Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 394 (2022) 114895
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A three-field Banach spaces-based mixed formulation for the unsteady Brinkman–Forchheimer equations[☆]

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Received 16 September 2021; received in revised form 15 March 2022; accepted 16 March 2022
Available online 6 April 2022

Abstract

We propose and analyze a new mixed formulation for the Brinkman–Forchheimer equations for unsteady flows. Besides the velocity, our approach introduces the velocity gradient and a pseudostress tensor as further unknowns. As a consequence, we obtain a three-field Banach spaces-based mixed variational formulation, where the aforementioned variables are the main unknowns of the system. We establish existence and uniqueness of a solution to the weak formulation, and derive the corresponding stability bounds, employing classical results on nonlinear monotone operators. We then propose a semidiscrete continuous-in-time approximation on simplicial grids based on the Raviart–Thomas elements of degree $k \geq 0$ for the pseudostress tensor and discontinuous piecewise polynomials of degree k for the velocity and the velocity gradient. In addition, by means of the backward Euler time discretization, we introduce a fully discrete finite element scheme. We prove well-posedness and derive the stability bounds for both schemes, and under a quasi-uniformity assumption on the mesh, we establish the corresponding error estimates. We provide several numerical results verifying the theoretical rates of convergence and illustrating the performance and flexibility of the method for a range of domain configurations and model parameters.

MSC: 65N30; 65N12; 65N15; 35Q79; 80A20; 76R05; 76D07

Keywords: Unsteady Brinkman–Forchheimer equations; Mixed finite element methods; Banach spaces

1. Introduction

In this work we study mathematical and computational modeling of fast flows in highly porous media using the unsteady Brinkman–Forchheimer equations. Such flows occur in a wide range of applications, among which

☆ This work has been partially supported by ANID-Chile through the projects ACE210010, Centro de Modelamiento Matematico (FB210005), FONDECYT project 1200666, project PAI77190084 of the PAI Program: Convocatoria Nacional Subvención a la Instalación en la Academia, and BECAS/DOCTORADO NACIONAL 21180900; by Universidad del Bío-Bío through VRIP-UBB project 2120173 GI/C; by the Department of Mathematics, University of Pittsburgh; and by NSF grants DMS 1818775 and DMS 2111129.
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https://doi.org/10.1016/j.cma.2022.114895
0045-7825/

we highlight predicting and controlling processes arising in chemical, petroleum and environmental engineering. Fast flows in the subsurface may occur in fractured or vuggy aquifers or reservoirs, as well as near injection and production wells during groundwater remediation or hydrocarbon production. The widely used Darcy's law is not suitable for flows through media with high porosity or with high Reynolds number. To overcome this limitation, an alternative is to employ the Forchheimer law [1], which accounts for faster flows by including a nonlinear inertial term. We refer the reader to [2–7] for previous works on the numerical solution of the Forchheimer model. Another possible option is the Brinkman model [8], which describes Stokes flows through a set of obstacles, and therefore it can be applied for highly porous media. Depending on its parameters, it can model flows in either the Stokes and Darcy regimes. Various numerical methods for the Brinkman model have been developed that are robust in both limits, see, e.g., [9] and references therein.

The Brinkman–Forchheimer model (see, e.g., [10–14]), which combines the advantages of both models, has been used to model fast flows in highly porous media. Up to the authors' knowledge, one of the first works in analyzing the unsteady Brinkman–Forchheimer equations is [10], where stability of solutions in the L^2-norm is established. This result is extended to the H^1-norm in [11]. In [12], well-posedness for a velocity–pressure variational formulation is established, whereas, a perturbed compressible system that approximates the Brinkman–Forchheimer equations is proposed and analyzed in [13]. There, a fully discrete numerical scheme is developed that combines a semi-implicit Euler scheme with the lowest-order Raviart–Thomas elements for the spatial discretization. In [15], a pressure stabilization method and its finite element approximation are developed and analyzed. The Brinkman–Forchheimer model is coupled with a variable porosity Darcy model and applied for simulating wormhole propagation in [16]. More recently, a mixed pseudostress–velocity formulation is analyzed in [14], where existence and uniqueness of a solution are established for the weak formulation in a Banach space framework. Semidiscrete continuous-in-time and fully discrete mixed finite element approximations are introduced and sub-optimal rates of convergence are established. In turn, in [17], the coupling of the steady Brinkman–Forchheimer and double-diffusion equations is analyzed. There, the velocity gradient, the pseudostress tensor, the temperature and concentration gradients, and a pair of flux vectors are introduced as further unknowns. As a consequence, a Banach space fully mixed variational formulation in each set of equations is obtained. Well-posedness of the solution of the continuous and discrete problems are proved by employing a fixed-point approach combined with classical results on nonlinear monotone operators and Babuška–Brezzi's theory in Banach spaces.

The purpose of the present work is to develop and analyze a new three-field mixed formulation of the unsteady Brinkman–Forchheimer problem and study a suitable conforming numerical discretization. To that end, unlike previous works and motivated by [17,18], we introduce the velocity gradient and a pseudostress tensor as additional unknowns besides the fluid velocity. The pressure is eliminated from the system and can be easily recovered through a simple postprocessing of the pseudostress. There are several advantages of this new approach, including the direct and accurate approximation of additional unknowns of physical interest, which are the velocity gradient and pseudostress tensors. The approximation of the pseudostress tensor in the $\mathbb{H}(\mathbf{div})$ space ensures compatible enforcement of momentum conservation. Moreover, our approach improves the suboptimal theoretical rates of convergence obtained in [14] for the pseudostress–velocity formulation under a quasi-uniformity assumption on the mesh. Compared to classical velocity–pressure formulations, which may not be suitable for both the Stokes and Darcy regimes in the Brinkman equation, our approach is robust in both regimes, which is illustrated in the numerical experiments. Two of the numerical examples also illustrate the capability of the method to resolve sharp velocity gradients in the presence of discontinuous spatially varying parameters in complex geometries.

We establish existence and uniqueness of a solution to the continuous weak formulation by employing techniques from [19,20], and [18], combined with the classical monotone operator theory in a Banach space setting. Stability for the weak solution is established by means of an energy estimate. We further develop semidiscrete continuous-in-time and fully discrete finite element approximations. The pseudostress tensor is approximated by the Raviart–Thomas elements of order $k \geq 0$, whereas, discontinuous piecewise polynomials of degree k are employed to approximate the velocity and the velocity gradient tensor. We make use of the backward Euler method for the discretization in time. Adapting the tools employed for the analysis of the continuous problem, we prove well-posedness of the discrete schemes and derive the corresponding stability estimates. We further perform error analysis for the semidiscrete and fully discrete schemes, establishing rates of convergence in space and time.

The rest of this work is organized as follows. The remainder of this section describes standard notation and functional spaces to be employed throughout the paper. In Section 2, we introduce the model problem and derive its three-field mixed variational formulation. Next, in Section 3 we establish the well-posedness of the weak formulation. The semidiscrete continuous-in-time scheme is introduced and analyzed in Section 4. Error estimates and rates of convergence are also derived. In Section 5, the fully discrete approximation is developed and analyzed. Finally, the performance of the method is illustrated in Section 6 with several numerical examples in 2D and 3D, thus verifying the aforementioned rates of convergence, as well as its flexibility to handle spatially varying parameters in complex geometries.

Preliminaries

Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^\mathrm{p}(\Omega)$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^\mathrm{p}(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\Omega) = \mathrm{L}^\mathrm{p}(\Omega)$. If $\mathrm{p} = 2$, we write $\mathrm{H}^\mathrm{s}(\Omega)$ in place of $\mathrm{W}^{\mathrm{s},2}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^\mathrm{s}(\Omega)}$. By $\mathbf{H}$ and $\mathbb{H}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space H. Moreover, given $T > 0$ and a separable Banach space V endowed with the norm $\|\cdot\|_\mathrm{V}$, we let $\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})$ be the space of classes of functions $f : (0,T) \to \mathrm{V}$ that are Bochner measurable and such that $\|f\|_{\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})} < \infty$, with

$$\|f\|^\mathrm{p}_{\mathrm{L}^\mathrm{p}(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^\mathrm{p}_\mathrm{V}\, dt, \quad \|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_\mathrm{V}.$$

In turn, for any vector field $\mathbf{v} := (v_i)_{i=1,d}$, we set the gradient and divergence operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,d} \quad \text{and} \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^d \frac{\partial v_j}{\partial x_j}.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,d}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,d}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^\mathrm{t} := (\tau_{ji})_{i,j=1,d}, \quad \mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^d \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^d \tau_{ij}\,\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^\mathrm{d} := \boldsymbol{\tau} - \frac{1}{d}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I},$$

where $\mathbf{I}$ is the identity tensor in $\mathrm{R}^{d\times d}$. For simplicity, in what follows we denote

$$(v,w)_\Omega := \int_\Omega v\,w, \quad (\mathbf{v},\mathbf{w})_\Omega := \int_\Omega \mathbf{v}\cdot\mathbf{w}, \quad (\boldsymbol{\tau},\boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}.$$

When no confusion arises, $|\cdot|$ will denote the Euclidean norm in R^d or $\mathrm{R}^{d\times d}$. Additionally, we introduce the Hilbert space

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Omega)\Big\},$$

equipped with the usual norm $\|\boldsymbol{\tau}\|^2_{\mathbb{H}(\mathbf{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{\mathbf{L}^2(\Omega)}$. In addition, in the sequel we will make use of the well-known Young's inequality, for $a, b \geq 0$, $1/\mathrm{p} + 1/\mathrm{q} = 1$, and $\delta > 0$,

$$a\,b \leq \frac{\delta^{\mathrm{p}/2}}{\mathrm{p}}\,a^\mathrm{p} + \frac{1}{\mathrm{q}\,\delta^{\mathrm{q}/2}}\,b^\mathrm{q}. \tag{1.1}$$

Finally, we end this section by mentioning that, throughout the rest of the paper, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. Continuous formulation

2.1. Model problem

In this work we are interested in approximating the solution of the unsteady Brinkman–Forchheimer equations (see for instance [11–15]). More precisely, given the body force term $\mathbf{f}$ and a suitable initial data $\mathbf{u}_0$, the aforementioned system of equations is given by

$$\begin{aligned} & \frac{\partial \mathbf{u}}{\partial t} - \nu\, \Delta \mathbf{u} + \alpha\, \mathbf{u} + \mathtt{F}\, |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} + \nabla p \,=\, \mathbf{f}\,, \quad \mathrm{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega \times (0,T]\,, \\ & \mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T]\,, \quad \mathbf{u}(0) \,=\, \mathbf{u}_0 \quad \text{in} \quad \Omega\,, \quad (p,1)_\Omega \,=\, 0 \quad \text{in} \quad (0,T]\,, \end{aligned} \tag{2.1}$$

where the unknowns are the velocity field $\mathbf{u}$ and the scalar pressure p. In addition, the constant $\nu > 0$ is the Brinkman coefficient, $\alpha > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient and $\mathrm{p} \in [3,4]$ is given.

Now, in order to derive our weak formulation, we first rewrite (2.1) as an equivalent first-order set of equations. To that end, unlike [14] and inspired by [17,18], we introduce the velocity gradient and pseudostress tensors as further unknowns, that is

$$\mathbf{t} := \nabla \mathbf{u}, \quad \boldsymbol{\sigma} := \nu\, \mathbf{t} - p\, \mathbf{I} \quad \text{in} \quad \Omega \times (0,T]\,. \tag{2.2}$$

In this way, applying the trace operator to $\mathbf{t}$ and $\boldsymbol{\sigma}$, and utilizing the incompressibility condition $\mathrm{div}(\mathbf{u}) = 0$ in $\Omega \times (0,T]$, one arrives at $\mathrm{tr}(\mathbf{t}) = 0$ in $\Omega \times (0,T]$ and

$$p \,=\, -\frac{1}{d}\,\mathrm{tr}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega \times (0,T]\,. \tag{2.3}$$

Hence, replacing back (2.3) in the second equation of (2.2), we find that our model problem (2.1) can be rewritten, equivalently, as the set of equations with unknowns $\mathbf{u}, \mathbf{t}$ and $\boldsymbol{\sigma}$, given by

$$\begin{aligned} & \mathbf{t} \,=\, \nabla \mathbf{u}\,, \quad \boldsymbol{\sigma}^{\mathrm{d}} \,=\, \nu\, \mathbf{t}\,, \quad \frac{\partial \mathbf{u}}{\partial t} + \alpha\, \mathbf{u} + \mathtt{F}\, |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \,=\, \mathbf{f} \quad \text{in} \quad \Omega \times (0,T]\,, \\ & \mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma \times (0,T]\,, \quad \mathbf{u}(0) \,=\, \mathbf{u}_0 \quad \text{in} \quad \Omega\,, \quad (\mathrm{tr}(\boldsymbol{\sigma}),1)_\Omega \,=\, 0 \quad \text{in} \quad (0,T]\,. \end{aligned} \tag{2.4}$$

At this point we stress that, as suggested by (2.3), p is eliminated from the formulation (2.4) and computed afterwards in terms of $\boldsymbol{\sigma}$ by using identity (2.3). This fact, justifies the last equation in (2.4), which is equivalent to imposing $(p,1)_\Omega = 0$ in $(0,T]$.

2.2. Variational formulation

In this section we derive our three-field Banach mixed variational formulation for the system (2.4). To that end, we proceed as in [17, Section 2.2] (see also [18,21,22] for similar approaches) and extend the analysis derived there to our current unsteady regime, considering a generalized version of the inertial term $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$, with $\mathrm{p} \in [3,4]$. In fact, multiplying the first, second and third equations of (2.4) by suitable test functions $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{v}$, respectively, integrating by parts and using the Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\Gamma \times (0,T]$, we get

$$(\mathbf{t}, \boldsymbol{\tau})_\Omega + (\mathbf{u}, \mathbf{div}(\boldsymbol{\tau}))_\Omega = 0\,, \tag{2.5}$$

$$\nu\,(\mathbf{t}, \mathbf{r})_\Omega - (\boldsymbol{\sigma}^{\mathrm{d}}, \mathbf{r})_\Omega = 0\,, \tag{2.6}$$

$$(\partial_t\, \mathbf{u}, \mathbf{v})_\Omega + \alpha\,(\mathbf{u}, \mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}, \mathbf{v})_\Omega - (\mathbf{div}(\boldsymbol{\sigma}), \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega\,, \tag{2.7}$$

for all $(\boldsymbol{\tau}, \mathbf{r}, \mathbf{v})$ in $\mathbb{X} \times \mathbb{Q} \times \mathbf{M}$, where $\mathbb{X}$, $\mathbb{Q}$ and $\mathbf{M}$ are spaces to be defined below.

We begin by noting that the first term in (2.6) is well defined for $\mathbf{t}, \mathbf{r} \in \mathbb{L}^2(\Omega)$, but due to the incompressibility condition $\mathrm{div}(\mathbf{u}) = \mathrm{tr}(\mathbf{t}) = 0$, it makes sense to look for $\mathbf{t}$, and consequently the test function $\mathbf{r}$, in

$$\mathbb{Q} := \Big\{ \mathbf{r} \in \mathbb{L}^2(\Omega): \quad \mathrm{tr}(\mathbf{r}) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{2.8}$$

This implies that (2.6) can be rewritten equivalently as

$$\nu\,(\mathbf{t}, \mathbf{r})_\Omega - (\boldsymbol{\sigma}, \mathbf{r})_\Omega \,=\, 0 \quad \forall\, \mathbf{r} \in \mathbb{Q}\,. \tag{2.9}$$

In addition, we note that the first and second terms in (2.5) and (2.6) (or (2.9)), respectively, are well defined if $\boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$. In turn, if $\mathbf{u}, \mathbf{v} \in \mathbf{L}^{\mathrm{p}}(\Omega)$, with $\mathrm{p} \in [3,4]$, then the first, second, and third terms in (2.7) are clearly well defined, which forces both $\mathbf{div}(\boldsymbol{\sigma})$ and $\mathbf{div}(\boldsymbol{\tau})$ to live in $\mathbf{L}^{\mathrm{q}}(\Omega)$, with $\mathrm{q} \in [4/3, 3/2]$ satisfying $1/\mathrm{p} + 1/\mathrm{q} = 1$. According to this, we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_{\mathrm{q}};\Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^{\mathrm{q}}(\Omega) \Big\},$$

equipped with the norm

$$\|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div}_q;\Omega)} := \|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{L}^q(\Omega)},$$

and deduce that Eqs. (2.5)–(2.7) and (2.9) are well defined if we choose the spaces $\mathbb{Q}$ as in (2.8) and

$$\mathbf{M} := \mathbf{L}^p(\Omega) \quad \text{and} \quad \mathbb{X} := \mathbb{H}(\mathbf{div}_q;\Omega)$$

with their respective norms: $\|\cdot\|_{\mathbb{Q}} := \|\cdot\|_{\mathbb{L}^2(\Omega)}$, $\|\cdot\|_{\mathbf{M}} := \|\cdot\|_{\mathbf{L}^p(\Omega)}$, and $\|\cdot\|_{\mathbb{X}} := \|\cdot\|_{\mathbb{H}(\mathbf{div}_q;\Omega)}$.

Now, for convenience of the subsequent analysis and similarly as in [22] (see also [17,18,23]) we consider the decomposition:

$$\mathbb{X} = \mathbb{X}_0 \oplus \mathrm{R}\,\mathbf{I},$$

where

$$\mathbb{X}_0 := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_q;\Omega) : \quad (\mathrm{tr}(\boldsymbol{\tau}),1)_\Omega = 0\right\};$$

that is, $\mathrm{R}\,\mathbf{I}$ is a topological supplement for $\mathbb{X}_0$. More precisely, each $\boldsymbol{\tau} \in \mathbb{X}$ can be decomposed uniquely as:

$$\boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\,\mathbf{I} \quad \text{with} \quad \boldsymbol{\tau}_0 \in \mathbb{X}_0 \quad \text{and} \quad c := \frac{1}{d\,|\Omega|}(\mathrm{tr}(\boldsymbol{\tau}),1)_\Omega \in \mathrm{R}.$$

Then, noticing that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{div}(\boldsymbol{\tau}_0)$ and employing the last equation of (2.4), we deduce that both $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can be considered hereafter in $\mathbb{X}_0$. Next, in order to write the above formulation in a more suitable way for the analysis to be developed below, we now set the notations

$$\underline{\mathbf{u}} := (\mathbf{u},\mathbf{t}), \quad \underline{\mathbf{v}} := (\mathbf{v},\mathbf{r}) \in \mathbf{M}\times\mathbb{Q},$$

with corresponding norm given by

$$\|\underline{\mathbf{v}}\| := \|\mathbf{v}\|_{\mathbf{M}} + \|\mathbf{r}\|_{\mathbb{Q}} \quad \forall\,\underline{\mathbf{v}} \in \mathbf{M}\times\mathbb{Q}.$$

Hence, the weak formulation associated with the unsteady Brinkman–Forchheimer system (2.4) reads: Given $\mathbf{f} : [0,T] \to \mathbf{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{M}$, find $(\underline{\mathbf{u}},\boldsymbol{\sigma}) : [0,T] \to (\mathbf{M}\times\mathbb{Q})\times\mathbb{X}_0$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for a.e. $t \in (0,T)$,

$$\begin{aligned} \frac{\partial}{\partial t}[\mathcal{E}(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] + [\mathcal{A}(\underline{\mathbf{u}}(t)),\underline{\mathbf{v}}] + [\mathcal{B}'(\boldsymbol{\sigma}(t)),\underline{\mathbf{v}}] &= [F(t),\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}} \in \mathbf{M}\times\mathbb{Q},\\ -[\mathcal{B}(\underline{\mathbf{u}}(t)),\boldsymbol{\tau}] &= 0 \quad \forall\,\boldsymbol{\tau} \in \mathbb{X}_0, \end{aligned} \tag{2.10}$$

where, the operators $\mathcal{E},\mathcal{A} : (\mathbf{M}\times\mathbb{Q}) \to (\mathbf{M}\times\mathbb{Q})'$, and $\mathcal{B} : (\mathbf{M}\times\mathbb{Q}) \to \mathbb{X}_0'$ are defined, respectively, as

$$[\mathcal{E}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := (\mathbf{u},\mathbf{v})_\Omega, \tag{2.11}$$

$$[\mathcal{A}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] := \alpha\,(\mathbf{u},\mathbf{v})_\Omega + \mathtt{F}\,(|\mathbf{u}|^{p-2}\mathbf{u},\mathbf{v})_\Omega + \nu\,(\mathbf{t},\mathbf{r})_\Omega, \tag{2.12}$$

$$[\mathcal{B}(\underline{\mathbf{v}}),\boldsymbol{\tau}] := -(\mathbf{v},\mathbf{div}(\boldsymbol{\tau}))_\Omega - (\mathbf{r},\boldsymbol{\tau})_\Omega, \tag{2.13}$$

and F is the bounded linear functional given by

$$[F,\underline{\mathbf{v}}] := (\mathbf{f},\mathbf{v})_\Omega. \tag{2.14}$$

In all the terms above, $[\cdot,\cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathcal{B}' : \mathbb{X}_0 \to \big(\mathbf{M}\times\mathbb{Q}\big)'$ be the adjoint of $\mathcal{B}$, which satisfies $[\mathcal{B}'(\boldsymbol{\tau}),\underline{\mathbf{v}}] = [\mathcal{B}(\underline{\mathbf{v}}),\boldsymbol{\tau}]$ for all $\underline{\mathbf{v}} \in \mathbf{M}\times\mathbb{Q}$ and $\boldsymbol{\tau} \in \mathbb{X}_0$.

3. Well-posedness of the model

In this section we establish the solvability of (2.10). To that end we first collect some previous results that will be used in the forthcoming analysis.

3.1. Preliminary results

We begin by recalling the key result [19, Theorem IV.6.1(b)], which will be used to establish the existence of a solution to (2.10).

Theorem 3.1. *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given for the real vector space E to its algebraic dual E^*, and let E_b' be the Hilbert space which is the dual of E with the seminorm*

$$|x|_b = \big(\mathcal{N}x(x)\big)^{1/2} \quad x \in E.$$

Let $\mathcal{M} \subset E \times E_b'$ be a relation with domain $\mathcal{D} = \Big\{x \in E \,:\, \mathcal{M}(x) \neq \emptyset\Big\}$.

Assume $\mathcal{M}$ is monotone and $Rg(\mathcal{N}+\mathcal{M}) = E_b'$. Then, for each $f \in \mathrm{W}^{1,1}(0,T;E_b')$ and for each $u_0 \in \mathcal{D}$, there is a solution u of

$$\frac{d}{dt}\big(\mathcal{N}u(t)\big) + \mathcal{M}\big(u(t)\big) \ni f(t) \quad a.e. \quad 0 < t < T, \tag{3.1}$$

with

$$\mathcal{N}u \in \mathrm{W}^{1,\infty}(0,T;E_b'), \quad u(t) \in \mathcal{D}, \quad \text{for all } 0 \le t \le T, \quad \text{and} \quad \mathcal{N}u(0) = \mathcal{N}u_0.$$

Note that E_b' must be understood as the dual space of $(E, |\cdot|_b)$.

In addition, in order to provide the range condition in Theorem 3.1 we will require the following abstract result (see [20, Theorem 3.1] for details).

Theorem 3.2. *Let X_1, X_2 and Y be separable and reflexive Banach spaces, X_1 and X_2 being uniformly convex, and set $X = X_1 \times X_2$. Let $a : X \to X'$ be a nonlinear operator, $b : \mathcal{L}(X, Y')$, and let V be the kernel of b, that is,*

$$V := \Big\{v \in X : \quad [b(v), q] = 0 \quad \forall\, q \in Y\Big\}.$$

Assume that

(i) *a is hemi-continuous, that is, for each $u, v \in X$ the real mapping*

$$J : \mathrm{R} \to \mathrm{R}, \quad t \to J(t) = [a(u + tv), v]$$

is continuous;

(ii) *there exist constants $L > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \ge 2$, such that*

$$\|a(u) - a(v)\|_{X'} \,\le\, L \sum_{j=1}^{2} \Big\{ \|u_j - v_j\|_{X_j} + \big(\|u_j\|_{X_j} + \|v_j\|_{X_j}\big)^{\mathrm{p}_j - 2} \|u_j - v_j\|_{X_j} \Big\}, \tag{3.2}$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X$;

(iii) *the family of operators $\Big\{a(\cdot + t) : V \to \mathrm{V}' \;:\; t \in X\Big\}$ is uniformly strictly monotone, that is there exist $\gamma > 0$ and $\mathrm{p}_1, \mathrm{p}_2 \ge 2$, such that*

$$[a(u + t) - a(v + t), u - v] \,\ge\, \gamma \Big\{ \|u_1 - v_1\|_{X_1}^{\mathrm{p}_1} + \|u_2 - v_2\|_{X_2}^{\mathrm{p}_2} \Big\},$$

for all $t \in X$, and for all $u = (u_1, u_2), v = (v_1, v_2) \in V$;

(iv) *there exist $\beta > 0$ such that*

$$\sup_{\substack{v \in X \\ v \neq 0}} \frac{[b(v), q]}{\|v\|_X} \,\ge\, \beta\, \|q\|_Y \quad \forall\, q \in Y.$$

Then, for each $(f, g) \in X' \times Y'$ there exists a unique $(u, p) \in X \times Y$ such that

$$\begin{aligned} [a(u), v] + [b(v), p] &= [f, v] \quad \forall\, v \in X\,, \\ [b(u), q] &= [g, q] \quad \forall\, q \in Y\,. \end{aligned}$$

Next, we establish the stability properties of the operators involved in (2.10). We begin by observing that the operators $\mathcal{E}, \mathcal{B}$ and the functional F are linear. In turn, from (2.11), (2.13) and (2.14), and employing Hölder and Cauchy–Schwarz inequalities, there hold

$$\big|[\mathcal{B}(\underline{\mathbf{v}}), \boldsymbol{\tau}]\big| \leq \|\underline{\mathbf{v}}\| \, \|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall\, (\underline{\mathbf{v}}, \boldsymbol{\tau}) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0 \,, \tag{3.3}$$

$$\big|[F, \underline{\mathbf{v}}]\big| \leq \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \leq |\Omega|^{(\mathrm{p}-2)/(2\,\mathrm{p})} \|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} \|\underline{\mathbf{v}}\| \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \,, \tag{3.4}$$

and

$$\big|[\mathcal{E}(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \leq |\Omega|^{(\mathrm{p}-2)/\mathrm{p}} \|\underline{\mathbf{u}}\| \, \|\underline{\mathbf{v}}\|, \quad [\mathcal{E}(\underline{\mathbf{v}}), \underline{\mathbf{v}}] = \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} \quad \forall\, \underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \,, \tag{3.5}$$

which implies that $\mathcal{B}$ and F are bounded and continuous, and $\mathcal{E}$ is bounded, continuous, and monotone. In addition, employing the Cauchy–Schwarz and Hölder inequalities, it is readily seen that the nonlinear operator $\mathcal{A}$ (cf. (2.12)) is bounded, that is

$$\big|[\mathcal{A}(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \leq \Big(\alpha\, |\Omega|^{(\mathrm{p}-2)/\mathrm{p}} \, \|\mathbf{u}\|_{\mathbf{M}} + \mathrm{F}\, \|\mathbf{u}\|^{\mathrm{p}-1}_{\mathbf{M}} + \nu\, \|\mathbf{t}\|_{\mathbb{Q}}\Big) \|\underline{\mathbf{v}}\| \,. \tag{3.6}$$

Finally, recalling the definition of the operators $\mathcal{E}, \mathcal{A}$, and $\mathcal{B}$ (cf. (2.11)–(2.13)), we stress that problem (2.10) can be written in the form of (3.1) with

$$E := \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0 \,, \quad u := \begin{pmatrix} \underline{\mathbf{u}} \\ \boldsymbol{\sigma} \end{pmatrix}, \quad \mathcal{N} := \begin{pmatrix} \mathcal{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \mathcal{M} := \begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix}. \tag{3.7}$$

Let $\mathbf{E}'_2$ be the Hilbert space that is the dual of $\mathbf{M} \times \mathbb{Q}$ with the seminorm induced by the operator $\mathcal{E} = \left(\begin{smallmatrix} I & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right)$ (cf. (2.11)), which is $|\underline{\mathbf{v}}|_{\mathcal{E}} = (\mathbf{v}, \mathbf{v})^{1/2}_{\Omega} = \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$ $\forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}$. Note that $\mathbf{E}'_2 = \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$. Then we define the spaces

$$E'_b := \big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\}\big) \times \{\mathbf{0}\}, \quad \mathcal{D} := \Big\{(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0 : \quad \mathcal{M}(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \in E'_b\Big\} \,. \tag{3.8}$$

In the next section we prove the hypotheses of Theorem 3.1 to establish the well-posedness of (2.10).

3.2. Range condition and initial data

We begin with the verification of the range condition in Theorem 3.1. Let us consider the resolvent system associated with (2.10): Find $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \in (\mathbf{M} \times \mathbb{Q}) \times \mathbb{X}_0$ such that

$$\begin{aligned} [(\mathcal{E} + \mathcal{A})(\underline{\mathbf{u}}), \underline{\mathbf{v}}] + [\mathcal{B}'(\boldsymbol{\sigma}), \underline{\mathbf{v}}] &= [\widehat{F}, \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \,, \\ [\mathcal{B}(\underline{\mathbf{u}}), \boldsymbol{\tau}] &= 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0 \,, \end{aligned} \tag{3.9}$$

where $\widehat{F} \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\} \subset \mathbf{M}' \times \{\mathbf{0}\}$ is a functional given by $\widehat{F}(\underline{\mathbf{v}}) := (\widehat{\mathbf{f}}, \mathbf{v})_{\Omega}$ for some $\widehat{\mathbf{f}} \in \mathbf{L}^2(\Omega)$. Next, a unique solution to (3.9) is established by employing Theorem 3.2. We stress that alternatively to Theorem 3.2, similar arguments developed in [17, Section 3.3] can be employed to establish the well-posedness of (3.9). We begin by observing that, thanks to the uniform convexity and separability of $\mathrm{L}^{\mathrm{p}}(\Omega)$ for $\mathrm{p} \in (1, +\infty)$, the spaces $\mathbf{M}, \mathbb{Q}$, and $\mathbb{X}_0$ are uniformly convex and separable as well.

We continue our analysis by proving that the nonlinear operator $\mathcal{E} + \mathcal{A}$ satisfies hypothesis (ii) of Theorem 3.2 with $\mathrm{p}_1 = \mathrm{p} \in [3, 4]$ and $\mathrm{p}_2 = 2$.

Lemma 3.3. *Let* $\mathrm{p} \in [3, 4]$. *Then, there exists* $L_{\mathrm{BF}} > 0$, *depending on* ν, F, *and* α, *such that*

$$\|(\mathcal{E} + \mathcal{A})(\underline{\mathbf{u}}) - (\mathcal{E} + \mathcal{A})(\underline{\mathbf{v}})\| \leq L_{\mathrm{BF}} \Big\{\|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}} + \|\mathbf{t} - \mathbf{r}\|_{\mathbb{Q}} + \big(\|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{v}\|_{\mathbf{M}}\big)^{\mathrm{p}-2} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}}\Big\} \,, \tag{3.10}$$

for all $\underline{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \underline{\mathbf{v}} = (\mathbf{v}, \mathbf{r}) \in \mathbf{M} \times \mathbb{Q}$.

Proof. Let $\underline{\mathbf{u}} = (\mathbf{u}, \mathbf{t}), \underline{\mathbf{v}} = (\mathbf{v}, \mathbf{r}) \in \mathbf{M} \times \mathbb{Q}$. Then, according to the definition of the operators $\mathcal{E}, \mathcal{A}$ (cf. (2.11), (2.12)), similarly to the boundedness estimates (3.5) and (3.6), using Hölder's and Cauchy–Schwarz inequalities, we find that

$$\begin{aligned} &\|(\mathcal{E} + \mathcal{A})(\underline{\mathbf{u}}) - (\mathcal{E} + \mathcal{A})(\underline{\mathbf{v}})\| \\ &\quad \leq (1 + \alpha)|\Omega|^{(\mathrm{p}-2)/\mathrm{p}} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}} + \mathrm{F}\, \|\, |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{\mathbf{M}'} + \nu\, \|\mathbf{t} - \mathbf{r}\|_{\mathbb{Q}} \,. \end{aligned} \tag{3.11}$$

In turn, applying [24, Lemma 2.1, eq. (2.1a)] to bound the second term on the right hand side of (3.11), we deduce that there exists $c_{\mathrm{p}} > 0$, depending only on $|\Omega|$ and p such that

$$\| \, |\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{v}|^{\mathrm{p}-2}\mathbf{v}\|_{\mathbf{M}'} \,\leq\, c_{\mathrm{p}} \left(\|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{v}\|_{\mathbf{M}}\right)^{\mathrm{p}-2} \|\mathbf{u}-\mathbf{v}\|_{\mathbf{M}} \,. \tag{3.12}$$

Thus, using (3.12) and (3.11), we obtain (3.10) with $L_{\mathrm{BF}} = \max\left\{(1+\alpha)|\Omega|^{(\mathrm{p}-2)/\mathrm{p}}, \mathrm{F}\, c_{\mathrm{p}}, \nu\right\}$, which completes the proof. □

Next, the following lemma shows that the operator $\mathcal{E} + \mathcal{A}$ satisfies hypothesis (iii) of Theorem 3.2 with $\mathrm{p}_1 = \mathrm{p} \in [3,4]$ and $\mathrm{p}_2 = 2$.

Lemma 3.4. *Let* $\mathrm{p} \in [3,4]$. *The family of operators* $\left\{(\mathcal{E}+\mathcal{A})(\cdot + \underline{\mathbf{z}}) : \mathbf{M}\times\mathbb{Q} \to (\mathbf{M}\times\mathbb{Q})' : \quad \underline{\mathbf{z}} \in \mathbf{M}\times\mathbb{Q}\right\}$ *is uniformly strictly monotone, that is, there exists* $\gamma_{\mathrm{BF}} > 0$*, such that*

$$\left[(\mathcal{E}+\mathcal{A})(\underline{\mathbf{u}}+\underline{\mathbf{z}}) - (\mathcal{E}+\mathcal{A})(\underline{\mathbf{v}}+\underline{\mathbf{z}}), \underline{\mathbf{u}}-\underline{\mathbf{v}}\right] \,\geq\, \gamma_{\mathrm{BF}} \left\{\|\mathbf{u}-\mathbf{v}\|^{\mathrm{p}}_{\mathbf{M}} + \|\mathbf{t}-\mathbf{r}\|^2_{\mathbb{Q}}\right\}, \tag{3.13}$$

for all $\underline{\mathbf{z}} = (\mathbf{z},\mathbf{s}) \in \mathbf{M}\times\mathbb{Q}$*, and for all* $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{t}), \underline{\mathbf{v}} = (\mathbf{v},\mathbf{r}) \in \mathbf{M}\times\mathbb{Q}$.

Proof. Let $\underline{\mathbf{z}} = (\mathbf{z},\mathbf{s}) \in \mathbf{M}\times\mathbb{Q}$ and $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{t}), \underline{\mathbf{v}} = (\mathbf{v},\mathbf{r}) \in \mathbf{M}\times\mathbb{Q}$. Then, from the definition of the operators $\mathcal{E}, \mathcal{A}$ (cf. (2.11), (2.12)), we get

$$\begin{aligned}&\left[(\mathcal{E}+\mathcal{A})(\underline{\mathbf{u}}+\underline{\mathbf{z}}) - (\mathcal{E}+\mathcal{A})(\underline{\mathbf{v}}+\underline{\mathbf{z}}), \underline{\mathbf{u}}-\underline{\mathbf{v}}\right] \\ &\quad= (1+\alpha)\|\mathbf{u}-\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \mathrm{F}\,(|\mathbf{u}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{v}+\mathbf{z}), \mathbf{u}-\mathbf{v})_\Omega + \nu\,\|\mathbf{t}-\mathbf{r}\|^2_{\mathbb{Q}}\,,\end{aligned} \tag{3.14}$$

where, employing [24, Lemma 2.1, eq. (2.1b)] to bound the second term in (3.14), we deduce that there exists $C_{\mathrm{p}} > 0$ depending only on $|\Omega|$ and p such that

$$(|\mathbf{u}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{u}+\mathbf{z}) - |\mathbf{v}+\mathbf{z}|^{\mathrm{p}-2}(\mathbf{v}+\mathbf{z}), \mathbf{u}-\mathbf{v})_\Omega \,\geq\, C_{\mathrm{p}}\,\|\mathbf{u}-\mathbf{v}\|^{\mathrm{p}}_{\mathbf{M}}\,. \tag{3.15}$$

Thus, replacing (3.15) back into (3.14), and bounding below the first term on the right-hand side of (3.14) by 0, we obtain

$$\left[(\mathcal{E}+\mathcal{A})(\underline{\mathbf{u}}+\underline{\mathbf{z}}) - (\mathcal{E}+\mathcal{A})(\underline{\mathbf{v}}+\underline{\mathbf{z}}), \underline{\mathbf{u}}-\underline{\mathbf{v}}\right] \,\geq\, C_{\mathrm{p}}\,\mathrm{F}\,\|\mathbf{u}-\mathbf{v}\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\mathbf{t}-\mathbf{r}\|^2_{\mathbb{Q}},$$

which gives (3.13) with $\gamma_{\mathrm{BF}} = \min\left\{C_{\mathrm{p}}\,\mathrm{F}, \nu\right\}$. □

Remark 3.1. We observe that, using similar arguments to [18, eq. (3.30)], the kernel of the operator $\mathcal{B}$ (cf. (2.13)) can be written as

$$\mathbf{V} = \left\{\underline{\mathbf{v}} = (\mathbf{v},\mathbf{r}) \in \mathbf{M}\times\mathbb{Q} : \quad \nabla\mathbf{v} = \mathbf{r} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega)\right\}. \tag{3.16}$$

In turn, since the strict monotonicity bound (3.13) holds on $\mathbf{M}\times\mathbb{Q}$, it is clear that it also holds on $\mathbf{V}$. Notice also that, alternatively to Lemma 3.4, and similarly to [17, Lemma 3.5], it is possible to prove that the family of operators $\left\{(\mathcal{E}+\mathcal{A})(\cdot + \underline{\mathbf{z}}) : \mathbf{V} \to \mathbf{V}' : \quad \underline{\mathbf{z}} \in \mathbf{M}\times\mathbb{Q}\right\}$ is uniformly strongly monotone, that is, there exists $\widetilde{\gamma}_{\mathrm{BF}} > 0$, such that

$$\left[(\mathcal{E}+\mathcal{A})(\underline{\mathbf{u}}+\underline{\mathbf{z}}) - (\mathcal{E}+\mathcal{A})(\underline{\mathbf{v}}+\underline{\mathbf{z}}), \underline{\mathbf{u}}-\underline{\mathbf{v}}\right] \,\geq\, \widetilde{\gamma}_{\mathrm{BF}}\,\|\underline{\mathbf{u}}-\underline{\mathbf{v}}\|^2,$$

for all $\underline{\mathbf{z}} = (\mathbf{z},\mathbf{s}) \in \mathbf{M}\times\mathbb{Q}$, and for all $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{t}), \underline{\mathbf{v}} = (\mathbf{v},\mathbf{r}) \in \mathbf{V}$.

We end the verification of the hypotheses of Theorem 3.2, with the corresponding inf–sup condition for the operator $\mathcal{B}$.

Lemma 3.5. *There exists a constant* $\beta > 0$ *such that*

$$\sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq\mathbf{0}}} \frac{[\mathcal{B}(\underline{\mathbf{v}}), \boldsymbol{\tau}]}{\|\underline{\mathbf{v}}\|} \,\geq\, \beta\,\|\boldsymbol{\tau}\|_{\mathbb{X}} \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0\,. \tag{3.17}$$

Proof. For the case $\mathrm{p} = 4$ and $\mathrm{q} = 4/3$ we refer the reader to [18, eq. (3.44), Lemma 3.3], whose proof can be easily extended to the case $\mathrm{p} \in [3, 4]$ and $\mathrm{q} \in [4/3, 3/2]$ satisfying $1/\mathrm{p} + 1/\mathrm{q} = 1$. Further details are omitted. □

Now, we are in a position of establishing the solvability of the resolvent system (3.9).

Lemma 3.6. *Given $\widehat{F} = (\widehat{\mathbf{f}}, \mathbf{0}) \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$, there exists a unique solution $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in (\mathbf{M} \times \mathbb{Q}) \times \mathbb{X}_0$ of the resolvent system (3.9).*

Proof. First, we recall from (3.3) and (3.4) that $\mathcal{B}$ and $\widehat{F}$ are linear and bounded. In turn, we note that Lemma 3.3 implies, in particular, that the nonlinear operator $\mathcal{E} + \mathcal{A}$ is hemi-continuous, that is, for each $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}$, the mapping

$$J : \mathrm{R} \to \mathrm{R}, \quad z \mapsto J(z) := [(\mathcal{E} + \mathcal{A})(\underline{\mathbf{u}} + z\, \underline{\mathbf{v}}), \underline{\mathbf{v}}]$$

is continuous. In this way, as a consequence of Lemmas 3.3, 3.4, and 3.5, and a straightforward application of Theorem 3.2, we conclude the result. □

We end this section by establishing a suitable initial condition result, which is necessary to apply Theorem 3.1 to our context.

Lemma 3.7. *Assume that the initial condition $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$, where*

$$\mathbf{H} := \Big\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \quad \Delta \mathbf{v} \in \mathbf{L}^2(\Omega) \quad \textit{and} \quad \operatorname{div}(\mathbf{v}) = 0 \quad \textit{in} \quad \Omega \Big\}. \tag{3.18}$$

Then, there exists $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbb{Q} \times \mathbb{X}_0$ such that $\underline{\mathbf{u}}_0 := (\mathbf{u}_0, \mathbf{t}_0)$ and $\boldsymbol{\sigma}_0$ satisfy

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} \in \Big(\mathbf{L}^2(\Omega) \times \{\mathbf{0}\} \Big) \times \{\mathbf{0}\}. \tag{3.19}$$

Proof. We proceed similarly to the proof of [14, Lemma 3.6]. Given $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$, we can define $\mathbf{t}_0 := \nabla \mathbf{u}_0$ and $\boldsymbol{\sigma}_0 := \nu\, \mathbf{t}_0$, which satisfy

$$\operatorname{tr}(\mathbf{t}_0) = 0, \quad \mathbf{div}(\boldsymbol{\sigma}_0) = \nu\, \Delta \mathbf{u}_0, \quad \text{and} \quad \operatorname{tr}(\boldsymbol{\sigma}_0) = 0 \quad \text{in} \quad \Omega. \tag{3.20}$$

Notice that $\mathbf{t}_0 \in \mathbb{Q}$ and $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}; \Omega) \subset \mathbb{X}_0$, with $\mathbb{H}_0(\mathbf{div}; \Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega) : (\operatorname{tr}(\boldsymbol{\tau}), 1)_\Omega = 0 \Big\}$. Next, integrating by parts the identity $\mathbf{t}_0 = \nabla \mathbf{u}_0$ and proceeding similarly to (2.5), we obtain

$$-[\mathcal{B}(\underline{\mathbf{u}}_0), \boldsymbol{\tau}] = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0.$$

Hence, given $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$ (cf. (3.18)), multiplying the identity $\nu\, \mathbf{t}_0 = \boldsymbol{\sigma}_0$ and the second equation in (3.20) by $\mathbf{r} \in \mathbb{Q}$ and $\mathbf{v} \in \mathbf{M}$, respectively, and after minor algebraic manipulation we deduce that

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \underline{\mathbf{u}}_0 \\ \boldsymbol{\sigma}_0 \end{pmatrix} = \begin{pmatrix} F_0 \\ \mathbf{0} \end{pmatrix}, \tag{3.21}$$

where, $F_0 = (\mathbf{f}_0, \mathbf{0})$ and

$$(\mathbf{f}_0, \mathbf{v})_\Omega := (-\nu\, \Delta \mathbf{u}_0 + \alpha\, \mathbf{u}_0 + \mathrm{F}\, |\mathbf{u}_0|^{\mathrm{p}-2} \mathbf{u}_0, \mathbf{v})_\Omega.$$

Using the additional regularity of $\mathbf{u}_0$ and the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^{2(\mathrm{p}-1)}(\Omega)$, with $\mathrm{p} \in [3, 4]$, we obtain

$$\begin{aligned} \big|(\mathbf{f}_0, \mathbf{v})_\Omega\big| &\le \Big\{ \nu\, \|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \alpha\, \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \mathrm{F}\, \|\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{L}^{2(\mathrm{p}-1)}(\Omega)} \Big\} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \\ &\le C \Big\{ \|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{H}^1(\Omega)} \Big\} \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{3.22}$$

Thus, $F_0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$ so then (3.19) holds, completing the proof. □

Remark 3.2. The assumption on the initial condition $\mathbf{u}_0$ in (3.18) is not necessary for all the results that follow but we shall assume it from now on for simplicity. A similar assumption to $\mathbf{u}_0$ is also made in [14, Lemma 3.6] (see also [12, eq. (2.2)]). Note also that $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0)$ satisfying (3.19) is not unique.

3.3. Main result

We now establish the well-posedness of problem (2.10).

Theorem 3.8. *For each compatible initial data* $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$ *constructed in Lemma 3.7 and each* $\mathbf{f} \in \mathrm{W}^{1,1}(0, T; \mathbf{L}^2(\Omega))$*, there exists a unique* $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, T] \to \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$ *solution to (2.10), such that* $\mathbf{u} \in \mathrm{W}^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$ *and* $(\mathbf{u}(0), \mathbf{t}(0), \boldsymbol{\sigma}^{\mathrm{d}}(0)) = (\mathbf{u}_0, \mathbf{t}_0, \boldsymbol{\sigma}_0^{\mathrm{d}})$.

Proof. We recall that (2.10) fits in the framework of Theorem 3.1 with the definitions (3.7) and (3.8). Note that $\mathcal{N}$ is linear, symmetric and monotone since $\mathcal{E}$ is (cf. (3.5)). In addition, since $\mathcal{A}$ is strictly monotone, it is not difficult to see that $\mathcal{M}$ is monotone. On the other hand, from Lemma 3.6 we know that given $(\widehat{F}, \mathbf{0}) \in E_b'$ with $\widehat{F} = (\widehat{\mathbf{f}}, \mathbf{0})$, there is a unique $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$, such that $(\widehat{F}, \mathbf{0}) = (\mathcal{N} + \mathcal{M})(\underline{\mathbf{u}}, \boldsymbol{\sigma})$ which implies $Rg(\mathcal{N} + \mathcal{M}) = E_b'$. Finally, considering $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$ (cf. (3.18)), from a straightforward application of Lemma 3.7 we are able to find $(\mathbf{t}_0, \boldsymbol{\sigma}_0) \in \mathbb{Q} \times \mathbb{X}_0$ such that $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0) \in \mathcal{D}$. Therefore, applying Theorem 3.1 to our context, we conclude the existence of a solution $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma})$ to (2.10), with $\mathbf{u} \in \mathrm{W}^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$ and $\mathbf{u}(0) = \mathbf{u}_0$.

We next show that the solution of (2.10) is unique. To that end, let $(\underline{\mathbf{u}}_i, \boldsymbol{\sigma}_i)$, with $i \in \{1, 2\}$, be two solutions corresponding to the same data. Then, taking (2.10) with $(\underline{\mathbf{v}}, \boldsymbol{\tau}) = (\underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2, \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \in \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$, we deduce that

$$\frac{1}{2}\,\partial_t \,\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + [\mathcal{A}(\underline{\mathbf{u}}_1) - \mathcal{A}(\underline{\mathbf{u}}_2), \underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2] \,=\, 0,$$

which together with the strict monotonicity bound of $\mathcal{A}$ (cf. (3.13)), yields

$$\frac{1}{2}\,\partial_t \,\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + C_{\mathrm{p}}\,\mathrm{F}\,\|\mathbf{u}_1 - \mathbf{u}_2\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\mathbf{t}_1 - \mathbf{t}_2\|^2_{\mathbb{Q}} \,\leq\, 0.$$

Integrating in time from 0 to $t \in (0, T]$, and using $\mathbf{u}_1(0) = \mathbf{u}_2(0)$, we obtain

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \Big(\|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_1 - \mathbf{u}_2\|^{\mathrm{p}}_{\mathbf{M}} + \|\mathbf{t}_1 - \mathbf{t}_2\|^2_{\mathbb{Q}}\Big)\,ds \,\leq\, 0\,. \tag{3.23}$$

Therefore, it follows from (3.23) that $\mathbf{u}_1(t) = \mathbf{u}_2(t)$ and $\mathbf{t}_1(t) = \mathbf{t}_2(t)$ for all $t \in (0, T]$. Next, from the inf–sup condition of the operator $\mathcal{B}$ (cf. (3.17)) and the first equation of (2.10), we get

$$\begin{aligned}\beta\,\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{\mathbb{X}} \,&\leq\, \sup_{\substack{\underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \\ \underline{\mathbf{v}} \neq \mathbf{0}}} \frac{[\mathcal{B}'(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|} \\ &=\, -\sup_{\substack{\underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q} \\ \underline{\mathbf{v}} \neq \mathbf{0}}} \frac{[\partial_t\,\mathcal{E}(\underline{\mathbf{u}}_1 - \underline{\mathbf{u}}_2), \underline{\mathbf{v}}] + [\mathcal{A}(\underline{\mathbf{u}}_1) - \mathcal{A}(\underline{\mathbf{u}}_2), \underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|} \,=\, 0\,,\end{aligned}$$

which implies that $\boldsymbol{\sigma}_1(t) = \boldsymbol{\sigma}_2(t)$ for all $t \in (0, T]$, and therefore (2.10) has a unique solution.

Finally, since Theorem 3.1 implies that $\mathcal{M}(u) \in \mathrm{L}^\infty(0, T; E_b')$, we can take $t \to 0$ in all equations without time derivatives in (2.10). Using that the initial data $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0) = ((\mathbf{u}_0, \mathbf{t}_0), \boldsymbol{\sigma}_0)$ satisfies the same equations at $t = 0$ (cf. (3.21)), and that $\mathbf{u}(0) = \mathbf{u}_0$, we obtain

$$\begin{aligned}\nu\,(\mathbf{t}(0) - \mathbf{t}_0, \mathbf{r})_\Omega - (\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_0, \mathbf{r})_\Omega \,&=\, 0 \quad \forall\, \mathbf{r} \in \mathbb{Q}\,, \\ (\mathbf{t}(0) - \mathbf{t}_0, \boldsymbol{\tau})_\Omega \,&=\, 0 \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0\,.\end{aligned} \tag{3.24}$$

Thus, taking $\mathbf{r} = \mathbf{t}(0) - \mathbf{t}_0$ and $\boldsymbol{\tau} = \boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_0$ in (3.24) we deduce that $\mathbf{t}(0) = \mathbf{t}_0$. In addition, from the latter and testing the first equation in (3.24) with $\mathbf{r} = (\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_0)^{\mathrm{d}} \in \mathbb{Q}$ implies that $\boldsymbol{\sigma}^{\mathrm{d}}(0) = \boldsymbol{\sigma}_0^{\mathrm{d}}$, completing the proof. □

We conclude this section with the corresponding stability bounds for the solution of (2.10).

Theorem 3.9. *Let* $\mathrm{p} \in [3, 4]$*. Assume that* $\mathbf{f} \in \mathrm{W}^{1,1}(0, T; \mathbf{L}^2(\Omega)) \cap \mathrm{L}^{2(\mathrm{p}-1)}(0, T; \mathbf{L}^2(\Omega))$*, and* $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$ *satisfying (3.19). Then, there exist constants* $C_{\mathrm{BF},1}, C_{\mathrm{BF},2} > 0$ *only depending on* $|\Omega|, \nu, \alpha, \mathrm{F}$*, and* β*, such that*

$$\begin{aligned}&\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\mathbf{t}\|_{\mathrm{L}^2(0,T;\mathbb{Q})} + \|\boldsymbol{\sigma}\|_{\mathrm{L}^2(0,T;\mathbb{X})} \\ &\leq\, C_{\mathrm{BF},1}\,\Big\{\|\mathbf{f}\|^{\mathrm{p}-1}_{\mathrm{L}^{2(\mathrm{p}-1)}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^{\mathrm{p}/2}_{\mathbf{M}} + \|\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{H}^1(\Omega)}\Big\}\end{aligned} \tag{3.25}$$

and

$$\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})} \,\leq\, C_{\mathrm{BF},2}\left\{\|\mathbf{f}\|^{2/\mathrm{p}}_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|_{\mathbf{M}} + \|\mathbf{u}_0\|^{2/\mathrm{p}}_{\mathbf{H}^1(\Omega)}\right\}. \tag{3.26}$$

Proof. We follow an analogous reasoning to the proof of [14, Theorem 3.3]. We begin by choosing $(\underline{\mathbf{v}},\boldsymbol{\tau}) = (\underline{\mathbf{u}},\boldsymbol{\sigma})$ in (2.10), to get

$$\frac{1}{2}\,\partial_t\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + [\mathcal{A}(\underline{\mathbf{u}}),\underline{\mathbf{u}}] = (\mathbf{f},\mathbf{u})_\Omega.$$

Next, from the definition of the operator $\mathcal{A}$ (cf. (2.12)), using Cauchy–Schwarz and Young's inequalities (cf. (1.1)), we obtain

$$\frac{1}{2}\,\partial_t\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,\|\mathbf{u}\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\mathbf{t}\|^2_{\mathbb{Q}} \,\leq\, \frac{\delta}{2}\,\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2\,\delta}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.27}$$

In turn, noting from the second row of (2.10) that $\underline{\mathbf{u}} = (\mathbf{u},\mathbf{t})$ belongs to $\mathbf{V}$ (cf. (3.16)), we know that $\mathbf{t} = \nabla\mathbf{u}$ and $\mathbf{u}\in\mathbf{H}^1_0(\Omega)$, which combined with the Sobolev embedding from $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^\mathrm{p}(\Omega)$, with $\mathrm{p}\in[3,4]$, implies

$$\frac{\alpha}{2}\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{2}\,\|\mathbf{t}\|^2_{\mathbb{Q}} \,\geq\, \frac{\min\{\alpha,\nu\}}{2}\left(\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\nabla\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)}\right) \,\geq\, \frac{\min\{\alpha,\nu\}}{2\,\|\mathbf{i}_\mathrm{p}\|^2}\,\|\mathbf{u}\|^2_{\mathbf{M}}, \tag{3.28}$$

where $\mathbf{i}_\mathrm{p}$ is the embedding operator. Combining the above with (3.27) and choosing $\delta = 1/\alpha$, yields

$$\partial_t\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\min\{\alpha,\nu\}}{\|\mathbf{i}_\mathrm{p}\|^2}\,\|\mathbf{u}\|^2_{\mathbf{M}} + \nu\,\|\mathbf{t}\|^2_{\mathbb{Q}} \,\leq\, \frac{1}{\alpha}\,\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}. \tag{3.29}$$

Notice that, in order to simplify the stability bound, we have neglected the term $\mathtt{F}\|\mathbf{u}\|^{\mathrm{p}}_{\mathbf{M}}$ on the left hand side of (3.27). Integrating (3.29) from 0 to $t\in(0,T]$, we obtain

$$\|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t\left(\|\mathbf{u}\|^2_{\mathbf{M}} + \|\mathbf{t}\|^2_{\mathbb{Q}}\right)ds \,\leq\, C_1\left\{\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\,ds + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)}\right\}, \tag{3.30}$$

with $C_1 := \max\left\{1,\frac{1}{\alpha}\right\}\|\mathbf{i}_\mathrm{p}\|^2\left(\min\left\{\alpha,\nu,\|\mathbf{i}_\mathrm{p}\|^2,\nu\,\|\mathbf{i}_\mathrm{p}\|^2\right\}\right)^{-1}$.

On the other hand, from the inf–sup condition of $\mathcal{B}$ (cf. (3.17)), the first equation of (2.10), and the stability bounds of $F,\mathcal{E},\mathcal{A}$ (cf. (3.5), (3.4) and (3.6)), we deduce that

$$\begin{aligned}\beta\,\|\boldsymbol{\sigma}\|_{\mathbb{X}} \,&\leq\, \sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq\mathbf{0}}}\frac{[\mathcal{B}'(\boldsymbol{\sigma}),\underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|} \,=\, \sup_{\substack{\underline{\mathbf{v}}\in\mathbf{M}\times\mathbb{Q}\\ \underline{\mathbf{v}}\neq\mathbf{0}}}\frac{[F,\underline{\mathbf{v}}] - [\partial_t\,\mathcal{E}(\underline{\mathbf{u}}),\underline{\mathbf{v}}] - [\mathcal{A}(\underline{\mathbf{u}}),\underline{\mathbf{v}}]}{\|\underline{\mathbf{v}}\|}\\ &\leq\, C_2\left(\|\mathbf{f}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{u}\|^{\mathrm{p}-1}_{\mathbf{M}} + \|\mathbf{t}\|_{\mathbb{Q}} + \|\partial_t\,\mathbf{u}\|_{\mathbf{L}^2(\Omega)}\right),\end{aligned} \tag{3.31}$$

where $C_2 := \max\left\{|\Omega|^{(\mathrm{p}-2)/(2\mathrm{p})},|\Omega|^{(\mathrm{p}-2)/\mathrm{p}},\alpha\,|\Omega|^{(\mathrm{p}-2)/\mathrm{p}},\mathtt{F},\nu\right\}$. In turn, using (3.29), the Sobolev embedding of $\mathbf{L}^\mathrm{p}(\Omega)$ into $\mathbf{L}^2(\Omega)$, with $\mathrm{p}>2$, the Young inequality (cf. (1.1)), and simple algebraic computations, we are able to find that

$$\begin{aligned}&\partial_t\,\|\mathbf{u}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} = (\mathrm{p}-1)\|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{L}^2(\Omega)}\,\partial_t\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\,\|\mathbf{u}\|^2_{\mathbf{M}}\\ &\quad\leq\,\widetilde{C}_3\,\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\,\|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{M}} \,\leq\, C(\mathrm{p})\,\widetilde{C}_3^{\,\mathrm{p}-1}\,\|\mathbf{f}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\,\|\mathbf{u}\|^{2(\mathrm{p}-1)}_{\mathbf{M}},\end{aligned} \tag{3.32}$$

with $\widetilde{C}_3 := \frac{(\mathrm{p}-1)|\Omega|^{(\mathrm{p}-2)^2/\mathrm{p}}\,\|\mathbf{i}_\mathrm{p}\|^2}{\alpha\,\min\{\|\mathbf{i}_\mathrm{p}\|^2,\alpha,\nu\}}$ and $C(\mathrm{p}) := \frac{(2(\mathrm{p}-2))^{\mathrm{p}-2}}{(\mathrm{p}-1)^{\mathrm{p}-1}}$, which, similarly to (3.30), implies

$$\|\mathbf{u}(t)\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{u}\|^{2(\mathrm{p}-1)}_{\mathbf{M}}\,ds \,\leq\, C_3\left\{\int_0^t \|\mathbf{f}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)}\,ds + \|\mathbf{u}(0)\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)}\right\}, \tag{3.33}$$

where $C_3 := \max\{1,2\,C(\mathrm{p})\,\widetilde{C}_3^{\,\mathrm{p}-1}\}$. Then, taking square in (3.31), integrating from 0 to $t\in(0,T]$, and using (3.30) and (3.33), we get

$$\begin{aligned}\int_0^t \|\boldsymbol{\sigma}\|^2_{\mathbb{X}}\,ds \,\leq\, C_4\,\Bigg\{&\int_0^t\left(\|\mathbf{f}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\right)ds\\ &+ \|\mathbf{u}(0)\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\partial_t\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,ds\Bigg\},\end{aligned} \tag{3.34}$$

with $C_4 := 10\,C_2^2\,\beta^{-2}\max\{1, C_1, C_3\}$ (cf. (3.30), (3.31), (3.33)). Next, in order to bound the last term in (3.34), we differentiate in time the second equation of (2.10), choose $(\underline{\mathbf{v}}, \boldsymbol{\tau}) = ((\partial_t\,\mathbf{u}, \partial_t\,\mathbf{t}), \boldsymbol{\sigma})$, and employ Cauchy–Schwarz and Young's inequalities, to obtain

$$\frac{1}{2}\,\partial_t\left(\alpha\,\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + \frac{2\,\mathrm{F}}{\mathrm{p}}\,\|\mathbf{u}\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\mathbf{t}\|^2_{\mathbb{Q}}\right) + \|\partial_t\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} \,\le\, \frac{1}{2}\,\|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\,\|\partial_t\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}.$$

Integrating from 0 to $t \in (0, T]$, we get

$$\frac{2\,\mathrm{F}}{\mathrm{p}}\,\|\mathbf{u}(t)\|^{\mathrm{p}}_{\mathbf{M}} + \int_0^t \|\partial_t\,\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\,ds \,\le\, C_5\left\{\int_0^t \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\,ds + \|\mathbf{u}(0)\|^{\mathrm{p}}_{\mathbf{M}} + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}(0)\|^2_{\mathbb{Q}}\right\}, \tag{3.35}$$

with $C_5 := \max\big\{1, \alpha, 2\,\mathrm{F}/\mathrm{p}, \nu\big\}$. Then, combining (3.35) with (3.34), yields

$$\begin{aligned}\int_0^t \|\boldsymbol{\sigma}\|^2_{\mathbb{X}}\,ds \,\le\, C_4\,(1 + C_5)\,\Bigg\{&\int_0^t\left(\|\mathbf{f}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}\|^2_{\mathbf{L}^2(\Omega)}\right)ds \\ &+ \|\mathbf{u}(0)\|^{\mathrm{p}}_{\mathbf{M}} + \|\mathbf{u}(0)\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}(0)\|^2_{\mathbb{Q}}\Bigg\},\end{aligned} \tag{3.36}$$

which, combined with (3.30) and the fact that $(\mathbf{u}(0), \mathbf{t}(0)) = (\mathbf{u}_0, \mathbf{t}_0)$, with $\mathbf{t}_0 = \nabla\mathbf{u}_0$ in Ω (cf. Lemma 3.7 and Theorem 3.8), implies (3.25) with

$$C_{\mathrm{BF},1} := \Big(8\max\{C_1, C_4\,(1 + C_5)\}\Big)^{1/2},$$

and C_1, C_4, and C_5 defined in (3.30), (3.34), and (3.35), respectively. In addition, (3.35) yields (3.26) with

$$C_{\mathrm{BF},2} := \left(\frac{\mathrm{p}}{2\,\mathrm{F}}\,\max\left\{1, \alpha, \frac{2\,\mathrm{F}}{\mathrm{p}}, \nu\right\}\right)^{1/\mathrm{p}},$$

concluding the proof. □

Remark 3.3. The stability bound (3.25) can be derived alternatively without using the fact that $\underline{\mathbf{u}} = (\mathbf{u}, \mathbf{t})$ belongs to $\mathbf{V}$ (cf. (3.16)) and the corresponding coercivity bound (3.28), but in that case the strict monotonicity bound (3.13) should be employed and consequently the expression on the right-hand side of (3.25) would be more complicated, involving other terms related to $\mathrm{p} \in [3, 4]$. We also note that (3.26) will be employed in the next section to deal with the nonlinear term associated to the operator $\mathcal{A}$ (cf. (2.12)), which is necessary to obtain the corresponding error estimate. It is important to mention that the constants $C_{\mathrm{BF},1}$ and $C_{\mathrm{BF},2}$ (cf. (3.25), (3.26)) are not explicitly computable since they depend on the theoretical constants $\|\mathbf{i}_{\mathrm{p}}\|$ and β, respectively.

Remark 3.4. The analysis developed in this section can be easily extended to the problem (2.4) with non-homogeneous Dirichlet boundary condition, $\mathbf{u} = \mathbf{u}_{\mathrm{D}}$ on $\Gamma \times (0, T]$. To that end, (2.10) has to be rewritten as follows: given $\mathbf{f} : [0, T] \to \mathbf{L}^2(\Omega)$, $\mathbf{u}_{\mathrm{D}} : [0, T] \to \mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{u}_0 \in \mathbf{M} \cap \mathbf{H}$ (cf. (3.18)), find $(\underline{\mathbf{u}}, \boldsymbol{\sigma}) = ((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, T] \to \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$, such that $\mathbf{u}(0) = \mathbf{u}_0$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned}\frac{\partial}{\partial t}\,[\mathcal{E}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{A}(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] + [\mathcal{B}'(\boldsymbol{\sigma}(t)), \underline{\mathbf{v}}] \quad &= \quad [F(t), \underline{\mathbf{v}}] \quad \forall\, \underline{\mathbf{v}} \in \mathbf{M} \times \mathbb{Q}\,, \\ -\,[\mathcal{B}(\underline{\mathbf{u}}(t)), \boldsymbol{\tau}] \quad &= \quad [G(t), \boldsymbol{\tau}] \quad \forall\, \boldsymbol{\tau} \in \mathbb{X}_0\,,\end{aligned}$$

where the functional $G \in \mathbb{X}_0'$ is given by $[G, \boldsymbol{\tau}] = \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle_\Gamma$, with $\langle \cdot, \cdot\rangle_\Gamma$ denoting the duality between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. We refer the reader to [22, Lemma 3.5] for the proof that $\boldsymbol{\tau}\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$ for all $\boldsymbol{\tau} \in \mathbb{X}_0$ in the case $\mathrm{p} = 4$ and $\mathrm{q} = 4/3$. The proof can be extended to the case $\mathrm{p} \in [3, 4]$ and $\mathrm{q} \in [4/3, 3/2]$ satisfying $1/\mathrm{p} + 1/\mathrm{q} = 1$, after slight adaptations. Then, we reformulate the problem as a parabolic problem for $\mathbf{u}$, and proceed as in [25, eq. (4.14), Section 4.1].

Remark 3.5. We stress that the analysis developed in this section can be adapted to prove the well-posedness of related unsteady problems such as the Navier–Stokes–Brinkman or the Boussinesq equations. In particular, for the Boussinesq equations, the range condition stated in Lemma 3.6 reduces to solving the stationary Boussinesq problem by employing a fixed-point strategy as in [18]. This is a topic of an ongoing research.

4. Semidiscrete continuous-in-time approximation

In this section we introduce and analyze the semidiscrete continuous-in-time approximation of (2.10). We analyze its solvability by employing the strategy developed in Section 3. Finally, we derive the error estimates and obtain the corresponding rates of convergence.

4.1. Existence and uniqueness of a solution

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω consisting of triangles K (when $d = 2$) or tetrahedra K (when $d = 3$) of diameter h_K, and define the mesh-size $h := \max\{h_K : K \in \mathcal{T}_h\}$. In turn, given an integer $l \geq 0$ and a subset S of R^d, we denote by $\mathrm{P}_l(S)$ the space of polynomials of total degree at most l defined on S. Hence, for each integer $k \geq 0$ and for each $K \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order k as

$$\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\,\mathbf{x},$$

where $\mathbf{x} := (x_1, \ldots, x_d)^{\mathrm{t}}$ is a generic vector of R^d, $\widetilde{\mathrm{P}}_k(K)$ is the space of polynomials of total degree equal to k defined on K, and, according to the convention in Section 1, we set $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^d$ and $\mathbb{P}_k(K) := [\mathrm{P}_k(K)]^{d\times d}$. In this way, introducing the finite element subspaces:

$$\begin{aligned}
\mathbf{M}_h &:= \Big\{\mathbf{v}_h \in \mathbf{M} : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\
\mathbb{Q}_h &:= \Big\{\mathbf{r}_h \in \mathbb{Q} : \quad \mathbf{r}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\},\\
\mathbb{X}_h &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{X} : \quad \mathbf{c}^{\mathrm{t}}\boldsymbol{\tau}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, \mathbf{c} \in \mathrm{R}^n \quad \forall\, K \in \mathcal{T}_h\Big\}, \qquad \mathbb{X}_{0,h} := \mathbb{X}_h \cap \mathbb{X}_0\,,
\end{aligned} \tag{4.1}$$

and denoting from now on

$$\underline{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h), \quad \underline{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{r}_h) \in \mathbf{M}_h \times \mathbb{Q}_h\,,$$

the semidiscrete continuous-in-time problem associated with (2.10) reads: Find $(\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h) : [0, T] \to \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ such that, for a.e. $t \in (0, T)$,

$$\begin{aligned}
\frac{\partial}{\partial t}[\mathcal{E}(\underline{\mathbf{u}}_h), \mathbf{v}_h] + [\mathcal{A}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\sigma}_h] &= [F, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,,\\
-[\mathcal{B}(\underline{\mathbf{u}}_h), \boldsymbol{\tau}_h] &= 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,.
\end{aligned} \tag{4.2}$$

As initial condition we take $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0}) = ((\mathbf{u}_{h,0}, \mathbf{t}_{h,0}), \boldsymbol{\sigma}_{h,0})$ to be a suitable approximations of $(\underline{\mathbf{u}}_0, \boldsymbol{\sigma}_0)$, the solution of (3.21), that is, we chose $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0})$ solving

$$\begin{aligned}
[\mathcal{A}(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\sigma}_{h,0}] &= [F_0, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,,\\
-[\mathcal{B}(\underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] &= 0 \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,,
\end{aligned} \tag{4.3}$$

with $F_0 \in \mathbf{L}^2(\Omega) \times \{\mathbf{0}\}$ being the right-hand side of (3.21). This choice is necessary to guarantee that the discrete initial datum is compatible in the sense of Lemma 3.7, which is needed for the application of Theorem 3.1. Notice that the well-posedness of problem (4.3) follows from similar arguments to the proof of Lemma 3.6. In addition, taking $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\underline{\mathbf{u}}_h, \boldsymbol{\sigma}_h)$ in (4.3), we deduce from the definition of the operator $\mathcal{A}$ (cf. (2.12)) and the continuity bound of F_0 (cf. (3.22)) that, there exists a constant $C_0 > 0$, depending only on $|\Omega|, \nu, \alpha$, and $\mathtt{F}$, and hence independent of h, such that

$$\|\mathbf{u}_{h,0}\|^{\mathrm{p}}_{\mathbf{M}} + \|\mathbf{u}_{h,0}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{t}_{h,0}\|^2_{\mathbb{Q}} \leq C_0 \left\{\|\mathbf{u}_0\|^{2(\mathrm{p}-1)}_{\mathbf{H}^1(\Omega)} + \|\Delta \mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)}\right\}. \tag{4.4}$$

In this way, the well-posedness of (4.2) follows analogously to its continuous counterpart provided in Theorem 3.8. More precisely, we begin by introducing the discrete kernel of the operator $\mathcal{B}$, that is,

$$\mathbf{V}_h := \Big\{\underline{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{r}_h) \in \mathbf{M}_h \times \mathbb{Q}_h : \quad (\mathbf{v}_h, \mathbf{div}(\boldsymbol{\tau}_h))_\Omega + (\mathbf{r}_h, \boldsymbol{\tau}_h)_\Omega = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\Big\}. \tag{4.5}$$

Then, we derive from [18, Section 5] the following two properties, the first one being the discrete inf–sup condition of $\mathcal{B}$ and the second one an auxiliary result that will be used to obtain the stability bound (4.10).

Lemma 4.1. *There exist positive constants $\widetilde{\beta}$ and C_{d}, such that*

$$\sup_{\substack{\underline{\mathbf{v}}_h\in\mathbf{M}_h\times\mathbb{Q}_h\\ \underline{\mathbf{v}}_h\neq\mathbf{0}}}\frac{[\mathcal{B}(\underline{\mathbf{v}}_h),\boldsymbol{\tau}_h]}{\|\underline{\mathbf{v}}_h\|}\,\geq\,\widetilde{\beta}\,\|\boldsymbol{\tau}_h\|_{\mathbb{X}}\quad\forall\,\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}\tag{4.6}$$

and

$$\|\mathbf{r}_h\|_{\mathbb{Q}}\,\geq\,C_{\mathrm{d}}\,\|\mathbf{v}_h\|_{\mathbf{M}}\quad\forall\,(\mathbf{v}_h,\mathbf{r}_h)\in\mathbf{V}_h\,.\tag{4.7}$$

Proof. For the case $\mathrm{p}=3$ and $\mathrm{q}=3/2$ we refer the reader to [17, Lemma 4.1], whose proof can be easily extended to the case $\mathrm{p}\in[3,4]$ and $\mathrm{q}\in[4/3,3/2]$ satisfying $1/\mathrm{p}+1/\mathrm{q}=1$. In what follows we provide some details just for sake of completeness. We begin by introducing the discrete space $Z_{0,h}$ defined by

$$Z_{0,h}:=\Big\{\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}:\quad[\mathcal{B}(\mathbf{v}_h,\mathbf{0}),\boldsymbol{\tau}_h]=(\mathbf{v}_h,\mathbf{div}(\boldsymbol{\tau}_h))_\Omega=0\quad\forall\,\mathbf{v}_h\in\mathbf{M}_h\Big\},$$

which, according to the fact that $\mathbf{div}(\mathbb{X}_{0,h})\subseteq\mathbf{M}_h$, becomes

$$Z_{0,h}=\Big\{\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}:\quad\mathbf{div}(\boldsymbol{\tau}_h)=0\quad\text{in}\quad\Omega\Big\}.$$

Next, by using the abstract equivalence result provided by [18, Lemma 5.1], we deduce that (4.6) and (4.7) are jointly equivalent to the existence of positive constants β_1 and β_2, independent of h, such that there hold

$$\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}}\frac{[\mathcal{B}(\mathbf{v}_h,\mathbf{0}),\boldsymbol{\tau}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}}=\sup_{\substack{\boldsymbol{\tau}_h\in\mathbb{X}_{0,h}\\ \boldsymbol{\tau}_h\neq\mathbf{0}}}\frac{(\mathbf{v}_h,\mathbf{div}(\boldsymbol{\tau}_h))_\Omega}{\|\boldsymbol{\tau}_h\|_{\mathbb{X}}}\,\geq\,\beta_1\,\|\mathbf{v}_h\|_{\mathbf{M}}\quad\forall\,\mathbf{v}_h\in\mathbf{M}_h\tag{4.8}$$

and

$$\sup_{\substack{\mathbf{r}_h\in\mathbb{Q}_h\\ \mathbf{r}_h\neq\mathbf{0}}}\frac{[\mathcal{B}(\mathbf{0},\mathbf{r}_h),\boldsymbol{\tau}_h]}{\|\mathbf{r}_h\|_{\mathbb{Q}}}=\sup_{\substack{\mathbf{r}_h\in\mathbb{Q}_h\\ \mathbf{r}_h\neq\mathbf{0}}}\frac{(\mathbf{r}_h,\boldsymbol{\tau}_h)_\Omega}{\|\mathbf{r}_h\|_{\mathbb{Q}}}\,\geq\,\beta_2\,\|\boldsymbol{\tau}_h\|_{\mathbb{X}}\quad\forall\,\boldsymbol{\tau}_h\in Z_{0,h}\,.\tag{4.9}$$

Then, we observe that (4.8) follows from a slight adaptation of [22, Lemma 4.3] (see also [18, eq. (5.45)]). Furthermore, recalling from [23, Lemma 2.3] that there exists a constant $c_1>0$, depending only on Ω, such that

$$c_1\,\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)}\,\leq\,\|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{\mathbb{L}^2(\Omega)}+\|\mathbf{div}(\boldsymbol{\tau})\|^2_{\mathbf{L}^2(\Omega)}\quad\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),$$

and using the fact that $\boldsymbol{\tau}_h^{\mathrm{d}}\in\mathbb{Q}_h$ for each $\boldsymbol{\tau}_h\in Z_{0,h}$ (see the proof of [23, Theorem 3.3] for details), we easily get (4.9) with $\beta_2=c_1^{1/2}$. □

Next, we address the discrete counterparts of Lemmas 3.3 and 3.4, whose proofs, being almost verbatim of the continuous ones, are omitted.

Lemma 4.2. *Let $\mathrm{p}\in[3,4]$. The family of operators $\Big\{(\mathcal{E}+\mathcal{A})(\cdot+\underline{\mathbf{z}}_h):\mathbf{M}_h\times\mathbb{Q}_h\to(\mathbf{M}_h\times\mathbb{Q}_h)':\quad\underline{\mathbf{z}}_h\in\mathbf{M}_h\times\mathbb{Q}_h\Big\}$ is uniformly strongly monotone with the same constant $\gamma_{\mathrm{BF}}>0$ from (3.13), that is, there holds*

$$\Big[(\mathcal{E}+\mathcal{A})(\underline{\mathbf{u}}_h+\underline{\mathbf{z}}_h)-(\mathcal{E}+\mathcal{A})(\underline{\mathbf{v}}_h+\underline{\mathbf{z}}_h),\underline{\mathbf{u}}_h-\underline{\mathbf{v}}_h\Big]\,\geq\,\gamma_{\mathrm{BF}}\Big\{\|\mathbf{u}_h-\mathbf{v}_h\|^{\mathrm{p}}_{\mathbf{M}}+\|\mathbf{t}_h-\mathbf{r}_h\|^2_{\mathbb{Q}}\Big\},$$

for each $\underline{\mathbf{z}}_h=(\mathbf{z}_h,\mathbf{s}_h)\in\mathbf{M}_h\times\mathbb{Q}_h$, and for all $\underline{\mathbf{u}}_h=(\mathbf{u}_h,\mathbf{t}_h)$, $\underline{\mathbf{v}}_h=(\mathbf{v}_h,\mathbf{r}_h)\in\mathbf{M}_h\times\mathbb{Q}_h$. In addition, the operator $\mathcal{E}+\mathcal{A}:(\mathbf{M}_h\times\mathbb{Q}_h)\to(\mathbf{M}_h\times\mathbb{Q}_h)'$ is continuous in the sense of (3.10), with the same constant L_{BF}.

We are now in a position to establish the semi-discrete continuous in time analogue of Theorems 3.8 and 3.9.

Theorem 4.3. *Let $\mathrm{p}\in[3,4]$. For each compatible initial data $(\underline{\mathbf{u}}_{h,0},\boldsymbol{\sigma}_{h,0}):=((\mathbf{u}_{h,0},\mathbf{t}_{h,0}),\boldsymbol{\sigma}_{h,0})$ satisfying (4.3) and $\mathbf{f}\in\mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega))$, there exists a unique $(\underline{\mathbf{u}}_h,\boldsymbol{\sigma}_h)=((\mathbf{u}_h,\mathbf{t}_h),\boldsymbol{\sigma}_h):[0,T]\to\big(\mathbf{M}_h\times\mathbb{Q}_h\big)\times\mathbb{X}_{0,h}$ solution to (4.2), satisfying $\mathbf{u}_h\in\mathrm{W}^{1,\infty}(0,T;\mathbf{M}_h)$ and $(\mathbf{u}_h(0),\mathbf{t}_h(0))=(\mathbf{u}_{h,0},\mathbf{t}_{h,0})$. Moreover, assuming that $\mathbf{u}_0\in\mathbf{M}\cap\mathbf{H}$ satisfies (3.19) and that $\mathbf{f}\in\mathrm{L}^{2(\mathrm{p}-1)}(0,T;\mathbf{L}^2(\Omega))$, there exist constants $\widehat{C}_{\mathrm{BF},1},\widehat{C}_{\mathrm{BF},2}>0$ depending only on $|\Omega|$, ν, α, $\mathtt{F}$, and*

$\widetilde{\beta}$, such that

$$
\begin{aligned}
&\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_h\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\mathbf{t}_h\|_{\mathrm{L}^2(0,T;\mathbb{Q})} + \|\boldsymbol{\sigma}_h\|_{\mathrm{L}^2(0,T;\mathbb{X})} \\
&\quad \le \widehat{C}_{\mathrm{BF},1} \Big\{ \|\mathbf{f}\|^{\mathrm{p}-1}_{\mathrm{L}^{2(\mathrm{p}-1)}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\mathbf{u}_0\|^{(\mathrm{p}-1)^2}_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{H}^1(\Omega)} + \|\Delta\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{L}^2(\Omega)} + \|\Delta\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \Big\},
\end{aligned} \tag{4.10}
$$

and

$$
\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})} \le \widehat{C}_{\mathrm{BF},2}\Big\{ \|\mathbf{f}\|^{2/\mathrm{p}}_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^{2(\mathrm{p}-1)/\mathrm{p}}_{\mathbf{H}^1(\Omega)} + \|\Delta\mathbf{u}_0\|^{2/\mathrm{p}}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^{2/\mathrm{p}}_{\mathbf{L}^2(\Omega)} \Big\}. \tag{4.11}
$$

Proof. According to Lemma 4.2, the discrete inf–sup condition for $\mathcal{B}$ provided by (4.6) (cf. Lemma 4.1), and considering that $(\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0})$ satisfies (4.3), the proof of existence and uniqueness of solution of (4.2) with $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbf{M}_h)$ and $\mathbf{u}_h(0) = \mathbf{u}_{h,0}$, follows similarly to the proof of Theorem 3.8 by applying Theorem 3.1. Moreover, from the discrete version of (3.24), we deduce that $\mathbf{t}_h(0) = \mathbf{t}_{h,0}$. Notice that, it is not possible to prove that $\boldsymbol{\sigma}^{\mathrm{d}}_h(0) = \boldsymbol{\sigma}^{\mathrm{d}}_{h,0}$ since $(\boldsymbol{\sigma}_h(0) - \boldsymbol{\sigma}_{h,0})^{\mathrm{d}}$ does not belong to $\mathbb{Q}_h$.

On the other hand, noticing from the second row of (4.2) that $\underline{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h) : [0,T] \to \mathbf{V}_h$ (cf. (4.5)), employing (4.7) to obtain the discrete version of (3.30), using the fact that $(\mathbf{u}_h(0), \mathbf{t}_h(0)) = (\mathbf{u}_{h,0}, \mathbf{t}_{h,0})$ and estimate (4.4) to obtain the discrete versions of (3.30)–(3.36), we proceed as in the proof of Theorem 3.9 and derive (4.10) and (4.11), thus completing the proof. □

We stress that, alternatively to the analysis developed in Theorem 4.3, one can prove that (4.2) is well-posed by using classical arguments of the theory of differential–algebraic equations (DAEs) as in [26, Section 3.1].

4.2. Error analysis

Now we derive suitable error estimates for the semidiscrete scheme (4.2). To that end, in what follows we assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations, which implies that the following inverse inequality holds (see, for instance,[27, Corollary 1.141]):

$$
\text{for } 1 \le \mathrm{p} \le \mathrm{q} < \infty, \quad \|\xi\|_{\mathrm{L}^{\mathrm{q}}(\Omega)} \le C h^{d(\frac{1}{\mathrm{q}} - \frac{1}{\mathrm{p}})} \|\xi\|_{\mathrm{L}^{\mathrm{p}}(\Omega)}, \tag{4.12}
$$

for all piecewise polynomial functions ξ with $C > 0$ independent of h.

Now we introduce some notations and state a couple of previous results. First, we recall the discrete inf–sup condition of $\mathcal{B}$ (cf. (4.6)), and a classical result on mixed methods (see, for instance [23, eq. (2.89) in Theorem 2.6]) ensure the existence of a constant $C > 0$, independent of h, such that:

$$
\inf_{\underline{\mathbf{v}}_h \in \mathbf{V}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \le C \inf_{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\|. \tag{4.13}
$$

Now, in order to obtain the theoretical rates of convergence for the discrete scheme (4.2), we recall the approximation properties of the finite element subspaces $\mathbf{M}_h$, $\mathbb{Q}_h$, and $\mathbb{X}_h$ (cf. (4.1)), that can be found in [23,27,28], and [21, Section 3.1] (see also [18, Section 5]).

$(\mathbf{AP}^{\mathbf{u}}_h)$ For each $l \in [0, k+1]$ and for each $\mathbf{v} \in \mathbf{W}^{l,\mathrm{p}}(\Omega)$, there holds

$$
\inf_{\mathbf{v}_h \in \mathbf{M}_h} \|\mathbf{v} - \mathbf{v}_h\|_{\mathbf{M}} \le C\, h^l\, \|\mathbf{v}\|_{\mathbf{W}^{l,\mathrm{p}}(\Omega)}.
$$

$(\mathbf{AP}^{\mathbf{t}}_h)$ For each $l \in [0, k+1]$ and for each $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{Q}$, there holds

$$
\inf_{\mathbf{r}_h \in \mathbb{Q}_h} \|\mathbf{r} - \mathbf{r}_h\|_{\mathbb{Q}} \le C\, h^l\, \|\mathbf{r}\|_{\mathbb{H}^l(\Omega)}.
$$

$(\mathbf{AP}^{\boldsymbol{\sigma}}_h)$ For each $l \in (0, k+1]$ and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{X}_0$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,\mathrm{q}}(\Omega)$, there holds

$$
\inf_{\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbb{X}} \le C\, h^l \Big\{ \|\boldsymbol{\tau}\|_{\mathbb{H}^l(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{W}^{l,\mathrm{q}}(\Omega)} \Big\}.
$$

Owing to (4.13) and $(\mathbf{AP}^{\mathbf{u}}_h)$, $(\mathbf{AP}^{\mathbf{t}}_h)$ and $(\mathbf{AP}^{\boldsymbol{\sigma}}_h)$, it follows that, under an extra regularity assumption on the exact solution (to be specified below in Theorem 4.6), there exist positive constants $C(\underline{\mathbf{u}})$, $C(\partial_t\,\underline{\mathbf{u}})$, $C(\boldsymbol{\sigma})$, and $C(\partial_t\,\boldsymbol{\sigma})$, depending on $\mathbf{u}, \mathbf{t}$ and $\boldsymbol{\sigma}$, respectively, such that

$$\begin{aligned} &\inf_{\underline{\mathbf{v}}_h\in \mathbf{V}_h} \|\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \,\le\, C(\underline{\mathbf{u}})\,h^l\,, \quad \inf_{\underline{\mathbf{v}}_h\in \mathbf{V}_h} \|\partial_t\,\underline{\mathbf{u}} - \underline{\mathbf{v}}_h\| \,\le\, C(\partial_t\,\underline{\mathbf{u}})\,h^l\,, \\ &\inf_{\boldsymbol{\tau}_h\in \mathbb{X}_{0,h}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbb{X}} \,\le\, C(\boldsymbol{\sigma})\,h^l\,, \quad \text{and} \quad \inf_{\boldsymbol{\tau}_h\in \mathbb{X}_{0,h}} \|\partial_t\,\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbb{X}} \,\le\, C(\partial_t\,\boldsymbol{\sigma})\,h^l\,. \end{aligned} \tag{4.14}$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_{\underline{\mathbf{u}}} = (\mathbf{e}_{\mathbf{u}}, \mathbf{e}_{\mathbf{t}}) = (\mathbf{u}-\mathbf{u}_h, \mathbf{t}-\mathbf{t}_h)$, and $\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h$. Next, given arbitrary $\widehat{\underline{\mathbf{v}}}_h := (\widehat{\mathbf{v}}_h, \widehat{\mathbf{r}}_h) : [0,T] \to \mathbf{V}_h$ (cf. (4.5)) and $\widehat{\boldsymbol{\tau}}_h : [0,T] \to \mathbb{X}_{0,h}$, as usual, we shall then decompose the errors into

$$\mathbf{e}_{\underline{\mathbf{u}}} = \boldsymbol{\delta}_{\underline{\mathbf{u}}} + \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\delta}_{\mathbf{t}}) + (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{t}})\,, \quad \mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\delta}_{\boldsymbol{\sigma}} + \boldsymbol{\eta}_{\boldsymbol{\sigma}}\,, \tag{4.15}$$

with

$$\begin{aligned} &\boldsymbol{\delta}_{\mathbf{u}} = \mathbf{u} - \widehat{\mathbf{v}}_h\,, \quad \boldsymbol{\delta}_{\mathbf{t}} = \mathbf{t} - \widehat{\mathbf{r}}_h\,, \quad \boldsymbol{\delta}_{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \widehat{\boldsymbol{\tau}}_h\,, \\ &\boldsymbol{\eta}_{\mathbf{u}} = \widehat{\mathbf{v}}_h - \mathbf{u}_h\,, \quad \boldsymbol{\eta}_{\mathbf{t}} = \widehat{\mathbf{r}}_h - \mathbf{t}_h\,, \quad \boldsymbol{\eta}_{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h\,. \end{aligned} \tag{4.16}$$

In addition, we stress for later use that $\partial_t\,\underline{\mathbf{v}}_h : [0,T] \to \mathbf{V}_h$ for each $\underline{\mathbf{v}}_h(t) \in \mathbf{V}_h$ (cf. (4.5)). In fact, given $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) : [0,T] \to \mathbf{V}_h \times \mathbb{X}_{0,h}$, after simple algebraic computations, we obtain

$$[\mathcal{B}(\partial_t\,\underline{\mathbf{v}}_h), \boldsymbol{\tau}_h] \,=\, \partial_t\big([\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\tau}_h]\big) - [\mathcal{B}(\underline{\mathbf{v}}_h), \partial_t\,\boldsymbol{\tau}_h] \,=\, 0\,, \tag{4.17}$$

where, the latter is obtained by observing that $\partial_t\,\boldsymbol{\tau}_h(t) \in \mathbb{X}_{0,h}$.

In this way, by subtracting the discrete and continuous problems (2.10) and (4.2), respectively, we obtain the following system:

$$\begin{aligned} \frac{\partial}{\partial\,t}\,[\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}(\underline{\mathbf{u}}) - \mathcal{A}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}_{\boldsymbol{\sigma}}] \quad &= \quad 0 \quad \forall\,\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h, \\ [\mathcal{B}(\mathbf{e}_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] \quad &= \quad 0 \quad \forall\,\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}. \end{aligned} \tag{4.18}$$

We now establish the main result of this section, namely, the theoretical rate of convergence of the discrete scheme (4.2). To that end, we first establish two preliminary results.

Lemma 4.4. *Let* $((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}) : [0,T] \to \big(\mathbf{M}\times\mathbb{Q}\big)\times\mathbb{X}_0$ *with* $\mathbf{u} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))$ *and* $((\mathbf{u}_h,\mathbf{t}_h),\boldsymbol{\sigma}_h) : [0,T] \to \big(\mathbf{M}_h\times\mathbb{Q}_h\big)\times\mathbb{X}_{0,h}$ *with* $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbf{M}_h)$*, be the unique solutions of the continuous and semidiscrete problems* (2.10) and (4.2)*, respectively. Let* $\mathrm{p} \in [3,4]$*. Then, there exists* $C > 0$ *depending only on* $|\Omega|$*,* ν*,* α*,* $\mathtt{F}$*,* C_{d}*, and data, such that*

$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\mathbf{t}}\|_{\mathrm{L}^2(0,T;\mathbb{Q})} \,\le\, C\,\Psi(\underline{\mathbf{u}},\boldsymbol{\sigma})\,, \tag{4.19}$$

where

$$\begin{aligned} \Psi(\underline{\mathbf{u}},\boldsymbol{\sigma}) := \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^\infty(0,T;\mathbf{M}\times\mathbb{Q})} + \|\partial_t\,\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{M}\times\mathbb{Q})} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^{\mathrm{p}-1}_{\mathrm{L}^{2(\mathrm{p}-1)}(0,T;\mathbf{M}\times\mathbb{Q})} \\ + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\mathrm{L}^2(0,T;\mathbf{M}\times\mathbb{Q})} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^{\mathrm{p}-1} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\| + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|_{\mathbb{X}}\,. \end{aligned} \tag{4.20}$$

Proof. First, adding and subtracting suitable terms in (4.18) with $\underline{\mathbf{v}}_h = \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{t}}) : [0,T] \to \mathbf{V}_h$ (cf. (4.5)) and $\boldsymbol{\tau}_h = \boldsymbol{\eta}_{\boldsymbol{\sigma}} : [0,T] \to \mathbb{X}_{0,h}$, and employing the strict monotonicity bound of $\mathcal{A}$ (cf. (3.13)) and the fact that $\boldsymbol{\eta}_{\underline{\mathbf{u}}}(t) \in \mathbf{V}_h$, thus $[\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\boldsymbol{\sigma}}] = 0$, we deduce that

$$\begin{aligned} &\frac{1}{2}\,\partial_t\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,C_{\mathrm{p}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}} \\ &\quad \le -(\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \alpha(\boldsymbol{\delta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \mathtt{F}(|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\widehat{\mathbf{v}}_h|^{\mathrm{p}-2}\widehat{\mathbf{v}}_h, \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \nu(\boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega - [\mathcal{B}(\boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\delta}_{\boldsymbol{\sigma}}]\,. \end{aligned} \tag{4.21}$$

Next, using again the fact that $\boldsymbol{\eta}_{\underline{\mathbf{u}}}(t) = (\boldsymbol{\eta}_{\mathbf{u}}, \boldsymbol{\eta}_{\mathbf{t}})(t) \in \mathbf{V}_h$, we deduce from (4.7) that $C_{\mathrm{d}}\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \le \|\boldsymbol{\eta}_{\mathbf{t}}\|_{\mathbb{Q}}$. Thus, using (3.12), the continuity bound of the operator $\mathcal{B}$ (cf. (3.3)), the Cauchy–Schwarz, Hölder and Young's inequalities (cf. (1.1)), and neglecting the term $\|\boldsymbol{\eta}_{\mathbf{u}}\|^{\mathrm{p}}_{\mathbf{M}}$ in (4.21) to obtain a simplified error estimate, we obtain

$$\begin{aligned}
&\frac{1}{2}\,\partial_t\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_{\mathrm{d}}^2\,\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \frac{\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\\
&\leq \|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} + \alpha\,\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,c_{\mathrm{p}}\left(\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}} + 2\,\|\mathbf{u}\|_{\mathbf{M}}\right)^{\mathrm{p}-2}\|\boldsymbol{\delta}_{\mathbf{u}}\|_{\mathbf{M}}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}}\\
&+\nu\,\|\boldsymbol{\delta}_{\mathbf{t}}\|_{\mathbb{Q}}\|\boldsymbol{\eta}_{\mathbf{t}}\|_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\|\boldsymbol{\eta}_{\underline{\mathbf{u}}}\|\\
&\leq C_1\left(\|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \left(1 + \|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\right)\|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{t}}\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\right)\\
&+\frac{1}{2}\left(\alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_{\mathrm{d}}^2\,\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \frac{\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\right),
\end{aligned}$$

where C_1 is a positive constant depending on $|\Omega|$, ν, α, $\mathtt{F}$, and C_{d}, which yields

$$\begin{aligned}
&\partial_t\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \frac{C_{\mathrm{d}}^2\,\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \frac{\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\\
&\leq 2\,C_1\left(\|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \left(1 + \|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\right)\|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{t}}\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\right).
\end{aligned}\tag{4.22}$$

Integrating (4.22) from 0 to $t \in (0, T]$, recalling that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ is bounded by data (cf. (3.26)), we find that

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t\left(\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\right)ds\\
&\leq C_2\left\{\int_0^t\left(\|\partial_t\,\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{t}}\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\right)ds + \|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)}\right\},
\end{aligned}\tag{4.23}$$

with $C_2 > 0$ depending only on $|\Omega|$, ν, α, $\mathtt{F}$, C_{d}, and data.

Next, in order to bound the last term in (4.23), we subtract the continuous and discrete initial condition problems (3.21) and (4.3), to obtain the error system:

$$\begin{aligned}
[\mathcal{A}(\underline{\mathbf{u}}_0) - \mathcal{A}(\underline{\mathbf{u}}_{h,0}), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_{h,0}] &= 0 \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,,\\
-\,[\mathcal{B}(\underline{\mathbf{u}}_0 - \underline{\mathbf{u}}_{h,0}), \boldsymbol{\tau}_h] &= 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,.
\end{aligned}$$

Then, proceeding as in (4.22), recalling from Theorems 3.8 and 4.3 that $(\mathbf{u}(0), \mathbf{t}(0)) = (\mathbf{u}_0, \mathbf{t}_0)$ and $(\mathbf{u}_h(0), \mathbf{t}_h(0)) = (\mathbf{u}_{h,0}, \mathbf{t}_{h,0})$, respectively, we get

$$\|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\underline{\mathbf{u}}}(0)\|^2 \,\leq\, \widehat{C}_0\left(\|\boldsymbol{\delta}_{\mathbf{u}_0}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}}\right),\tag{4.24}$$

where, similarly to (4.16), we denote $\boldsymbol{\delta}_{\underline{\mathbf{u}}_0} = (\boldsymbol{\delta}_{\mathbf{u}_0}, \boldsymbol{\delta}_{\mathbf{t}_0}) = (\mathbf{u}_0 - \widehat{\mathbf{v}}_h(0), \mathbf{t}_0 - \widehat{\mathbf{r}}_h(0))$ and $\boldsymbol{\delta}_{\boldsymbol{\sigma}_0} = \boldsymbol{\sigma}_0 - \widehat{\boldsymbol{\tau}}_h(0)$, with arbitrary $(\widehat{\mathbf{v}}_h(0), \widehat{\mathbf{r}}_h(0)) \in \mathbf{V}_h$ and $\widehat{\boldsymbol{\tau}}_h(0) \in \mathbb{X}_{0,h}$, and $\widehat{C}_0$ is a positive constant depending only on $|\Omega|$, ν, α, $\mathtt{F}$, and C_{d}. Thus, combining (4.23) and (4.24), and using the error decomposition (4.15), we obtain (4.19) and conclude the proof. □

Lemma 4.5. *Let* $((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, T] \to \left(\mathbf{M} \times \mathbb{Q}\right) \times \mathbb{X}_0$ *with* $\mathbf{u} \in \mathrm{W}^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$ *and* $((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h) : [0, T] \to \left(\mathbf{M}_h \times \mathbb{Q}_h\right) \times \mathbb{X}_{0,h}$ *with* $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0, T; \mathbf{M}_h)$*, be the unique solutions of the continuous and semidiscrete problems* (2.10) and (4.2)*, respectively. Let* $\mathrm{p} \in [3, 4]$*. Assume that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations. Then, there exists* $C > 0$ *depending only on* $|\Omega|$*,* ν*,* α*,* $\mathtt{F}$*,* $\widetilde{\beta}$*, and data, such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})} \,\leq\, C\,h^{-d(\mathrm{p}-2)/(2\mathrm{p})}\left\{\Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma}) + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^\infty(0,T;\mathbb{X})} + \|\partial_t\,\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})}\right\},\tag{4.25}$$

with $\Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma})$ *defined in* (4.20)*.*

Proof. We begin by observing that the discrete inf–sup condition of $\mathcal{B}$ (cf. (4.6)), combined with the first equation of (4.18), and the continuity bounds of $\mathcal{E}, \mathcal{A}, \mathcal{B}$ (cf. (3.5) (3.10), (3.3)), there holds

$$\begin{aligned}
\widetilde{\beta}\,\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}} &\leq \sup_{\substack{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\\ \underline{\mathbf{v}}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\eta}_{\boldsymbol{\sigma}}]}{\|\underline{\mathbf{v}}_h\|}\\
&= -\sup_{\substack{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\\ \underline{\mathbf{v}}_h \neq \mathbf{0}}} \frac{[\partial_t\,\mathcal{E}(\mathbf{e}_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}(\underline{\mathbf{u}}) - \mathcal{A}(\underline{\mathbf{u}}_h), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\delta}_{\boldsymbol{\sigma}}]}{\|\underline{\mathbf{v}}_h\|}\\
&\leq \widetilde{C}_3\left(\|\partial_t\,\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}} + \left(\|\mathbf{u}\|_{\mathbf{M}} + \|\mathbf{u}_h\|_{\mathbf{M}}\right)^{\mathrm{p}-2}\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{e}_{\mathbf{t}}\|_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\right),
\end{aligned}$$

with $\widetilde{C}_3 > 0$ depending only on $|\Omega|, \nu, \alpha$, and $\mathtt{F}$. Then, taking square in the above inequality, integrating from 0 to $t \in (0, T]$, recalling that both $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ are bounded by data (cf. (3.26), (4.11)), and employing (4.19), we deduce that

$$\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|^2_{\mathrm{L}^2(0,T;\mathbb{X})} \leq C_3 \left\{ \Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma})^2 + \int_0^t \|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} ds \right\}, \tag{4.26}$$

with $\Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma})$ defined in (4.20) and $C_3 > 0$ depending on $|\Omega|, \nu, \alpha, \mathtt{F}, \widetilde{\beta}$, and data. Next, in order to bound the last term in (4.26), we choose $\underline{\mathbf{v}}_h = \partial_t \boldsymbol{\eta}_{\underline{\mathbf{u}}} = (\partial_t \boldsymbol{\eta}_{\mathbf{u}}, \partial_t \boldsymbol{\eta}_{\mathbf{t}})$ in the first equation of (4.18), to find that

$$\begin{aligned}\frac{1}{2}\,\partial_t\Big(\alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\Big) + \|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} &= -(\partial_t \boldsymbol{\delta}_{\mathbf{u}}, \partial_t \boldsymbol{\eta}_{\mathbf{u}})_\Omega - \alpha\,(\boldsymbol{\delta}_{\mathbf{u}}, \partial_t \boldsymbol{\eta}_{\mathbf{u}})_\Omega \\ - \mathtt{F}\,(|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u} - |\mathbf{u}_h|^{\mathrm{p}-2}\mathbf{u}_h, \partial_t \boldsymbol{\eta}_{\mathbf{u}})_\Omega &+ (\partial_t \boldsymbol{\eta}_{\mathbf{u}}, \mathbf{div}(\boldsymbol{\delta}_{\boldsymbol{\sigma}}))_\Omega - \nu\,(\boldsymbol{\delta}_{\mathbf{t}}, \partial_t \boldsymbol{\eta}_{\mathbf{t}})_\Omega + (\partial_t \boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega\,.\end{aligned}$$

Notice that $[\mathcal{B}(\partial_t \boldsymbol{\eta}_{\underline{\mathbf{u}}}), \boldsymbol{\eta}_{\boldsymbol{\sigma}}] = 0$ since $\boldsymbol{\eta}_{\underline{\mathbf{u}}}(t) \in \mathbf{V}_h$ (cf. (4.17)). Then, using the identities

$$(\boldsymbol{\delta}_{\mathbf{t}}, \partial_t \boldsymbol{\eta}_{\mathbf{t}})_\Omega = \partial_t\,(\boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega - (\partial_t \boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega \quad \text{and} \quad (\partial_t \boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega = \partial_t\,(\boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega - (\boldsymbol{\eta}_{\mathbf{t}}, \partial_t \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega,$$

in combination with the Cauchy–Schwarz, Hölder and Young's inequalities, the continuity bound of $\mathcal{A}$ (cf. (3.10)), the quasi-uniformity assumption on the family mesh $\{\mathcal{T}_h\}_{h>0}$ and the inverse inequality $\|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{M}} \leq c\,h^{-d(\mathrm{p}-2)/(2\mathrm{p})}\,\|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}$ (cf. (4.12)), with $\boldsymbol{\eta}_{\mathbf{u}}(t) \in \mathbf{M}_h$, we obtain

$$\begin{aligned}&\frac{1}{2}\,\partial_t\left(\alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}}\right) + \|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad\leq C_4\,h^{-d(\mathrm{p}-2)/\mathrm{p}}\,C(\mathbf{u}, \mathbf{u}_h)\left(\|\partial_t \boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\right) \\ &\quad + \frac{1}{2}\,\|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \partial_t\left((\boldsymbol{\eta}_{\mathbf{t}}, \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega - \nu\,(\boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega\right) + \nu\,(\partial_t \boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega - (\boldsymbol{\eta}_{\mathbf{t}}, \partial_t \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega\,,\end{aligned}$$

with

$$C(\mathbf{u}, \mathbf{u}_h) := 1 + \|\mathbf{u}\|^{2(\mathrm{p}-2)}_{\mathbf{M}} + \|\mathbf{u}_h\|^{2(\mathrm{p}-2)}_{\mathbf{M}}$$

and $C_4 > 0$ depending on $|\Omega|, \nu, \alpha, \mathtt{F}, \widetilde{\beta}$, and data. Thus, integrating from 0 to $t \in (0, T]$, we find that

$$\begin{aligned}&\frac{1}{2}\left(\alpha\,\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \nu\,\|\boldsymbol{\eta}_{\mathbf{t}}(t)\|^2_{\mathbb{Q}} + \int_0^t \|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} ds\right) \\ &\quad\leq C_4\,h^{-d(\mathrm{p}-2)/\mathrm{p}} \int_0^t C(\mathbf{u}, \mathbf{u}_h)\Big(\|\partial_t \boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\mathbf{e}_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\Big)\,ds \\ &\quad + \Big((\boldsymbol{\eta}_{\mathbf{t}}(t), \boldsymbol{\delta}_{\boldsymbol{\sigma}}(t))_\Omega - \nu\,(\boldsymbol{\delta}_{\mathbf{t}}(t), \boldsymbol{\eta}_{\mathbf{t}}(t))_\Omega\Big) + \int_0^t \Big(\nu\,(\partial_t \boldsymbol{\delta}_{\mathbf{t}}, \boldsymbol{\eta}_{\mathbf{t}})_\Omega - (\boldsymbol{\eta}_{\mathbf{t}}, \partial_t \boldsymbol{\delta}_{\boldsymbol{\sigma}})_\Omega\Big)\,ds \\ &\quad + \frac{\alpha}{2}\,\|\boldsymbol{\eta}_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \frac{\nu}{2}\,\|\boldsymbol{\eta}_{\mathbf{t}}(0)\|^2_{\mathbb{Q}} - \Big((\boldsymbol{\eta}_{\mathbf{t}}(0), \boldsymbol{\delta}_{\boldsymbol{\sigma}}(0))_\Omega - \nu\,(\boldsymbol{\delta}_{\mathbf{t}}(0), \boldsymbol{\eta}_{\mathbf{t}}(0))_\Omega\Big)\,.\end{aligned}$$

Then, using Cauchy–Schwarz and Young's inequalities, recalling that $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ and $\|\mathbf{u}_h\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$ are bounded by data (cf. (3.26) and (4.11)), employing estimates (4.23), (4.24) and (4.19), and some algebraic manipulations, we deduce that

$$\begin{aligned}&\|\boldsymbol{\eta}_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{t}}(t)\|^2_{\mathbb{Q}} + \int_0^t \|\partial_t \boldsymbol{\eta}_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} ds \\ &\quad\leq C_5\,\Bigg\{\,h^{-d(\mathrm{p}-2)/\mathrm{p}}\,\Psi(\underline{\mathbf{u}}, \boldsymbol{\sigma})^2 + \|\boldsymbol{\delta}_{\mathbf{t}}(t)\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}(t)\|^2_{\mathbb{X}} + \int_0^t \Big(\|\partial_t \boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\partial_t \boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{L}^2(\Omega)}\Big)\,ds \\ &\quad + \int_0^t \Big(\|\boldsymbol{\delta}_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}}\Big)\,ds + \|\boldsymbol{\delta}_{\mathbf{u}_0}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}_0}\|^2 + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}_0}\|^2_{\mathbb{X}}\Bigg\},\end{aligned} \tag{4.27}$$

with $C_5 > 0$ depending on $|\Omega|, \nu, \alpha, \mathtt{F}, \widetilde{\beta}$, and data. Thus, combining (4.26) and (4.27), using the error decomposition (4.15) and considering sufficiently small values of h, we obtain (4.25) concluding the proof. □

We now establish the main convergence result. Notice that, optimal and sub-optimal rates of convergences of order $\mathcal{O}(h^l)$ and $\mathcal{O}(h^{l-d(\mathrm{p}-2)/(2\mathrm{p})})$ are confirmed for $(\mathbf{u}, \mathbf{t})$ and $\boldsymbol{\sigma}$, respectively.

Theorem 4.6. *Let* $((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}) : [0, T] \to \big(\mathbf{M} \times \mathbb{Q}\big) \times \mathbb{X}_0$ *with* $\mathbf{u} \in \mathrm{W}^{1,\infty}(0, T; \mathbf{L}^2(\Omega))$ *and* $((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h) : [0, T] \to \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ *with* $\mathbf{u}_h \in \mathrm{W}^{1,\infty}(0, T; \mathbf{M}_h)$*, be the unique solutions of the continuous and semidiscrete* *problems (2.10) and (4.2), respectively. Assume further that* $\{\mathcal{T}_h\}_{h>0}$ *is a family of quasi-uniform triangulations and that there exists* $l \in (0, k+1]$, *such that* $\mathbf{u} \in \mathbf{W}^{l,\mathrm{p}}(\Omega)$, $\mathbf{t} \in \mathbb{H}^l(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega)$, *and* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,\mathrm{q}}(\Omega)$, *with* $\mathrm{p} \in [3,4]$ *and* $\mathrm{q} \in [4/3, 3/2]$ *satisfying* $1/\mathrm{p} + 1/\mathrm{q} = 1$. *Then, there exist* $C_1(\underline{\mathbf{u}}, \boldsymbol{\sigma}), C_2(\underline{\mathbf{u}}, \boldsymbol{\sigma}) > 0$ *depending only on* $C(\underline{\mathbf{u}})$, $C(\partial_t\, \underline{\mathbf{u}})$, $C(\boldsymbol{\sigma})$, $C(\partial_t\, \boldsymbol{\sigma})$, $|\Omega|$, ν, α, $\mathtt{F}$, C_{d}, $\widetilde{\beta}$, *and data, such that*

$$\|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{L}^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\mathbf{t}}\|_{\mathrm{L}^2(0,T;\mathbb{Q})} \,\le\, C_1(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \left(h^l + h^{l\,(\mathrm{p}-1)}\right) \tag{4.28}$$

and

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{X})} \,\le\, C_2(\underline{\mathbf{u}}, \boldsymbol{\sigma})\, h^{-d(\mathrm{p}-2)/(2\mathrm{p})} \left(h^l + h^{l\,(\mathrm{p}-1)}\right). \tag{4.29}$$

Proof. The assertion of the theorem follows straightforwardly from Lemmas 4.4 and 4.5, by considering the fact that $\widehat{\underline{\mathbf{v}}}_h : [0,T] \to \mathbf{V}_h$ and $\boldsymbol{\tau}_h : [0,T] \to \mathbb{X}_{0,h}$ are arbitrary, taking infimum in (4.19) and (4.25) over the corresponding discrete subspaces $\mathbf{V}_h$ and $\mathbb{X}_{0,h}$, and applying the approximation properties (4.14). □

Remark 4.1. The rates of convergences obtained in (4.28)–(4.29) improve the ones obtained in [14, Theorem 4.4] for the pseudostress–velocity formulation. More precisely, an additional order of convergence $h^{l\,(\mathrm{p}-2)/2(\mathrm{p}-1)}$ is gained, which illustrate one of the advantage of our three-field mixed formulation (4.2). We also note that in the steady state case of (2.4) the error estimate (4.29) does not include the term $h^{-d(\mathrm{p}-2)/(2\mathrm{p})}$ because the global inverse inequality is not necessary to bound $\|\boldsymbol{\eta}_{\boldsymbol{\sigma}}\|_{\mathbb{X}}$ (see [17, Section 5] for details of the case $\mathrm{p} = 3$).

Remark 4.2. Alternatively to Lemmas 4.4 and 4.5, the theoretical rates of convergence obtained in (4.28) and (4.29) (cf. Theorem 4.6) can be obtained by employing a Strang-type lemma for the generic problem (3.1) and then applying that result to (2.10) and (4.2). To that end, similar arguments to the ones developed in [17, Lemma 5.1] would be required.

5. Fully discrete approximation

In this section we introduce and analyze a fully discrete approximation of (2.10) (cf. (4.2)). To that end, for the time discretization we employ the backward Euler method. Let Δt be the time step, $T = N\Delta t$, and let $t_n = n\Delta t$, $n = 0, \ldots, N$. More precisely, we let $d_t u^n = (\Delta t)^{-1}(u^n - u^{n-1})$ be the first order (backward) discrete time derivative, where $u^n := u(t_n)$. Then the fully discrete method reads: given $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$ and $(\underline{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) = (\underline{\mathbf{u}}_{h,0}, \boldsymbol{\sigma}_{h,0})$ satisfying (4.3) find $(\underline{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) = ((\mathbf{u}_h^n, \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n) \in \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$, $n = 1, \ldots, N$, such that

$$\begin{aligned} d_t\,[\mathcal{E}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{A}(\underline{\mathbf{u}}_h^n), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\sigma}_h^n] &= [F^n, \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h\,, \\ -\,[\mathcal{B}(\underline{\mathbf{u}}_h^n), \boldsymbol{\tau}_h] &= 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,, \end{aligned} \tag{5.1}$$

where $[F^n, \underline{\mathbf{v}}_h] := (\mathbf{f}^n, \mathbf{v}_h)_\Omega$.

In what follows, given a separable Banach space V endowed with the norm $\|\cdot\|_{\mathrm{V}}$, we make use of the following discrete in time norms

$$\|u\|^{\mathrm{p}}_{\ell^{\mathrm{p}}(0,T,\mathrm{V})} := \Delta t \sum_{n=1}^N \|u^n\|^{\mathrm{p}}_{\mathrm{V}} \quad \text{and} \quad \|u\|_{\ell^\infty(0,T,\mathrm{V})} := \max_{0\le n\le N} \|u^n\|_{\mathrm{V}}\,. \tag{5.2}$$

Next, we state the main results for method (5.1).

Theorem 5.1. *Let* $\mathrm{p} \in [3,4]$. *For each* $(\underline{\mathbf{u}}_h^0, \boldsymbol{\sigma}_h^0) = ((\mathbf{u}_{h,0}, \mathbf{t}_{h,0}), \boldsymbol{\sigma}_{h,0})$ *satisfying (4.3) and* $\mathbf{f}^n \in \mathbf{L}^2(\Omega)$, $n = 1, \ldots, N$, *there exists a unique solution* $(\underline{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n) = ((\mathbf{u}_h^n, \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n) \in \big(\mathbf{M}_h \times \mathbb{Q}_h\big) \times \mathbb{X}_{0,h}$ *to (5.1). Moreover, under a suitable extra regularity assumption on the data, there exist constants* $\widetilde{C}_{\mathrm{BF},1}, \widetilde{C}_{\mathrm{BF},2} > 0$ *depending only on* $|\Omega|, \nu, \alpha, \mathtt{F}$, *and* $\widetilde{\beta}$, *such that*

$$\begin{aligned} &\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \Delta t\, \|d_t \mathbf{u}_h\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_h\|_{\ell^2(0,T;\mathbf{M})} + \|\mathbf{t}_h\|_{\ell^2(0,T;\mathbb{Q})} + \|\boldsymbol{\sigma}_h\|_{\ell^2(0,T;\mathbb{X})} \\ &\quad \le\, \widetilde{C}_{\mathrm{BF},1} \,\Big\{ \|\mathbf{f}\|^{\mathrm{p}-1}_{\ell^{2(\mathrm{p}-1)}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} \\ &\quad + \|\mathbf{u}_0\|^{(\mathrm{p}-1)^2}_{\mathbf{H}^1(\Omega)} + \|\mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{H}^1(\Omega)} + \|\Delta \mathbf{u}_0\|^{\mathrm{p}-1}_{\mathbf{L}^2(\Omega)} + \|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} \Big\} \end{aligned} \tag{5.3}$$

and

$$\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{M})} \leq \widetilde{C}_{\mathrm{BF},2}\left\{\|\mathbf{f}\|^{2/\mathrm{p}}_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_0\|^{2(\mathrm{p}-1)/\mathrm{p}}_{\mathbf{H}^1(\Omega)} + \|\Delta\mathbf{u}_0\|^{2/\mathrm{p}}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^{2/\mathrm{p}}_{\mathbf{L}^2(\Omega)}\right\}. \tag{5.4}$$

Proof. First, we note that at each time step the well-posedness of the fully discrete problem (5.1), with $n = 1,\ldots,N$, follows from similar arguments to the proof of Lemma 3.6 (see also [17, Section 3.3] for the case $\mathrm{p}=3$).

On the other hand, the derivation of (5.3) and (5.4) can be obtained similarly as in the proof of Theorem 3.9. In fact, we choose $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) = (\underline{\mathbf{u}}_h^n, \boldsymbol{\sigma}_h^n)$ in (5.1), use the identity

$$(d_t\,\mathbf{u}_h^n, \mathbf{u}_h^n)_\Omega = \frac{1}{2}\,d_t\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\,\Delta t\,\|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)}, \tag{5.5}$$

the definition of the operator $\mathcal{A}$ (cf. (2.12)), and the Cauchy–Schwarz and Young's inequalities (cf. (1.1)), to obtain

$$\begin{aligned}&\frac{1}{2}\,d_t\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\,\Delta t\,\|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \alpha\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \mathrm{F}\,\|\mathbf{u}_h^n\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\mathbf{t}_h^n\|^2_{\mathbb{Q}}\\ &\quad\leq \frac{\delta}{2}\,\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{2\,\delta}\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned} \tag{5.6}$$

In turn, noting from the second row of (5.1) that $\underline{\mathbf{u}}_h^n = (\mathbf{u}_h^n, \mathbf{t}_h^n) \in \mathbf{V}_h$ (cf. (4.5)), with $n = 1,\ldots,N$, using the estimate (4.7), and choosing $\delta = \dfrac{1}{2\,\alpha}$, we obtain

$$d_t\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \Delta t\,\|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + C_{\mathrm{d}}^2\,\nu\,\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \nu\,\|\mathbf{t}_h^n\|^2_{\mathbb{Q}} \leq \frac{1}{2\,\alpha}\,\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)}. \tag{5.7}$$

Notice that, in order to simplify the stability bound, we have neglected the term $\|\mathbf{u}_h^n\|^{\mathrm{p}}_{\mathbf{M}}$ on the left-hand side of (5.6). Thus summing up over the time index $n = 1,\ldots,m$, with $m = 1,\ldots,N$, in (5.7) and multiplying by Δt, we get

$$\begin{aligned}&\|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} + (\Delta t)^2\sum_{n=1}^m \|d_t\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \Delta t\sum_{n=1}^m\left(\|\mathbf{u}_h^n\|^2_{\mathbf{M}} + \|\mathbf{t}_h^n\|^2_{\mathbb{Q}}\right)\\ &\quad\leq C_1\left\{\Delta t\sum_{n=1}^m\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^0\|^2_{\mathbf{L}^2(\Omega)}\right\},\end{aligned} \tag{5.8}$$

with $C_1 := \max\{1, \frac{1}{2\alpha}\}\left(\min\{1,\nu,C_{\mathrm{d}}^2\,\nu\}\right)^{-1}$.

On the other hand, from the discrete inf–sup condition of $\mathcal{B}$ (cf. (4.6)) and the first equation of (5.1), we deduce that

$$\widetilde{\beta}\,\|\boldsymbol{\sigma}_h^n\|_{\mathbb{X}} \leq C_2\left\{\|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^n\|_{\mathbf{M}} + \|\mathbf{u}_h^n\|^{\mathrm{p}-1}_{\mathbf{M}} + \|\mathbf{t}_h^n\|_{\mathbb{Q}} + \|d_t\mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}\right\}, \tag{5.9}$$

with $C_2 > 0$ defined in (3.31). In turn, using Young's inequality (cf. (1.1)), we readily obtain

$$\|\mathbf{u}_h^{n-1}\|^2_{\mathbf{L}^2(\Omega)}\,\|\mathbf{u}_h^n\|^{2(\mathrm{p}-2)}_{\mathbf{L}^2(\Omega)} \leq \frac{1}{\mathrm{p}-1}\,\|\mathbf{u}_h^{n-1}\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \frac{\mathrm{p}-2}{\mathrm{p}-1}\,\|\mathbf{u}_h^n\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)},$$

which, together with (5.7), the fact that $\mathbf{L}^{\mathrm{p}}(\Omega)$ is continuously embedded into $\mathbf{L}^2(\Omega)$, with $\mathrm{p}\in[3,4]$, the Young inequality (cf. (1.1)), and simple algebraic computations, imply

$$\begin{aligned}&d_t\,\|\mathbf{u}_h^n\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^n\|^{2(\mathrm{p}-1)}_{\mathbf{M}} \leq (\mathrm{p}-1)\|\mathbf{u}_h^n\|^{2(\mathrm{p}-2)}_{\mathbf{L}^2(\Omega)}\,d_t\,\|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^n\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\,\|\mathbf{u}_h^n\|^2_{\mathbf{M}}\\ &\quad\leq \widetilde{C}_3\,\|\mathbf{f}^n\|^2_{\mathbf{L}^2(\Omega)}\,\|\mathbf{u}_h^n\|^{2(\mathrm{p}-2)}_{\mathbf{M}} \leq C(\mathrm{p})\,\widetilde{C}_3^{\mathrm{p}-1}\,\|\mathbf{f}^n\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \frac{1}{2}\,\|\mathbf{u}_h^n\|^{2(\mathrm{p}-1)}_{\mathbf{M}},\end{aligned}$$

with $\widetilde{C}_3$ and $C(\mathrm{p})$ defined in (3.32), which, similarly to (5.8), yields

$$\|\mathbf{u}_h^m\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \Delta t\sum_{n=1}^m\|\mathbf{u}_h^n\|^{2(\mathrm{p}-1)}_{\mathbf{M}} \leq C_3\left\{\Delta t\sum_{n=1}^m\|\mathbf{f}^n\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_h^0\|^{2(\mathrm{p}-1)}_{\mathbf{L}^2(\Omega)}\right\}, \tag{5.10}$$

with C_3 defined in (3.33). Then, taking square in (5.9), using (5.8) and (5.10), we deduce the analogous estimate of (3.34), that is

$$\begin{aligned} \Delta t \sum_{n=1}^{m} \|\boldsymbol{\sigma}_h^n\|_{\mathbb{X}}^2 \,\le\, C_4 \,\Bigg\{\, \Delta t \sum_{n=1}^{m} \left(\|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^{2(\mathrm{p}-1)} + \|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^2 \right) \\ + \|\mathbf{u}_h^0\|_{\mathbf{L}^2(\Omega)}^{2\,(\mathrm{p}-1)} + \|\mathbf{u}_h^0\|_{\mathbf{L}^2(\Omega)}^2 + \Delta t \sum_{n=1}^{m} \|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \,\Bigg\}, \quad \text{with } \ m = 1, \ldots, N \,, \end{aligned} \tag{5.11}$$

with $C_4 := 10\, C_2^2\, \widetilde{\beta}^{-2} \max\{1, C_1, C_3\}$ (cf. (5.8), (3.31), (3.33)). Next, in order to bound the last term in (5.11), we choose $(\underline{\mathbf{v}}_h, \boldsymbol{\tau}_h) = ((d_t\, \mathbf{u}_h^n, d_t\, \mathbf{t}_h^n), \boldsymbol{\sigma}_h^n)$ in (5.1), apply some algebraic manipulation, and employ the Cauchy–Schwarz and Young's inequalities, to obtain

$$\begin{aligned} \|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2}\, d_t \left(\alpha\, \|\mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 + \nu\, \|\mathbf{t}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \right) + \mathrm{F}\, (|\mathbf{u}_h^n|^{\mathrm{p}-2} \mathbf{u}_h^n, d_t \mathbf{u}_h^n)_\Omega \\ + \frac{1}{2}\, \Delta t \left(\alpha\, \|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 + \nu\, \|d_t \mathbf{t}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \right) \,\le\, \frac{1}{2} \|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^2 + \frac{1}{2} \|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \,. \end{aligned} \tag{5.12}$$

In turn, employing Hölder and Young's inequalities, we have

$$\left| (|\mathbf{u}_h^n|^{\mathrm{p}-2} \mathbf{u}_h^n, \mathbf{u}_h^{n-1})_\Omega \right| \,\le\, \frac{\mathrm{p}-1}{\mathrm{p}} \|\mathbf{u}_h^n\|_{\mathbf{M}}^{\mathrm{p}} + \frac{1}{\mathrm{p}} \|\mathbf{u}_h^{n-1}\|_{\mathbf{M}}^{\mathrm{p}},$$

which implies

$$(|\mathbf{u}_h^n|^{\mathrm{p}-2} \mathbf{u}_h^n, d_t \mathbf{u}_h^n)_\Omega \,\ge\, \frac{(\Delta t)^{-1}}{\mathrm{p}} \left(\|\mathbf{u}_h^n\|_{\mathbf{M}}^{\mathrm{p}} - \|\mathbf{u}_h^{n-1}\|_{\mathbf{M}}^{\mathrm{p}} \right) = \frac{1}{\mathrm{p}}\, d_t\, \|\mathbf{u}_h^n\|_{\mathbf{M}}^{\mathrm{p}} \,. \tag{5.13}$$

Thus, combining (5.12) with (5.13), summing up over the time index $n = 1, \ldots, m$, with $m = 1, \ldots, N$ and multiplying by Δt, we get

$$\begin{aligned} \frac{2\mathrm{F}}{\mathrm{p}} \|\mathbf{u}_h^m\|_{\mathbf{M}}^{\mathrm{p}} + \Delta t \sum_{n=1}^{m} \|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)}^2 \\ \le\, C_5 \left\{ \Delta t \sum_{n=1}^{m} \|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}_h^0\|_{\mathbf{M}}^{\mathrm{p}} + \|\mathbf{u}_h^0\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{t}_h^0\|_{\mathbb{Q}}^2 \right\}, \end{aligned} \tag{5.14}$$

with $C_5 := \max\big\{1, \alpha, 2\,\mathrm{F}/\mathrm{p}, \nu\big\}$ (cf. (3.35)). Then, combining (5.11) and (5.14) yields

$$\begin{aligned} \Delta t \sum_{n=1}^{m} \|\boldsymbol{\sigma}_h^n\|_{\mathbb{X}}^2 \,\le\, C_4\, (1 + C_5) \,\Bigg\{\, \Delta t \sum_{n=1}^{m} \left(\|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{f}^n\|_{\mathbf{L}^2(\Omega)}^{2(\mathrm{p}-1)} \right) \\ + \|\mathbf{u}_h^0\|_{\mathbf{L}^2(\Omega)}^{2(\mathrm{p}-1)} + \|\mathbf{u}_h^0\|_{\mathbf{M}}^{\mathrm{p}} + \|\mathbf{u}_h^0\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{t}_h^0\|_{\mathbb{Q}}^2 \,\Bigg\}, \quad \text{with } \ m = 1, \ldots, N \end{aligned}$$

which, combined with (5.8), the fact that $(\mathbf{u}_h^0, \mathbf{t}_h^0) = (\mathbf{u}_{h,0}, \mathbf{t}_{h,0})$ and the estimate (4.4), implies (5.3), with

$$\widetilde{C}_{\mathrm{BF},1} := \left(20 \max\{C_1, C_4\,(1 + C_5)\} \max\left\{1, C_0, 3^{\mathrm{p}-2}\, C_0^{\mathrm{p}-1}\right\}\right)^{1/2},$$

where C_0, C_1, C_4, and C_5 are the constants defined in (4.4), (5.8), (5.11), and (5.14), respectively. In addition, (4.4) and (5.14) yield (5.4), with

$$\widetilde{C}_{\mathrm{BF},2} := \left(\frac{\mathrm{p}}{2\,\mathrm{F}} \max\left\{1, \alpha, \frac{2\,\mathrm{F}}{\mathrm{p}}, \nu\right\} \max\{1, C_0\}\right)^{1/\mathrm{p}},$$

which concludes the proof. □

We observe that, similarly to (3.25) and (3.26), the constants $\widetilde{C}_{\mathrm{BF},1}$ and $\widetilde{C}_{\mathrm{BF},2}$ (cf. (5.3), (5.4)) are not explicitly computable since they depend on the theoretical constants $\|\mathbf{i}_{\mathrm{p}}\|$, C_{d} and $\widetilde{\beta}$.

Now, we proceed by establishing the corresponding rates of convergence for the fully discrete scheme (5.1). To that end, as in Section 4.2, we assume that $\{\mathcal{T}_h\}_{h>0}$ is a family of quasi-uniform triangulations and write

$\mathbf{e}^n_{\underline{\mathbf{u}}} = (\mathbf{e}^n_{\mathbf{u}}, \mathbf{e}^n_{\mathbf{t}}) = (\mathbf{u}^n - \mathbf{u}^n_h, \mathbf{t}^n - \mathbf{t}^n_h)$, and $\mathbf{e}^n_{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^n - \boldsymbol{\sigma}^n_h$. Next, given arbitrary $\widehat{\underline{\mathbf{v}}}^n_h := (\widehat{\mathbf{v}}^n_h, \widehat{\mathbf{r}}^n_h) \in \mathbf{V}_h$ (cf. (4.5)) and $\widehat{\boldsymbol{\tau}}^n_h \in \mathbb{X}_{0,h}$, with $n = 1, \ldots, N$, we decompose the errors into

$$\mathbf{e}^n_{\underline{\mathbf{u}}} = \boldsymbol{\delta}^n_{\underline{\mathbf{u}}} + \boldsymbol{\eta}^n_{\underline{\mathbf{u}}} = (\boldsymbol{\delta}^n_{\mathbf{u}}, \boldsymbol{\delta}^n_{\mathbf{t}}) + (\boldsymbol{\eta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{t}}), \quad \mathbf{e}^n_{\boldsymbol{\sigma}} = \boldsymbol{\delta}^n_{\boldsymbol{\sigma}} + \boldsymbol{\eta}^n_{\boldsymbol{\sigma}}, \tag{5.15}$$

with

$$\boldsymbol{\delta}^n_{\mathbf{u}} = \mathbf{u}^n - \widehat{\mathbf{v}}^n_h, \quad \boldsymbol{\delta}^n_{\mathbf{t}} = \mathbf{t}^n - \widehat{\mathbf{r}}^n_h, \quad \boldsymbol{\delta}^n_{\boldsymbol{\sigma}} = \boldsymbol{\sigma}^n - \widehat{\boldsymbol{\tau}}^n_h,$$

$$\boldsymbol{\eta}^n_{\mathbf{u}} = \widehat{\mathbf{v}}^n_h - \mathbf{u}^n_h, \quad \boldsymbol{\eta}^n_{\mathbf{t}} = \widehat{\mathbf{r}}^n_h - \mathbf{t}^n_h, \quad \boldsymbol{\eta}^n_{\boldsymbol{\sigma}} = \widehat{\boldsymbol{\tau}}^n_h - \boldsymbol{\sigma}^n_h.$$

Thus, subtracting the fully discrete problem (5.1) from the continuous counterparts (2.10) at each time step $n = 1, \ldots, N$, we obtain the following error system:

$$\begin{aligned} d_t\,[\mathcal{E}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] + [\mathcal{A}(\underline{\mathbf{u}}^n) - \mathcal{A}(\underline{\mathbf{u}}^n_h), \underline{\mathbf{v}}_h] + [\mathcal{B}(\underline{\mathbf{v}}_h), \mathbf{e}^n_{\boldsymbol{\sigma}}] &= (r_n(\mathbf{u}), \mathbf{v}_h)_\Omega, \\ [\mathcal{B}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \boldsymbol{\tau}_h] &= 0. \end{aligned} \tag{5.16}$$

for all $\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h$ and $\boldsymbol{\tau}_h \in \mathbb{X}_{0,h}$, where $r_n(\mathbf{u})$ denotes the difference between the time derivative and its discrete analog, that is

$$r_n(\mathbf{u}) = d_t \mathbf{u}^n - \partial_t \mathbf{u}(t_n).$$

In addition, we recall from [29, Lemma 4] that for sufficiently smooth $\mathbf{u}$, there holds

$$\|r_n(\mathbf{u})\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} \le C(\partial_{tt}\mathbf{u})\Delta t\,, \quad \text{with} \quad C(\partial_{tt}\mathbf{u}) := C\|\partial_{tt}\mathbf{u}\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}\,. \tag{5.17}$$

Then, using discrete-in-time arguments as in the proof of Theorem 5.1 and the estimate (5.17), the derivation of the theoretical rate of convergence of the fully discrete scheme (5.1) follows similarly to the proof of Theorem 4.6,.

We stress for later use that $d_t\,\underline{\mathbf{v}}^n_h \in \mathbf{V}_h$, when $\underline{\mathbf{v}}^n_h \in \mathbf{V}_h$ (cf. (4.5)), for each $n = 1, \ldots, N$. In fact, given $\underline{\mathbf{v}}^n_h \in \mathbf{V}_h$, with $n = 1, \ldots, N$, assuming $\underline{\mathbf{v}}^0_h \in \mathbf{V}_h$ and using the linearity of the operator $\mathcal{B}$, we obtain

$$[\mathcal{B}(d_t\,\underline{\mathbf{v}}^n_h), \boldsymbol{\tau}_h] = \frac{1}{\Delta t}\Big([\mathcal{B}(\underline{\mathbf{v}}^n_h), \boldsymbol{\tau}_h] - [\mathcal{B}(\underline{\mathbf{v}}^{n-1}_h), \boldsymbol{\tau}_h]\Big) = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{X}_{0,h}\,. \tag{5.18}$$

Next, for the sake of the presentation of the rate of convergence of the fully discrete scheme (5.1), we first establish two preliminary lemmas.

Lemma 5.2. *Let* $\mathrm{p} \in [3, 4]$. *Then, for the solution of the fully discrete problem* (5.1) *there exists* $C > 0$ *depending only on* $|\Omega|$, ν, α, $\mathtt{F}$, C_{d}, *and data, such that*

$$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \Delta t\,\|d_t\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\mathbf{t}}\|_{\ell^2(0,T;\mathbb{Q})} \le C\,\widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma})\,, \tag{5.19}$$

where

$$\begin{aligned} \widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma}) := {} & \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\ell^\infty(0,T;\mathbf{M}\times\mathbb{Q})} + \Delta t\,\|d_t\boldsymbol{\delta}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|d_t\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\ell^2(0,T;\mathbf{M}\times\mathbb{Q})} + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|^{\mathrm{p}-1}_{\ell^{2(\mathrm{p}-1)}(0,T;\mathbf{M}\times\mathbb{Q})} \\ & + \|\boldsymbol{\delta}_{\underline{\mathbf{u}}}\|_{\ell^2(0,T;\mathbf{M}\times\mathbb{Q})} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} + \|r_n(\mathbf{u})\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\delta}^0_{\underline{\mathbf{u}}}\|^{\mathrm{p}-1} + \|\boldsymbol{\delta}^0_{\underline{\mathbf{u}}}\| + \|\boldsymbol{\delta}^0_{\boldsymbol{\sigma}}\|_{\mathbb{X}}\,. \end{aligned} \tag{5.20}$$

Proof. Similarly as in the proof of Theorem 4.6, adding and subtracting suitable terms in (5.16) with $\underline{\mathbf{v}}_h = \boldsymbol{\eta}^n_{\underline{\mathbf{u}}} = (\boldsymbol{\eta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{t}}) \in \mathbf{V}_h$ and $\boldsymbol{\tau}_h = \boldsymbol{\eta}^n_{\boldsymbol{\sigma}} \in \mathbb{X}_{0,h}$, with $n = 1, \ldots, N$, and employing the strict monotonicity of $\mathcal{A}$ (cf. (3.15)), we deduce that

$$\begin{aligned} & (d_t\,\boldsymbol{\eta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega + \alpha\,\|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{F}\,C_{\mathrm{p}}\,\|\boldsymbol{\eta}^n_{\mathbf{u}}\|^{\mathrm{p}}_{\mathbf{M}} + \nu\,\|\boldsymbol{\eta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} \\ & \quad \le -(d_t\,\boldsymbol{\delta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega - \alpha\,(\boldsymbol{\delta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega - \mathtt{F}\,(|\mathbf{u}^n|^{\mathrm{p}-2}\mathbf{u}^n - |\widehat{\mathbf{v}}^n_h|^{\mathrm{p}-2}\widehat{\mathbf{v}}^n_h, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega \\ & \quad - \nu\,(\boldsymbol{\delta}^n_{\mathbf{t}}, \boldsymbol{\eta}^n_{\mathbf{t}})_\Omega - [\mathcal{B}(\boldsymbol{\eta}^n_{\underline{\mathbf{u}}}), \boldsymbol{\delta}^n_{\boldsymbol{\sigma}}] + (r_n(\mathbf{u}), \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega\,. \end{aligned}$$

Notice that $[\mathcal{B}(\boldsymbol{\eta}^n_{\underline{\mathbf{u}}}), \boldsymbol{\eta}^n_{\boldsymbol{\sigma}}] = 0$ since $\boldsymbol{\eta}^n_{\underline{\mathbf{u}}} \in \mathbf{V}_h$, $n = 1, \ldots, N$. In addition, using the identity (5.5), the fact that $(\boldsymbol{\eta}^n_{\mathbf{u}}, \boldsymbol{\eta}^n_{\mathbf{t}}) \in \mathbf{V}_h$ (cf. (4.7)), the continuity bound of $\mathcal{B}$ (cf. (3.3)), and similar arguments employed to derive (4.22), we obtain

$$\begin{aligned} & d_t\,\|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \Delta t\,\|d_t\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} \\ & \quad \le C_1\left\{\|d_t\,\boldsymbol{\delta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}^n_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \big(1 + \|\mathbf{u}^n\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\big)\|\boldsymbol{\delta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}^n_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|r_n(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}\right\}, \end{aligned} \tag{5.21}$$

with $C_1 > 0$ depending on $|\Omega|, \nu, \alpha, \mathtt{F}$, and C_{d}. Thus, summing up over the time index $n = 1, \ldots, m$, with $m = 1, \ldots, N$, in (5.21) and multiplying by Δt, we get

$$\begin{aligned} &\|\boldsymbol{\eta}^m_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + (\Delta t)^2 \sum_{n=1}^{m} \|d_t \boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \Delta t \sum_{n=1}^{m} \Big(\|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\eta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} \Big) \\ &\leq C_2\, \Delta t \sum_{n=1}^{m} \Bigg\{ \|d_t\, \boldsymbol{\delta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}^n_{\mathbf{u}}\|^{2(\mathrm{p}-1)}_{\mathbf{M}} + \big(1 + \|\mathbf{u}^n\|^{2(\mathrm{p}-2)}_{\mathbf{M}}\big) \|\boldsymbol{\delta}^n_{\mathbf{u}}\|^2_{\mathbf{M}} \\ &+ \|\boldsymbol{\delta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} + \|\boldsymbol{\delta}^n_{\boldsymbol{\sigma}}\|^2_{\mathbb{X}} + \|r_n(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)} \Bigg\} + \|\boldsymbol{\eta}^0_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}, \end{aligned} \tag{5.22}$$

with $C_2 > 0$ depending on $|\Omega|, \nu, \alpha, \mathtt{F}$, and C_{d}. Thus, using (4.24) and the error decomposition (5.15) to bound $\|\boldsymbol{\eta}^0_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)}$, noting that $\|\mathbf{u}\|_{\ell^\infty(0,T;\mathbf{M})}$ is bounded by $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$, which is bounded by data (cf. (3.26)), we obtain (5.19) completing the proof. □

Lemma 5.3. *Let* $\mathrm{p} \in [3, 4]$. *Then, for the solution of the fully discrete problem* (5.1) *there exists* $C > 0$ *depending only on* $|\Omega|$, ν, α, $\mathtt{F}$, $\widetilde{\beta}$, *and data, such that*

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} \leq C\, h^{-d(\mathrm{p}-2)/(2\mathrm{p})} \Big\{ \widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma}) + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\ell^\infty(0,T;\mathbb{X})} + \|d_t\, \boldsymbol{\delta}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} \Big\}, \tag{5.23}$$

with $\widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma})$ *defined in* (5.20).

Proof. We proceed as in Lemma 4.5, using the discrete inf–sup condition of $\mathcal{B}$ (cf. (4.6)), the first equation of (5.16), and the continuity bound of $\mathcal{E}, \mathcal{A}, \mathcal{B}$ (cf. (3.5), (3.10), (3.3)), to deduce that

$$\begin{aligned} \widetilde{\beta}\, \|\boldsymbol{\eta}^n_{\boldsymbol{\sigma}}\|_{\mathbb{X}} &\leq \sup_{\substack{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h \\ \underline{\mathbf{v}}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\eta}^n_{\boldsymbol{\sigma}}]}{\|\underline{\mathbf{v}}_h\|} \\ &= \sup_{\substack{\underline{\mathbf{v}}_h \in \mathbf{M}_h \times \mathbb{Q}_h \\ \underline{\mathbf{v}}_h \neq \mathbf{0}}} \frac{-[d_t\, \mathcal{E}(\mathbf{e}^n_{\underline{\mathbf{u}}}), \underline{\mathbf{v}}_h] - [\mathcal{A}(\underline{\mathbf{u}}^n) - \mathcal{A}(\underline{\mathbf{u}}^n_h), \underline{\mathbf{v}}_h] - [\mathcal{B}(\underline{\mathbf{v}}_h), \boldsymbol{\delta}^n_{\boldsymbol{\sigma}}] + (r_n(\mathbf{u}), \mathbf{v}_h)_\Omega}{\|\underline{\mathbf{v}}_h\|} \\ &\leq C_3 \Big(\|d_t\, \mathbf{e}^n_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)} + \big(\|\mathbf{u}^n\|_{\mathbf{M}} + \|\mathbf{u}^n_h\|_{\mathbf{M}}\big)^{\mathrm{p}-2} \|\mathbf{e}^n_{\mathbf{u}}\|_{\mathbf{M}} + \|\mathbf{e}^n_{\underline{\mathbf{u}}}\| + \|\boldsymbol{\delta}^n_{\boldsymbol{\sigma}}\|_{\mathbb{X}} + \|r_n(\mathbf{u})\|_{\mathbf{L}^2(\Omega)} \Big). \end{aligned}$$

Then, taking square in the above inequality, summing up over the time index $n = 1, \ldots, m$, with $m = 1, \ldots, N$, multiplying by Δt, noting that $\|\mathbf{u}\|_{\ell^\infty(0,T;\mathbf{M})}$ is bounded by $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$, which in turn is bounded by data, as well as $\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{M})}$ (cf. (3.26) and (5.3)), and employing (5.19), we get

$$\|\boldsymbol{\eta}^n_{\boldsymbol{\sigma}}\|^2_{\ell^2(0,T;\mathbb{X})} \leq C_4 \Big\{ \widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma})^2 + \Delta t \sum_{n=1}^{N} \|d_t\, \boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \Big\}, \tag{5.24}$$

with $\widehat{\Psi}(\underline{\mathbf{u}}, \boldsymbol{\sigma})$ defined in (5.20) and $C_4 > 0$ depending on $|\Omega|$, ν, α, $\mathtt{F}$, $\widetilde{\beta}$, and data. Next, in order to bound the last term on the right-hand side of (5.24), we choose $\underline{\mathbf{v}}_h = (d_t\, \boldsymbol{\eta}^n_{\mathbf{u}}, d_t\, \boldsymbol{\eta}^n_{\mathbf{t}})$ in the first equation of (5.16) and use the identity (5.5), and the fact that $\boldsymbol{\eta}^n_{\underline{\mathbf{u}}} \in \mathbf{V}_h$ (cf. (5.18)), which implies $[\mathcal{B}(d_t\, \boldsymbol{\eta}^n_{\underline{\mathbf{u}}}), \boldsymbol{\eta}^n_{\boldsymbol{\sigma}}] = 0$, to find that

$$\begin{aligned} &\frac{1}{2}\, d_t \Big(\alpha\, \|\boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\, \|\boldsymbol{\eta}_{\mathbf{t}}\|^2_{\mathbb{Q}} \Big) + \frac{1}{2}\, \Delta t \Big(\alpha\, \|d_t \boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \nu\, \|d_t \boldsymbol{\eta}^n_{\mathbf{t}}\|^2_{\mathbb{Q}} \Big) + \|d_t\, \boldsymbol{\eta}^n_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} \\ &= -(d_t\, \boldsymbol{\delta}^n_{\mathbf{u}}, d_t\, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega - \alpha\, (\boldsymbol{\delta}^n_{\mathbf{u}}, d_t\, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega - \mathtt{F}\, (|\mathbf{u}^n|^{\mathrm{p}-2}\mathbf{u}^n - |\mathbf{u}^n_h|^{\mathrm{p}-2}\mathbf{u}^n_h, d_t\, \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega \\ &+ (d_t\, \boldsymbol{\eta}^n_{\mathbf{u}}, \mathbf{div}(\boldsymbol{\delta}^n_{\boldsymbol{\sigma}}))_\Omega + (r_n(\mathbf{u}), d_t \boldsymbol{\eta}^n_{\mathbf{u}})_\Omega - \nu\, (\boldsymbol{\delta}^n_{\mathbf{t}}, d_t\, \boldsymbol{\eta}^n_{\mathbf{t}})_\Omega + (d_t\, \boldsymbol{\eta}^n_{\mathbf{t}}, \boldsymbol{\delta}^n_{\boldsymbol{\sigma}})_\Omega\,. \end{aligned}$$

Then, using the identities

$$(\boldsymbol{\delta}^n_{\mathbf{t}}, d_t\, \boldsymbol{\eta}^n_{\mathbf{t}})_\Omega = d_t \big(\boldsymbol{\delta}^n_{\mathbf{t}}, \boldsymbol{\eta}^n_{\mathbf{t}}\big)_\Omega - (d_t\, \boldsymbol{\delta}^n_{\mathbf{t}}, \boldsymbol{\eta}^{n-1}_{\mathbf{t}})_\Omega, \quad \text{and} \quad (d_t\, \boldsymbol{\eta}^n_{\mathbf{t}}, \boldsymbol{\delta}^n_{\boldsymbol{\sigma}})_\Omega = d_t \big(\boldsymbol{\eta}^n_{\mathbf{t}}, \boldsymbol{\delta}^n_{\boldsymbol{\sigma}}\big)_\Omega - (\boldsymbol{\eta}^{n-1}_{\mathbf{t}}, d_t\, \boldsymbol{\delta}^n_{\boldsymbol{\sigma}})_\Omega,$$

with $n = 1, \ldots, N$, in combination with Cauchy–Schwarz, Hölder and Young's inequalities (cf. (1.1)), the continuity bound (3.12), and the fact that $\|d_t\, \boldsymbol{\eta}_{\mathbf{u}}^n\|_{\mathbf{M}} \le c\, h^{-d(\mathrm{p}-2)/(2\mathrm{p})}\, \|d_t\, \boldsymbol{\eta}_{\mathbf{u}}^n\|_{\mathbf{L}^2(\Omega)}$, with $\boldsymbol{\eta}_{\mathbf{u}}^n \in \mathbf{M}_h$ (cf. (4.12)), we obtain

$$\begin{aligned}
&\frac{1}{2}\, d_t \left(\alpha\, \|\boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} + \nu \|\boldsymbol{\eta}_{\mathbf{t}}^n\|^2_{\mathbb{Q}}\right) + \frac{1}{2}\, \Delta t \left(\alpha\, \|d_t \boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} + \nu\, \|d_t \boldsymbol{\eta}_{\mathbf{t}}^n\|^2_{\mathbb{Q}}\right) + \|d_t\, \boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le\, C_5\, h^{-d(\mathrm{p}-2)/\mathrm{p}}\, \widehat{C}(\mathbf{u}^n, \mathbf{u}_h^n) \left(\|d_t\, \boldsymbol{\delta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\mathbf{u}}^n\|^2_{\mathbf{M}} + \|\mathbf{e}_{\mathbf{u}}^n\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}^n\|^2_{\mathbb{X}} + \|r_n(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}\right) \\
&\quad\quad + \frac{1}{2}\, \|d_t\, \boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} + d_t \left((\boldsymbol{\eta}_{\mathbf{t}}^n, \boldsymbol{\delta}_{\boldsymbol{\sigma}}^n)_\Omega - \nu\, (\boldsymbol{\delta}_{\mathbf{t}}^n, \boldsymbol{\eta}_{\mathbf{t}}^n)_\Omega\right) + \nu\, (d_t\, \boldsymbol{\delta}_{\mathbf{t}}^n, \boldsymbol{\eta}_{\mathbf{t}}^{n-1})_\Omega - (\boldsymbol{\eta}_{\mathbf{t}}^{n-1}, d_t\, \boldsymbol{\delta}_{\boldsymbol{\sigma}}^n)_\Omega\,,
\end{aligned}$$

where

$$\widehat{C}(\mathbf{u}^n, \mathbf{u}_h^n) \,:=\, 1 + \|\mathbf{u}^n\|_{\mathbf{M}}^{2(\mathrm{p}-2)} + \|\mathbf{u}_h^n\|_{\mathbf{M}}^{2(\mathrm{p}-2)},$$

and C_5 is a positive constant depending on $|\Omega|, \alpha$ and $\mathtt{F}$. Thus, summing up over the time index $n = 1, \ldots, m$, with $m = 1, \ldots, N$, and multiplying by Δt, using Cauchy–Schwarz and Young's inequalities, and minor algebraic manipulations, we get

$$\begin{aligned}
&\|\boldsymbol{\eta}_{\mathbf{u}}^m\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\eta}_{\mathbf{t}}^m\|^2_{\mathbb{Q}} + (\Delta t)^2 \sum_{n=1}^m \left(\|d_t \boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} + \|d_t \boldsymbol{\eta}_{\mathbf{t}}^n\|^2_{\mathbb{Q}}\right) + \Delta t \sum_{n=1}^m \|d_t\, \boldsymbol{\eta}_{\mathbf{u}}^n\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le C_6\, h^{-d(\mathrm{p}-2)/\mathrm{p}}\, \Delta t \sum_{n=1}^m \widehat{C}(\mathbf{u}^n, \mathbf{u}_h^n) \left(\|d_t\, \boldsymbol{\delta}_{\mathbf{u}}^n\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\mathbf{u}}^n\|^2_{\mathbf{M}} + \|\mathbf{e}_{\mathbf{u}}^n\|^2_{\mathbf{M}} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}^n\|^2_{\mathbb{X}} + \|r_n(\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}\right) \\
&\quad\quad + C_7 \left\{ \|\boldsymbol{\delta}_{\mathbf{t}}^m\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}^m\|^2_{\mathbb{X}} + \Delta t \sum_{n=1}^m \left(\|d_t\, \boldsymbol{\delta}_{\mathbf{t}}^n\|^2_{\mathbf{L}^2(\Omega)} + \|d_t \boldsymbol{\delta}_{\boldsymbol{\sigma}}^n\|^2_{\mathbb{X}}\right) + \|\boldsymbol{\delta}_{\mathbf{t}}^0\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\delta}_{\boldsymbol{\sigma}}^0\|^2_{\mathbb{X}} \right. \\
&\quad\quad \left. + \Delta t \sum_{n=1}^{m-1} \|\boldsymbol{\eta}_{\mathbf{t}}^n\|^2_{\mathbb{Q}} + \|\boldsymbol{\eta}_{\mathbf{u}}^0\|^2_{\mathbf{L}^2(\Omega)} + \big(1 + \Delta t\big) \|\boldsymbol{\eta}_{\mathbf{t}}^0\|^2_{\mathbb{Q}} \right\},
\end{aligned} \tag{5.25}$$

with $C_6, C_7 > 0$ depending on $|\Omega|, \nu, \alpha$ and $\mathtt{F}$. Thus, using the error decomposition (5.15), combining (5.22) and (5.25), employing (4.24) to bound the terms $\|\boldsymbol{\eta}_{\mathbf{u}}^0\|_{\mathbf{L}^2(\Omega)}, \|\boldsymbol{\eta}_{\mathbf{t}}^0\|_{\mathbb{Q}}$, noting again that $\|\mathbf{u}\|_{\ell^\infty(0,T;\mathbf{M})}$ is bounded by $\|\mathbf{u}\|_{\mathrm{L}^\infty(0,T;\mathbf{M})}$, which together with $\|\mathbf{u}_h\|_{\ell^\infty(0,T;\mathbf{M})}$ are bounded by data (cf. (3.26) and (5.3)), and considering sufficiently small values of h, we obtain (5.23) concluding the proof. □

Theorem 5.4. *Let the assumptions of Theorem 4.6 hold, with* $\mathrm{p} \in [3, 4]$. *Then, for the solution of the fully discrete problem (5.1) there exist* $\widehat{C}_1(\underline{\mathbf{u}}, \boldsymbol{\sigma}), \widehat{C}_2(\underline{\mathbf{u}}, \boldsymbol{\sigma}) > 0$ *depending only on* $C(\underline{\mathbf{u}})$, $C(\partial_t\, \underline{\mathbf{u}})$, $C(\partial_{tt} \mathbf{u})$, $C(\boldsymbol{\sigma})$, $C(\partial_t\, \boldsymbol{\sigma})$, $|\Omega|$, ν, α, $\mathtt{F}$, $\widetilde{\beta}$, *and data, such that*

$$\begin{aligned}
&\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \Delta t\, \|d_t \mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\ell^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\mathbf{t}}\|_{\ell^2(0,T;\mathbb{Q})} \\
&\quad \le\, \widehat{C}_1(\underline{\mathbf{u}}, \boldsymbol{\sigma}) \left(h^l + h^{l\,(\mathrm{p}-1)} + \Delta t\right)
\end{aligned} \tag{5.26}$$

and

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})} \,\le\, \widehat{C}_2(\underline{\mathbf{u}}, \boldsymbol{\sigma})\, h^{-d(\mathrm{p}-2)/(2\mathrm{p})} \left(h^l + h^{l\,(\mathrm{p}-1)} + \Delta t\right). \tag{5.27}$$

Proof. The assertion of the theorem follows straightforwardly from Lemmas 5.2 and 5.3, the estimate (5.17), the fact that $\widehat{\underline{\mathbf{v}}}_h^n \in \mathbf{V}_h$ and $\boldsymbol{\tau}_h^n \in \mathbb{X}_{0,h}$, with $n = 0, 1, \ldots, N$, are arbitrary, taking infimum in (5.19) and (5.23), over the corresponding discrete subspaces $\mathbf{V}_h$ and $\mathbb{X}_{0,h}$, and using (5.17) and the approximation properties (4.14). □

Remark 5.1. For the fully discrete scheme (5.1) we have considered the backward Euler method only for the sake of simplicity. The analysis developed in Section 5 can be adapted to other time discretizations, such as BDF schemes or the Crank–Nicolson method.

6. Numerical results

In this section we present four numerical results that illustrate the performance of the fully discrete method (5.1) on a set of quasi-uniform triangulations of the respective domains, considering the finite element subspaces defined by (4.1) (cf. Section 4.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ and $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$, respectively. Our implementation is based on a `FreeFem++` code [30], in conjunction with the direct linear solver `UMFPACK` [31]. We handle the nonlinearly using a Newton–Raphson algorithm with a fixed tolerance $\mathsf{tol} = 1\mathrm{E}-06$. As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^{m+1}$ and $\mathbf{coeff}^{m}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|}{\|\mathbf{coeff}^{m+1}\|} \leq \mathsf{tol},$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$, with DOF denoting the total number of degrees of freedom defined by the finite element subspaces $\mathbf{M}_h$, $\mathbb{Q}_h$ and $\mathbb{X}_{0,h}$ (cf. (4.1)).

We stress that, according to the notation used for the fully discrete norm (5.2), and besides the unknowns $\mathbf{u}, \mathbf{t}$, and $\boldsymbol{\sigma}$, we are also able to compute the pressure error:

$$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} = \left\{\Delta t \sum_{n=1}^{N} \|p^n - p_h^n\|^2_{\mathrm{L}^2(\Omega)}\right\}^{1/2},$$

where, p_h^n stands for the post-processed pressure suggested by the identity (2.3), that is

$$p_h^n = -\frac{1}{d}\,\mathrm{tr}(\boldsymbol{\sigma}_h^n) \quad \text{with } n = 1, \ldots, N\,. \tag{6.1}$$

It follows that

$$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} = \frac{1}{d}\,\|\mathrm{tr}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} \leq \frac{1}{\sqrt{d}}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\ell^2(0,T;\mathbb{X})},$$

which shows that the rate of convergence for p is at least the one for $\boldsymbol{\sigma}$. This is indeed confirmed by the numerical results reported below.

The examples considered in this section are described next. In all of them, and for the sake of simplicity, we choose $\nu = 1$. In addition, the condition $(\mathrm{tr}(\boldsymbol{\sigma}_h^n), 1)_\Omega = 0$ is implemented using a scalar Lagrange multiplier (adding one row and one column to the matrix system that solves (5.1) for $\mathbf{u}_h^n$, $\mathbf{t}_h^n$, and $\boldsymbol{\sigma}_h^n$).

Examples 1 and 2 are used to corroborate the rate of convergence in two and three dimensional domains, respectively. The total simulation time for these examples is $T = 0.01$ and the time step is $\Delta t = 10^{-3}$. The time step is sufficiently small, so that the time discretization error does not affect the convergence rates. On the other hand, Examples 3 and 4 are used to analyze the behavior of the method when different Darcy and Forchheimer coefficients are considered in different scenarios. For these cases, the total simulation time and the time step are considered as $T = 1$ and $\Delta t = 10^{-2}$, respectively.

Example 1 (*2D Domain with Different Values of the Parameter* p)**.** In this test we corroborate the convergence for the space discretization using an analytical solution and also study the performance of the numerical method with respect to the total error and different values of the power p in the inertial term $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$ (cf. (2.4)). The domain is the square $\Omega = (0,1)^2$. First, we consider $\mathrm{p} = 4$, $\alpha = 1$, $\mathrm{F} = 10$, and the data $\mathbf{f}$ and the initial condition $\mathbf{u}_0$ are defined by means of the exact solution given by the smooth functions

$$\mathbf{u} = \exp(t)\begin{pmatrix}\sin(\pi x)\cos(\pi y)\\ -\cos(\pi x)\sin(\pi y)\end{pmatrix}, \quad p = \exp(t)\cos(\pi x)\sin\left(\frac{\pi y}{2}\right).$$

Notice that the given exact solution $\mathbf{u}$ is non-homogeneous on the boundary so that the right-hand side must be adjusted properly as described in Remark 3.4.

In Fig. 6.1 we display the solution obtained with the mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation with mesh size $h = 0.0128$ and 39,146 triangle elements (actually representing $979,674$ DOF) at time $T = 0.01$. Note that we are able to compute not only the original unknowns, but also the pressure field through the formula (6.1). Tables 6.1 and 6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the average number of Newton iterations. The results illustrate that the optimal and sub-optimal spatial rates of convergence $\mathcal{O}(h^{k+1})$ and $\mathcal{O}(h^{k+1/2})$ for $(\mathbf{u}, \mathbf{t})$ and $\boldsymbol{\sigma}$, respectively, provided by Theorem 5.4 (see also Theorem 4.6) are attained for $d = 2$, $\mathrm{p} = 4$, and $k = 0, 1$. Moreover, the numerical results suggest optimal rate of convergence $\mathcal{O}(h^{k+1})$ for

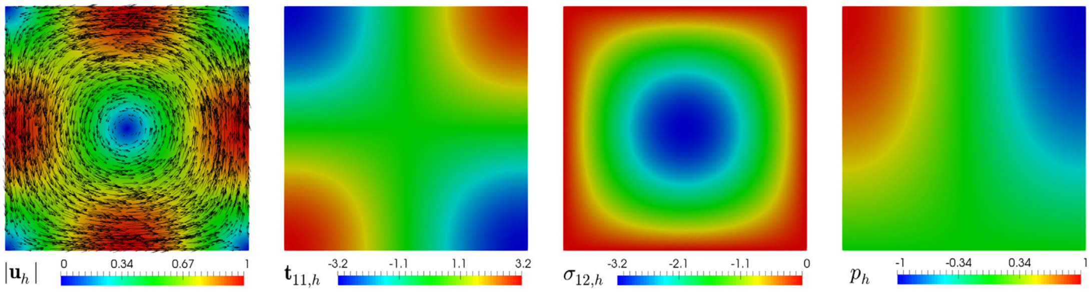

Fig. 6.1. Example 1, computed magnitude of the velocity, velocity gradient component, pseudostress tensor component, and pressure field.

Table 6.1
Example 1, number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation of the Brinkman–Forchheimer model with $\mathtt{p} = 4$ and $\mathtt{F} = 10$.

DOF	h	$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{M})}$		$\|\mathbf{e_t}\|_{\ell^2(0,T;\mathbb{Q})}$		$\|\mathbf{e}_\sigma\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		iter
		Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	
304	0.3727	2.02E−01	–	2.51E−02	–	9.23E−02	–	4.99E−01	–	4.31E−02	–	2.3
1248	0.1964	8.73E−02	1.3069	1.09E−02	1.2964	4.48E−02	1.1299	1.88E−01	1.5214	1.88E−02	1.2980	2.2
4896	0.0970	4.38E−02	0.9772	5.48E−03	0.9806	2.24E−02	0.9782	8.60E−02	1.1116	8.30E−03	1.1563	2.2
19 456	0.0478	2.13E−02	1.0183	2.65E−03	1.0294	1.14E−02	0.9617	3.96E−02	1.0954	3.46E−03	1.2360	2.2
77 648	0.0245	1.08E−02	1.0188	1.35E−03	1.0115	5.66E−03	1.0427	1.96E−02	1.0539	1.76E−03	1.0131	2.2
313 680	0.0128	5.35E−03	1.0755	6.67E−04	1.0769	2.80E−03	1.0790	9.65E−03	1.0865	8.44E−04	1.1264	2.2

Table 6.2
Example 1, number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation of the Brinkman–Forchheimer model with $\mathtt{p} = 4$ and $\mathtt{F} = 10$.

DOF	h	$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{M})}$		$\|\mathbf{e_t}\|_{\ell^2(0,T;\mathbb{Q})}$		$\|\mathbf{e}_\sigma\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		iter
		Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	
932	0.3727	5.71E−02	–	5.53E−03	–	3.56E−02	–	6.52E−01	–	6.34E−02	–	2.7
3864	0.1964	1.39E−02	2.2117	1.31E−03	2.2546	8.44E−03	2.2489	1.83E−01	1.9865	1.14E−02	2.6740	2.3
15 228	0.0970	3.46E−03	1.9675	3.23E−04	1.9787	2.07E−03	1.9902	4.98E−02	1.8416	1.85E−03	2.5816	2.2
60 656	0.0478	8.76E−04	1.9398	8.10E−05	1.9561	5.24E−04	1.9431	1.31E−02	1.8874	3.99E−04	2.1684	2.2
242 362	0.0245	2.20E−04	2.0693	2.04E−05	2.0646	1.29E−04	2.0955	3.38E−03	2.0269	6.53E−05	2.7076	2.2
979 674	0.0128	5.35E−05	2.1671	4.91E−06	2.1801	3.07E−05	2.2019	8.10E−04	2.1911	1.23E−05	2.5544	2.2

Table 6.3
Example 1, number of degrees of freedom, mesh sizes, total errors, rates of convergences, and average number of Newton iterations for the $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation of the Brinkman–Forchheimer model, considering $\mathrm{p} \in \{3.0, 3.2, 3.4, 3.6, 3.8, 4.0\}$ and $\mathtt{F} = 10$.

DOF	h	p = 3.0			p = 3.2			p = 3.4		
		$\mathbf{e}_{\mathtt{total}}$	Rate	iter	$\mathbf{e}_{\mathtt{total}}$	Rate	iter	$\mathbf{e}_{\mathtt{total}}$	Rate	iter
304	0.3727	5.20E−01	–	2.1	5.17E−01	–	2.2	5.14E−01	–	2.3
1248	0.1964	1.99E−01	1.4991	2.1	1.98E−01	1.5005	2.1	1.97E−01	1.5017	2.2
4896	0.0970	9.18E−02	1.0978	2.1	9.11E−02	1.0992	2.1	9.05E−02	1.1004	2.2
19 456	0.0478	4.26E−02	1.0834	2.1	4.23E−02	1.0839	2.1	4.20E−02	1.0844	2.2
77 648	0.0245	2.11E−02	1.0500	2.1	2.10E−02	1.0507	2.1	2.08E−02	1.0514	2.2
313 680	0.0128	1.04E−02	1.0846	2.1	1.03E−02	1.0849	2.1	1.02E−02	1.0852	2.2

p = 3.6			p = 3.8			p = 4.0		
$\mathbf{e}_{\mathtt{total}}$	Rate	iter	$\mathbf{e}_{\mathtt{total}}$	Rate	iter	$\mathbf{e}_{\mathtt{total}}$	Rate	iter
5.12E−01	–	2.3	5.10E−01	–	2.3	5.08E−01	–	2.3
1.96E−01	1.5027	2.2	1.95E−01	1.5035	2.2	1.94E−01	1.5042	2.2
8.99E−02	1.1015	2.2	8.95E−02	1.1025	2.2	8.91E−02	1.1034	2.2
4.17E−02	1.0849	2.2	4.15E−02	1.0854	2.2	4.13E−02	1.0859	2.2
2.07E−02	1.0519	2.2	2.05E−02	1.0524	2.2	2.04E−02	1.0529	2.2
1.02E−02	1.0854	2.2	1.01E−02	1.0857	2.2	1.01E−02	1.0859	2.2

all the unknowns. The Newton's method exhibits a behavior independent of the mesh size, converging in average of 2.2 iterations in all cases. On the other hand, in Table 6.3 we show the behavior of our method with respect to the total error

$$\mathbf{e}_{\mathtt{total}} := \left(\|\mathbf{e}_{\mathbf{u}}\|^2_{\ell^2(0,T;\mathbf{M})} + \|\mathbf{e}_{\mathbf{t}}\|^2_{\ell^2(0,T;\mathbb{Q})} + \|\mathbf{e}_{\boldsymbol{\sigma}}\|^2_{\ell^2(0,T;\mathbb{X})} \right)^{1/2},$$

considering $\alpha = 1, \mathtt{F} = 10$, and different powers $\mathrm{p} \in \{3.0, 3.2, 3.4, 3.6, 3.8, 4.0\}$ in the inertial term $|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$ (cf. (2.4)), polynomial degree $k = 0$, and different mesh sizes h. Here we observe that the method provides optimal rate of convergence independently of p.

Example 2 (*Convergence Against Smooth Exact Solutions in a 3D Domain*)**.** In our second example, we consider the cube domain $\Omega = (0, 1)^3$ and the exact solution:

$$\mathbf{u} = \exp(t) \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}, \quad p = \exp(t)\,(x - 0.5)^3 \sin(y + z).$$

Similarly to the first example, we consider the parameters $\mathrm{p} = 4, \alpha = 1$, and $\mathtt{F} = 10$, whereas the right-hand side function $\mathbf{f}$ is computed from (2.1) using the above solution. In addition, the model problem is complemented with the appropriate Dirichlet boundary condition and initial data.

The numerical solutions at time $T = 0.01$ are shown in Fig. 6.2, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation with mesh size $h = 0.0786$ and 34,992 tetrahedral elements (actually representing 600, 696 DOF). The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.4. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$ for all the unknowns, which, in particular, is better than the theoretical suboptimal rate of convergence $\mathcal{O}(h^{1/4})$ provided by (5.27) in Theorem 5.4 (see also Theorem 4.6) for $\boldsymbol{\sigma}$ with $d = 3, \mathrm{p} = 4$, and $k = 0$.

Example 3 (*Flow Through Porous Media with Channel Network*)**.** In our third example, inspired by [32, Section 5.2.4], we focus on a flow through a porous medium with a channel network. We consider the square domain $\Omega = (-1, 1)^2$ with an internal channel network denoted as Ω_c, which is described in the first plot of Fig. 6.3. First, we consider the Brinkman–Forchheimer model (2.4) in the whole domain Ω, with inertial power $\mathrm{p} = 4$ but with different values of the parameters α and $\mathtt{F}$ for the interior and the exterior of the channel, that is,

$$\alpha = \begin{cases} 1 & \text{in } \Omega_c \\ 1000 & \text{in } \Omega \setminus \Omega_c \end{cases} \quad \text{and} \quad \mathtt{F} = \begin{cases} 10 & \text{in } \Omega_c \\ 1 & \text{in } \Omega \setminus \Omega_c \end{cases}.$$

Table 6.4
Example 2, number of degrees of freedom, mesh sizes, errors, rates of convergences, and average number of Newton iterations for the mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation of the Brinkman–Forchheimer model with $\mathrm{p} = 4$ and $\mathtt{F} = 10$.

DOF	h	$\|\mathbf{e_u}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e_u}\|_{\ell^2(0,T;\mathbf{M})}$		$\|\mathbf{e_t}\|_{\ell^2(0,T;\mathbb{Q})}$	
		Error	Rate	Error	Rate	Error	Rate
888	0.7071	4.50E−01	–	5.73E−02	–	2.93E−01	–
2916	0.4714	3.11E−01	1.2964	3.96E−02	0.9106	1.93E−01	1.0284
22 680	0.2357	1.60E−01	0.9806	2.06E−02	0.9394	9.54E−02	1.0179
137 940	0.1286	8.81E−02	1.0294	1.14E−02	0.9831	5.18E−02	1.0068
600 696	0.0786	5.39E−02	1.0115	6.97E−03	0.9943	3.16E−02	1.0020

$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\ell^2(0,T;\mathbb{X})}$		$\|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))}$		iter
Error	Rate	Error	Rate	
2.70E−00	–	1.98E−01	–	3.1
1.40E−00	1.6237	1.14E−01	1.3593	2.8
5.49E−01	1.3470	5.03E−02	1.1810	2.3
2.67E−01	1.1900	2.26E−02	1.3220	2.2
1.54E−01	1.1178	1.10E−02	1.4654	2.2

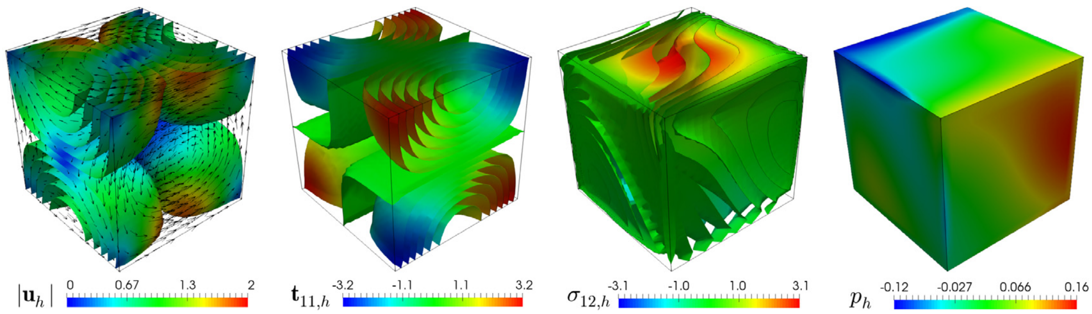

Fig. 6.2. Example 2, computed magnitude of the velocity, velocity gradient component, pseudostress tensor component, and pressure field.

The parameter choice corresponds to a high permeability ($\alpha = 1$) in the channel and increased inertial effect ($\mathtt{F} = 10$), compared to low permeability ($\alpha = 1000$) in the porous medium and reduced inertial effect ($\mathtt{F} = 1$). In addition, the body force term is $\mathbf{f} = \mathbf{0}$, the initial condition is zero, and the boundaries conditions are

$$\mathbf{u}\cdot\mathbf{n} = 0.2, \quad \mathbf{u}\cdot\mathbf{t} = 0 \quad \text{on} \quad \Gamma_{\text{left}}, \quad \boldsymbol{\sigma}\,\mathbf{n} = (0,\,0) \quad \text{on} \quad \Gamma \setminus \Gamma_{\text{left}},$$

which corresponds to inflow on the left boundary and zero stress outflow on the rest of the boundary.

In Fig. 6.3 we display the computed magnitude of the velocity, velocity gradient tensor, and pseudostress tensor at times $T = 0.01$ and $T = 1$, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0$ approximation on a mesh with 27,287 triangle elements (actually representing 218, 561 DOF). As expected, we observe faster flow through the channel network, with a significant velocity gradient across the interface between the channel and the porous medium. The pseudostress is more diffused since it includes the pressure field. This example illustrates the ability of the Brinkman–Forchheimer model to handle heterogeneous media using spatially varying parameters, as well as the ability of our three-field mixed finite element method to resolve sharp velocity gradients in the presence of strong jump discontinuity of the parameters. We further study the robustness of the method with respect to the physical parameters. In Fig. 6.4 we display the computed magnitude of the velocity with the setting $\alpha = 1000$, $\mathtt{F} = 1$ in the porous medium and $\mathtt{F} \in \{10, 100, 1000, 10000\}$, $\alpha \in \{10, 100\}$ in the channel. The top two rows are with $\mathrm{p} = 3$ and the bottom two rows are with $\mathrm{p} = 4$. We observe that in both cases with $\mathrm{p} = 3$ or $\mathrm{p} = 4$ the inertial term $\mathtt{F}\,|\mathbf{u}|^{\mathrm{p}-2}\mathbf{u}$ has the effect of reducing the velocity on the channel, with the velocity decreasing when $\mathtt{F}$ is increased. This effect is higher when $\mathrm{p} = 3$ and $\mathtt{F} \in \{1000, 10000\}$. Furthermore, comparing the results with $\alpha = 10$ and 100, we observe that the higher value of α results in smaller velocity. This study illustrates that the method produces stable and physically reasonable results for a wide range of the physical parameters in both the Stokes and the Darcy regimes.

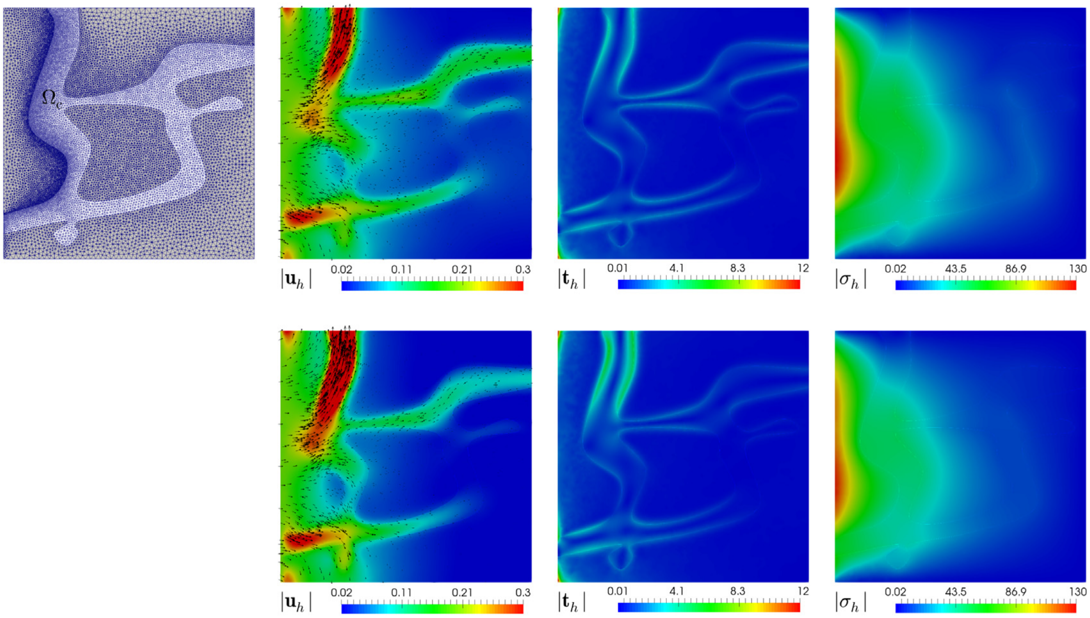

Fig. 6.3. Example 3, domain configuration, computed magnitude of the velocity, velocity gradient tensor, and pseudostress tensor at time $T = 0.01$ (top plots), and at time $T = 1$ (bottom plots).

Example 4 (*Flow Through Porous Media with Fracture Network*)**.** In our last example, inspired by [32, Section 5.2.5], we focus on flows through porous media with fracture network. We consider the square domain $\Omega = (-1, 1)^2$ with an internal network of thin fractures (denoted as Ω_{f}) that intersect at sharp angles, as shown in the first plot of Fig. 6.5. Similarly to Example 3, we consider the Brinkman–Forchheimer model (2.4) in the whole domain Ω, with inertial power $\mathrm{p} = 4$ but with different values of the parameters α and $\mathtt{F}$ for the interior and the exterior of the fracture, that is,

$$\alpha = \begin{cases} 1 & \text{in} \quad \Omega_{\mathrm{f}} \\ 1000 & \text{in} \quad \overline{\Omega} \setminus \Omega_{\mathrm{f}} \end{cases} \quad \text{and} \quad \mathtt{F} = \begin{cases} 10 & \text{in} \quad \Omega_{\mathrm{f}} \\ 1 & \text{in} \quad \overline{\Omega} \setminus \Omega_{\mathrm{f}} \end{cases}. \tag{6.2}$$

In turn, the body force term is $\mathbf{f} = \mathbf{0}$, the initial condition is zero, and the boundaries conditions are

$$\boldsymbol{\sigma}\,\mathbf{n} = \begin{cases} (-0.5(y-1),\ 0) & \text{on} \quad \Gamma_{\text{left}}\,, \\ (0,\ -0.5(x-1)) & \text{on} \quad \Gamma_{\text{bottom}}\,, \end{cases} \qquad \boldsymbol{\sigma}\,\mathbf{n} = (0,\ 0) \quad \text{on} \quad \Gamma_{\text{right}} \cup \Gamma_{\text{top}}\,, \tag{6.3}$$

which drives the flow in a diagonal direction from the left-bottom corner to the right-top corner of the square domain Ω.

In Fig. 6.5 we display the computed magnitude of the velocity, velocity gradient tensor, and pseudostress tensor at times $T = 0.01$ and $T = 1$, which, due to the challenging geometry of the fracture region, were built using the $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1$ approximation on a mesh with 48,891 triangle elements (actually representing $1,222,689$ DOF). We note that the velocity in the fractures is higher than the velocity in the porous medium, due to smaller fractures thickness and the parameter setting (6.2). Also, the velocity is higher in branches of the network where the fluid enters from the left-bottom corner and decreases toward the right-top corner of the domain. In addition, we observe a sharp velocity gradient across the interfaces between the fractures and the porous medium. The pseudostress is consistent with the boundary conditions (6.3) and, similarly to the channel network example, it is more diffused since it includes the pressure field. This example illustrates the ability of the method to provide accurate resolution and numerically stable results for heterogeneous inclusions with high aspect ratio and complex geometry, as presented in the network of thin fractures.

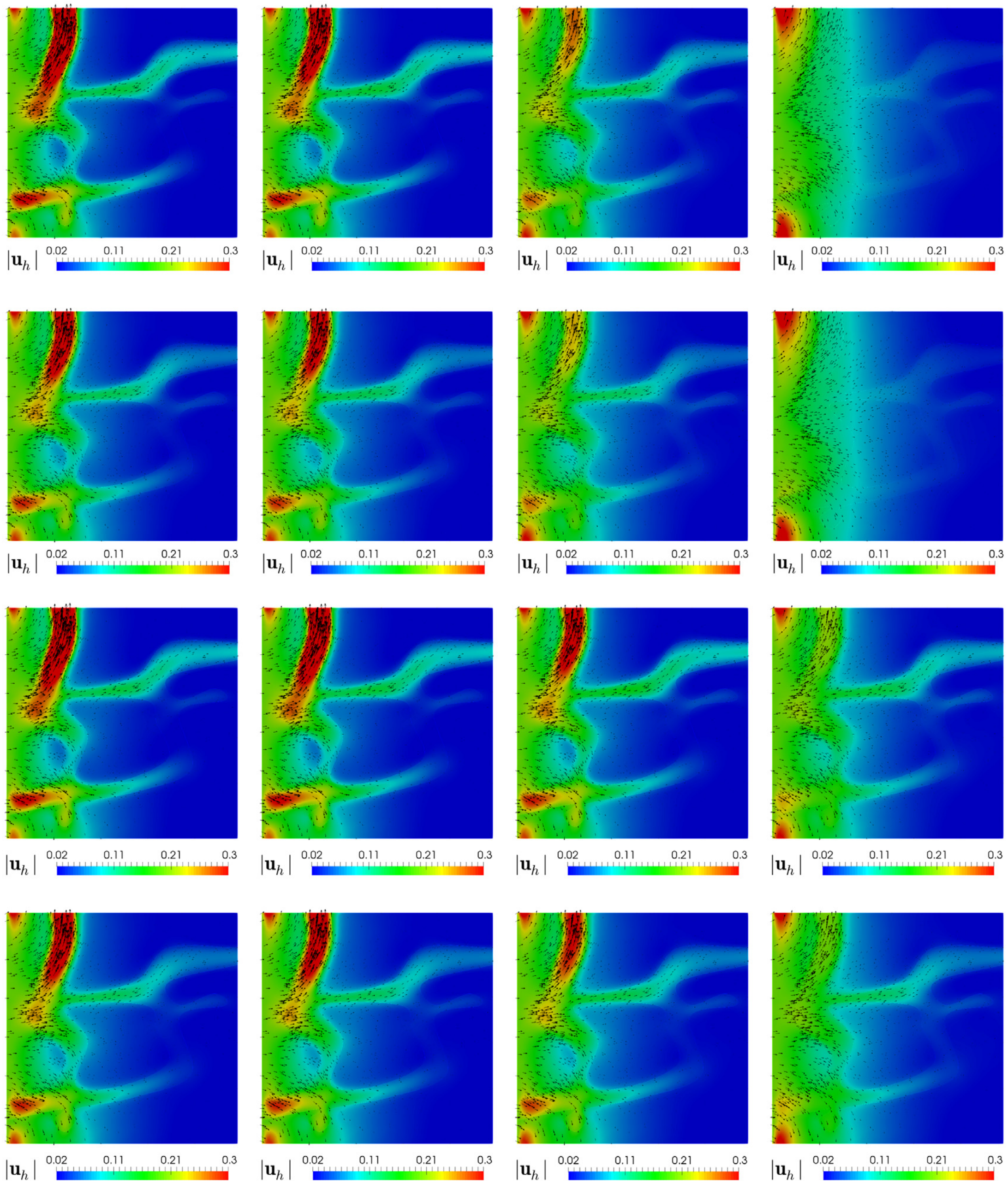

Fig. 6.4. Example 3, computed magnitude of the velocity with p = 3 and channel setting $\mathtt{F} \in \{10, 100, 1000, 10000\}$ with $\alpha = 10$ and 100 (first and second rows, respectively), and p = 4 with channel setting $\mathtt{F} \in \{10, 100, 1000, 10000\}$ with $\alpha = 10$ and 100 (third and fourth rows, respectively).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

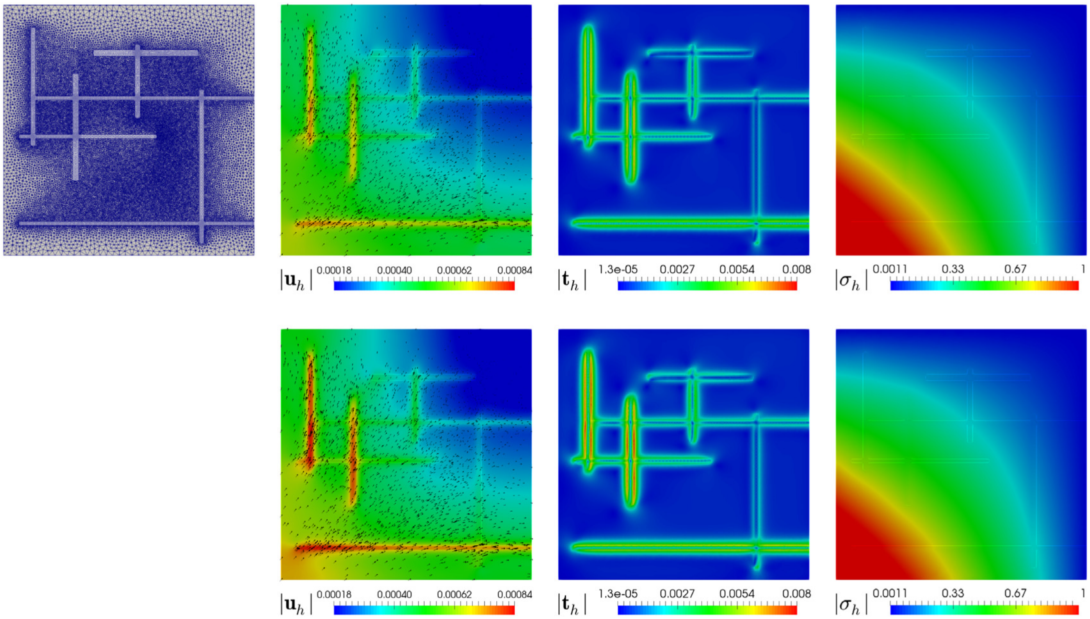

Fig. 6.5. Example 4, domain configuration, computed magnitude of the velocity, velocity gradient tensor, and pseudostress tensor at time $T = 0.01$ (top plots), and at time $T = 1$ (bottom plots).

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