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A POSTERIORI ERROR ESTIMATES FOR THE MORTAR MIXED FINITE ELEMENT METHOD*

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Abstract. Several a posteriori error estimators for mortar mixed finite element discretizations of elliptic equations are derived. A residual-based estimator provides optimal upper and lower bounds for the pressure error. An efficient and reliable estimator for the velocity and mortar pressure error is also derived, which is based on solving local (element) problems in a higher-order space. The interface flux-jump term that appears in the estimators can be used as an indicator for driving an adaptive process for the mortar grids only.

Key words. mixed finite element, mortar finite element, nonmatching grids, a posteriori error estimates, adaptive mesh refinement

AMS subject classifications. 65N12, 65N15, 65N30, 65N50

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1. Introduction.

We consider the second order elliptic problem written as a system of two first order equations

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{1.2}$$

$$p = g \quad \text{on } \Gamma_D, \tag{1.3}$$

$$\mathbf{u} \cdot \nu = 0 \quad \text{on } \Gamma_N, \tag{1.4}$$

where $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3, is a multiblock domain with a boundary $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$, measure $(\Gamma_D) > 0$, ν is the outward unit normal on $\partial\Omega$, and K is a symmetric, uniformly positive definite tensor satisfying, for some $0 < k_0 \le k_1 < \infty$,

$$k_0\xi^T\xi \le \xi^T K(x)\xi \le k_1\xi^T\xi \quad \forall x \in \Omega \quad \forall \xi \in \mathbf{R}^d. \tag{1.5}$$

In flow in porous media the above system models single-phase flow where p is the pressure, $\mathbf{u}$ is the Darcy velocity, and K represents the permeability divided by the viscosity.

A number of papers in recent years have studied the numerical solution of the above and related problems on multiblock domains with nonmatching grids across the interfaces. This growing interest is driven by the flexibility provided by the multiblock paradigm. Complicated geometries can be modeled as unions of relatively simple subdomains with locally constructed grids. Local features of the solution such as

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corner singularities or large gradients can be resolved by finer grids in the local region. Large scale features such as geological faults and layers in subsurface flow can be modeled with nonmatching grids. Moreover, the resulting algebraic problem can be efficiently solved via parallel domain decomposition algorithms.

In a multiblock formulation, the equations are imposed locally on each subdomain and appropriate interface matching conditions are enforced on the interfaces. The use of mortar finite elements to impose the interface conditions is a popular approach due to its excellent stability and accuracy. For the use of mortars, the reader is referred to [10, 8, 20] and references therein for Galerkin finite element and finite volume methods, and to [40, 3, 9] in the context of mixed finite element methods.

An integral part of any successful computational method is the development of a posteriori error estimators and adaptive mesh refinement strategies. Although there is an enormous amount of literature on a posteriori error estimation and adaptivity on conforming grids (seminal works include [5, 7, 2, 35]), few papers deal with this issue for mortar finite element methods. In the case of Galerkin finite elements, error estimators have been developed in [37, 38, 30]. Even fewer results are available for the mortar mixed finite element method. Goal-oriented estimates and adaptivity are developed in [6]. Computational results from [36, 29] indicate that a judicious choice of mortar grids can lead to an accurate solution at low computational cost, but no rigorous justification is given. The goal of this paper is to develop a posteriori error estimators and an adaptive mesh refinement strategy for the mortar mixed finite element method.

Previous works on error estimation for mixed finite element methods on conforming grids include [11, 12, 17, 24, 39, 25]. In [11], mesh-dependent norms are utilized to obtain optimal residual-based error estimators. Estimators based on superconvergence error estimates are developed in [12, 24]. In [17, 39], the Helmholtz decomposition is used to derive optimal residual-based error estimators in the natural pressure and velocity norms. Hierarchical estimates and implicit estimates based on solving local problems are also investigated in [39]. Only the three-dimensional results are given in [25], where a duality argument is employed to obtain residual-based estimates. However, the velocity bounds derived there depend on a saturation assumption that may not hold in general.

In this paper we derive a posteriori error estimates that provide lower and upper bounds for the pressure, velocity, and mortar error in two and three dimensions. According to the widely accepted terminology, an estimator is referred to as *reliable* if it provides an upper bound of the error, whereas it is called *efficient* if it gives a lower bound. We employ a duality-type argument to obtain an efficient and reliable residual-based estimator for the pressure error. In addition to the usual element residual terms, the estimator involves a flux-jump term and a mortar pressure difference term on subdomain interfaces. A closely related estimator of the velocity and mortar error is also derived, which provides an optimal upper bound, but suboptimal (yet sharp) lower bound. We then proceed to derive an optimal efficient and reliable implicit estimator for the velocity based on solving local (element) problems in a higher-order space. Throughout the paper we make several reasonable saturation assumptions which are motivated by known a priori error estimates.

It was observed in [36, 29] that varying the mortar degrees of freedom while keeping the subdomain grids fixed has a substantial effect on the convergence of the interface algorithm employed to solve the algebraic system. At the same time mortar grids that are too coarse lead to deterioration of the accuracy of the method. Therefore, finding "optimal" mortar grids for given subdomain grids is an important question. The flux-jump term that appears in all estimators provides a stand-alone indicator of the nonconformity error in the mortar discretization. It can be used to drive an adaptive mesh refinement process for the mortar grids.

The rest of the paper is organized as follows. In the next section the mortar mixed finite element method is defined along with its equivalent interface formulation. In section 3, the residual-based error estimators are derived and analyzed. The implicit estimator for the velocity is developed in section 4. Computational results are presented in section 5, followed by some remarks and conclusions in section 6.

2. Formulation of the method and preliminaries. We will make use of the following standard notation. For a subdomain $G \subset \mathbf{R}^d$, the $L^2(G)$ inner product (or duality pairing) and norm are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_G$, respectively, for scalar and vector valued functions. The Sobolev spaces $W^k_p(G)$, $k \in \mathbf{R}$, $1 \le p \le \infty$, are defined in the usual way [1] with the usual norm $\|\cdot\|_{k,p,G}$. Let $\|\cdot\|_{k,G}$ be the norm of the Hilbert space $H^k(G) = W^k_2(G)$. We omit G in the subscript if $G = \Omega$. For a section of a subdomain boundary $S \subset \cup_{i=1}^n \partial\Omega_i$ we write $\langle\cdot,\cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively.

We assume that problem (1.1)–(1.4) is H^2-regular, i.e., there exists a positive constant C depending only on K and Ω such that

$$\|p\|_2 \le C(\|f\| + \|g\|_{3/2,\Gamma_D}). \tag{2.1}$$

We refer the reader to [23, 26, 21] for sufficient conditions for H^2-regularity.

To give the weak formulation of (1.1)–(1.4) we recall the usual velocity space [16]

$$H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

with a norm

$$\|\mathbf{v}\|_{H(\mathrm{div})} = (\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2)^{1/2}$$

and define

$$\bar{\mathbf{V}} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}\cdot\nu = 0 \text{ on } \Gamma_N\}.$$

A weak solution of (1.1)–(1.4) is $\mathbf{u} \in \bar{\mathbf{V}}$, $p \in L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u},\mathbf{v}) = (p,\nabla\cdot\mathbf{v}) - \langle g, \mathbf{v}\cdot\nu\rangle_{\Gamma_D}, \quad \mathbf{v} \in \bar{\mathbf{V}}, \tag{2.2}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \quad w \in L^2(\Omega). \tag{2.3}$$

It is well known (see, e.g., [16, 32]) that (2.2) and (2.3) have a unique solution.

Let $\Omega = \cup_{i=1}^n \Omega_i$ be a union of nonoverlapping subdomains. Let

$$\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j, \quad \Gamma = \cup_{i,j=1}^n \Gamma_{i,j}, \quad \Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \backslash \partial\Omega.$$

Let

$$\mathbf{V}_i = \{\mathbf{v} \in H(\mathrm{div};\Omega_i) : \mathbf{v}\cdot\nu_i \in L^2(\partial\Omega_i) \text{ and } \mathbf{v}\cdot\nu_i = 0 \text{ on } \partial\Omega_i \cap \Gamma_N\}, \quad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \quad W = \bigoplus_{i=1}^n W_i = L^2(\Omega), \quad M = L^2(\Gamma).$$

It is easy to see that if the solution of (2.2) and (2.3) satisfies $\mathbf{u}\cdot\nu|_\Gamma \in L^2(\Gamma)$ and $p \in H^1(\Omega)$, then for $1 \le i \le n$

$$(K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i} = (p, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\cap\Gamma_D}, \quad \mathbf{v}\in\mathbf{V}_i, \tag{2.4}$$

$$(\nabla\cdot\mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_i, \tag{2.5}$$

$$\sum_{i=1}^{n}\langle \mathbf{u}\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0, \quad \mu \in M, \tag{2.6}$$

where $\lambda = p|_\Gamma$. (2.4)–(2.6) imply that $(\mathbf{u}, p, \lambda) \in \mathbf{V}\times W\times M$ satisfy

$$A(\mathbf{u},p,\lambda;\mathbf{v},w,\mu) = L(\mathbf{v},w,\mu) \quad \forall\ (\mathbf{v},w,\mu)\in\mathbf{V}\times W\times M, \tag{2.7}$$

where

$$\begin{aligned} &A(\mathbf{u},p,\lambda;\mathbf{v},w,\mu)\\ &\quad = \sum_{i=1}^{n}\big((K^{-1}\mathbf{u},\mathbf{v})_{\Omega_i} - (p,\nabla\cdot\mathbf{v})_{\Omega_i} + \langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} + \sigma(\nabla\cdot\mathbf{u}, w)_{\Omega_i} - \sigma\langle\mathbf{u}\cdot\nu_i,\mu\rangle_{\Gamma_i}\big) \end{aligned}$$

and

$$L(\mathbf{v},w,\mu) = \sigma(f,w) - \langle g, \mathbf{v}\cdot\nu\rangle_{\Gamma_D}.$$

Here $\sigma = 1$ or $\sigma = -1$. If $\sigma = -1$, $A(\cdot;\cdot)$ is a symmetric bilinear form, which we denote by $A^s(\cdot;\cdot)$. If $\sigma = 1$, we denote $A(\cdot;\cdot)$ by $A^c(\cdot;\cdot)$ and note that

$$A^c(\mathbf{v},w,\mu;\mathbf{v},w,\mu) = (K^{-1}\mathbf{v},\mathbf{v});$$

thus $A^c(\cdot;\cdot)$ is nonsymmetric, but coercive. Note that the solution does not depend on the choice of σ.

Let $\{\mathcal{T}_{h,i}\}_h$ be a family of finite element partitions of Ω_i, $1\le i\le n$. Let, for any $E\in\mathcal{T}_{h,i}$, $h_E = \mathrm{diam}(E)$ and let

$$h_i = \max_{E\in\mathcal{T}_{h,i}} h_E, \quad h = \max_{1\le i\le n} h_i.$$

Define ρ_E to be the largest diameter of a ball contained in $\overline{E}$. We require that each subdomain grid satisfies the nondegeneracy condition

$$\max_{E\in\mathcal{T}_{h,i}} \frac{h_E}{\rho_E} \le c_0,$$

where the constant c_0 is independent of h_i. The partitions $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ may be nonmatching along $\Gamma_{i,j}$. Let $\mathcal{T}_h = \cup_{i=1}^n \mathcal{T}_{h,i}$ and let $\mathcal{E}_h$ be the union of all interior edges (faces) not including the interfaces and the outer boundary. Let

$$\mathbf{V}_{h,i}\times W_{h,i} \subset \mathbf{V}_i\times W_i$$

be any of the usual mixed finite element spaces (i.e., the RTN spaces [34, 31, 27], BDM spaces [15], BDFM spaces [14], BDDF spaces [13], or CD spaces [18]). The order of the spaces is assumed to be the same on every subdomain. Let

$$\mathbf{V}_h = \bigoplus_{i=1}^{n}\mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^{n} W_{h,i}.$$

Note that this choice leads to a nonconforming approximation since the normal components of vectors in $\mathbf{V}_h$ do not have to be continuous across Γ. Throughout the paper we will abuse notation when using the $H(\mathrm{div})$-norm. In particular, for $\mathbf{v} \in H(\mathrm{div};\Omega_i)$, $i = 1, \dots, n$,

$$\|\mathbf{v}\|_{H(\mathrm{div})} = \left(\|\mathbf{v}\|^2 + \sum_{i=1}^{n} \|\nabla \cdot \mathbf{v}\|_{\Omega_i}^2 \right)^{1/2},$$

and for $\mathbf{v} \in H(\mathrm{div}; E)$, $E \in \mathcal{T}_h$,

$$\|\mathbf{v}\|_{H(\mathrm{div})} = \left(\|\mathbf{v}\|^2 + \sum_{E \in \mathcal{T}_h} \|\nabla \cdot \mathbf{v}\|_{E}^2 \right)^{1/2}.$$

For all of the above spaces

$$\nabla \cdot \mathbf{V}_{h,i} = W_{h,i}$$

and there exists a projection operator $\Pi_{h,i}$ of $(H^1(\Omega_i))^d$ onto $\mathbf{V}_{h,i}$ satisfying for any $\mathbf{q} \in (H^1(\Omega_i))^d$

$$(\nabla \cdot (\Pi_{h,i}\mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \quad w \in W_{h,i}, \tag{2.8}$$

$$\langle (\mathbf{q} - \Pi_{h,i}\mathbf{q}) \cdot \nu_i, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{2.9}$$

Let $\Pi_h : \bigoplus (H^1(\Omega_i))^d \to \mathbf{V}_h$ be such that $\Pi_h \mathbf{q}|_{\Omega_i} = \Pi_{h,i}\mathbf{q}$ for all $\mathbf{q} \in \bigoplus (H^1(\Omega_i))^d$.

Let the mortar interface mesh $\mathcal{T}_{h,i,j}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$ and let $\mathcal{T}^{\Gamma,h} = \cup_{1 \le i < j \le n} \mathcal{T}_{h,i,j}$. For any $\tau \in \mathcal{T}_{h,i,j}$, let

$$E_\tau = \cup (E \in \mathcal{T}_h : \partial E \cap \tau \neq \emptyset).$$

We will assume that there exist constants c_1 and c_2 such that

$$c_1 h_E \le h_\tau \le c_2 h_E \quad \forall E \in E_\tau, \tag{2.10}$$

where the notation $h_S = \mathrm{diam}(S)$ is used. Denote by $M_{h,i,j} \subset L^2(\Gamma_{i,j})$ the mortar space on $\Gamma_{i,j}$ containing at least either the continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathcal{T}_{h,i,j}$, where k is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu$. More precisely, if $d = 3$ and e is a triangle of the mesh, we take $M_{h,i,j}|_e = P_{k+1}(e)$, the set of polynomials of degree less than or equal to $k+1$ on e. If e is a rectangle, we take $M_{h,i,j}|_e = Q_{k+1}(e)$, the set of polynomials on e for which the degree in each variable separately is less than or equal to $k+1$. Now let

$$M_h = \bigoplus_{1 \le i < j \le n} M_{h,i,j}$$

be the mortar finite element space on Γ.

In the mortar mixed finite element method for approximating (2.4)–(2.6) we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_h \in M_h$ such that, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \cap \Gamma_D}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{2.11}$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}, \tag{2.12}$$

$$\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in M_h. \tag{2.13}$$

(2.13) enforces weak (with respect to the mortar space M_h) continuity of the flux across the block interfaces. Existence and uniqueness of a solution of (2.11)–(2.13) are shown in [40, 3] along with optimal convergence and superconvergence for both pressure and velocity under the assumption that for all $\mu \in M_{h,i,j}$ there exists a constant C independent of h such that

$$\|\mu\|_{\Gamma_{i,j}} \le C(\|\mathcal{Q}_{h,i}\mu\|_{\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{\Gamma_{i,j}}), \tag{2.14}$$

where $\mathcal{Q}_{h,i} : L^2(\partial\Omega_i) \to \mathbf{V}_{h,i} \cdot \nu_i|_{\partial\Omega_i}$ is the L^2-orthogonal projection satisfying for any $\phi \in L^2(\partial\Omega_i)$

$$\langle \phi - \mathcal{Q}_{h,i}\phi, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} = 0 \quad \forall \mathbf{v} \in \mathbf{V}_{h,i}. \tag{2.15}$$

Remark 2.1. Condition (2.14) imposes a limit on the number of mortar degrees of freedom and is easily satisfied in practice [40, 28].

We recall some a priori error estimates from [3] which will later motivate some of the saturation assumptions needed in the a posteriori error analysis. Herein l is associated with the degree of the polynomials in W_h and $\|\cdot\|_{d_h}$ is a mortar space norm defined in the next subsection. Throughout the paper C denotes a generic constant independent of h.

THEOREM 2.1. *For the solution of* (2.11)–(2.13) *if* (2.14) *holds, then*

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \le C \sum_{i=1}^{n} \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i} h^r, \quad 1 \le r \le l+1,$$

$$\|\mathbf{u} - \mathbf{u}_h\| \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i}) h^r, \quad 1 \le r \le k+1,$$

$$\|p - p_h\| \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i} + \|\nabla \cdot \mathbf{u}\|_{r,\Omega_i}) h^r, \quad 1 \le r \le \min(k+1, l+1),$$

$$\|\lambda - \lambda_h\|_{d_h} \le C \sum_{i=1}^{n} (\|p\|_{r+1,\Omega_i} + \|\mathbf{u}\|_{r,\Omega_i}) h^r, \quad 1 \le r \le k+1.$$

2.1. Interface formulation. Method (2.11)–(2.13) can be reduced to an equivalent interface (mortar) problem. We recall this interface formulation from [22, 40, 3], as it will be used in estimating the mortar error.

Define $d_h : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ for $\varphi, \mu \in L^2(\Gamma)$ by

$$d_h(\varphi, \mu) = \sum_{i=1}^{n} d_{h,i}(\varphi, \mu) = -\sum_{i=1}^{n} \langle \mathbf{u}_h^*(\varphi) \cdot \nu_i, \mu \rangle_{\Gamma_i}, \tag{2.16}$$

where $(\mathbf{u}_h^*(\varphi), p_h^*(\varphi)) \in \mathbf{V}_h \times W_h$ solve, for $1 \le i \le n$,

$$(K^{-1}\mathbf{u}_h^*(\varphi), \mathbf{v})_{\Omega_i} = (p_h^*(\varphi), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \varphi, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{2.17}$$

$$(\nabla \cdot \mathbf{u}_h^*(\varphi), w)_{\Omega_i} = 0, \quad w \in W_{h,i}. \tag{2.18}$$

Define $g_h : L^2(\Gamma) \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^{n} g_{h,i}(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}}_h \cdot \nu_i, \mu \rangle_{\Gamma_i},$$

where $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solve, for $1 \le i \le n$,

$$\begin{aligned}(K^{-1}\bar{\mathbf{u}}_h, \mathbf{v})_{\Omega_i} &= (\bar{p}_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\cap\Gamma_D}, \quad \mathbf{v}\in\mathbf{V}_{h,i},\\ (\nabla\cdot\bar{\mathbf{u}}_h, w)_{\Omega_i} &= (f, w)_{\Omega_i}, \quad w\in W_{h,i}.\end{aligned}$$

Then $(\mathbf{u}_h, p_h, \lambda_h)$ satisfies

$$d_h(\lambda_h, \mu) = g_h(\mu) \quad \forall \mu\in M_h, \quad \mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h.$$

It is easy to see from (2.16) and (2.17) that

$$d_{h,i}(\varphi,\varphi) = (K^{-1}\mathbf{u}_h^*(\varphi), \mathbf{u}_h^*(\varphi))_{\Omega_i}, \tag{2.19}$$

which implies that $d_h(\cdot,\cdot)$ is positive semidefinite in $M\times M$ and, assuming (2.14), positive definite in $M_h\times M_h$. We define the norm in M_h:

$$\|\mu\|_{d_h} := d_h(\mu,\mu)^{1/2}.$$

It is shown in [40, 28] for RT_0 rectangular elements and very general hanging interface nodes and mortar grid configurations satisfying (2.14) that

$$\sum_{\tau\in\mathcal{T}^{\Gamma,h}} \|\mu\|_{1/2,\tau}^2 \le C d_h(\mu,\mu) \quad \forall\mu\in M_h. \tag{2.20}$$

The proofs in [40, 28] can be generalized in a relatively straightforward way to the other mixed finite element spaces under consideration and to higher-order elements.

The following construction will also be useful in the analysis of the mortar error. Define, for $\varphi\in L^2(\Gamma)$,

$$\mathbf{u}_h(\varphi) = \mathbf{u}_h^*(\varphi) + \bar{\mathbf{u}}_h, \quad p_h(\varphi) = p_h^*(\varphi) + \bar{p}_h.$$

We note that $(\mathbf{u}_h(\varphi), p_h(\varphi)) \in \mathbf{V}_h\times W_h$ satisfy, for $1\le i\le n$,

$$\begin{aligned}(K^{-1}\mathbf{u}_h(\varphi), \mathbf{v})_{\Omega_i} &= (p_h(\varphi), \nabla\cdot\mathbf{v})_{\Omega_i} - \langle\varphi, \mathbf{v}\cdot\nu\rangle_{\Gamma_i}\\ &\qquad - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\cap\Gamma_D}, \quad \mathbf{v}\in\mathbf{V}_{h,i}, \qquad (2.21)\\ (\nabla\cdot\mathbf{u}_h(\varphi), w)_{\Omega_i} &= (f,w)_{\Omega_i}, \quad w\in W_{h,i}. \qquad (2.22)\end{aligned}$$

In particular, $\mathbf{u}_h(\lambda_h) = \mathbf{u}_h$ and $p_h(\lambda_h) = p_h$.

The a priori error bounds from Theorem 2.1 motivate the following assumption on the mortar error.

Saturation assumption. There exists a constant γ such that

$$|||\lambda-\lambda_h||| := \left(\sum_{\tau\in\mathcal{T}^{\Gamma,h}} h_\tau^{-1}\|\lambda-\lambda_h\|_\tau^2\right)^{1/2} \le \gamma\|\mathbf{u}-\mathbf{u}_h\|. \tag{2.23}$$

For further justification of (2.23), note that $|||\lambda-\lambda_h|||$ is closely related to the discrete $H^{1/2}(\Gamma)$ norm and, by (2.20), to $\|\lambda-\lambda_h\|_{d_h}$. Now, assuming that

$$\|\mathbf{u}-\mathbf{u}_h(\lambda)\| \le \gamma_1\|\mathbf{u}-\mathbf{u}_h\|,$$

which is reasonable, since $\mathbf{u}_h(\lambda)$ is the numerical solution based on the true interface data, we have, using (2.19),

$$\begin{aligned}C\|\lambda-\lambda_h\|_{d_h} &\le \|\mathbf{u}_h^*(\lambda)-\mathbf{u}_h^*(\lambda_h)\| = \|\mathbf{u}_h(\lambda)-\mathbf{u}_h(\lambda_h)\| = \|\mathbf{u}_h(\lambda)-\mathbf{u}_h\|\\ &\le \|\mathbf{u}-\mathbf{u}_h(\lambda)\| + \|\mathbf{u}-\mathbf{u}_h\| \le (1+\gamma_1)\|\mathbf{u}-\mathbf{u}_h\|.\end{aligned}$$

Remark 2.2. Condition (2.14) is necessary for the solvability and accuracy of the method and for the validity of (2.23). See [40, 28] for examples of grids that satisfy (2.14). Note that (2.14) excludes the case of matching subdomain grids and a mortar grid that coincides with them. In the case of matching subdomain grids, the mortar grid has to be at least twice as coarse as their trace on the interface. Another possibility in the case of matching grids is to use the standard Lagrange multipliers from the hybrid mixed method [4], in which case the conforming mixed method solution is recovered. This trivial case is not a special case of the mortar mixed finite element method, since the mortar spaces consist of polynomials of one degree higher than the Lagrange multipliers.

2.2. Residual representation and orthogonality of error. Using the notation from (2.7), the solution of (2.11)–(2.13) $(\mathbf{u}_h, p_h, \lambda_h) \in \mathbf{V}_h \times W_h \times M_h$ satisfies

$$A(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v}, w, \mu) = L(\mathbf{v}, w, \mu) \quad \forall\ (\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h. \tag{2.24}$$

Our goal is to derive a posteriori estimates of the error functions

$$\xi = \mathbf{u} - \mathbf{u}_h, \quad \eta = p - p_h, \quad \text{and} \quad \delta = \lambda - \lambda_h.$$

Using (2.7), $(\xi, \eta, \delta) \in \mathbf{V} \times W \times M$ satisfies the residual equation

$$A(\xi, \eta, \delta; \mathbf{v}, w, \mu) = L(\mathbf{v}, w, \mu) - A(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v}, w, \mu) \quad \forall\ (\mathbf{v}, w, \mu) \in \mathbf{V} \times W \times M, \tag{2.25}$$

which, together with (2.24), implies the orthogonality condition

$$A(\xi, \eta, \delta; \mathbf{v}, w, \mu) = 0 \quad \forall\ (\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h. \tag{2.26}$$

2.3. Approximation properties. We present below some of the approximation properties of the finite element spaces. In addition to the operators defined above, we will make use of the interpolant $\mathcal{I}_h$ in the mortar space M_h, and the L^2-projection onto W_h, defined as

$$(w - \hat{w}, w_h) = 0 \quad \forall w_h \in W_h.$$

The following approximation properties hold true. For all $E \in \mathcal{T}_h$, $\tau \in \mathcal{T}^{\Gamma,h}$, $e \in \mathcal{T}_{h,i}|_{\partial\Omega_i}$, and smooth enough functions $\mathbf{v}$, w, and μ,

$$\|\mathbf{v} - \Pi_h \mathbf{v}\|_E \le C h_E \|\mathbf{v}\|_{1,E}, \tag{2.27}$$

$$\|(\mathbf{v} - \Pi_h \mathbf{v}) \cdot \nu_E\|_{\partial E} \le C h_E^s \|\mathbf{v} \cdot \nu_E\|_{s,\partial E}, \quad s = 0, 1/2, \tag{2.28}$$

$$\|w - \hat{w}\|_E \le C h_E \|w\|_{1,E}, \tag{2.29}$$

$$\|\mu - \mathcal{I}_h \mu\|_\tau \le C h_\tau^{3/2} \|\mu\|_{3/2,\tau}, \tag{2.30}$$

$$\|\mu - \mathcal{Q}_{h,i} \mu\|_e \le C h_e \|\mu\|_{1,e}. \tag{2.31}$$

Bound (2.27) can be found in [16, 33]; bounds (2.28)–(2.31) are standard interpolation and L^2-projection approximation results [19].

2.4. Some useful inequalities. In the analysis below we will make use of the trace inequalities

$$\forall E \in \mathcal{T}_h, \quad e \in \partial E, \quad \|\phi\|_e \le C(h_E^{-1/2} \|\phi\|_E + h_E^{1/2} \|\nabla \phi\|_E) \quad \forall \phi \in H^1(E), \tag{2.32}$$

$$(2.33) \qquad \forall E \in \mathcal{T}_h, \quad e \in \partial E, \quad \|\phi\|_{1/2,e} \le C\|\phi\|_{1,E} \quad \forall \phi \in H^1(E),$$

$$(2.34) \qquad \forall E \in \mathcal{T}_h, \quad e \in \partial E, \quad \|\mathbf{v}\cdot\nu\|_e \le C h_E^{-1/2}\|\mathbf{v}\|_E \quad \forall \mathbf{v} \in \mathbf{V}_h,$$

and the well-known inequality

$$(2.35) \qquad ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2 \quad \forall \epsilon > 0.$$

3. Residual-based error estimators. In this section we derive upper and lower bounds on the error in terms of local residuals. The resulting estimators are often called explicit estimators as they involve only residual terms that depend explicitly on the input data and the computed solution and do not require the solution of additional finite element problems.

3.1. Upper bounds. Let, for all $E \in \mathcal{T}_h$, $\tau \in \mathcal{T}^{\Gamma,h}$,

$$(3.1) \qquad \omega_E^2 = \|K^{-1}\mathbf{u}_h + \nabla p_h\|_E^2 h_E^2 + \|f - \nabla\cdot\mathbf{u}_h\|_E^2 h_E^2 + \|\lambda_h - p_h\|_{\partial E\cap\Gamma}^2 h_E,$$

$$(3.2) \qquad \omega_\tau^2 = \|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau^3,$$

where for any $\mathbf{v} \in \mathbf{V}$, $\mathbf{v}|_{\Omega_i} = \mathbf{v}_i$,

$$[\mathbf{v}\cdot\nu]|_{\Gamma_{i,j}} = \mathbf{v}_i\cdot\nu_i + \mathbf{v}_j\cdot\nu_j$$

is the jump operator. We first derive an upper bound on the pressure error η.

THEOREM 3.1. *There exists a constant C independent of h such that*

$$\|\eta\|^2 \le C\Bigg\{\sum_{E\in\mathcal{T}_h}\omega_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\omega_\tau^2 + \sum_{e\in\mathcal{T}_h|_{\Gamma_D}}\|g - \mathcal{Q}_h g\|_e^2 h_e\Bigg\}.$$

Proof. The proof is based on a duality argument. Consider the auxiliary problem

$$\begin{aligned} -\nabla\cdot K\nabla w &= \eta \quad \text{in } \Omega,\\ w &= 0 \quad \text{on } \Gamma_D,\\ K\nabla w\cdot\nu &= 0 \quad \text{on } \Gamma_N. \end{aligned}$$

The elliptic regularity assumption (2.1) implies that

$$(3.3) \qquad \|w\|_2 \le C\|\eta\|.$$

Let $\mathbf{v} = -K\nabla w$ and $\mu = w|_\Gamma$. With (2.7), $(\mathbf{v}, w, \mu)$ satisfy

$$A^s(\mathbf{v}, w, \mu; \tilde{\mathbf{v}}, \tilde{w}, \tilde{\mu}) = -(\eta, \tilde{w}) \quad \forall\ (\tilde{\mathbf{v}}, \tilde{w}, \tilde{\mu}) \in \mathbf{V}\times W\times M.$$

Then, using (2.26) and (2.26),

$$\begin{aligned} \|\eta\|^2 &= -A^s(\mathbf{v}, w, \mu; \xi, \eta, \delta) = -A^s(\xi, \eta, \delta; \mathbf{v}, w, \mu)\\ &= -A^s(\xi, \eta, \delta; \mathbf{v} - \Pi_h\mathbf{v}, w - \hat{w}, \mu - \mathcal{I}_h\mu)\\ &= A^s(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v} - \Pi_h\mathbf{v}, w - \hat{w}, \mu - \hat{\mu}) + (f, w - \hat{w}) + \langle g, (\mathbf{v} - \Pi_h\mathbf{v})\cdot\nu\rangle_{\Gamma_D}\\ &= \sum_{E\in\mathcal{T}_h}\big((K^{-1}\mathbf{u}_h, \mathbf{v} - \Pi_h\mathbf{v})_E - (p_h, \nabla\cdot(\mathbf{v} - \Pi_h\mathbf{v}))_E - (\nabla\cdot\mathbf{u}_h, w - \hat{w})_E\big)\\ &\quad + \sum_{i=1}^n\big(\langle\lambda_h, (\mathbf{v} - \Pi_h\mathbf{v})\cdot\nu_i\rangle_{\Gamma_i} + \langle\mathbf{u}_h\cdot\nu_i, \mu - \mathcal{I}_h\mu\rangle_{\Gamma_i}\big)\\ &\quad + (f, w - \hat{w}) + \langle g, (\mathbf{v} - \Pi_h\mathbf{v})\cdot\nu\rangle_{\Gamma_D}. \end{aligned}$$

Applying Green's formula and using (2.9),

$$\begin{aligned}\|\eta\|^2 = &\sum_{E\in\mathcal{T}_h} \big((K^{-1}\mathbf{u}_h + \nabla p_h, \mathbf{v} - \Pi_h \mathbf{v})_E + (f - \nabla\cdot\mathbf{u}_h, w - \hat{w})_E\big)\\ &+ \sum_{i=1}^{n} \big(\langle \lambda_h - p_h, (\mathbf{v} - \Pi_h\mathbf{v})\cdot\nu_i\rangle_{\Gamma_i} + \langle \mathbf{u}_h\cdot\nu_i, \mu - \mathcal{I}_h\mu\rangle_{\Gamma_i}\big)\\ &+ \langle g - \mathcal{Q}_h g, (\mathbf{v} - \Pi_h\mathbf{v})\cdot\nu\rangle_{\Gamma_D}.\end{aligned}$$

Using the Cauchy–Schwartz inequality and the approximation properties (2.27)–(2.31),

$$\begin{aligned}\|\eta\|^2 \le C\Bigg\{ &\sum_{E\in\mathcal{T}_h} \big(\|K^{-1}\mathbf{u}_h + \nabla p_h\|_E h_E \|\mathbf{v}\|_{1,E} + \|f - \nabla\cdot\mathbf{u}_h\|_E h_E \|w\|_{1,E}\\ &+ \|\lambda_h - p_h\|_{\partial E\cap\Gamma} h_E^{1/2}\|\mathbf{v}\|_{1,E}\big) + \sum_{\tau\in\mathcal{T}^{\Gamma,h}} \|[\mathbf{u}_h\cdot\nu]\|_\tau h_\tau^{3/2}\|\mu\|_{3/2,\tau}\\ &+ \sum_{e\in\mathcal{T}_h|_{\Gamma_D}} \|g - \mathcal{Q}_h g\|_e h_e^{1/2}\|\mathbf{v}\|_{1/2,e}\Bigg\}.\end{aligned}$$

An application of the discrete Cauchy–Schwartz inequality, the trace inequality (2.33), and (3.3) completes the proof. ∎

Remark 3.1. Because of the approximation property (2.31) of $\mathcal{Q}_h$ the last term in the bound of Theorem 3.1 is of higher order than the other terms. Therefore, its effect becomes negligible for small h.

To derive a bound on $\xi = \mathbf{u} - \mathbf{u}_h$ we need a saturation assumption. Let $\mathbf{V}'_h$, W'_h, and M'_h be the finite element spaces of one order higher than $\mathbf{V}_h$, W_h, and M_h, respectively. Let $\mathbf{u}'_h \in \mathbf{V}'_h$, $p'_h \in W'_h$, and $\lambda'_h \in M'_h$ be the mortar mixed finite element solution in these higher-order spaces (see (2.11)–(2.13)). The a priori error estimates from Theorem 2.1 motivate the following.

Saturation assumption. There exist constants $\beta < 1$, $\beta_{\text{div}} < 1$, and $\beta_p < \infty$ such that

$$\|\mathbf{u} - \mathbf{u}'_h\| \le \beta\|\mathbf{u} - \mathbf{u}_h\|, \tag{3.4}$$

$$\|\nabla\cdot(\mathbf{u} - \mathbf{u}'_h)\| \le \beta_{\text{div}}\|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|, \tag{3.5}$$

$$\|p - p'_h\| \le \beta_p\|p - p_h\|. \tag{3.6}$$

Let

$$\xi' = \mathbf{u}'_h - \mathbf{u}_h, \quad \eta' = p'_h - p_h, \quad \text{and} \quad \delta' = \lambda'_h - \lambda_h.$$

Similar to (2.26) and (2.26), we have that $(\xi', \eta', \delta') \in \mathbf{V}'_h \times W'_h \times M'_h$ satisfy the residual equation

$$\begin{aligned}A(\xi', \eta', \delta'; \mathbf{v}'_h, w'_h, \mu'_h) = L(\mathbf{v}'_h, w'_h, \mu'_h) - A(\mathbf{u}_h, p_h, \lambda_h; \mathbf{v}'_h, w'_h, \mu'_h)&\\ \forall\ (\mathbf{v}'_h, w'_h, \mu'_h) \in \mathbf{V}'_h \times W'_h \times M'_h&\end{aligned} \tag{3.7}$$

and the orthogonality condition

$$A(\xi', \eta', \delta'; \mathbf{v}, w, \mu) = 0 \quad \forall\ (\mathbf{v}, w, \mu) \in \mathbf{V}_h \times W_h \times M_h. \tag{3.8}$$

The bounds on ξ and δ will be expressed in terms of weighted local residuals, for all $E \in \mathcal{T}_h$, $\tau \in \mathcal{T}^{\Gamma,h}$,

$$\begin{aligned}\tilde{\omega}_E^2 &= h_E^{-2}\omega_E^2 = \|K^{-1}\mathbf{u}_h + \nabla p_h\|_E^2 + \|f - \nabla\cdot\mathbf{u}_h\|_E^2 + \|\lambda_h - p_h\|_{\partial E\cap\Gamma}^2 h_E^{-1},\\ \tilde{\omega}_\tau^2 &= h_\tau^{-2}\omega_\tau^2 = \|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau.\end{aligned}$$

THEOREM 3.2. *Assume that the saturation assumptions* (2.23) *and* (3.4) *hold. Then there exists a constant C independent of β such that*

$$\|\xi\|^2_{H(\mathrm{div})} \le \frac{C}{(1-\beta)^2}\left\{ \sum_{E\in\mathcal{T}_h} \tilde{\omega}_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}} \tilde{\omega}_\tau^2 + \sum_{e\in\mathcal{T}_h|_{\Gamma_D}} \|g-\mathcal{Q}_h g\|_e^2 h_e^{-1}\right\}.$$

Proof. The bound on $\|\nabla\cdot\xi\|$ is trivial. Indeed, for all $E\in\mathcal{T}_h$,

$$\|\nabla\cdot\xi\|_E = \|f-\nabla\cdot\mathbf{u}_h\|_E \le \tilde{\omega}_E.$$

To bound $\|\xi\|$, since (3.4) implies that

$$\|\xi\| \le \frac{1}{1-\beta}\|\xi'\|, \tag{3.9}$$

it is enough to bound $\|\xi'\|$. Using (3.8) and (3.7),

$$\begin{aligned}
\|K^{-1/2}\xi'\|^2 &= A^c(\xi',\eta',\delta';\xi',\eta',\delta') = A^c(\xi',\eta',\delta';\xi'-\Pi_h\xi',\eta',\delta')\\
&= L^c(\xi'-\Pi_h\xi',\eta',\delta') - A^c(\mathbf{u}_h,p_h,\lambda_h;\xi'-\Pi_h\xi',\eta',\delta')\\
&= -\sum_{E\in\mathcal{T}_h}\big((K^{-1}\mathbf{u}_h,\xi'-\Pi_h\xi')_E - (p_h,\nabla\cdot(\xi'-\Pi_h\xi'))_E + (\nabla\cdot\mathbf{u}_h,\eta')_E\big)\\
&\quad -\sum_{i=1}^n\big(\langle\lambda_h,(\xi'-\Pi_h\xi')\cdot\nu_i\rangle_{\Gamma_i} - \langle\mathbf{u}_h\cdot\nu_i,\delta'\rangle_{\Gamma_i}\big)\\
&\quad + (f,\eta') - \langle g,(\xi'-\Pi_h\xi')\cdot\nu\rangle_{\Gamma_D}.
\end{aligned}$$

The use of Green's formula and (2.9) gives

$$\begin{aligned}
\|K^{-1/2}\xi'\|^2 = &-\sum_{E\in\mathcal{T}_h}\big((K^{-1}\mathbf{u}_h+\nabla p_h,\xi'-\Pi_h\xi')_E + (\nabla\cdot\mathbf{u}_h - f,\eta')_E\big)\\
&-\sum_{i=1}^n\big(\langle\lambda_h-p_h,(\xi'-\Pi_h\xi')\cdot\nu_i\rangle_{\Gamma_i} - \langle\mathbf{u}_h\cdot\nu_i,\delta'\rangle_{\Gamma_i}\big)\\
&-\langle g-\mathcal{Q}_h g,(\xi'-\Pi_h\xi')\cdot\nu\rangle_{\Gamma_D} = T_1+\cdots+T_5.
\end{aligned}\tag{3.10}$$

For T_1, using the Cauchy–Schwartz inequality, (2.27), the inverse inequality, and (2.35), we have

$$|(K^{-1}\mathbf{u}_h+\nabla p_h,\xi'-\Pi_h\xi')_E| \le C\left(\frac{1}{4\epsilon_1}\|K^{-1}\mathbf{u}_h+\nabla p_h\|_E^2 + \epsilon_1\|\xi'\|_E^2\right). \tag{3.11}$$

Similarly for T_2,

$$|(\nabla\cdot\mathbf{u}_h - f,\eta')_E| \le \frac{1}{2}\|\nabla\cdot\mathbf{u}_h - f\|_E^2 + \frac{1}{2}\|\eta'\|_E^2. \tag{3.12}$$

To bound T_3, the use of (2.28) with $s=0$ gives, for $e\in\Gamma_i$, $e\in\partial E$,

$$|\langle\lambda_h-p_h,(\xi'-\Pi_h\xi')\cdot\nu_i\rangle_e| \le C\left(\frac{1}{4\epsilon_3}\|\lambda_h-p_h\|_e^2 h_E^{-1} + \epsilon_3\|\xi'\|_E^2\right). \tag{3.13}$$

Similarly for T_5,

$$|\langle g-\mathcal{Q}_h g,(\xi'-\Pi_h\xi')\cdot\nu\rangle_e| \le C\left(\frac{1}{4\epsilon_5}\|g-\mathcal{Q}_h g\|_e^2 h_e^{-1} + \epsilon_5\|\xi'\|_E^2\right). \tag{3.14}$$

Finally for T_4, using (2.35),

$$
\begin{aligned}
(3.15)\qquad \left|\sum_{i=1}^{n}\langle \mathbf{u}_h\cdot\nu_i,\delta'\rangle_{\Gamma_i}\right| &= \left|\sum_{\tau\in\mathcal{T}^{\Gamma,h}}\langle[\mathbf{u}_h\cdot\nu],\delta'\rangle_\tau\right| \le \sum_{\tau\in\mathcal{T}^{\Gamma,h}} h_\tau^{1/2}\|[\mathbf{u}_h\cdot\nu]\|_\tau h_\tau^{-1/2}\|\delta'\|_\tau \\
&\le \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\left(\frac{1}{4\epsilon_4}\|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau + \epsilon_4\|\delta'\|_\tau^2 h_\tau^{-1}\right).
\end{aligned}
$$

Combining (1.5) with (3.10)–(3.15) for small enough ϵ_1, ϵ_3, and ϵ_5,

(3.16)

$$
\begin{aligned}
\|\xi'\|^2 \le C\Bigg\{ &\sum_{E\in\mathcal{T}_h}(\|K^{-1}\mathbf{u}_h+\nabla p_h\|_E^2 + \|f-\nabla\cdot\mathbf{u}_h\|_E^2 + \|\lambda_h - p_h\|_{\partial E\cap\Gamma}^2 h_E^{-1} + \|\eta'\|_E^2) \\
&+ \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\left(\frac{1}{4\epsilon_4}\|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau + \epsilon_4\|\delta'\|_\tau^2 h_\tau^{-1}\right) + \sum_{e\in\mathcal{T}_h|_{\Gamma_D}}\|g-\mathcal{Q}_h g\|_e^2 h_e^{-1}\Bigg\}.
\end{aligned}
$$

Because of (3.6), the bound on $\|\eta\|$ from Theorem 3.1 applies to $\|\eta'\|$ as well. It remains to estimate $|||\delta'|||^2 = \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\|\delta'\|_\tau^2 h_\tau^{-1}$. Using (2.23) (with a constant γ' in the case of the higher-order spaces) and (3.4), we have

$$
\begin{aligned}
(3.17)\qquad |||\delta'||| &\le |||\lambda-\lambda_h||| + |||\lambda-\lambda_h'||| \le \gamma\|\mathbf{u}-\mathbf{u}_h\| + \gamma'\|\mathbf{u}-\mathbf{u}_h'\| \\
&\le (\gamma+\gamma'\beta)\|\mathbf{u}-\mathbf{u}_h\|.
\end{aligned}
$$

Using (3.17), (3.9) and taking ϵ_4 in (3.16) small enough completes the proof. ∎

3.2. Lower bounds. Next, we establish lower bounds on the error, which indicate that the residual error estimators can be used effectively in an adaptive mesh refinement algorithm.

THEOREM 3.3. *There exists a constant C independent of h such that*

$$
(3.18)\qquad \sum_{E\in\mathcal{T}_h}\omega_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\omega_\tau^2 \le C\left(\|\eta\|^2 + \sum_{E\in\mathcal{T}_h} h_E^2\|\xi\|_{H(\mathrm{div};E)}^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}} h_\tau\|\delta\|_\tau^2\right)
$$

and, assuming that the saturation assumption (2.23) *holds,*

$$
(3.19)\qquad \sum_{E\in\mathcal{T}_h}\tilde{\omega}_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\tilde{\omega}_\tau^2 \le C\left(\sum_{E\in\mathcal{T}_h} h_E^{-2}\|\eta\|_E^2 + \|\xi\|_{H(\mathrm{div})}^2\right).
$$

Moreover, the following local bounds hold for any $E\in\mathcal{T}_h$, $e\in\partial E$, and $\tau\in\mathcal{T}^{\Gamma,h}$:

$$
(3.20)\qquad \|K^{-1}\mathbf{u}_h+\nabla p_h\|_E^2 h_E^2 + \|f-\nabla\cdot\mathbf{u}_h\|_E^2 h_E^2 \le C(\|\eta\|_E^2 + \|\xi\|_{H(\mathrm{div};E)}^2 h_E^2),
$$

$$
(3.21)\qquad \|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau^3 \le C\|\xi\|_{H(\mathrm{div};E_\tau)}^2 h_\tau^2,
$$

$$
(3.22)\qquad \|\lambda_h - p_h\|_e^2 h_E \le C(\|\eta\|_E^2 + \|\xi\|_{H(\mathrm{div};E)}^2 h_E^2 + \|\delta\|_e^2 h_E).
$$

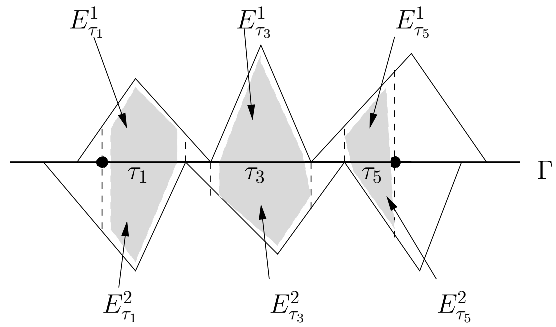

FIG. 3.1. *Construction of* E_{τ_k}.

Proof. It has been shown in [17], using a bubble function argument, that

$$\|K^{-1}\mathbf{u}_h + \nabla p_h\|_E h_E \le C(\|\eta\|_E + \|\xi\|_E h_E),$$

which, combined with

$$\|f - \nabla \cdot \mathbf{u}_h\|_E h_E = \|\nabla \cdot \xi\|_E h_E,$$

gives (3.20). To prove (3.21), consider any $\tau \in \mathcal{T}^{\Gamma,h}$. Let τ be divided by the intersection of the two traces of $\mathcal{T}_h$ on Γ into elements $\tau_1, \dots, \tau_l$. Because of (2.10) there exists $c > 0$ such that

$$h_{\tau_k} \ge c h_\tau, \quad k = 1, \dots, l. \tag{3.23}$$

Next, let us translate any point in τ_k in both directions orthogonal to Γ until an interior edge (face) of an element of $\mathcal{T}_h$ is reached. Let E_{τ_k} be the union of all such trajectories. Figure 3.1 illustrates this construction in the case of triangular grids in $\mathbf{R}^2$, where the neighboring domains are Ω_1 and Ω_2. Note that

$$E_{\tau_k} = E^1_{\tau_k} \cup E^2_{\tau_k},$$

where $E^i_{\tau_k}$, $i = 1, 2$, is a subset of an element of $\mathcal{T}_{h,i}$. Let φ_k be a continuous piecewise linear bubble function such that $0 \le \varphi_k(x) \le 1$ in E_{τ_k}, $\varphi_k = 1$ at the gravity center of τ_k and $\varphi_k = 0$ on ∂E_{τ_k}. Such a function can be easily constructed by decomposing E_{τ_k} into triangles if $d = 2$ or tetrahedra if $d = 3$. We also need an extension of $[\mathbf{u}_h \cdot \nu]_{\tau_k}$ to E_{τ_k}. Given $\psi \in H^{1/2}(\tau_k)$, define $\mathcal{R}\psi \in H^1(E_{\tau_k})$ such that $\mathcal{R}\psi$ is constant along lines perpendicular to Γ. Let

$$\phi_k = \varphi_k \mathcal{R}[\mathbf{u}_h \cdot \nu]_{\tau_k} \in H^1(E_{\tau_k}).$$

Note that $\phi_k = 0$ on ∂E_{τ_k}. Using a scaling argument similar to the one in [35] it can be shown that

$$\|\phi_k\|_{\tau_k} \le \|[\mathbf{u}_h \cdot \nu]\|_{\tau_k}, \tag{3.24}$$

$$\|\nabla \phi_k\|_{E_{\tau_k}} \le C h_{\tau_k}^{-1} \|\phi_k\|_{E_{\tau_k}}, \tag{3.25}$$

$$\|\phi_k\|_{E_{\tau_k}} \le C h_{\tau_k}^{1/2} \|\phi_k\|_{\tau_k}, \tag{3.26}$$

$$C\|[\mathbf{u}_h \cdot \nu]\|^2_{\tau_k} \le \langle \phi_k, [\mathbf{u}_h \cdot \nu] \rangle_{\tau_k}. \tag{3.27}$$

Using (3.27) and that $[\mathbf{u}\cdot\nu] = 0$,

$$C\|[\mathbf{u}_h\cdot\nu]\|_{\tau_k}^2 \le \langle \mathbf{u}_{h,1}\cdot\nu_1 + \mathbf{u}_{h,2}\cdot\nu_2, \phi_k\rangle_{\tau_k} = \langle(\mathbf{u}_{h,1}-\mathbf{u})\cdot\nu_1,\phi_k\rangle_{\tau_k} + \langle(\mathbf{u}_{h,2}-\mathbf{u})\cdot\nu_2,\phi_k\rangle_{\tau_k}. \tag{3.28}$$

Using Green's formula for the first term on the right-hand side, we have

$$\begin{aligned} \left|\langle\xi_1\cdot\nu_1,\phi_k\rangle_{\tau_k}\right| &= \left|(\nabla\phi_k,\xi_1)_{E^1_{\tau_k}} + (\phi_k,\nabla\cdot\xi_1)_{E^1_{\tau_k}}\right| \\ &\le \|\nabla\phi_k\|_{E^1_{\tau_k}}\|\xi_1\|_{E^1_{\tau_k}} + \|\phi_k\|_{E^1_{\tau_k}}\|\nabla\cdot\xi_1\|_{E^1_{\tau_k}} \\ &\le Ch_{\tau_k}^{-1}\|\phi_k\|_{E^1_{\tau_k}}\|\xi_1\|_{E^1_{\tau_k}} + \|\phi_k\|_{E^1_{\tau_k}}\|\nabla\cdot\xi_1\|_{E^1_{\tau_k}} \\ &\le C(h_{\tau_k}^{-1/2}\|\xi_1\|_{E^1_{\tau_k}} + h_{\tau_k}^{1/2}\|\nabla\cdot\xi_1\|_{E^1_{\tau_k}})\|[\mathbf{u}_h\cdot\nu]\|_{\tau_k}, \end{aligned} \tag{3.29}$$

where we have used (3.25) for the second inequality and (3.26), (3.24) for the third inequality. The second term on the right-hand side of (3.28) can be bounded similarly in terms of $\|\xi_2\|_{H(\mathrm{div};E^2_{\tau_k})}$. A combination of (3.28), (3.29), and (3.23) gives (3.21).

It remains to show (3.22). By the triangle inequality,

$$\|\lambda_h - p_h\|_e \le \|\lambda_h-\lambda\|_e + \|p-p_h\|_e. \tag{3.30}$$

For the second term on the right-hand side we employ the trace inequality (2.32)

$$\begin{aligned} \|p-p_h\|_e &\le C\big(h_E^{-1/2}\|p-p_h\|_E + h_E^{1/2}\|\nabla(p-p_h)\|_E\big) \\ &\le C\big(h_E^{-1/2}\|p-p_h\|_E + h_E^{1/2}\|K^{-1}\mathbf{u}_h + \nabla p_h\|_E + h_E^{1/2}\|K^{-1}(\mathbf{u}-\mathbf{u}_h)\|_E\big) \\ &\le C\big(h_E^{-1/2}\|\eta\|_E + h_E^{1/2}\|\xi\|_{H(\mathrm{div};E)}\big), \end{aligned} \tag{3.31}$$

using (3.20) for the last inequality. A combination of (3.30), (3.31), and (2.10) completes the proof of (3.22). The global bound (3.18) follows immediately from (3.20) to (3.22), using (2.10), and so does (3.19), using (2.23). ☐

Remark 3.2. The last two terms in (3.18) are of higher order, so $\|\eta\|$ dominates for small enough h. Therefore, this bound, combined with Theorem 3.1, implies that $\sum_{E\in\mathcal{T}_h}\omega_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\omega_\tau^2$ is an efficient and reliable estimator for the pressure error. Because of the negative power of h in the first term on the right-hand side of (3.19), the estimator $\sum_{E\in\mathcal{T}_h}\tilde\omega_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\tilde\omega_\tau^2$ provides only a suboptimal bound for the velocity error.

4. Error estimators based on solving local problems. In this section we derive an implicit error estimator which requires solving local (element) boundary value problems. These problems approximate the local residual equations satisfied by the true error. The motivation for considering implicit estimators comes from the unknown generic constants that appear in the explicit estimators, as well as the suboptimality in the lower bound for the velocity error. We show that the implicit estimator provides both optimal upper and lower bounds of the error.

4.1. Global approximation to the error. Similar to the approach in [39], we first construct a global approximation to the error based on higher-order finite element spaces. Using (2.4)–(2.6), the true error satisfies the residual equations:

$$
\begin{aligned}
&(K^{-1}\xi, \mathbf{v})_{\Omega_i} - (\eta, \nabla\cdot\mathbf{v})_{\Omega_i} + \langle \delta, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} = -\langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega_i\cap\Gamma_D} \\
&\quad - (K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} + (p_h, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} \equiv r(\mathbf{v}), \quad \mathbf{v}\in\mathbf{V}_i, \qquad (4.1)
\end{aligned}
$$

$$(\nabla\cdot\xi, w)_{\Omega_i} = (f - \nabla\cdot\mathbf{u}_h, w)_{\Omega_i}, \quad w\in W_i, \tag{4.2}$$

$$\sum_{i=1}^{n}\langle \xi\cdot\nu_i, \mu\rangle_{\Gamma_i} = -\sum_{i=1}^{n}\langle \mathbf{u}_h\cdot\nu_i, \mu\rangle_{\Gamma_i}, \quad \mu\in M. \tag{4.3}$$

Recall from the previous section that $\mathbf{V}_h'\times W_h'\times M_h'$ are the mortar mixed finite element spaces of one order higher than $\mathbf{V}_h\times W_h\times M_h$ and $(\xi',\eta',\delta')\in\mathbf{V}_h'\times W_h'\times M_h'$ is the finite element approximation to (ξ,η,δ) satisfying

$$(K^{-1}\xi', \mathbf{v})_{\Omega_i} - (\eta', \nabla\cdot\mathbf{v})_{\Omega_i} + \langle \delta', \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} = r(\mathbf{v}), \quad \mathbf{v}\in\mathbf{V}_{h,i}', \tag{4.4}$$

$$(\nabla\cdot\xi', w)_{\Omega_i} = (f - \nabla\cdot\mathbf{u}_h, w)_{\Omega_i}, \quad w\in W_{h,i}', \tag{4.5}$$

$$\sum_{i=1}^{n}\langle \xi'\cdot\nu_i, \mu\rangle_{\Gamma_i} = -\sum_{i=1}^{n}\langle \mathbf{u}_h\cdot\nu_i, \mu\rangle_{\Gamma_i}, \quad \mu\in M_h'. \tag{4.6}$$

Note that (4.4)–(4.6) implies that $(\mathbf{u}_h' = \mathbf{u}_h + \xi', p_h' = p_h + \eta', \lambda_h' = \lambda_h + \delta')$ is the finite element approximation to $(\mathbf{u}, p, \lambda)$ in $\mathbf{V}_h'\times W_h'\times M_h'$ satisfying

$$(K^{-1}\mathbf{u}_h', \mathbf{v})_{\Omega_i} = (p_h', \nabla\cdot\mathbf{v})_{\Omega_i} - \langle \lambda_h', \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\cap\Gamma_D}, \quad \mathbf{v}\in\mathbf{V}_{h,i}', \tag{4.7}$$

$$(\nabla\cdot\mathbf{u}_h', w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w\in W_{h,i}', \tag{4.8}$$

$$\sum_{i=1}^{n}\langle \mathbf{u}_h'\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0, \quad \mu\in M_h'. \tag{4.9}$$

The saturation assumptions (3.4) and (3.5) imply

$$(1-\beta)\|\xi\| \le \|\xi'\| \le (1+\beta)\|\xi\|, \tag{4.10}$$

$$(1-\beta_{\mathrm{div}})\|\nabla\cdot\xi\| \le \|\nabla\cdot\xi'\| \le (1+\beta_{\mathrm{div}})\|\nabla\cdot\xi\|, \tag{4.11}$$

so it is enough to estimate ξ'.

4.2. Local (element) approximation to the error. For any $E\in\mathcal{T}_h$, the true error satisfies the local equations:

$$(K^{-1}\xi, \mathbf{v})_E - (\eta, \nabla\cdot\mathbf{v})_E = r_E(\mathbf{v}) - \langle p, \mathbf{v}\cdot\nu_E\rangle_{\partial E}, \quad \mathbf{v}\in\mathbf{V}(E), \tag{4.12}$$

$$(\nabla\cdot\xi, w)_E = (f - \nabla\cdot\mathbf{u}_h, w)_E, \quad w\in W(E), \tag{4.13}$$

where

$$r_E(\mathbf{v}) = -(K^{-1}\mathbf{u}_h, \mathbf{v})_E + (p_h, \nabla\cdot\mathbf{v})_E.$$

We construct a higher-order local approximation of the error by solving element subproblems: find $\psi'\in\mathbf{V}_h'(E)$ and $\theta'\in W_h'(E)$ such that

$$(K^{-1}\psi', \mathbf{v})_E - (\theta', \nabla\cdot\mathbf{v})_E = r_E(\mathbf{v}) - \langle p_A, \mathbf{v}\cdot\nu_E\rangle_{\partial E}, \quad \mathbf{v}\in\mathbf{V}_h'(E), \tag{4.14}$$

$$(\nabla\cdot\psi', w)_E = (f - \nabla\cdot\mathbf{u}_h, w)_E, \quad w\in W_h'(E), \tag{4.15}$$

where $p_A = g$ on Γ_D, $p_A = \lambda_h$ on $\partial E \cap \Gamma$, and $p_A = \tilde{p}_h$ on $\partial E \cap \mathcal{E}_h$, where $\tilde{p}_h$ is the Lagrange multiplier for $\mathbf{V}_h$ and W_h defined as

$$\langle \tilde{p}_h, \mathbf{v}\cdot\nu_E\rangle_{\partial E} = -(K^{-1}\mathbf{u}_h, \mathbf{v})_E + (p_h, \nabla\cdot\mathbf{v})_E, \quad \mathbf{v}\in\mathbf{V}_h(E). \tag{4.16}$$

Let $\tilde{p}'$ be the Lagrange multiplier for the higher-order spaces $\mathbf{V}_h'$ and W_h' satisfying

$$\langle \tilde{p}', \mathbf{v}\cdot\nu_E\rangle_{\partial E} = -(K^{-1}\mathbf{u}_h', \mathbf{v})_E + (p_h', \nabla\cdot\mathbf{v})_E, \quad \mathbf{v}\in V_h'(E). \tag{4.17}$$

We make the following.

Saturation assumption. There exists a constant σ such that

$$\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|\tilde{p}' - \tilde{p}_h\|_e^2\right)^{1/2} \le \sigma\|\mathbf{u}-\mathbf{u}_h\|. \tag{4.18}$$

Assumption (4.18) is motivated by the a priori error estimate for the Lagrange multiplier [16]

$$\left(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|\bar{p} - \tilde{p}_h\|_e^2\right)^{1/2} \le Ch^{k+1},$$

where $\bar{p}$ is the L^2-projection of p onto $\mathbf{V}_h\cdot\nu|_{\mathcal{E}_h}$.

THEOREM 4.1. *Assume that the saturation assumptions* (2.23), (3.4), (3.5), *and* (4.18) *hold. Then there exist constants C_1 and C_2 independent of β and β_{div} such that*

$$\begin{aligned} C_1\Bigg(\|\psi'\|_{H(\text{div})} + \sum_{\tau\in\mathcal{T}^{\Gamma,h}} \|[\mathbf{u}_h\cdot\nu]\|_\tau h_\tau^{1/2}\Bigg) &\le \|\xi\|_{H(\text{div})} \\ &\le \frac{C_2}{1-\beta_{max}}\Bigg(\|\psi'\|_{H(\text{div})} + \sum_{\tau\in\mathcal{T}^{\Gamma,h}} \|[\mathbf{u}_h\cdot\nu]\|_\tau h_\tau^{1/2}\Bigg), \end{aligned} \tag{4.19}$$

where $\beta_{max} = \max\{\beta, \beta_{\text{div}}\}$.

Proof. We first note that (4.5) and (4.15) imply that on every $E\in\mathcal{T}_h$,

$$\nabla\cdot\psi' = \nabla\cdot\xi'. \tag{4.20}$$

Taking $\mathbf{v} = \psi' - \xi'$ in (4.14) and summing over all elements, we have

$$\begin{aligned} &\sum_{E\in\mathcal{T}_h}\big((K^{-1}(\psi'-\xi'), \psi'-\xi')_E - (\theta'-\eta', \nabla\cdot(\psi'-\xi'))_E\big) \\ &\quad= \sum_{E\in\mathcal{T}_h}\big(-(K^{-1}\xi', \psi'-\xi')_E + (\eta', \nabla\cdot(\psi'-\xi'))_E \\ &\qquad + r_E(\psi'-\xi') - \langle p_A, (\psi'-\xi')\cdot\nu_E\rangle_{\partial E}\big) \\ &\quad= \sum_{E\in\mathcal{T}_h}\big(-(K^{-1}\mathbf{u}_h', \psi'-\xi')_E + (p_h', \nabla\cdot(\psi'-\xi'))_E - \langle\tilde{p}_h, (\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h} \\ &\qquad - \langle g, (\psi'-\xi')\cdot\nu\rangle_{\partial E\cap\Gamma_D} - \langle\lambda_h, (\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\Gamma}\big) \\ &\quad= \sum_{E\in\mathcal{T}_h}\big(\langle\tilde{p}'-\tilde{p}_h, (\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h} + \langle\lambda_h'-\lambda_h, (\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\Gamma}\big), \end{aligned} \tag{4.21}$$

using (4.7) and (4.17) for the last equality. For the first term on the right-hand side, using the saturation assumption (4.18) and (2.34), we have

$$\begin{aligned}\left|\sum_{E\in\mathcal{T}_h}\langle\tilde{p}'-\tilde{p}_h,(\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h}\right| &\le \sum_{e\in\mathcal{E}_h}h_e^{-1/2}\|\tilde{p}'-\tilde{p}_h\|_e h_e^{1/2}\|(\psi'-\xi')\cdot\nu_e\|_e \\ &\le C\|\xi\|\|\psi'-\xi'\|.\end{aligned} \tag{4.22}$$

For the second term on the right-hand side of (4.21) we write, using (2.34), (2.23), and (3.17),

$$\begin{aligned}\left|\sum_{E\in\mathcal{T}_h}\langle\lambda_h'-\lambda_h,(\psi'-\xi')\cdot\nu_E\rangle_{\partial E\cap\Gamma}\right| &= \left|\sum_{i=1}^n\langle\delta',(\psi'-\xi')\cdot\nu_i\rangle_{\Gamma_i}\right| \\ &\le \sum_{\tau\in\mathcal{T}^{\Gamma,h}} h_\tau^{-1/2}\|\delta'\|_\tau h_\tau^{1/2}\|[(\psi'-\xi')\cdot\nu]\|_\tau \\ &\le |||\delta'|||\,\|\psi'-\xi'\|\le C\|\xi\|\|\psi'-\xi'\|.\end{aligned} \tag{4.23}$$

Combining (4.21)–(4.23) and using (4.20), we obtain

$$\|\psi'-\xi'\|\le C\|\xi\|,$$

which implies, using the triangle inequality and (4.10),

$$\|\psi'\|\le C\|\xi\|. \tag{4.24}$$

Taking $w=\nabla\cdot\psi'$ in (4.15) immediately gives

$$\|\nabla\cdot\psi'\|\le\|\nabla\cdot\xi\|,$$

which, combined with (4.24), implies

$$\|\psi'\|_{H(\mathrm{div})}\le C\|\xi\|_{H(\mathrm{div})}.$$

Combining the above bound with (3.21) completes the proof of the left inequality in (4.19). To show the right inequality in (4.19), taking $\mathbf{v}=\xi'$ in (4.4), and using (4.14), we have

$$(K^{-1}(\xi'-\psi'),\xi')=\sum_{i=1}^n\big((\eta'-\theta',\nabla\cdot\xi')_{\Omega_i}-\langle\delta',\xi'\cdot\nu_i\rangle_{\Gamma_i}\big). \tag{4.25}$$

For the first term on the right-hand side of (4.25) we use (4.20) and the argument from (4.21) to obtain

$$\begin{aligned}\sum_{i=1}^n(\eta'-\theta',\nabla\cdot\xi')_{\Omega_i} &= \sum_{E\in\mathcal{T}_h}(\eta'-\theta',\nabla\cdot\psi')_E \\ &= \big(K^{-1}(\xi'-\psi'),\psi'\big)-\sum_{E\in\mathcal{T}_h}\big(\langle\tilde{p}'-\tilde{p}_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h} \\ &\quad+\langle\lambda_h'-\lambda_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\Gamma}\big),\end{aligned} \tag{4.26}$$

which, combined with (4.25), implies

$$\begin{aligned}&\big(K^{-1}(\xi'-\psi'),\xi'-\psi'\big)=-\langle\delta',\xi'\cdot\nu_i\rangle_{\Gamma_i} \\ &\quad-\sum_{E\in\mathcal{T}_h}\big(\langle\tilde{p}'-\tilde{p}_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h}+\langle\lambda_h'-\lambda_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\Gamma}\big).\end{aligned} \tag{4.27}$$

For the first term on the right-hand side we have, using (4.6),

$$\left|\sum_{i=1}^{n}\langle\delta',\xi'\cdot\nu_i\rangle_{\Gamma_i}\right| = \left|\sum_{i=1}^{n}\langle\delta',\mathbf{u}_h\cdot\nu_i\rangle_{\Gamma_i}\right| \le \epsilon_1\|\xi\|^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\frac{1}{4\epsilon_1}\|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau, \tag{4.28}$$

where the inequality is obtained using the argument in (3.15) and (3.17). The last two terms on the right-hand side of (4.27) can be bounded similarly to (4.22) and (4.23):

$$\left|\sum_{E\in\mathcal{T}_h}\left(\langle\tilde{p}'-\tilde{p}_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\mathcal{E}_h} + \langle\lambda_h'-\lambda_h,\psi'\cdot\nu_E\rangle_{\partial E\cap\Gamma}\right)\right| \le C\left(\epsilon_2\|\xi\|^2 + \frac{1}{4\epsilon_2}\|\psi'\|^2\right). \tag{4.29}$$

Combining (4.27)–(4.29),

$$\|\xi'-\psi'\|^2 \le C\left((\epsilon_1+\epsilon_2)\|\xi\|^2 + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\frac{1}{4\epsilon_1}\|[\mathbf{u}_h\cdot\nu]\|_\tau^2 h_\tau + \frac{1}{4\epsilon_2}\|\psi'\|^2\right),$$

which implies, using the triangle inequality, (4.10), and taking ϵ_1 and ϵ_2 small enough,

$$\|\xi\| \le \frac{C}{1-\beta}\left(\|\psi'\| + \sum_{\tau\in\mathcal{T}^{\Gamma,h}}\|[\mathbf{u}_h\cdot\nu]\|_\tau h_\tau^{1/2}\right).$$

An application of (4.11) and (4.20) completes the proof. ◻

5. Numerical results. In this section we test the performance of the residual-based error estimator. The estimator is used as a local error indicator that drives an adaptive mesh refinement process. The following algorithm describes the adaptive procedure.

ALGORITHM.

1. Solve the problem on a coarse (both subdomain and mortar) grid.
2. For each subdomain Ω_i
   (a) Compute
   $$\omega_i = \left(\sum_{E\in\mathcal{T}_{h,i}}\omega_E^2 + \sum_{\tau\in\mathcal{T}^{\Gamma_i,h}}\omega_\tau^2\right)^{1/2}.$$
   (b) If $\omega_i > 0.5\max_{1\le j\le n}\omega_j$, refine $\mathcal{T}_{h,i}$.
   (c) If any neighboring subdomain grid has been refined two times more than Ω_i, refine $\mathcal{T}_{h,i}$.
3. For each interface $\Gamma_{i,j}$, if either Ω_i or Ω_j has been refined, refine $\mathcal{T}_{h,i,j}$.
4. Solve the problem on the refined grid. If either the desired error tolerance or the maximum refinement level has been reached, exit; otherwise go to step 2.

Several comments are in order. First, we employ the pressure error estimator based on ω_E and ω_τ, defined in (3.1) and (3.2), since it provides an efficient and reliable estimate of the L^2 pressure error, due to Theorems 3.1 and 3.3. Second, the refinement rule 2(c) is needed to reduce the effect of discretization error due to large ratios between grid sizes in neighboring subdomains. Third, according to rule 3, mortar grids are refined if either adjacent subdomain grid is refined.

In the examples below, the subdomains are discretized by the lowest-order Raviart–Thomas spaces. Discontinuous piecewise linear mortar spaces are used on the interfaces.

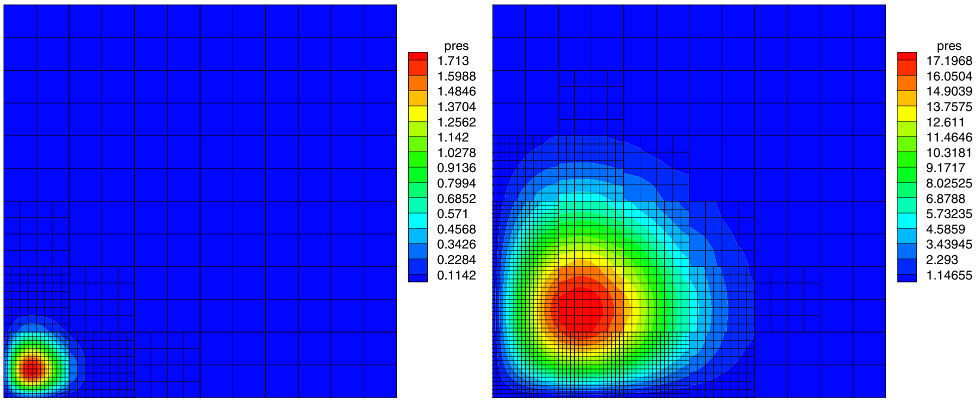

FIG. 5.1. *Computed pressure on the fourth grid level for examples* 1 *and* 2.

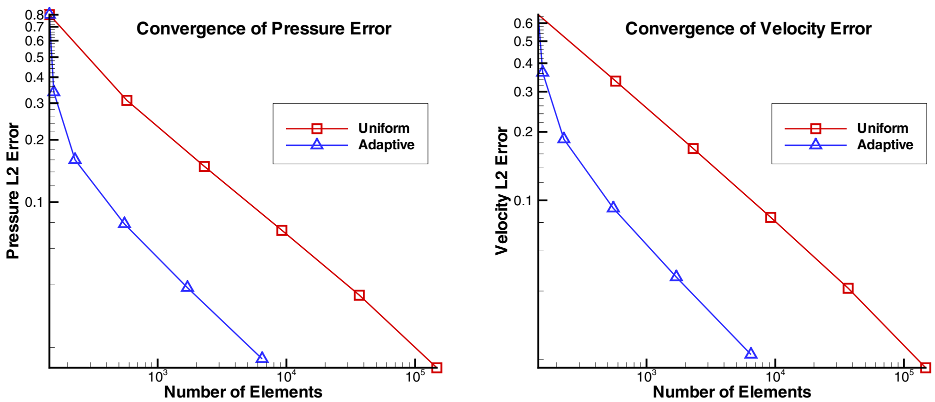

FIG. 5.2. *Convergence of pressure and velocity error for example* 1.

We first illustrate the above algorithm for several two-dimensional problems. In all examples the domain is the unit square, decomposed into 6×6 subdomains. The coarse grid in each subdomain is 2×2 with a single mortar element on each interface. In the first two examples we test problems with boundary layers. The true pressure solution is

$$p(x, y) = 1000\, x\, y\, e^{-k(x^2+y^2)},$$

where $k = 100$ in example 1 and $k = 10$ in example 2. In both cases $K = I$. The computed pressure after three refinements is shown in Figure 5.1. We note that in both cases the grids are appropriately refined along the boundary layers. In the second example the exponential drop is less steep. This causes an extended boundary layer, which is resolved by a strip of fine subdomain grids along the boundary. In Figure 5.2, the pressure and velocity errors in example 1 are plotted as functions of the total number of finite elements. The convergence of the error for all other examples is similar and is not shown. We observe that the adaptive solution needs about 20 times fewer elements to provide the same accuracy.

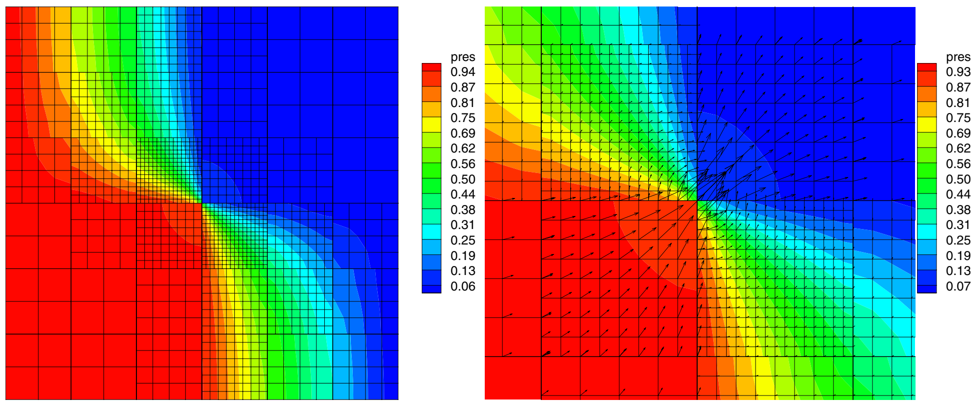

FIG. 5.3. *Computed solution on the fourth grid level for example* 3. *Left: pressure on the full grid. Right: pressure and velocity near the singularity.*

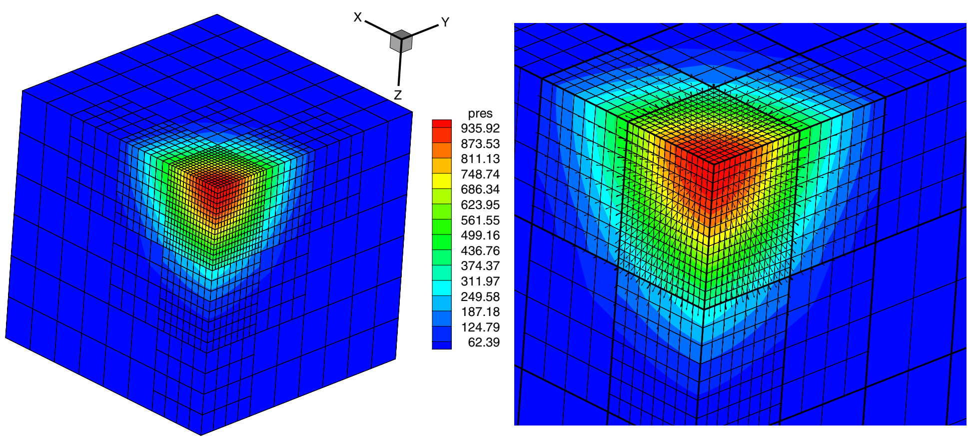

FIG. 5.4. *Computed solution on the fourth grid level for example* 4. *Left: pressure on the full grid. Right: pressure and velocity zoom.*

In the next example, motivated by the modeling flow in heterogeneous porous media, we test a problem with a discontinuous permeability tensor K. The domain is divided into four subregions by the lines $x = 0.5$ and $y = 0.5$. The permeability is $K = 100I$ in the lower-left and upper-right regions and $K = I$ in the other two regions. Dirichlet boundary conditions $p = 1$ on the left and $p = 0$ on the right and no-flow boundary conditions on the top and bottom force the flow from left to right. It is known for the true solution that $p \in H^{1+\alpha}$ for some $0 < \alpha < 1$ with singularity occurring at the cross-point. The computed solution after three refinements is shown in Figure 5.3. As expected the grids are finest near the singularity and are also refined in the low permeability region to resolve the high pressure gradient. Some of the grids in the high permeability region are refined as well, due to the refinement rule 2(c).

Finally, a three-dimensional example is presented. The unit cube is divided into $4 \times 4 \times 3$ subdomains. The true pressure

$$p = 1000\, e^{-10(x^2+y^2+z^2)}$$

exhibits a steep exponential decay near the origin. The computed pressure and velocity on the fourth grid level are given in Figure 5.4. The steep pressure gradient and large velocity are well resolved by the fine computational grids near the origin.

6. Conclusions. In this paper, several two- and three-dimensional a posteriori error estimators for mortar mixed finite element methods for elliptic equations have been derived. A residual-based error estimator provides optimal upper and lower bounds for the pressure error. A closely related error estimator for the velocity gives an optimal upper bound, but suboptimal lower bound. The negative power of h that appears is due to the different order of derivatives involved in the L^2-norm and the $H(\mathrm{div})$-norm. An efficient and reliable implicit estimator for the velocity is also derived, which is based on solving local (element) problems. All estimators include a term that measures the jump of flux across subdomain interfaces. This term provides a measure of nonconformity in the mortar discretization. In cases where the subdomain grids are fixed and optimal mortar grids need to be obtained, this flux-jump term can be used to drive an adaptive process for the mortar grids independently of the subdomain grids.

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