1.6E-03	0.93
0.0250	2.9E-03	1.04	3.4E-03	1.04	2.0E-04	1.13	7.8E-04	1.07
0.0136	1.4E-03	1.14	1.7E-03	1.15	9.4E-05	1.23	3.9E-04	1.15
0.0072	7.1E-04	1.08	8.4E-04	1.10	4.7E-05	1.09	2.0E-04	1.02

h_p	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$		$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$		$\|\mathbf{e}_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$		$\|\mathbf{e}_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$		$\|\mathbf{e}_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.2828	2.7E-01	–	4.3E-02	–	3.4E-02	–	1.0E-01	–	7.5E-02	–
0.1646	1.4E-01	1.27	2.2E-02	1.23	9.4E-03	2.38	5.2E-02	1.27	3.8E-02	1.25
0.0779	6.7E-02	0.97	1.1E-02	0.96	2.2E-03	1.96	2.5E-02	1.00	1.9E-02	0.93
0.0434	3.4E-02	1.17	5.4E-03	1.19	5.8E-04	2.25	1.2E-02	1.24	9.4E-03	1.22
0.0227	1.7E-02	1.06	2.7E-03	1.07	2.0E-04	1.68	5.9E-03	1.08	4.7E-03	1.07
0.0124	8.4E-03	1.15	1.4E-03	1.15	8.1E-05	1.48	2.9E-03	1.15	2.4E-03	1.14

$\|\mathbf{e}_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$		h_{tf}	$\|\mathbf{e}_{\boldsymbol{\varphi}}\|_{\ell^2(0,T;\boldsymbol{\Lambda}_f)}$		h_{tp}	$\|\mathbf{e}_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\boldsymbol{\Lambda}_s)}$		$\|\mathbf{e}_{\lambda}\|_{\ell^2(0,T;\Lambda_p)}$	
error	rate		Error	Rate		Error	Rate	Error	Rate
2.7E-04	–	1/8	1.6E-03	–	1/5	1.6E-02	–	6.9E-03	–
1.4E-04	1.23	1/16	3.7E-04	2.11	1/10	5.7E-03	1.49	2.5E-03	1.49
6.7E-05	0.96	1/32	1.3E-04	1.45	1/20	1.2E-03	2.31	8.5E-04	1.52
3.4E-05	1.19	1/64	4.6E-05	1.54	1/40	3.4E-04	1.76	3.0E-04	1.50
1.7E-05	1.07	1/128	1.2E-05	1.96	1/80	1.1E-04	1.62	1.1E-04	1.50
8.4E-06	1.15	1/256	3.6E-06	1.70	1/160	2.2E-05	2.34	3.7E-05	1.54

7.2 Example 2: coupled surface and subsurface flows

In this example, we simulate coupling of surface and subsurface flows, which could be used to describe the interaction between a river and an aquifer. We consider the domain $\Omega = (0, 2) \times (-1, 1)$. We associate the upper half with the river flow modeled by Stokes equations, while the lower half represents the flow in the aquifer governed by the Biot system. The appropriate interface conditions are enforced along the interface $y = 0$. In this example we focus on the qualitative behavior of the solution and use unit physical parameters:

$$\mu = 1, \quad \alpha_p = 1, \quad \lambda_p = 1, \quad \mu_p = 1, \quad s_0 = 1, \quad \mathbf{K} = \mathbf{I}, \quad \alpha_{\mathrm{BJS}} = 1.$$

Table 2 Example 1, errors and convergence rates with $\mathbf{P}_1^{dc} - \mathbf{P}_1^{dc} - \mathrm{P}_1^{dc}$ Lagrange multipliers

h_f	$\|\mathbf{e}_{\boldsymbol{\sigma}_f}\|_{\ell^2(0,T;\mathbb{X}_f)}$		$\|\mathbf{e}_{\mathbf{u}_f}\|_{\ell^2(0,T;\mathbf{V}_f)}$		$\|\mathbf{e}_{\boldsymbol{\gamma}_f}\|_{\ell^2(0,T;\mathbb{Q}_f)}$		$\|\mathbf{e}_{p_f}\|_{\ell^2(0,T;\mathrm{L}^2(\Omega_f))}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.1964	2.2E-02	–	2.7E-02	–	2.4E-03	–	6.1E-03	–
0.0997	1.2E-02	0.94	1.4E-02	1.00	9.7E-04	1.31	3.1E-03	1.02
0.0487	5.7E-03	0.99	6.8E-03	0.99	4.2E-04	1.16	1.6E-03	0.92
0.0250	2.8E-03	1.04	3.4E-03	1.04	2.0E-04	1.13	7.8E-04	1.07
0.0136	1.4E-03	1.14	1.7E-03	1.15	9.4E-05	1.23	3.9E-04	1.15
0.0072	7.1E-04	1.08	8.4E-04	1.09	4.7E-05	1.09	2.0E-04	1.02

h_p	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{X}_p)}$		$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^2(0,T;\mathbf{V}_s)}$		$\|\mathbf{e}_{\boldsymbol{\gamma}_p}\|_{\ell^2(0,T;\mathbb{Q}_p)}$		$\|\mathbf{e}_{\mathbf{u}_p}\|_{\ell^2(0,T;\mathbf{V}_p)}$		$\|\mathbf{e}_{p_p}\|_{\ell^\infty(0,T;\mathrm{W}_p)}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
0.2828	2.7E-01	–	4.3E-02	–	3.4E-02	–	1.0E-01	–	7.5E-02	–
0.1646	1.4E-01	1.27	2.2E-02	1.23	9.4E-03	2.39	5.2E-02	1.26	3.8E-02	1.25
0.0779	6.7E-02	0.97	1.1E-02	0.96	2.2E-03	1.96	2.5E-02	1.00	1.9E-02	0.93
0.0434	3.4E-02	1.17	5.4E-03	1.19	5.8E-04	2.25	1.2E-02	1.24	9.4E-03	1.22
0.0227	1.7E-02	1.06	2.7E-03	1.07	2.0E-04	1.67	5.9E-03	1.08	4.7E-03	1.07
0.0124	8.4E-03	1.15	1.4E-03	1.15	8.1E-05	1.48	2.9E-03	1.15	2.4E-03	1.14

$\|\mathbf{e}_{\boldsymbol{\eta}_p}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega_p))}$		h_{tf}	$\|\mathbf{e}_{\boldsymbol{\varphi}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$		h_{tp}	$\|\mathbf{e}_{\boldsymbol{\theta}}\|_{\ell^2(0,T;\mathbf{L}^2(\Gamma_{fp}))}$		$\|\mathbf{e}_{\lambda}\|_{\ell^2(0,T;\mathrm{L}^2(\Gamma_{fp}))}$	
Error	Rate		Error	Rate		Error	Rate	Error	Rate
2.7E-04	–	1/8	4.1E-04	–	1/5	7.9E-03	–	1.1E-03	–
1.4E-04	1.23	1/16	2.0E-04	1.04	1/10	2.9E-03	1.46	3.1E-04	1.87
6.7E-05	0.96	1/32	2.4E-05	3.07	1/20	5.7E-04	2.34	7.7E-05	2.01
3.4E-05	1.19	1/64	6.4E-06	1.89	1/40	1.5E-04	1.89	1.9E-05	2.00
1.7E-05	1.07	1/128	1.6E-06	1.97	1/80	3.8E-05	2.01	4.9E-06	1.98
8.4E-06	1.15	1/256	4.0E-07	2.02	1/160	9.0E-06	2.09	1.2E-06	2.09

The body forces, the external source, and the initial conditions are set to zero:

$$\mathbf{f}_f = \mathbf{0}, \quad q_f = 0, \quad \mathbf{f}_p = \mathbf{0}, \quad q_p = 0, \quad p_{p,0} = 0, \quad \text{and} \quad \boldsymbol{\sigma}_{p,0} = \mathbf{0}.$$

The flow is driven through a parabolic fluid velocity on the left boundary of the fluid region with boundary conditions specified as follows:

$$\begin{aligned}
&\mathbf{u}_f = (-40y(y-1), \ \ 0)^t && \text{on } \Gamma_{f,left}, \\
&\mathbf{u}_f = \mathbf{0} && \text{on } \Gamma_{f,top}, \\
&\boldsymbol{\sigma}_f \mathbf{n}_f = \mathbf{0} && \text{on } \Gamma_{f,right}, \\
&p_p = 0 \quad \text{and} \quad \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0} && \text{on } \Gamma_{p,bottom}, \\
&\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{and} \quad \mathbf{u}_s = \mathbf{0} && \text{on } \Gamma_{p,left} \cup \Gamma_{p,right}.
\end{aligned}$$

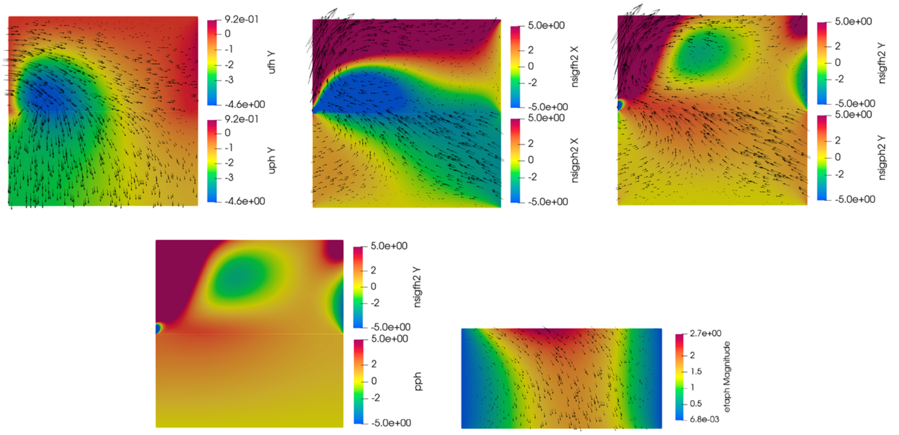


Fig. 2 Example 2, computed solution at $T = 3$. Top left: velocities $\mathbf{u}_{fh}$ and $\mathbf{u}_{ph}$ (arrows), $\mathbf{u}_{fh,2}$ and $\mathbf{u}_{ph,2}$ (color). Top middle and right: negative stresses $-(\boldsymbol{\sigma}_{fh,12}, \boldsymbol{\sigma}_{fh,22})^{\mathrm{t}}$ and $-(\boldsymbol{\sigma}_{ph,12}, \boldsymbol{\sigma}_{ph,22})^{\mathrm{t}}$ (arrows); middle: $-\boldsymbol{\sigma}_{fh,12}$ and $-\boldsymbol{\sigma}_{ph,12}$ (color); right: $-\boldsymbol{\sigma}_{fh,22}$ and $-\boldsymbol{\sigma}_{ph,22}$ (color). Bottom left: negative Stokes stress $-\boldsymbol{\sigma}_{fh,22}$ and Darcy pressure p_{ph}. Bottom right: displacement $\boldsymbol{\eta}_{ph}$ (arrows) and its magnitude (color)

Here $\Gamma_{f,left}$ denotes the section of the boundary of the fluid region that is on the left side of the domain Ω, $\{0\} \times (0, 1)$. The rest of the boundary sections are defined in a similar manner. To avoid inconsistency between the initial data and the non-zero boundary condition for $\mathbf{u}_f$, we start with $\mathbf{u}_f = \mathbf{0}$ on $\Gamma_{f,left}$ at $t = 0$ and gradually increase it to reach $\mathbf{u}_f = (-40y(y-1), \;\; 0)^t$ at $t = 0.3$. In this way, all boundary conditions and source terms are zero at $t = 0$ and we can take zero initial data for all variables that are compatible in the sense that all equations without time derivatives hold at $t = 0$. We remind the reader that, while the existence of compatible initial data for all variables is needed for the well posedness and accuracy of the numerical method, only the initial conditions $p_{p,0}$ and $\boldsymbol{\sigma}_{p,0}$ are used in the scheme. The simulation is run for a total time $T = 3$ with a time step $\Delta t = 0.06$. The computed solution is presented in Fig. 2.

From the velocity plot (top left), we see that the flow in the Stokes region is moving primarily from left to right, driven by the parabolic inflow condition, with some of the fluid percolating downward into the poroelastic medium due to the zero pressure at the bottom, which simulates gravity. The mass conservation $\mathbf{u}_f \cdot \mathbf{n}_f + \big(\partial_t \boldsymbol{\eta}_p + \mathbf{u}_p\big) \cdot \mathbf{n}_p = 0$ on the interface with $\mathbf{n}_p = (0, 1)^{\mathrm{t}}$ indicates the continuity of the second components of the fluid velocity and Darcy velocity when the displacement becomes steady, which is observed from the color plot of the vertical velocity. The stress plots (top middle and right) illustrate the ability of our fully mixed formulation to compute accurate $\mathbb{H}(\mathbf{div})$ stresses in both the fluid and poroelastic regions, without the need for numerical differentiation. In addition, the conservation of momentum $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}$ and balance of normal stress $(\boldsymbol{\sigma}_f \mathbf{n}_f) \cdot \mathbf{n}_f = -p_p$ imply that $\boldsymbol{\sigma}_{f,12} = \boldsymbol{\sigma}_{p,12}, \boldsymbol{\sigma}_{f,22} = \boldsymbol{\sigma}_{p,22}$ and $-\boldsymbol{\sigma}_{f,22} = p_p$ on the interface. These conditions are verified from the top middle and right color plots, as well as the bottom left plot. Furthermore, the arrows in the stress plots are formed by the second columns of the stresses, whose traces on the interface are $\boldsymbol{\sigma}_f \mathbf{n}_f$ and $-\boldsymbol{\sigma}_p \mathbf{n}_p$, respectively. For visualization purpose, the Stokes stress is scaled by a factor of $1/5$ compared to the poroelastic stress, due to large difference in their magnitudes away from the interface. Nevertheless, the continuity of the vector field across the interface is evident, consistent with the conservation of momentum condition $\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_p \mathbf{n}_p = \mathbf{0}$. The overall qualitative behavior of the computed stresses is consistent with the specified boundary and interface conditions. In particular, we observe large fluid stress along the top boundary due to the no slip condition, as well as along the interface due to the slip with friction condition. The singularity near the lower left corner of the Stokes region is due to the mismatch in boundary conditions between the fluid and poroelastic regions. Finally, the last plot shows that the inflow from the Stokes region causes deformation of the poroelastic medium.

7.3 Example 3: irregularly shaped fluid-filled cavity

This example features highly irregularly shaped cavity motivated by modeling flow through vuggy or naturally fractured reservoirs or aquifers. It uses physical units and realistic parameter values taken from the reservoir engineering literature [39]:

$$\mu = 10^{-6}\ \text{kPa s}, \quad \alpha_p = 1, \quad \lambda_p = 5/18 \times 10^7\ \text{kPa}, \quad \mu_p = 5/12 \times 10^7\ \text{kPa},$$
$$s_0 = 6.89 \times 10^{-2}\ \text{kPa}^{-1}, \quad \mathbf{K} = 10^{-8} \times \mathbf{I}\ \text{m}^2, \quad \alpha_{\mathtt{BJS}} = 1.$$

We emphasize that the problem features very small permeability and storativity, as well as large Lamé parameters. These are parameter regimes that are known to lead locking in modeling of the Biot system of poroelasticity [47, 63]. The domain is $\Omega = (0, 1) \times (0, 1)$, with a large fluid-filled cavity in the interior. The boundary of the cavity is defined as a union of curved segments designed to give irregular geometry with sharp corners. The mesh files are available in https://github.com/tongtongli1/MSMFE_cavity-mesh. The body forces, the external source, and the initial conditions are set as follows:

$$\mathbf{f}_f = \mathbf{0}, \quad q_f = 0, \quad \mathbf{f}_p = \mathbf{0}, \quad q_p = 0, \quad p_{p,0} = 1000, \quad \text{and} \quad \boldsymbol{\sigma}_{p,0} = -\alpha_p\, p_{p,0}\, \mathbf{I}.$$

The flow is driven from left to right via a pressure drop of 1 kPa, with boundary conditions specified as follows:

$$\begin{aligned}
&\boldsymbol{\sigma}_f \mathbf{n}_f \cdot \mathbf{n}_f = 1000, \quad \mathbf{u}_f \cdot \mathbf{t}_f = 0 \quad \text{on} \quad \Gamma_{f,right},\\
&p_p = 1001 \quad \text{on} \quad \Gamma_{p,left}, \quad p_p = 1000 \quad \text{on} \quad \Gamma_{p,right},\\
&\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom},\\
&\boldsymbol{\sigma}_p \mathbf{n}_p = -\alpha_p\, p_p\, \mathbf{n}_p \quad \text{on} \quad \Gamma_{p,left} \cup \Gamma_{p,right},\\
&\mathbf{u}_s = \mathbf{0} \quad \text{on} \quad \Gamma_{p,top} \cup \Gamma_{p,bottom}.
\end{aligned}$$

Here $\Gamma_{f,right}$ denotes the section of the boundary of the fluid region that is on the right side of the domain Ω, $\{1\} \times (0, 1)$. The rest of the boundary sections are defined

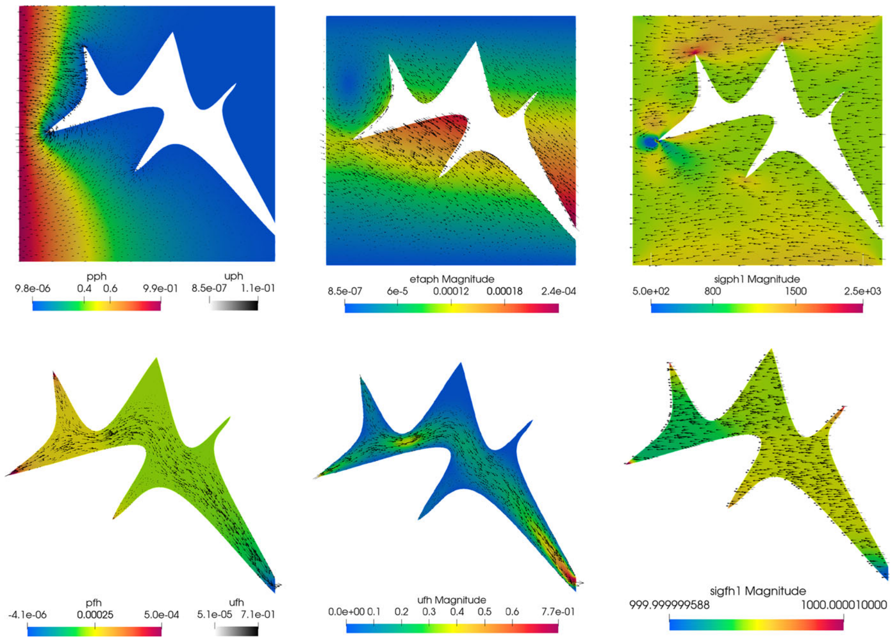


Fig. 3 Example 3, computed solution at $T = 10$ s. Top left: Darcy velocity (arrows) and pressure (color). Top middle: displacement (arrows) and its magnitude (color). Top right: first row of the poroelastic stress tensor (arrows) and its magnitude (color). Bottom left: Stokes velocity (arrows) and pressure (color). Bottom middle: Stokes velocity (arrows) and its magnitude (color). Bottom right: first row of the Stokes stress (arrows) and its magnitude (color)

in a similar manner. The total simulation time is $T = 10$ s with a time step of size $\Delta t = 0.05$ s. To avoid inconsistency between the initial and boundary conditions for p_p, we start with $p_p = 1000$ on $\Gamma_{p,left}$ at $t = 0$ and gradually increase it to reach $p_p = 1001$ at $t = 0.5$ s. Similar adjustment is done for $\boldsymbol{\sigma}_p \mathbf{n}_p$. In this way there exist compatible initial data for all variables, which combine the prescribed initial conditions $p_{p,0} = 1000$ and $\boldsymbol{\sigma}_{p,0} = -\alpha_p\, p_{p,0}\, \mathbf{I}$ with zero data for the rest of the variables. We again remind the reader that only the initial conditions $p_{p,0} = 1000$ and $\boldsymbol{\sigma}_{p,0} = -\alpha_p\, p_{p,0}\, \mathbf{I}$ are used in the scheme.

The simulation results at the final time $T = 10$ s are shown in Fig. 3. In the top plots, we present the Darcy pressure and Darcy velocity vector, the displacement vector with its magnitude, and the first row of the poroelastic stress with its magnitude. Since the pressure variation is small relative to its value, for visualization purpose we plot its difference from the reference pressure, $p_p - 1000$. The Darcy velocity and the pressure drop are largest in the region between the left inflow boundary and the cavity. The displacement is largest around the cavity, due to the large fluid velocity within the cavity and the slip with friction interface condition. The poroelastic stress exhibits singularities near some of the sharp tips of the cavity. The bottom plots show the fluid pressure and velocity vector, the velocity vector with its magnitude, and the first row of the fluid stress with its magnitude. Similarly to the Darcy pressure, we plot $p_f - 1000$. A channel-like flow profile is clearly visible within the cavity, with the largest velocity along a central path away from the cavity walls. The fluid pressure is decreasing from left to right along the central path of the cavity. Consistent with the poroelastic stress, the fluid stress near the tips of the cavity is relatively larger. We emphasize that, despite the locking regime of the parameters, the computed solution is free of locking and spurious oscillations. This example illustrates the ability of our method to handle computationally challenging problems with physically realistic parameters in poroelastic locking regimes.

8 Conclusions

In this paper we present and analyze the first, to the best of our knowledge, fully dual mixed formulation of the quasi-static Stokes-Biot model, and its mixed finite element approximation, using a velocity-pressure Darcy formulation, a weakly symmetric stress-displacement-rotation elasticity formulation, and a weakly symmetric stress-velocity-vorticity Stokes formulation. Essential-type interface conditions are imposed via suitable Lagrange multipliers. The numerical method features accurate stresses and Darcy velocity with local mass and momentum conservation. Furthermore, a new multipoint stress-flux mixed finite element method is developed that allows for local elimination of the Darcy velocity, the fluid and poroelastic stresses, the vorticity, and the rotation, resulting in a reduced positive definite cell-centered pressure-velocities-traces system. The theoretical results are complemented by a series of numerical experiments that illustrate the convergence rates for all variables in their natural norms, as well as the ability of the method to simulate physically realistic problems motivated by applications to coupled surface-subsurface flows and flows in fractured poroelastic media with parameter values in locking regimes. A possible future direction is to consider the extension of the multipoint stress-flux mixed finite element method to higher order, employing techniques developed in [4, 30] for Darcy flow. One could also consider displacement-based discontinuous Galerkin methods, which are known to exhibit anti-locking behavior for the Biot system of poroelasticity [54].

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