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# Balancing domain decomposition for mortar mixed finite element methods

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SUMMARY

The balancing domain decomposition method for mixed finite elements by Cowsar, Mandel, and Wheeler is extended to the case of mortar mixed finite elements on non-matching multiblock grids. The algorithm involves an iterative solution of a mortar interface problem with one local Dirichlet solve and one local Neumann solve per subdomain on each iteration. A coarse solve is used to guarantee that the Neumann problems are consistent and to provide global exchange of information across subdomains. Quasi-optimal condition number bounds are derived, which are independent of the jump in coefficients between subdomains. Numerical experiments confirm the theoretical results. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: mixed finite element; mortar finite element; non-matching grids; balancing domain decomposition

## 1. INTRODUCTION

A growing number of papers in recent years deal with the numerical modelling of partial differential equations on non-matching grids. This generality, often referred to as a multiblock approach, allows for modelling complex geometries by representing them as unions of simpler locally discretized subdomains (blocks). The computational grids need not match across interfaces, which allows for modelling internal boundaries and for efficient treatment of spatially and temporally varying physical processes. A typical example is modelling large scale geological structures such as faults and layers, and high gradients near wells in flow in porous media.

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In a multiblock formulation the underlying equations hold locally on each subdomain. This includes the possibility of multiphysics formulations where different physical processes and different mathematical models may be associated with different blocks, e.g. coupling single-phase flow with multiphase flow in reservoir modelling [1, 2]. Appropriate discretization methods are applied locally on the subdomains. Physically meaningful and mathematically consistent matching conditions are imposed on the interfaces. Mortar finite elements have been successfully employed for discretely imposing these interface conditions when coupling discretizations based on Galerkin finite elements [3–5], mixed finite elements [6–8], and finite volume elements [9].

In this work we consider mixed finite element methods for subdomain discretizations. Mixed methods owe their popularity to their local (element-wise) mass conservation property and the simultaneous and accurate approximation of two variables of physical interest, e.g. pressure and velocity in fluid flow. The mortar mixed method can be viewed as an extension to non-matching grids of the partially hybridized form of the mixed method where Lagrange multiplier pressures are introduced on the inter-block boundaries [10–12].

This paper deals with the problem of solving efficiently the algebraic system arising in mortar mixed finite element discretizations of elliptic equations. A non-overlapping domain decomposition algorithm developed for matching grids by Glowinski and Wheeler [12, 13] and later extended to non-matching grids [6, 14] is employed as a solver. The method reduces the global system to an interface problem which is symmetric and positive definite in the case of elliptic equations and can be solved iteratively via a preconditioned conjugate gradient method. This approach is very suitable for parallel implementation since the dominant cost is solving subdomain problems.

The feasibility of the domain decomposition solver depends critically on the rate of convergence of the interface iteration and ultimately on the conditioning of the interface operator. The goal of this paper is to extend to the case of non-matching multiblock grids the balancing preconditioner for mixed finite elements developed by Cowsar *et al.* [15]. Other substructuring preconditioners for mortar finite elements can be found in References [16–18]. The balancing domain decomposition method was introduced by Mandel [19] for Galerkin finite elements and later analysed by Mandel and Brezina [20]. The algorithm is based on the Neumann–Neumann preconditioner [21–23] and involves an iterative solution of the interface problem with one local Dirichlet solve (action of the operator) and one local Neumann solve (action of the preconditioner) per subdomain on each iteration. A coarse problem is added to guarantee that the Neumann problems are consistent which also provides global exchange of information across subdomains. The condition number analysis in Reference [15] pivots around a characterization of the interface bilinear form as a $H^{1/2}$-norm of an interpolant of the Dirichlet interface data. A key ingredient in our analysis is a similar characterization for the mortar bilinear form and the mortar interface data (see (30) below). Our theoretical results for the mortar balancing preconditioner provide, as in the case of matching grids, a quasi-optimal condition number bound $O((1+\log(H/h))^2)$ which is independent of the jump in coefficients between subdomains. Here $h$ is the discretization parameter and $H$ is the characteristic size of the subdomains. This bound also indicates very weak dependence on the number of subdomains which is confirmed experimentally in Section 5. Our analysis depends on assumption on the grids (33) which is closely related to a solvability condition (15) for the mortar spaces introduced in Reference [6] (see also Reference [7]). This assumption is justified in Appendix A for Raviart–Thomas subdomain

 

discretizations of lowest order [24] on fairly general grids and it is easy to satisfy in practice (see Remark 4.1).

The rest of the paper is organized as follows. Mortar mixed finite element methods for second-order elliptic equations are presented in Section 2. The non-overlapping domain decomposition method and the balancing preconditioner are described in Section 3. Section 4 is devoted to the analysis of the condition number of the preconditioned operator. In Section 5 the behaviour of the preconditioner is illustrated by a series of numerical experiments. A technical lemma needed in the analysis is proven in Appendix A.

## 2. MORTAR MIXED FINITE ELEMENT METHODS

We consider the following second-order elliptic problem written as a system of two first-order equations:

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega \tag{1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega \tag{2}$$

$$p = g \quad \text{on } \partial\Omega \tag{3}$$

where $\Omega = \bigcup_{i=1}^{n} \Omega_i \subset \mathbf{R}^d$, $d = 2$ or 3, is a multiblock domain and $K$ is a symmetric, uniformly positive definite tensor. This system models, among many other applications, single-phase flow in porous media where $p$ is the pressure and $\mathbf{u}$ is the velocity. The Dirichlet boundary conditions are considered merely for simplicity. The subdomains $\Omega_i$ are assumed to be non-overlapping and shape-regular with diameters $O(H)$, i.e. there exists a reference domain $\hat{\Omega}$ with a diameter $O(1)$ and bijective mappings $F_i$ such that

$$\Omega_i = F_i(\hat{\Omega}), \quad \|DF_i\| \leqslant CH, \quad \|DF_i^{-1}\| \leqslant CH^{-1}$$

Let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \bigcup_{i,j=1}^{n} \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \backslash \partial\Omega$ denote interior block interfaces. We assume that there exist positive constants $c$, $C$, and $\alpha_i$ such that

$$c\alpha_i \xi^{\mathrm{T}}\xi \leqslant \xi^{\mathrm{T}} K(x)\xi \leqslant C\alpha_i \xi^{\mathrm{T}}\xi, \quad \forall \xi \in \mathbf{R}^d, \ \forall x \in \Omega_i, \ i = 1, \ldots, n \tag{4}$$

We will make use of the following standard notation. For a subdomain $G \subset \mathbf{R}^d$, the $L^2(G)$ inner product (or duality pairing) and norm are denoted by $(\cdot,\cdot)_G$ and $\|\cdot\|_G$, respectively, for scalar and vector valued functions. For a section of a subdomain boundary $S \subset \bigcup_{i=1}^{n} \partial\Omega_i$ we write $\langle \cdot,\cdot \rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively. We omit $G(S)$ in the subscript if $G = \Omega$ ($S = \Gamma$). Let us also denote, for any symmetric and positive definite operator $B$ on $\Gamma$, the $B$-induced inner product and norm by $\langle u, v \rangle_B = \langle Bu, v \rangle$ and $\|u\|_B = (\langle u, u \rangle_B)^{1/2}$, respectively. We will also make use of the scaled Sobolev norms

$$\|w\|_{1,G}^2 = |w|_{1,G}^2 + \frac{1}{H^2}\|w\|_G^2, \quad \|w\|_{1/2,S}^2 = |w|_{1/2,S}^2 + \frac{1}{H}\|w\|_S^2$$

where

$$|w|_{1,G}^2 = \int_G |\nabla w(x)|^2 \, \mathrm{d}x, \quad |w|_{1/2,S}^2 = \int_S \int_S \frac{|w(t) - w(s)|^2}{|t - s|^d} \, \mathrm{d}t \, \mathrm{d}s$$

 

Throughout the paper the constants $c$ and $C$ will denote generic constants independent of $h$, $H$, and $\alpha_i$.

The velocity and pressure functional spaces for the mixed weak formulation of (1)–(3) are defined as usual [11] to be

$$\mathbf{V} = H(\mathrm{div};\Omega) = \{\mathbf{v} \in (L^2(\Omega))^d:\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\}, \quad W = L^2(\Omega)$$

with norms

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2)^{1/2}, \quad \|w\|_W = \|w\|$$

A weak solution of (1)–(3) is a pair $\mathbf{u} \in \mathbf{V}$, $p \in W$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla\cdot\mathbf{v}) - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega}, \quad \mathbf{v} \in H(\mathrm{div};\Omega) \tag{5}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \quad w \in L^2(\Omega) \tag{6}$$

where $\nu$ is the outward unit normal vector on $\partial\Omega$. It is well known (see, e.g. References [11, 25]) that (5)–(6) has a unique solution.

We also consider an alternative domain decomposition variational formulation. Let

$$\mathbf{V}_i = H(\mathrm{div};\Omega_i), \quad W_i = L^2(\Omega_i)$$

If the solution $(\mathbf{u}, p)$ of (5)–(6) belongs to $H^1(\Omega)\times L^2(\Omega)$, it is easy to see that it satisfies, for $1 \leqslant i \leqslant n$,

$$(K^{-1}\mathbf{u}, \mathbf{v})_{\Omega_i} = (p, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle p, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_i \tag{7}$$

$$(\nabla\cdot\mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_i \tag{8}$$

$$\sum_{i=1}^{n} \langle \mathbf{u}\cdot\nu_i, \mu\rangle_{\Gamma_i} = 0, \quad \mu \in L^2(\Gamma) \tag{9}$$

where $\nu_i$ is the outer unit normal to $\partial\Omega_i$.

Let $\mathcal{T}_{h,i}$ be a conforming, quasi-uniform finite element partition of $\Omega_i$, $1 \leqslant i \leqslant n$. The subdomain partitions $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not match on $\Gamma_{i,j}$. Let $\mathcal{T}_h = \bigcup_{i=1}^n \mathcal{T}_{h,i}$ denote the global finite element partition. Let

$$\mathbf{V}_{h,i}\times W_{h,i} \subset \mathbf{V}_i \times W_i$$

be any of the usual mixed finite element spaces defined on $\mathcal{T}_{h,i}$ (see Reference [11, Section III.3]), the RT spaces [24, 26], the BDM spaces [27], the BDFM spaces [28], the BDDF spaces [29], or the CD spaces [30]. It is known for these spaces [11] that

$$\nabla\cdot\mathbf{V}_{h,i} = W_{h,i}$$

and that there exists a projection $\Pi_i$ of $(H^1(\Omega_i))^d$ onto $\mathbf{V}_{h,i}$, satisfying amongst other properties that for any $\mathbf{q} \in (H^1(\Omega_i))^d$,

$$(\nabla\cdot(\Pi_i\mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \quad w \in W_{h,i} \tag{10}$$

$$\langle(\mathbf{q} - \Pi_i\mathbf{q})\cdot\nu_i, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i} \tag{11}$$

 

The most commonly used mixed spaces are the Raviart–Thomas spaces of lowest order $\mathrm{RT}_0$ [5, 24, 26]. In this case

$$\mathbf{V}_h(E) = \{\mathbf{v} = (v_1, v_2) \text{ or } \mathbf{v} = (v_1, v_2, v_3) : v_l = \alpha_l + \beta_l x_l; \alpha_l, \beta_l \in \mathbf{R}, \ l = 1, \ldots, d\}$$

$$W_h(E) = \{w = \text{const}\}$$

Note that for any element $E \in \mathscr{T}_h$, the degrees of freedom for a vector $\mathbf{v} \in \mathbf{V}_h(E)$ can be specified by the values of its normal components $\mathbf{v} \cdot v_E$ at the midpoints of all edges (faces) of $E$, where $v_E$ is the outward unit normal vector on $\partial E$. The degree of freedom for a function $w \in W_h(E)$ is its value at the center of $E$.

The velocity and pressure mixed finite element spaces on $\Omega$ are defined as follows:

$$\mathbf{V}_h = \bigoplus_{i=1}^{n} \mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^{n} W_{h,i}$$

Note that this choice leads to a non-conforming approximation since $\mathbf{V}_h \not\subset \mathbf{V}$.

Let $\mathscr{T}_{h,i,j}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$. Let $k$ be associated with the degree of the polynomials in $\mathbf{V}_h \cdot v$. Denote by $M_{h,i,j} \subset L^2(\Gamma_{i,j})$ the mortar finite element space on $\Gamma_{i,j}$ containing at least either the continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathscr{T}_{h,i,j}$. For example, in the case of $\mathrm{RT}_0$, $M_{h,i,j}$ is the space of piecewise linear (bilinear, if $d = 3$ and the grids are hexahedral) polynomials on $\mathscr{T}_{h,i,j}$. Let

$$M_h = \bigoplus_{1 \leqslant i < j \leqslant n} M_{h,i,j}$$

be the mortar finite element space on $\Gamma$.

In the mortar mixed finite element approximation of (5)–(6) we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_h \in M_h$ such that, for $1 \leqslant i \leqslant n$,

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot v_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot v_i \rangle_{\partial\Omega_i \setminus \Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i} \tag{12}$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i} \tag{13}$$

$$\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot v_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in M_h \tag{14}$$

It is clear from (7) and (12) that $\lambda_h \in M_h$ is an approximation to the pressure $p$ on $\Gamma$. Equation (14) enforces weak (with respect to the mortar space $M_h$) continuity of flux across the block interfaces. Existence and uniqueness of a solution to (12)–(14) are shown in References [6, 7] along with optimal convergence and superconvergence for both pressure and velocity under the assumption that for all $\mu \in M_{h,i,j}$ there exists a constant $C$ independent of $h$ such that

$$\|\mu\|_{\Gamma_{i,j}} \leqslant C(\|\mathscr{Q}_{h,i}\mu\|_{\Gamma_{i,j}} + \|\mathscr{Q}_{h,j}\mu\|_{\Gamma_{i,j}}) \tag{15}$$

where $\mathscr{Q}_{h,i} : L^2(\Gamma_i) \to \mathbf{V}_{h,i} \cdot v_i|_{\Gamma_i}$ is the $L^2$-orthogonal projection satisfying for any $\phi \in L^2(\Gamma_i)$

$$\langle \phi - \mathscr{Q}_{h,i}\phi, \ \mathbf{v} \cdot v_i \rangle_{\Gamma_i} = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{h,i} \tag{16}$$

 

*Remark 2.1*
The condition (15) imposes a limit on the number of mortar degrees of freedom and is easily satisfied in practice [6]. In the case of $RT_0$ spaces, (15) holds under the assumption on the grids in Lemma A.1, as it can be seen from the proof.

It will be convenient to treat the local operators $\mathcal{Q}_{h,i}$ as operators from $M_h$ to $\mathbf{V}_{h,i}\cdot\nu_i|_{\Gamma_i}$, implicitly assuming that, for a function $\mu\in M_h$, $\mathcal{Q}_{h,i}\mu=\mathcal{Q}_{h,i}N_i\mu$, where $N_i: M_h\to M_{h,i}\equiv M_h|_{\Gamma_i}$ is the restriction operator. Similarly $\mathcal{Q}_{h,i}^{\mathrm{T}}$, the $L^2$-orthogonal projector from $\mathbf{V}_{h,i}\cdot\nu_i|_{\Gamma_i}$ onto $M_{h,i}$, will be understood as an operator from $\mathbf{V}_{h,i}\cdot\nu_i|_{\Gamma_i}$ to $M_h$, implicitly assuming that $\mathcal{Q}_{h,i}^{\mathrm{T}}\mathbf{v}_i\cdot\nu_i = N_i^{\mathrm{T}}\mathcal{Q}_{h,i}^{\mathrm{T}}\mathbf{v}_i\cdot\nu_i$, where $N_i^{\mathrm{T}}: M_{h,i}\to M_h$ is the extension-by-zero operator. In addition, given a function in $M_{h,i}$ we assume by default that it is extended by zero to the whole $M_h$.

## 3. BALANCING DOMAIN DECOMPOSITION

We employ a non-overlapping domain decomposition method for the efficient parallel solution of the algebraic system that arises in the mortar mixed finite element discretization (12)–(14). The algorithm is based on the method originally developed by Glowinski and Wheeler in Reference [12] for mixed methods on conforming grids. It reduces the global system to a symmetric and positive definite interface mortar problem which can be solved by a preconditioned conjugate gradient iteration. We formulate the balancing interface preconditioner extending the work of Cowsar *et al.* [15] to mortar mixed finite element methods on non-matching grids.

### 3.1. Reduction to an interface problem

Define bilinear forms $a_{h,i}: L^2(\Gamma)\times L^2(\Gamma)\to\mathbf{R}$, $1\leqslant i\leqslant n$, and $a_h: L^2(\Gamma)\times L^2(\Gamma)\to\mathbf{R}$ by

$$a_{h,i}(\lambda,\mu)=-\langle\mathbf{u}_{h,i}^*(\lambda)\cdot\nu_i,\mu\rangle_{\Gamma_i},\quad a_h(\lambda,\mu)=\sum_{i=1}^{n}a_{h,i}(\lambda,\mu) \tag{17}$$

where, for $\lambda\in L^2(\Gamma)$, $(\mathbf{u}_{h,i}^*(\lambda),p_{h,i}^*(\lambda))\in\mathbf{V}_{h,i}\times W_{h,i}$, $1\leqslant i\leqslant n$, solve

$$(K^{-1}\mathbf{u}_{h,i}^*(\lambda),\mathbf{v})_{\Omega_i}=(p_{h,i}^*(\lambda),\nabla\cdot\mathbf{v})_{\Omega_i}-\langle\lambda,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i},\quad \mathbf{v}\in\mathbf{V}_{h,i} \tag{18}$$

$$(\nabla\cdot\mathbf{u}_{h,i}^*(\lambda),w)_{\Omega_i}=0,\quad w\in W_{h,i} \tag{19}$$

Define a linear functional $g_h: L^2(\Gamma)\to\mathbf{R}$ by

$$g_h(\mu)=\sum_{i=1}^{n}\langle\bar{\mathbf{u}}_{h,i}\cdot\nu_i,\mu\rangle_{\Gamma_i} \tag{20}$$

where $(\bar{\mathbf{u}}_{h,i}\bar{p}_{h,i})\in\mathbf{V}_{h,i}\times W_{h,i}$, $1\leqslant i\leqslant n$, solve

$$(K^{-1}\bar{\mathbf{u}}_{h,i},\mathbf{v})_{\Omega_i}=(\bar{p}_{h,i},\nabla\cdot\mathbf{v})_{\Omega_i}-\langle g,\mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma_i},\quad \mathbf{v}\in\mathbf{V}_{h,i} \tag{21}$$

$$(\nabla\cdot\bar{\mathbf{u}}_{h,i},w)_{\Omega_i}=(f,w)_{\Omega_i},\quad w\in W_{h,i} \tag{22}$$

 

It is straightforward to show (see Reference [12]) that the solution $(\mathbf{u}_h, p_h, \lambda_h)$ of (12)–(14) satisfies

$$a_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in M_h \tag{23}$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h \tag{24}$$

where $\mathbf{u}_h^*(\lambda) \in \mathbf{V}_h$ is such that $\mathbf{u}_h^*(\lambda)|_{\Omega_i} = \mathbf{u}_{h,i}^*(\lambda)$, with similar definitions for $p_h^*(\lambda)$, $\bar{\mathbf{u}}_h$, and $\bar{p}_h$.

We introduce linear maps $A_{h,i} : M_h \to M_h$, $i = 1, \ldots, n$, corresponding to the bilinear forms $a_{h,i}(\cdot, \cdot)$ and satisfying

$$\langle A_{h,i}\lambda, \mu \rangle = a_{h,i}(\lambda, \mu), \quad \forall \lambda, \mu \in M_h \tag{25}$$

Note that (17) and (25) imply

$$A_{h,i}\lambda = -\mathcal{Q}_{h,i}^{\mathrm{T}} \mathbf{u}_{h,i}^*(\lambda) \cdot \nu_i \tag{26}$$

hence the operators $A_{h,i}$ are Dirichlet-to-Neumann maps. It is clear from (18) that $\mathbf{u}_{h,i}^*(\lambda) = \mathbf{u}_{h,i}^*(\mathcal{Q}_{h,i}\lambda)$ which combined with (26) implies that

$$A_{h,i} = \mathcal{Q}_{h,i}^{\mathrm{T}} \overline{A_{h,i}} \mathcal{Q}_{h,i} \tag{27}$$

where $\overline{A_{h,i}}$ are the local non-mortar Dirichlet-to-Neumann maps from $\mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$ to $\mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$.

The interface problem (23) can now be written as

$$A_h \lambda = \bar{g}_h \tag{28}$$

where $A_h = \sum_{i=1}^n A_{h,i} : M_h \to M_h$ and $\bar{g}_h \in M_h$ is the Riesz representation of $g_h$. The operator $A_h$ is a mortar version of the Poincaré–Steklov operator [31]. It can be viewed algebraically as the Schur complement with respect to the mortar unknowns.

The following lemma has been shown in References [6, 7] (see also References [13, 15] for the conforming grids case).

*Lemma 3.1*

The interface bilinear form $a_h(\cdot, \cdot)$ is symmetric and positive semi-definite in $L^2(\Gamma) \times L^2(\Gamma)$. If (15) holds, then $a_h(\cdot, \cdot)$ is positive definite in $M_h \times M_h$.

The proof is based on the representation

$$a_{h,i}(\lambda, \mu) = (K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu))_{\Omega_i} \tag{29}$$

which follows easily from (17) and (18).

Another useful characterization for $a_{h,i}(\cdot, \cdot)$ has been shown in Reference [6] (see also Reference [15]). There exist positive constants $c$ and $C$ such that

$$c\alpha_i |\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu|^2_{1/2,\partial\Omega_i} \leqslant a_{h,i}(\mu, \mu) \leqslant C\alpha_i |\mathcal{I}^{\partial\Omega_i} \mathcal{Q}_{h,i}\mu|^2_{1/2,\partial\Omega_i}, \quad \forall \mu \in M_h \tag{30}$$

where $\alpha_i$ is the constant from (4) and $\mathcal{I}^{\partial\Omega_i}$ is a continuous piecewise linear interpolant on the trace of the $\mathcal{T}_{h,i}$ on the boundary introduced in Reference [15]. The interpolant $\mathcal{I}^{\partial\Omega_i}$ is defined in Appendix A for the case of $\mathrm{RT}_0$. See Reference [15] for a general definition.

 

Due to Lemma 3.1, the interface problem (28) can be solved using a preconditioned conjugate gradient (PCG) method. One evaluation of the operator $A_h : \lambda \to A_h\lambda$ is required on each PCG iteration. It involves the following steps.

1. Project ($L^2$-orthogonally) mortar data onto the subdomain grids
$$\lambda \overset{\mathcal{Q}_{h,i}}{\to} \mathcal{Q}_{h,i}\lambda$$
2. Solve in parallel subdomain problems (18)–(19) with Dirichlet data $\mathcal{Q}_{h,i}\lambda$, on the interior interfaces to compute the fluxes $\mathbf{u}^*_{h,i}(\lambda)\cdot\nu_i$.
3. Project the fluxes back to the mortar space
$$\mathbf{u}^*_{h,i}(\lambda)\cdot\nu_i \overset{Q^{\mathrm{T}}_{h,i}}{\to} \mathbf{u}^m_{h,i}$$
and compute the jump across each interface $\Gamma_{i,j}$
$$[\mathbf{u}^m_h]_{ij} = \mathbf{u}^m_{h,i} + \mathbf{u}^m_{h,j}$$

The projection Steps 1 and 3 are relatively inexpensive. The dominant cost is in Step 2.

### 3.2. *Balancing preconditioner*

The balancing preconditioner is based on the Neumann–Neumann preconditioner developed in [21–23]. The latter can be expressed in operator form as

$$B^{-1}_{NN} = \sum_{i=1}^{n} A^+_{h,i} \tag{31}$$

where $A^+_{h,i}$ is the Moore–Penrose pseudo-inverse of $A_{h,i}$. The evaluation of $B^{-1}_{NN}$ requires solving subdomain problems $A_{h,i}\lambda_i = r_i$ with Neumann boundary data $r_i$: find $\mathbf{u}_{h,i}\in V_{h,i}$, $p_{h,i}\in W_{h,i}$, $\lambda_i \in M_{h,i}$ such that

$$(K^{-1}\mathbf{u}_{h,i},\mathbf{v})_{\Omega_i} = (p_{h,i},\nabla\cdot\mathbf{v})_{\Omega_i} - \langle\lambda_i,\mathbf{v}\cdot\nu_i\rangle_{\Gamma_i}, \quad \mathbf{v}\in\mathbf{V}_{h,i}$$
$$(\nabla\cdot\mathbf{u}_{h,i},w)_{\Omega_i} = 0, \quad w\in W_{h,i}$$
$$\langle\mathbf{u}_{h,i}\cdot\nu_i,\mu\rangle_{\Gamma_i} = \langle r_i,\mu\rangle_{\Gamma_i}, \quad \mu\in M_{h,i}$$

The preconditioner (31) has two drawbacks: the local problems may not be solvable and the convergence deteriorates for large number of subdomains due to lack of global exchange of information. The balancing preconditioner [15, 19, 20] was developed to overcome these problems. The idea is to balance residuals so that local problems $A_{h,i}\lambda_i = r_i$ are solvable (modulo Null $A_{h,i}$) and the result does not depend on the specific choice of local solutions. We note that $A_{h,i}\lambda_i = r_i$ is solvable if

$$r_i \perp \mathrm{Null}\, A_{h,i} = \begin{cases} \{\mathrm{const}\} & \text{if full Neumann} \\ \emptyset & \text{otherwise} \end{cases}$$

Define a partition of unity $D_i$ such that $D_i\lambda$ is non-zero only on $\Gamma_i$ and

$$\sum_{i=1}^{n} D_i\lambda = \lambda, \quad \forall\lambda\in M_h \tag{32}$$

 

Define spaces $Z_i$ such that Null $A_{h,i} \subseteq Z_i$. We take $Z_i = \{\text{const}\}$ for $i = 1, \ldots, n$. The coarse space is defined as follows:

$$M_H = \left\{ \lambda \in M_h: \ \lambda = \sum_{i=1}^{n} D_i \zeta_i, \ \zeta_i \in Z_i \right\}$$

Clearly $\dim M_H \leqslant n$. A residual $r$ is said to be *balanced* (local problems are solvable) if

$$\langle r, \mu_H \rangle = 0, \quad \mu_H \in M_H$$

Balancing $r$ means replacing it with

$$r^{\text{bal}} = r - A_h r_H$$

where $r_H \in M_H$ is found by solving a coarse problem

$$a_h(r_H, \mu_H) = \langle r, \mu_H \rangle, \quad \mu_H \in M_H$$

*Algorithm* (Balancing Preconditioner)

Given $r \in M_h$, define $B_{\text{bal}}^{-1} r$ as follows:

1. Solve a coarse problem:

$$a_h(r_H, \mu_H) = \langle r, \mu_H \rangle, \quad \mu_H \in M_H$$

   and balance the residual:

$$r^{\text{bal}} = r - A_h r_H$$

2. Distribute $r^{\text{bal}}$ to subdomains: $r_i = D_i^{\mathrm{T}} r^{\text{bal}}$.
3. Solve local Neumann problems for $\lambda_i \in M_{h,i}$:

$$A_{h,i} \lambda_i = r_i$$

4. Average local solutions: $\lambda = \sum_{i=1}^{n} D_i \lambda_i$
5. Solve a coarse problem:

$$a_h(\lambda_H, \mu_H) = \langle r, \mu_H \rangle - a_h(\lambda, \mu_H), \quad \mu_H \in M_H$$

   and balance local solutions:

$$B_{\text{bal}}^{-1} r = \lambda + \lambda_H$$

Note that the coarse solves in Step 1 and Step 5 provide global exchange of information across subdomains. In addition, Step 1 guarantees that the local problems in Step 3 are solvable, and due to Step 5, the result of the preconditioner is independent of the specific choice of local solutions. The dominant cost is in Step 3 which requires solving subdomain problems in parallel. The preconditioning cost is comparable to the cost of performing one unpreconditioned iteration, thus one preconditioned iteration is twice as expensive as one unpreconditioned iteration.

 

## 4. ANALYSIS OF THE CONDITION NUMBER

We start with several technical lemmas. The first lemma establishes that the balancing preconditioner operator is symmetric and positive definite and gives an abstract bound on the condition number. The proof follows closely the proof of Theorem 3.2 in Reference [19] and is omitted here.

*Lemma 4.1*
$B_{\text{bal}}$ is symmetric and positive definite and

$$\text{cond}(B_{\text{bal}}^{-1}A_h) \leqslant \sup\left\{ \frac{\sum_{j=1}^n a_{h,j}(\sum_{i=1}^n D_i\lambda_i, \sum_{i=1}^n D_i\lambda_i)}{\sum_{i=1}^n a_{h,i}(\lambda_i,\lambda_i)} :\ \lambda_i \in M_{h,i} \text{ and } \lambda_i \perp \text{Null}\, A_{h,i} \right\}.$$

The proof of the following lemma which gives a bound on the condition number of the preconditioned system follows from the proof of Theorem 3.3 in Reference [20].

*Lemma 4.2*
For subdomain $\Omega_i$, define the weighting map $D_i$ as multiplication by edgewise (facewise if $d=3$) constants,

$$(D_i\lambda_i)(x) = \frac{\alpha_i}{\alpha_i+\alpha_j}\,\lambda_i(x), \quad x \in \Gamma_{i,j}$$

and assume that there exists a number $R$ so that

$$\frac{1}{\alpha_j}\, a_{h,j}(\lambda_i,\lambda_i) \leqslant \frac{1}{\alpha_i}\, R\, a_{h,i}(\lambda_i,\lambda_i)$$

for all $i,j=1,\ldots,n$ and all $\lambda_i \in M_{h,i}$ such that $\int_{\Gamma_i} \lambda_i\zeta_i\,\mathrm{d}s = 0$, $\forall \zeta_i \in \text{Null}\, A_{h,i}$. Then there exists a constant $C$ independent of $h$, $H$, and $R$ such that

$$\text{cond}(B_{\text{bal}}^{-1}A_h) \leqslant CR$$

We make the following explicit assumption about the computational grids. There exist positive constants $c$ and $C$ independent of $h$ and $H$ such that, for any $\lambda \in M_h$,

$$c\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda\|_{1/2,\Gamma_{i,j}} \leqslant \|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda\|_{1/2,\Gamma_{i,j}} \leqslant C\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda\|_{1/2,\Gamma_{i,j}}, \quad 1 \leqslant i,\ j \leqslant n \tag{33}$$

*Remark 4.1*
It is shown in Appendix A, Lemma A.1, that (33) holds in the case of $\text{RT}_0$ spaces under mild and easily satisfied in practice assumptions on the computational grids. These assumptions allow for a great amount of independence in constructing the subdomain grids, including large grid-size ratios (with constants possibly depending on these ratios).

The following lemma is an extension of Lemma 6.4 in Reference [15] to non-matching grids.

 

*Lemma 4.3*
Assume that (33) holds. Then there exists a constant $C$ independent of $H$ and $h$ such that

$$\|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i\|^2_{1/2,\partial\Omega_j}\leqslant C(1+\log(H/h))^2\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i\|^2_{1/2,\partial\Omega},\quad \forall\lambda_i\in M_{h,i}$$

*Proof*
By Lemma 4.3 of Reference [32], we have

$$\|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i\|^2_{1/2,\partial\Omega_j}\leqslant C(1+\log(H/h))^2\|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i\|^2_{1/2,\Gamma_{i,j}}$$

By (33),

$$\|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i\|_{1/2,\Gamma_{i,j}}\leqslant C\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i\|_{1/2,\Gamma_{i,j}}$$

Combining the above inequalities with the obvious inequality

$$\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i\|_{1/2,\Gamma_{i,j}}\leqslant\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i\|_{1/2,\partial\Omega_i}$$

completes the proof. □

We assume that $\partial\Omega_i\cap\partial\Omega$ is either empty or of size $O(H)$ so that the Poincaré inequality holds uniformly for all $\Omega_i$ and there exists a constant $C$ independent of $h$ and $H$ such that

$$\|w\|^2_{\Omega_i}\leqslant CH^2|w|^2_{1,\Omega_i},\quad \|w\|^2_{\partial\Omega_i}\leqslant CH|w|^2_{1/2,\partial\Omega_i}\tag{34}$$

for all $w\in H^1(\Omega_i)$ if $\partial\Omega_i\cap\partial\Omega$ is non-empty and for all $w\in H^1(\Omega_i)$, $\langle w,1\rangle_{\Gamma_i}=0$, if $\partial\Omega_i\cap\partial\Omega$ is empty.

We are now ready to state the main result.

*Theorem 4.1*
If (33) holds and the weights $D_i$ satisfy

$$(D_i\lambda)(x)=\frac{\alpha_i}{\alpha_i+\alpha_j}\lambda(x),\quad x\in\Gamma_{i,j},\quad \text{for all } \lambda\in M_h$$

then there exists a constant $C$ independent of $h$, $H$, and jumps in $K$, such that

$$\operatorname{cond}(B^{-1}_{\text{bal}}A_h)\leqslant C(1+\log(H/h))^2$$

*Proof*
Let $\lambda_i\in M_{h,i}$ be such that $\lambda_i\perp\operatorname{Null}A_{h,i}$. With (30) we have

$$\begin{aligned}a_{h,j}(\lambda_i,\lambda_i)&\leqslant C\alpha_j|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i|^2_{1/2,\partial\Omega_j}\leqslant C\alpha_j\|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda_i\|^2_{1/2,\partial\Omega_i}\\&\leqslant C\alpha_j(1+\log(H/h))^2\|\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i\|^2_{1/2,\partial\Omega_i}\end{aligned}$$

using Lemma 4.3 for the last inequality. Note that it easily follows from the definitions of $\mathscr{Q}_{h,i}$ and $\mathscr{I}^{\partial\Omega_i}$ that

$$\langle\mathscr{I}^{\partial\Omega_i}\mathscr{Q}_{h,i}\lambda_i,1\rangle_{\Gamma_i}=\langle\mathscr{Q}_{h,i}\lambda_i,1\rangle_{\Gamma_i}=\langle\lambda_i,1\rangle_{\Gamma_i}=0$$

 

so that the Poincaré inequality (34) implies

$$\|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\lambda_i\|^2_{1/2,\partial\Omega_i} \leqslant C|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\lambda_i|^2_{1/2,\partial\Omega_i}$$

Therefore we have

$$\begin{aligned} a_{h,j}(\lambda_i,\lambda_i) &\leqslant C\alpha_j(1+\log(H/h))^2|\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\lambda_i|^2_{1/2,\partial\Omega_i} \\ &\leqslant C\,\frac{\alpha_j}{\alpha_i}(1+\log(H/h))^2 a_{h,i}(\lambda_i,\lambda_i) \end{aligned}$$

using (30) for the last inequality. The proof is completed by applying Lemma 4.2. □

*Remark 4.2*
The above theorem implies in the case of non-matching grids a bound for the balancing preconditioner which is similar to the bounds obtained for matching grids [15, 20].

## 5. NUMERICAL RESULTS

We present four computational experiments confirming the theoretical results of Section 4 about the behaviour of the balancing preconditioner. In Examples 1 and 3 we study the dependence of the convergence rate on $h$ for a smooth and a highly heterogeneous problem, respectively. Example 2 is designed to investigate the effect of jumps in the coefficients. In Example 4 we consider the effect of the number of subdomains. In all cases one processor is assigned per subdomain. The runs in Examples 1–3 are performed on the unit square divided into four subdomains ($H = \frac{1}{2}$). The runs in Example 4 are performed on a sequence of domain decompositions ranging from $2 \times 2$ to $5 \times 5$ subdomains.

In Examples 1 and 3 the condition number and number of CG iterations with and without preconditioning are reported for several levels of grid refinements starting with the grids shown in Figure 1(A). The largest ratio of the subdomain grid sizes in these examples is $\frac{5}{2}$.

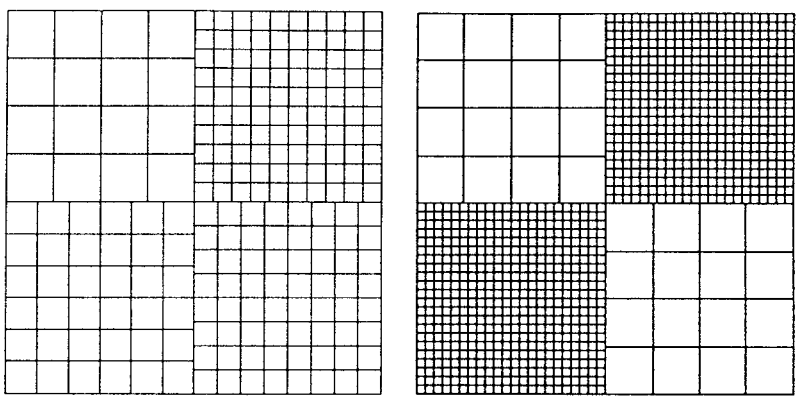


Figure 1. Initial grids for Example 1.

 

Table I. Condition number and number of iterations for Example 1.

| $1/h$ | BalCG | | CG | |
|---|---|---|---|---|
| | Cond. | Iter. | Cond. | Iter. |
| *Case 1* | | | | |
| 4 | 4.54520 | 11 | 9.0482 | 14 |
| 8 | 5.90177 | 11 | 17.0075 | 18 |
| 16 | 7.54221 | 12 | 33.5087 | 25 |
| 32 | 9.44828 | 12 | 66.7478 | 36 |
| *Case 2* | | | | |
| 4 | 2.30058 | 8 | 10.7644 | 12 |
| 8 | 2.83097 | 9 | 21.1122 | 19 |
| 16 | 3.37244 | 9 | 41.7829 | 29 |
| 32 | 4.11108 | 9 | 83.2505 | 41 |

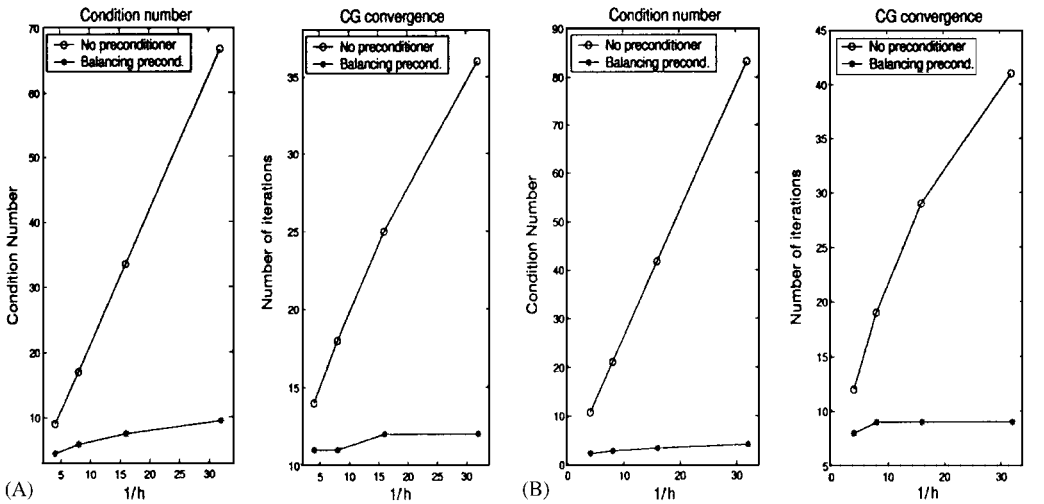


Figure 2. Condition number and number of iterations for Example 1. (A) Case 1 and (B) Case 2.

We also consider a second case in Example 1 where the ratio is $\frac{11}{2}$ (see Figure 1(B)). The mortars are chosen to be discontinuous (for Examples 1, 3 and 4) or continuous (for Example 2) piecewise linears on an interface grid obtained by coarsening by two the trace of the coarser of the neighbouring subdomain grids.

The problem in Example 1 has analytical solution $p(x, y) = x^3 y^2 + \sin(xy)$ and a smooth permeability tensor

$$K = \begin{pmatrix} 10 + 5\cos(xy) & 0 \\ 0 & 1 \end{pmatrix}$$

The condition number and number of iterations for both cases are given in Table I, Figure 2 and Figure 3. As expected from the theory, the condition number in the case of balancing preconditioner grows very slowly as $h$ gets smaller and the number of PCG iterations stays

 

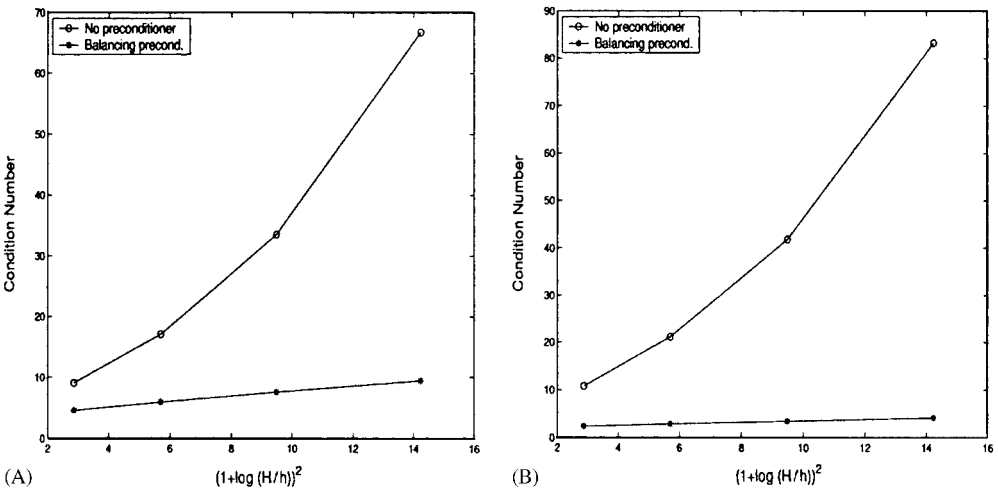


Figure 3. Dependence of the condition number on $(1+\log(H/h))^2$ in Example 1.
(A) Case 1 and (B) Case 2.

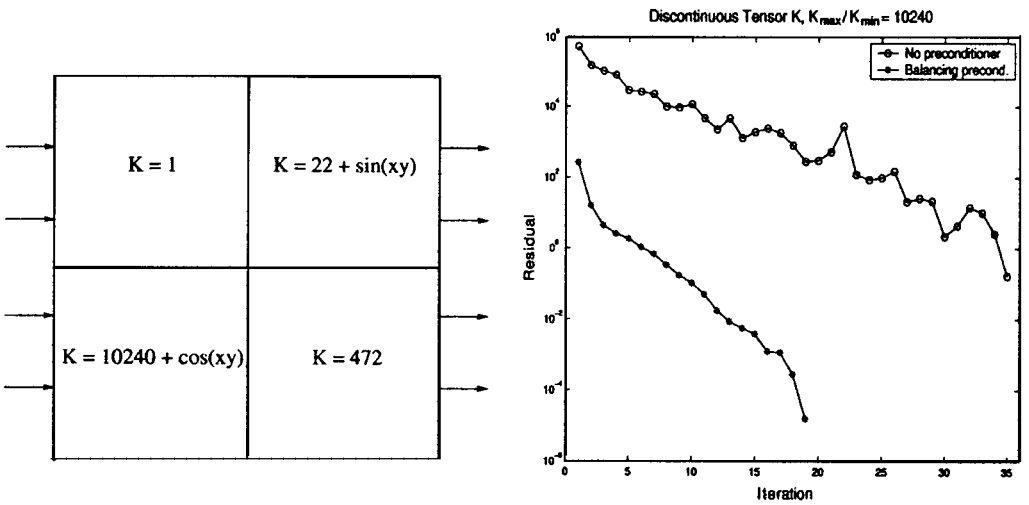


Figure 4. Permeability values (left) and residual reduction (right) for the initial level in Example 2.

almost the same. A comparison of the results from Case 1 and Case 2 indicates that the condition number and number of iterations are almost independent of the grid size ratio.

In Example 2 we study the dependence of the behaviour of the balancing preconditioner on jumps in the coefficient. A different permeability function is assigned on each subdomain as shown in Figure 4 (left). A series of runs is performed changing each function so that the jumps between subdomains get larger. The behaviour of the CG iteration is illustrated in Figure 5. We note that both the condition number and the number of iterations remain bounded when jumps become larger which is consistent with the bound given in Theorem 4.1.

 

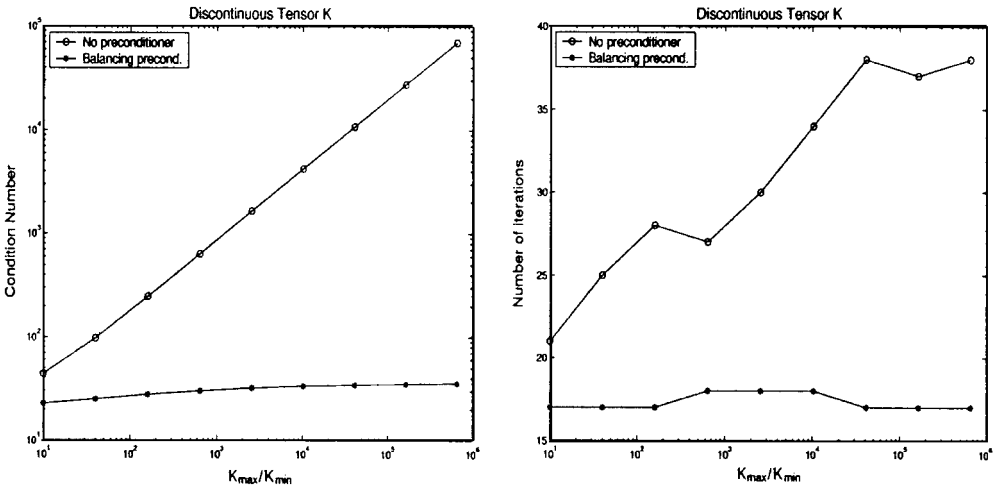


Figure 5. Dependence of CG convergence on jumps in coefficients in Example 2.

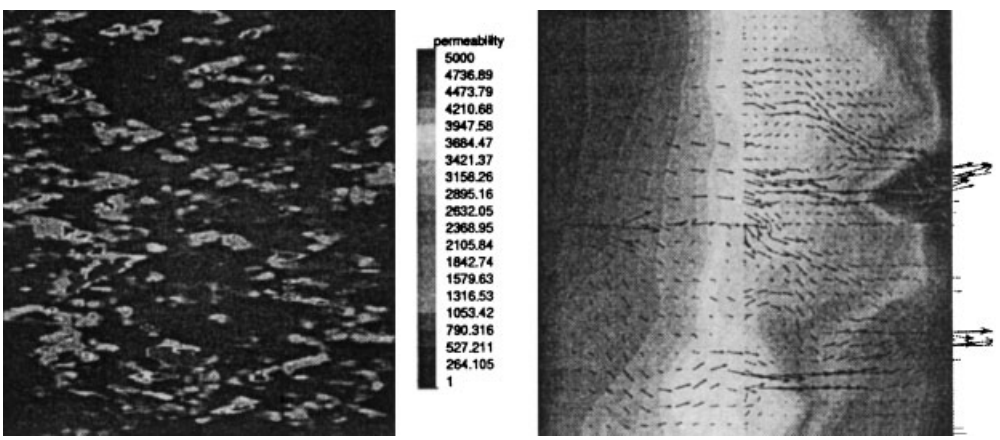


Figure 6. Permeability field and computed pressure (shade) and velocity (arrows) in Example 3.

We also compare in Figure 4 (right) the residual reductions in the unpreconditioned and the preconditioned CG iterations. The preconditioning accelerates the residual reduction and removes the oscillations observed in the unpreconditioned case.

In Example 3 we simulate flow through highly heterogeneous porous media. The permeability field and the computed solution on the first level of refinement (left-to-right flow is imposed through boundary conditions) are given in Figure 6. For each level of refinement the permeability field is projected onto the corresponding computational grids. The condition number and the number of iterations (see Table IIA and Figure 7) once again grow very slowly as $h$ gets smaller. On Figure 8(A) we compare the residual reductions in the unpreconditioned and the preconditioned CG iterations.

 

Table II. Condition number and number of iterations for Example 3 and 4.

| $1/h$ | BalCG | | CG | |
|---|---|---|---|---|
| | Cond. | Iter. | Cond. | Iter. |
| *A. Example 3* | | | | |
| 4 | 63.454 | 17 | 222.19 | 25 |
| 8 | 53.831 | 27 | 338.23 | 47 |
| 16 | 90.266 | 34 | 2313.30 | 94 |
| 32 | 192.754 | 38 | 14049.80 | 207 |
| 64 | 336.072 | 50 | 65824.20 | 492 |
| *B. Example 4* | | | | |
| 4 | 8.1659 | 13 | 38.257 | 23 |
| 9 | 12.2771 | 16 | 111.967 | 38 |
| 16 | 13.2133 | 19 | 225.357 | 55 |
| 25 | 13.4436 | 20 | 379.496 | 72 |

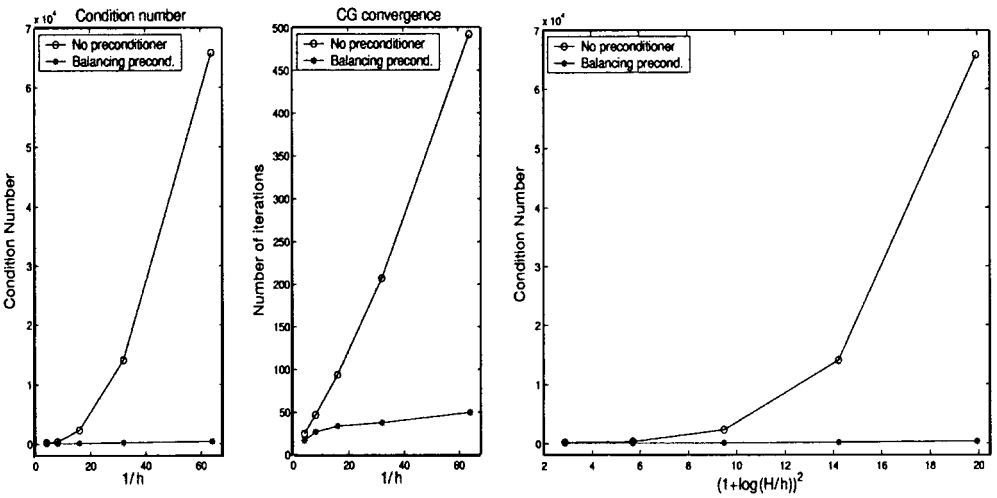


Figure 7. Condition number and number of iterations for Example 3.

In Example 4 we study the dependence of the behaviour of the balancing preconditioner on the number of subdomains. We solve a problem with analytical solution

$$p(x, y, z) = x^3 y^4 + x^2 + \sin(xy)\cos(y)$$

and a smooth permeability tensor

$$K = \begin{pmatrix} (x+1)^2 + y^2 & 0 & 0 \\ 0 & (x+1)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

on a sequence of domain decompositions from $2 \times 2$ to $5 \times 5$ subdomains. The ratio $H/h$ is kept constant with subdomain grids chosen $14 \times 18$ or $12 \times 20$ in a checkerboard fashion.

 

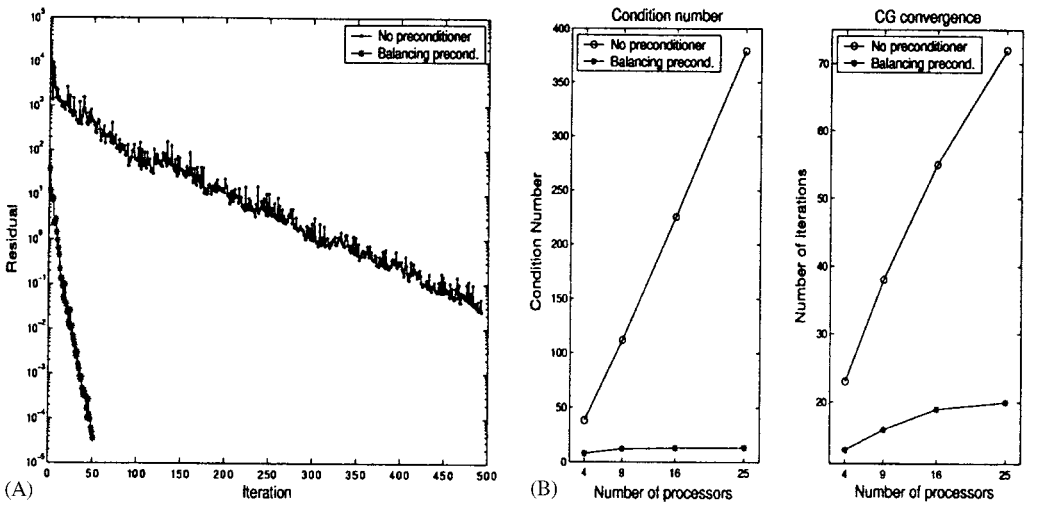


Figure 8. Performance of balancing preconditioner for Examples 3 and 4. (A) Residual reduction for Example 3 and (B) condition number and number of iterations for Example 4.

The condition number and number of iterations are given in Table II(B) and Figure 8(B). The results are consistent with the theory and indicate very good parallel scalability of the balancing preconditioner.

## APPENDIX A

Here we justify the assumption (33) on the computational grids. We show that (33) holds in the case of $RT_0$ rectangular subdomain discretizations for a fairly general grid configuration. We start by defining the piecewise linear interpolant $\mathcal{I}^{\partial\Omega_i}$ for the $RT_0$ spaces. More general definition is given in Reference [15]. Let $\hat{\mathcal{T}}_{h,i}$ be a refinement of $\mathcal{T}_{h,i}|_{\Gamma_i}$ (with vertices at the element centers (primary vertices) and the element vertices (secondary vertices) of $\mathcal{T}_{h,i}|_{\Gamma_i}$. Note that the primary vertices coincide with the degrees of freedom of $\mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$ and, correspondingly, the pressure Lagrange multipliers on $\Gamma_i$. Let $U_{h,i}$ be the space of continuous piecewise linear functions subordinate to the partition $\hat{\mathcal{T}}_{h,i}$. For $\phi \in \mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$, define $\mathcal{I}^{\partial\Omega_i}\phi \in U_{h,i}$ as follows:

$$\mathcal{I}^{\partial\Omega_i}\phi(x) = \begin{cases} \phi(x), & \text{if } x \text{ is a primary vertex of } \hat{\mathcal{T}}_{h,i} \\ \text{the area-weighted average of values of } \phi \text{ at all adjacent primary vertices,} & \text{if } x \text{ is a secondary vertex of } \hat{\mathcal{T}}_{h,i} \\ \text{the linear interpolation of vertex values,} & \text{if } x \text{ is not a vertex of } \hat{\mathcal{T}}_{h,i} \end{cases}$$

 

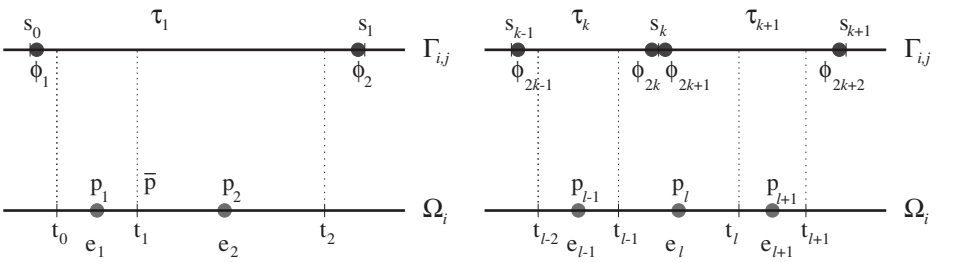


Figure A1. Grids on the mortar and neighbouring subdomain along the interface $\Gamma_{i,j}$.

To simplify the presentation below, we call two non-negative functions $f_1(\cdot)$ and $f_2(\cdot)$ with the same domain $D$ equivalent and write

$$f_1 \simeq f_2$$

if there exist positive constants $c$ and $C$ independent of $h$ and $H$ such that

$$c f_1(\phi) \leqslant f_2(\phi) \leqslant C f_1(\phi), \quad \forall \phi \in D$$

It is easy to see that for any $\hat{p} \in U_{h,i}$ and for any $\tau^i \in \hat{\mathscr{T}}_{h,i}$

$$|\hat{p}|^2_{1,\tau^i} \simeq |\tau^i|^{1-2/d} \sum_{\substack{\text{vertices} \\ v_l, v_k \in \tau^i}} (\hat{p}(v_l) - \hat{p}(v_k))^2 \tag{A1}$$

$$|\hat{p}|^2_{0,\tau^i} \simeq |\tau^i| \sum_{\substack{\text{vertices} \\ v_l \in \tau^i}} \hat{p}(v_l)^2 \tag{A2}$$

*Lemma A.1*
Consider $d = 2$, $\mathrm{RT}_0$ subdomain discretizations on rectangular grids, and discontinuous piecewise linear mortar spaces. Assume that every element of $\mathscr{T}_{h,i,j}$ contains at least one element of $\mathscr{T}_{h,i}|_{\Gamma_{i,j}}$ and that at least one element of $\mathscr{T}_{h,i,j}$ contains at least two elements of $\mathscr{T}_{h,i}|_{\Gamma_{i,j}}$. Assume that the same relation holds for $\mathscr{T}_{h,i,j}$ and $\mathscr{T}_{h,j}|_{\Gamma_{i,j}}$ (see Figure A1). Then, for any $\lambda \in M_h$,

$$\|\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i} \lambda\|_{1/2,\Gamma_{i,j}} \simeq \|\mathscr{I}^{\partial\Omega_j} \mathscr{Q}_{h,j} \lambda\|_{1/2,\Gamma_{i,j}}$$

*Proof*
First, consider an element $\tau_1$ of $\mathscr{T}_{h,i,j}$ that contains at least two elements, $e_1, e_2$, of $\mathscr{T}_{h,i}|_{\Gamma_{i,j}}$. Denote the vertices of $\tau_1$ by $v_1, v_2$ with coordinates $s_0, s_1$, respectively, and the coordinates of the endpoints of $e_1, e_2$ by $t_0, t_1, t_2$ (see Figure A1, left). Let, for a given $\lambda \in M_h$,

$$\lambda(v_m) = \phi_m, \quad m = 1, 2, \qquad p_l = \mathscr{Q}_{h,i} \lambda|_{e_l}, \quad l = 1, 2$$

By definition, the value of $\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i} \lambda$ at the midpoint of $e_l$ (primary vertex) is $p_l$. Denote by $\bar{p}$ the value of $I^{\partial\Omega_i} \mathscr{Q}_{h,i} \lambda$ at the secondary vertex between $e_1$ and $e_2$. We have

$$\bar{p} = \frac{|e_2| p_1 + |e_1| p_2}{|e_1| + |e_2|}, \quad p_1 - \bar{p} = \frac{|e_1|}{|e_1| + |e_2|}(p_1 - p_2), \quad \bar{p} - p_2 = \frac{|e_2|}{|e_1| + |e_2|}(p_1 - p_2)$$

 

hence it is enough to consider in (A1) and (A2) only the values of $\mathcal{I}^{\partial\Omega_i}\mathcal{Q}_{h,i}\lambda$ at the primary vertices. For $p_1, p_2$ we obtain

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = A_0 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad A_0 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \tag{A3}$$

where

$$\alpha_1 = \frac{2s_1 - t_0 - t_1}{2(s_1 - s_0)}, \quad \alpha_2 = \frac{t_0 + t_1 - 2s_0}{2(s_1 - s_0)}, \quad \beta_1 = \frac{2s_1 - t_1 - t_2}{2(s_1 - s_0)}, \quad \beta_2 = \frac{t_1 + t_2 - 2s_0}{2(s_1 - s_0)}$$

Note that $\alpha_1 + \alpha_2 = 1$, $\beta_1 + \beta_2 = 1$, and hence

$$\det A_0 = \alpha_1 - \beta_1 = \frac{t_2 - t_0}{2(s_1 - s_0)} \geqslant C > 0$$

which gives

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = A_0^{-1} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{A4}$$

where the elements of $A_0^{-1}$ do not depend on $h$. We also have

$$p_1 - p_2 = (\alpha_1 - \beta_1)(\phi_1 - \phi_2) \tag{A5}$$

Proceeding inductively, assume that we have expressed $p_1, \ldots, p_{l-1}$ in terms of $\phi_1, \ldots, \phi_{2k}$, as well as differences of two consecutive $p$'s in terms of differences of $\phi$'s and vice versa. Consider two neighbouring elements, $\tau_k, \tau_{k+1}$, of $\mathcal{T}_{h,i,j}$, each containing an element of $\mathcal{T}_{h,i}|_{\Gamma_{i,j}}$ ($e_{l-1}$ and $e_{l+1}$, respectively) (see Figure A1, right). Denote the vertices of $\tau_k$ and $\tau_{k+1}$ by $v_{2k-1}, v_{2k}$ and $v_{2k+1}, v_{2k+2}$, respectively, with co-ordinates $s_{k-1}, s_k, s_{k+1}$. Denote the coordinates of the endpoints of $e_{l-1}, e_l, e_{l+1}$ by $t_{l-2}, t_{l-1}, t_l, t_{l+1}$. Let

$$\lambda(v_m) = \phi_m, \quad m = 2k-1, \ldots, 2k+2, \qquad p_n = \mathcal{Q}_{h,i}\lambda|_{e_n}, \quad n = l-1, l, l+1$$

Then we have

$$\begin{aligned} p_{l-1} &= a_{l,1}\phi_{2k-1} + a_{l,2}\phi_{2k}, \\ p_l &= b_{l,1}\phi_{2k-1} + b_{l,2}\phi_{2k} + b_{l,3}\phi_{2k+1} + b_{l,4}\phi_{2k+2} \\ p_{l+1} &= c_{l,1}\phi_{2k+1} + c_{l,2}\phi_{2k+2} \end{aligned} \tag{A6}$$

where

$$\begin{aligned} a_{l,1} &= \frac{(2s_k - t_{l-2} - t_{l-1})}{2(s_k - s_{k-1})}, & a_{l,2} &= \frac{(t_{l-2} + t_{l-1} - 2s_{k-1})}{2(s_k - s_{k-1})} \\ b_{l,1} &= \frac{(s_k - t_{l-1})^2}{2(t_l - t_{l-1})(s_k - s_{k-1})}, & b_{l,2} &= \frac{(s_k - t_{l-1})(t_{l-1} + s_k - 2s_{k-1})}{2(t_l - t_{l-1})(s_k - s_{k-1})} \\ b_{l,3} &= \frac{(t_l - s_k)(2s_{k+1} - s_k - t_l)}{2(t_l - t_{l-1})(s_{k+1} - s_k)}, & b_{l,4} &= \frac{(t_l - s_k)^2}{2(t_l - t_{l-1})(s_{k+1} - s_k)} \\ c_{l,1} &= \frac{(2s_{k+1} - t_l - t_{l+1})}{2(s_{k+1} - s_k)}, & c_{l,2} &= \frac{(t_l + t_{l+1} - 2s_k)}{2(s_{k+1} - s_k)} \end{aligned}$$

 

Note that the coefficients in (A6) can be bounded above and below by constants independent of $h$ and dependent on the grid-size ratio. We can rewrite the last two equations of (A6) as

$$\begin{pmatrix} p_l \\ p_{l+1} \end{pmatrix} = \begin{pmatrix} b_{l,1}\phi_{2k-1} + b_{l,2}\phi_{2k} \\ 0 \end{pmatrix} + A_l \begin{pmatrix} \phi_{2k+1} \\ \phi_{2k+2} \end{pmatrix}, \quad A_l = \begin{pmatrix} b_{l,3} & b_{l,4} \\ c_{l,1} & c_{l,2} \end{pmatrix} \tag{A7}$$

Note that

$$a_{l,1} + a_{l,2} = 1, \quad b_{l,1} + b_{l,2} + b_{l,3} + b_{l,4} = 1, \quad c_{l,1} + c_{l,2} = 1 \tag{A8}$$

Since

$$\det A_l = \frac{(t_l - s_k)(t_{l+1} - s_k)}{2(t_l - t_{l-1})(s_{k+1} - s_k)} \geqslant C > 0$$

$$\begin{pmatrix} \phi_{2k+1} \\ \phi_{2k+2} \end{pmatrix} = A_l^{-1} \left[ \begin{pmatrix} p_l \\ p_{l+1} \end{pmatrix} - \begin{pmatrix} b_{l,1}\phi_{2k-1} + b_{l,2}\phi_{2k} \\ 0 \end{pmatrix} \right] \tag{A9}$$

Thus, $\phi_{2k+1}$ and $\phi_{2k+2}$ are linear combinations of $p_1, \ldots, p_{l+1}$ with coefficients independent of $h$. Using (A8), if follows from (A6) that

$$\begin{pmatrix} p_{l-1} - p_l \\ p_l - p_{l+1} \end{pmatrix} = (\phi_{2k-1} - \phi_{2k}) \begin{pmatrix} a_{l,1} - b_{l,1} \\ b_{l,1} \end{pmatrix} + B_l \begin{pmatrix} \phi_{2k} - \phi_{2k+1} \\ \phi_{2k+1} - \phi_{2k+2} \end{pmatrix} \tag{A10}$$

where

$$B_l = \begin{pmatrix} b_{l,3} + b_{l,4} & b_{l,4} \\ b_{l,1} + b_{l,2} & c_{l,2} - b_{l,4} \end{pmatrix} \tag{A11}$$

Using (A8), we get

$$\det B_l = b_{l,3}c_{l,2} - b_{l,4}c_{l,1} = \det A_l$$

so

$$\begin{pmatrix} \phi_{2k} - \phi_{2k+1} \\ \phi_{2k+1} - \phi_{2k+2} \end{pmatrix} = B_l^{-1} \left[ \begin{pmatrix} p_{l-1} - p_l \\ p_l - p_{l+1} \end{pmatrix} - (\phi_{2k-1} - \phi_{2k}) \begin{pmatrix} a_{l,1} - b_{l,1} \\ b_{l,1} \end{pmatrix} \right] \tag{A12}$$

Therefore, $(\phi_{2k} - \phi_{2k+1})$ and $(\phi_{2k+1} - \phi_{2k+2})$ can be expressed as linear combinations of the differences $(p_n - p_{n+1})$, $n = 1, \ldots, l$.

Applying the inequality $(a+b)^2 \leqslant 2(a^2 + b^2)$ and combining (A2), (A3), (A6), and (A9) implies

$$|\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i}\lambda|_{0,\Gamma_{i,j}} \simeq |\lambda|_{0,\Gamma_{i,j}} \tag{A13}$$

Combining (A1), (A5), (A10), and (A12) gives

$$|\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i}\lambda|_{1,\Gamma_{i,j}} \simeq |\lambda|_{1,\Gamma_{i,j}} \tag{A14}$$

Similar arguments imply

$$|\mathscr{I}^{\partial\Omega_j} \mathscr{Q}_{h,j}\lambda|_{0,\Gamma_{i,j}} \simeq |\lambda|_{0,\Gamma_{i,j}} \tag{A15}$$

 

and

$$|\mathscr{I}^{\partial\Omega_j}\mathscr{Q}_{h,j}\lambda|_{1,\Gamma_{i,j}} \simeq |\lambda|_{1,\Gamma_{i,j}} \tag{A16}$$

The interpolation theory of Sobolev spaces [33] and bounds (A13)–(A16) imply the statement of the lemma. □

*Remark A.1*

The proof of the above lemma is also valid in the case of continuous piecewise linear mortars.

*Remark A.2*

The above argument can be generalized to three-dimensional rectangular-type $RT_0$ discretizations in a relatively straightforward way.

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