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Multigrid on the interface for mortar mixed finite element methods for elliptic problems ☆

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Abstract

We consider mixed finite element approximations of second order elliptic equations on domains that can be described as a union of subdomains or blocks. We assume that the subdomain grids are locally defined and need not match across the block boundaries. Specially chosen mortar finite element spaces are introduced on the interfaces for approximating the scalar variable (pressure). The mortars also serve as Lagrange multipliers for imposing flux-matching conditions. The method is implemented by reducing the algebraic system to a positive definite interface problem in the mortar spaces. This problem is then solved using a multigrid on the interface with conjugate gradient smoothing. The algorithm is very efficient in a distributed parallel computing environment as only subdomain solves are required on each conjugate gradient iteration. The standard variational assumptions for the multigrid are not satisfied, since the interface bilinear forms vary from level to level. We present theoretical results for the convergence of the V-cycle and the W-cycle. Computational results in two- and three-dimensions are given to illustrate and confirm the theory. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

Mixed finite element methods are often used to approximate the solutions of second order elliptic equations due to their local mass conservation property and their direct approximation of the vector flux variable. In many applications the complexity of the domain geometry or the solution itself warrants using a multiblock domain structure, wherein the domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is decomposed into non-overlapping blocks or subdomains Ω_i, $i = 1, \ldots, n$, with grids defined independently on each block. On the $(d-1)$-dimensional interface Γ between subdomain blocks, the traces of the grids need not coincide. Two typical examples in subsurface porous medium applications are the modeling of faults, which are natural discontinuities in material properties, and the modeling of wells, the solution's response to which can be resolved often only by using locally refined grids.

We use notation appropriate for applications to porous media, and we consider a model problem. For the unknown pressure scalar function $p(\mathbf{x})$ and Darcy velocity vector function $\mathbf{u}(\mathbf{x})$, we consider the partial differential boundary value problem:

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$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{2}$$

$$p = g \quad \text{on } \partial\Omega, \tag{3}$$

where $K(\mathbf{x})$ is a symmetric, uniformly positive definite tensor representing the permeability divided by the viscosity and $f(\mathbf{x})$ represents the sources. We assume that each component of K is in $L^\infty(\Omega)$ and is smooth on each subdomain Ω_i, and that $f \in L^2(\Omega)$. The Dirichlet condition imposed is considered for simplicity; however, other boundary conditions can be handled easily.

A number of papers deal with the analysis and the implementation of mixed methods for our problem on conforming grids (see, e.g., [31,34,30,5,14,12,13,20,29,35,16,21,23,4,3] and the general references [15,32]). Mixed methods on nested locally refined grids are considered in [22,24], but these techniques rely heavily on the fact that the grids are nested and cannot be extended directly to arbitrary non-matching grids.

In [26] Glowinski and Wheeler introduced two domain decomposition algorithms for mixed finite element methods for elliptic equations (1)–(3). A key result was the formulation of the matching conditions at the interfaces of the subdomains as variational problems defined over trace spaces and thus reducing the mixed finite element saddle point problem to a positive definite interface problem. In method 1, fluxes are assumed to match and iterations are performed to match the pressure p. In method 2, the dual of method 1, pressures are assumed to match and iterations are performed to match fluxes. Conjugate gradient techniques are applied to both schemes to solve the interface problems.

A difficulty of the parallelization of method 1 on a distributed memory machine is the adjustment of the pressure solution over the subdomains. Each evaluation of the bilinear form requires the solution of a tridiagonal system of order at most $n - 1$. The order of the linear system varies depending on the boundary conditions imposed and the decomposition employed. In method 2, the pressure adjustment by a constant only arises for Neumann problems, and there is no linear system to solve.

In [25] a multilevel acceleration of method 1 is defined. Numerical experiments carried out on a sequential machine indicate that the number of V-cycles with conjugate gradient smoothing was practically independent of h despite the fact that the condition number is $\mathrm{O}(Hh)^{-1}$, where h denotes the mesh size and H the subdomain size.

In [19] a domain decomposition algorithm based on method 2 and incorporating an inner product modification and multilevel acceleration is defined and implemented for an arbitrary number of subdomains in two spatial dimensions. Numerical experiments on an Intel iPSC/860 indicate that the resulting algorithm is scalable due to an insensitivity in the number of V-cycles required for convergence to problem size and variation in coefficients.

In [18] two multigrid algorithms are discussed for three spatial dimensional problems, one a semi-coarsening multigrid algorithm [33] and the second an extension of the two-dimensional scheme developed in [19]. Both methods converge fast in terms of outer iterations for anisotropic models and models with strongly discontinuous coefficients. The global multigrid approach is more robust than the nested factorization subdomain solver employed in the domain decomposition algorithm. The advantages of the domain decomposition approach is that it required much less interprocessor communication per iteration and much less storage.

The goal of this paper is to discuss the extension of the above domain decomposition and multigrid algorithms to the case of non-matching multiblock grids and present theoretical and numerical results for their convergence.

Techniques have been developed to approximate elliptic problems on non-matching multiblock grids using Galerkin and spectral approximations in the blocks and tying these together through an approximation of the flux on Γ in a special finite element space called a mortar space [7,6]. In the case of mixed methods, mortar spaces are introduced for the interface pressure and used as Lagrange multipliers to impose weekly normal flux continuity. To achieve optimal convergence, the mortar spaces should consist of polynomials of one degree higher than the normal trace of the subdomain velocity spaces. This choice differs from the standard choice for Lagrange multipliers in the hybrid and macro-hybrid formulations of the mixed method [5,26]. For convergence analysis we refer to [36,2] in the case of affine triangulations and to [36] in the case of curvilinear logically rectangular grids.

Multiblock discretizations lead to algebraic systems suitable for efficient parallel solution algorithms. We extend the non-overlapping domain decomposition algorithm from [26,18] and reduce the discrete system to a symmetric and positive definite Schur-complement system in the mortar spaces.

Multigrid algorithms with conjugate gradient smoothing are used for the solution of the interface problem. Both mortar and subdomain grids are coarsened on each level. Evaluation of the mortar bilinear forms requires solving discrete subdomain problems on grids that vary from level to level. Therefore, the bilinear forms are non-inherited and the standard multigrid variational assumption is not satisfied. Our multigrid analysis is motivated by a general theory developed by Bramble et al. [9–11], where some of the standard assumptions are omitted. The main difficulties in our analysis are in proving the "regularity and approximation" assumption (see (35) below) in the case of mortar mixed discretizations on non-matching grids, as well as treating the non-stationary CG smoother. Our approach differs from the one in [8], where multigrid is applied to solve the global system arising in the mortar finite element method.

The rest of the paper is organized as follows. The mixed finite element method is presented in Section 2. In Section 3, we discuss the reduction of the computational problem to an interface problem and define multigrid algorithms on the interace. In Section 4, theoretical results are presented for these algorithms. Numerical results are given in Section 5.

2. Mortar mixed finite element methods

In the weak formulation of (1)–(3) we seek a pair $\mathbf{u} \in H(\mathrm{div};\Omega)$, $p \in L^2(\Omega)$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla\cdot\mathbf{v}) - \langle g, \mathbf{v}\cdot\nu\rangle_{\partial\Omega}, \quad \mathbf{v} \in H(\mathrm{div};\Omega), \tag{4}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \quad w \in L^2(\Omega). \tag{5}$$

Here $(\cdot,\cdot)_S$ denotes the $L^2(S)$ inner product and $\langle\cdot,\cdot\rangle_{\partial S}$ denotes the $L^2(\partial S)$ inner product or a duality pairing. Let $\|\cdot\|_S$ and $\|\cdot\|_{\partial S}$ be the associated L^2-norms. We omit S if $S = \Omega$. It is well known (see, e.g., [15]) that (4) and (5) have a unique solution.

Let $\Omega = \cup_{i=1}^n \Omega_i$ be decomposed into n non-overlapping subdomains Ω_i, and let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{1\leqslant i<j\leqslant n}\Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma = \partial\Omega_i \setminus \partial\Omega$. Let

$$\mathbf{V}_i = H(\mathrm{div};\Omega_i), \quad \mathbf{V} = \bigoplus_{i=1}^n \mathbf{V}_i$$

and

$$W_i = L^2(\Omega_i), \quad W = \bigoplus_{i=1}^n W_i = L^2(\Omega).$$

If the solution $(\mathbf{u}, p)$ of (4) and (5) belongs to $H(\mathrm{div};\Omega) \times H^1(\Omega)$, it is easy to see that it satisfies, for $1 \leqslant i \leqslant n$,

$$(K^{-1}\mathbf{u}, \mathbf{v})_{\Omega_i} = (p, \nabla\cdot\mathbf{v})_{\Omega_i} - \langle p, \mathbf{v}\cdot\nu_i\rangle_{\Gamma_i} - \langle g, \mathbf{v}\cdot\nu_i\rangle_{\partial\Omega_i\setminus\Gamma}, \quad \mathbf{v} \in \mathbf{V}_i, \tag{6}$$

$$(\nabla\cdot\mathbf{u}, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_i, \tag{7}$$

where ν_i is the outer unit normal to $\partial\Omega_i$.

Let $\mathscr{T}_{h,i}$ be a conforming, quasi-uniform finite element partition of Ω_i, $1 \leqslant i \leqslant n$, with $\mathscr{T}_{h,i}$ and $\mathscr{T}_{h,j}$ possibly non-matching on $\Gamma_{i,j}$. Let $\mathscr{T}_h = \cup_{i=1}^n \mathscr{T}_{h,i}$. Let

$$\mathbf{V}_{h,i} \times W_{h,i} \subset \mathbf{V}_i \times W_i$$

be any of the usual mixed finite element spaces, (i.e., the RT spaces [34,31,30]; BDM spaces [14]; BDFM spaces [13]; BDDF spaces [12], or CD spaces [16]). Let

$$\mathbf{V}_h = \bigoplus_{i=1}^{n} \mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^{n} W_{h,i}.$$

All of the spaces above satisfy

$$\nabla \cdot \mathbf{V}_{h,i} = W_{h,i}$$

and that there exists a projection operator Π_i onto $\mathbf{V}_{h,i}$, such that for any $\mathbf{q} \in (H^{1/2+\varepsilon}(\Omega_i))^d \cap \mathbf{V}_i$,

$$(\nabla \cdot (\Pi_i \mathbf{q} - \mathbf{q}), w)_{\Omega_i} = 0, \quad w \in W_{h,i}, \tag{8}$$

$$\langle (\mathbf{q} - \Pi_i \mathbf{q}) \cdot \nu_i, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{9}$$

Let $\mathscr{T}_{h,ij}$ be a quasi-uniform finite element partition of $\Gamma_{i,j}$. Denote by $M_{h,i,j} \subset L^2(\Gamma_{i,j})$ the space of either continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathscr{T}_{h,i,j}$, where k is associated with the degree of the polynomials in $\mathbf{V}_h \cdot \nu$. More precisely, if $d = 3$, on any boundary element K, $M_{h,i,j}|_K = P_{k+1}(K)$, if K is a triangle, and $M_{h,i,j}|_K = Q_{k+1}(K)$, if K is a rectangle. Here

$$P_{k+1} = \left\{ \sum_{0 \leqslant i+j \leqslant k+1} \alpha_{ij} x_1^i x_2^j : \alpha_{ij} \in \mathbf{R} \right\}, \quad Q_{k+1} = \left\{ \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \alpha_{ij} x_1^i x_2^j : \alpha_{ij} \in \mathbf{R} \right\}.$$

Let

$$M_h = \bigoplus_{1 \leqslant i < j \leqslant n} M_{h,i,j}.$$

In the mortar mixed finite element approximation of (4) and (5), we seek $\mathbf{u}_h \in \mathbf{V}_h$, $p_h \in W_h$, and $\lambda_h \in M_h$ such that, for $1 \leqslant i \leqslant n$,

$$(K^{-1}\mathbf{u}_h, \mathbf{v})_{\Omega_i} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{10}$$

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}, \tag{11}$$

$$\sum_{i=1}^{n} \langle \mathbf{u}_h \cdot \nu_i, \mu \rangle_{\Gamma_i} = 0, \quad \mu \in M_h. \tag{12}$$

Existence and uniqueness of a solution to (10)–(12) is shown in [36,2] under the assumption that, for any $\phi \in M_{h,i,j}$,

$$\mathscr{Q}_{h,i}\phi|_{\Gamma_{i,j}} = \mathscr{Q}_{h,j}\phi|_{\Gamma_{i,j}} = 0 \quad \text{implies that} \quad \phi|_{\Gamma_{i,j}} = 0, \tag{13}$$

where $\mathscr{Q}_{h,i} : L^2(\Gamma_i) \to \mathbf{V}_{h,i} \cdot \nu_i|_{\Gamma_i}$ is the L^2-projection satisfying for any $\phi \in L^2(\Gamma_i)$

$$\langle \phi - \mathscr{Q}_{h,i}\phi, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \tag{14}$$

The proof of the following convergence result can be found in [2], wherein l is associated with the degree of the polynomials in W_h.

Theorem 1. *For the solution of the mixed method* (10)–(12), *if*

$$\|\mu\|_{0,\Gamma_{i,j}} \leqslant C\Big(\|\mathscr{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathscr{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}} \Big) \quad \forall \mu \in M_h, \tag{15}$$

then there exists a positive constant C independent of h such that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leqslant C\sum_{i=1}^{n}\left(\|p\|_{s+1,\Omega_i}h^s + \|\mathbf{u}\|_{r,\Omega_i}h^r\right), \tag{16}$$

$$\|\hat{p} - p_h\|_0 \leqslant C\sum_{i=1}^{n}\left(\|p\|_{s+1,\Omega_i}h^{s+1} + \|\mathbf{u}\|_{r,\Omega_i}h^{r+1} + \|\nabla\cdot\mathbf{u}\|_{t,\Omega_i}h^{t+1}\right), \tag{17}$$

$$\|p - p_h\|_0 \leqslant C\sum_{i=1}^{n}\left(\|p\|_{s+1,\Omega_i}h^s + \|\mathbf{u}\|_{r,\Omega_i}h^{r+1} + \|\nabla\cdot\mathbf{u}\|_{t,\Omega_i}h^{t+1}\right), \tag{18}$$

$$\|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 \leqslant C\sum_{i=1}^{n}\|\nabla\cdot\mathbf{u}\|_{t,\Omega_i}h^t, \tag{19}$$

where $0 \leqslant s \leqslant l+1$, $1/2 < r \leqslant k+1$, $0 \leqslant t \leqslant l+1$, *and* $\hat{p} \in W_h$ *is the* L^2*-orthogonal projection of p.*

Moreover, if the tensor K is diagonal and the mixed finite element spaces are the Raviart–Thomas spaces on rectangular type grids:

$$|||\mathbf{u} - \mathbf{u}_h||| \leqslant C\sum_{i=1}^{n}\left(\|p\|_{s+3/2,\Omega_i}h^{s+1/2} + \|\mathbf{u}\|_{r+1/2,\Omega_i}h^{r+1/2}\right), \tag{20}$$

where $0 \leqslant s \leqslant l+1$, $1/2 < r \leqslant k+1$, *and* $|||\cdot|||$ *is a discrete approximation to the* L^2*-norm involving integration along Gaussian lines* (*see* [21,23,2] *for the exact definition*).

Remark 2. *The condition* (15) (*and subsequently* (13)) *on the mortar grids and spaces is easily satisfied in practice. It does not allow for the mortar space to be too rich compared to the subdomain grids. It has been shown* [36] *that it holds for either continuous or discontinuous mortar spaces if the the mortar grid on each interface is a coarsening by two in each direction of the trace of either one of the subdomain grids. This choice is reminiscent of the one in the case of standard or spectral finite element subdomain discretizations* [7,6].

Remark 3. *Bounds* (16), (18) *and* (19) *imply optimal convergence*, *bound* (17) *implies superconvergence for the pressure at the Gaussian points*, *and bound* (20) *implies superconvergence for the velocity along the Gaussian lines*, *with a slightly higher solution regularity requirement*.

Remark 4. *The choice of the mortar finite element space* M_h *consisting of piecewise polynomials of degree* $k+1$, *i.e.*, *one degree higher than the polynomials in* $\mathbf{V}_h \cdot \nu$, *is essential for the optimal convergence and superconvergence of the method. If only polynomials of degree k are used*, *which is the standard Lagrange multiplier choice for mixed methods on conforming grids* [5], *no superconvergence for the pressure*, *and only suboptimal convergence for the velocity* (*with a loss of* $O(h^{1/2})$), *can be shown. This is due to the approximation error on the interfaces. These theoretical estimates are in accordance with the numerically observed convergence rates.*

3. A non-overlapping domain decomposition algorithm

The algebraic system that arises in the mortar mixed finite element discretizations can be solved efficiently in parallel using non-overlapping domain decomposition techniques. In this section we formulate an algorithm based on the one originally developed in [26] for mixed methods on conforming grids. The method reduces the global system to a symmetric and positive definite interface mortar problem. We then formulate a multigrid algorithm for the solution of the interface problem.

3.1. Reduction to interface problem

Define a bilinear form $a_h : L^2(\Gamma) \times L^2(\Gamma) \to \mathbf{R}$ by

$$a_h(\lambda, \mu) = \sum_{i=1}^{n} a_{h,i}(\lambda, \mu) = -\sum_{i=1}^{n} \langle \mathbf{u}_h^*(\lambda) \cdot \nu_i, \mu \rangle_{\Gamma_i}, \tag{21}$$

where for $\lambda \in L^2(\Gamma)$, $(\mathbf{u}_h^*(\lambda), p_h^*(\lambda)) \in \mathbf{V}_h \times W_h$ solve, for $1 \leqslant i \leqslant n$,

$$\left(K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{v}\right)_{\Omega_i} = (p_h^*(\lambda), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \lambda, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{22}$$

$$(\nabla \cdot \mathbf{u}_h^*(\lambda), w)_{\Omega_i} = 0, \quad w \in W_{h,i}. \tag{23}$$

Define a linear functional $g_h : L^2(\Gamma) \to \mathbf{R}$ by

$$g_h(\mu) = \sum_{i=1}^{n} g_{h,i}(\mu) = \sum_{i=1}^{n} \langle \bar{\mathbf{u}}_h \cdot \nu_i, \mu \rangle_{\Gamma_i}, \tag{24}$$

where $(\bar{\mathbf{u}}_h, \bar{p}_h) \in \mathbf{V}_h \times W_h$ solve, for $1 \leqslant i \leqslant n$,

$$\left(K^{-1}\bar{\mathbf{u}}_h, \mathbf{v}\right)_{\Omega_i} = (\bar{p}_h, \nabla \cdot \mathbf{v})_{\Omega_i} - \langle g, \mathbf{v} \cdot \nu_i \rangle_{\partial\Omega_i \setminus \Gamma}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \tag{25}$$

$$(\nabla \cdot \bar{\mathbf{u}}_h, w)_{\Omega_i} = (f, w)_{\Omega_i}, \quad w \in W_{h,i}. \tag{26}$$

It is straightforward to show (see [26]) that the solution $(\mathbf{u}_h, p_h, \lambda_h)$ of (10)–(12) satisfies

$$a_h(\lambda_h, \mu) = g_h(\mu), \quad \mu \in M_h \tag{27}$$

with

$$\mathbf{u}_h = \mathbf{u}_h^*(\lambda_h) + \bar{\mathbf{u}}_h, \quad p_h = p_h^*(\lambda_h) + \bar{p}_h. \tag{28}$$

The following lemma has been shown in [36,2] (see also [19,17] for the conforming grids case).

Lemma 5. *The interface bilinear form $a_h(\cdot,\cdot)$ is symmetric and positive semidefinite on $L^2(\Gamma)$. If* (13) *holds, then $a_h(\cdot,\cdot)$ is positive definite on M_h.*

The proof is based on the representation

$$a_{h,i}(\lambda, \mu) = \left(K^{-1}\mathbf{u}_h^*(\lambda), \mathbf{u}_h^*(\mu)\right)_{\Omega_i}, \tag{29}$$

which follows easily from (22) and the definition of $a_h(\cdot,\cdot)$.

Another useful characterization for $a_{h,i}(\cdot,\cdot)$ has been shown in [17,36]. There exist positive constants c_1 and c_2 such that

$$c_1 |\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i}\mu|_{1/2,\partial\Omega_i} \leqslant a_{h,i}(\mu, \mu) \leqslant c_2 |\mathscr{I}^{\partial\Omega_i} \mathscr{Q}_{h,i}\mu|_{1/2,\partial\Omega_i} \quad \forall \mu \in M_h, \tag{30}$$

where $\mathscr{I}^{\partial\Omega_i}$ is a continuous piecewise linear interpolant on the trace of the $\mathscr{T}_{h,i}$ on the boundary. This relation has been employed in the convergence analysis for the mortar mixed finite element methods [36] and is instrumental in the multigrid convergence theory (see Appendix A).

3.2. Multigrid on the interface

Due to Lemma 5, a variety of iterative techniques can be used for the solution of (27). Here we employ a multigrid V-cycle and W-cycle. Let us consider a sequence of nested interface grids on $\Gamma_{i,j}$, $1 \leqslant i < j \leqslant n$,

$$\mathscr{T}_{h_1,i,j} \subset \mathscr{T}_{h_2,i,j} \subset \cdots \subset \mathscr{T}_{h_L,i,j} = \mathscr{T}_{h,i,j}.$$

We assume that, given $\mathscr{T}_{h_1,i,j}$, $\mathscr{T}_{h_k,i,j}$ is obtained from $\mathscr{T}_{h_{k-1},i,j}$ by connecting edge midpoints; therefore $h_k = 2 * h_{k+1}$, $k = 1, \ldots, L-1$. We associate with the grids a sequence of nested mortar finite element spaces,

$$M_{h_1,i,j} \subset M_{h_2,i,j} \subset \cdots \subset M_{h_L,i,j} = M_{h,i,j},$$

defined as in Section 2. Similarly, we consider a sequence of nested subdomain grids

$$\mathcal{T}_{h_1,i} \subset \mathcal{T}_{h_2,i} \subset \cdots \subset \mathcal{T}_{h_L,i} = \mathcal{T}_{h,i}$$

and corresponding nested mixed finite element spaces

$$\mathbf{V}_{h_1,i} \times W_{h_1,i} \subset \mathbf{V}_{h_2,i} \times W_{h_2,i} \subset \cdots \subset \mathbf{V}_{h_L,i} \times W_{h_L,i} = \mathbf{V}_{h,i} \times W_{h,i}.$$

To simplify notations, we will omit h from the subscripts for the rest of the paper. We next introduce a sequence of symmetric and positive definite bilinear forms,

$$a_k(\cdot,\cdot) : M_k \times M_k \to \mathbf{R}, \quad 1 \leqslant k \leqslant L,$$

defined as in (21). Note that the evaluation of $a_k(\cdot,\cdot)$ requires solving subdomain problems in spaces $\mathbf{V}_k \times W_k$.

Let $\langle \cdot,\cdot \rangle$ denote the $L^2(\Gamma)$-inner product and let $\|\cdot\|$ be the induced L^2-norm . On each level k we associate with $a_k(\cdot,\cdot)$ an operator $A_k : M_k \to M_k$ satisfying for any $\lambda \in M_k$:

$$\langle A_k(\lambda), \mu \rangle = a_k(\lambda, \mu) \quad \forall \mu \in M_k.$$

We note that $A_k = \sum_{i=1}^n A_{k,i}$, where $A_{k,i} : M_k \to M_k$ satisfy

$$\langle A_{k,i}(\lambda), \mu \rangle = a_{k,i}(\lambda, \mu) \quad \forall \mu \in M_k.$$

By (21),

$$A_{k,i}\lambda = -\mathscr{P}_i^{M_k} \mathbf{u}_k^*(\lambda) \cdot \nu_i, \tag{31}$$

where $\mathscr{P}_i^{M_k}$ is the $L^2(\partial\Omega_i)$-orthogonal projection onto M_k. The operator A_k is a mortar version of the Poincaré–Steklov operator [1]. It can be viewed algebraically as the Schur complement with respect to the mortar unknowns.

The multigrid algorithm uses the following intergrid transfer operators. Since the spaces are nested, the *coarse-to-fine* operator

$$I_k : M_{k-1} \to M_k$$

is taken to be the natural injection, i.e.,

$$I_k \mu = \mu \quad \forall \mu \in M_{k-1}.$$

The *fine-to-coarse* operator

$$Q_{k-1} : M_k \to M_{k-1}$$

is defined to be the transpose of I_k with respect to $\langle \cdot,\cdot \rangle$, i.e.,

$$\langle Q_{k-1}\lambda, \mu \rangle = \langle \lambda, I_k \mu \rangle = \langle \lambda, \mu \rangle \quad \forall \lambda \in M_k, \ \mu \in M_{k-1}.$$

Note that Q_{k-1} is the L^2-projection from M_k to M_{k-1}. In the analysis we will also need an a_k-orthogonal projection

$$P_{k-1} : M_k \to M_{k-1}$$

satisfying for any $\lambda \in M_k$:

$$a_{k-1}(P_{k-1}\lambda, \mu) = a_k(\lambda, I_k \mu) \quad \forall \mu \in M_{k-1}. \tag{32}$$

On each level k we employ $m(k)$ conjugate gradient (CG) smoothing iterations. We note that the matrix corresponding to $a_k(\cdot,\cdot)$ cannot be formed explicitly, hence stationary smoothers like Jacobi and Gauss–Seidel cannot be applied. It is well known that the mth CG iterate for solving $A_k \mu = g_k$ is of the form

$$\mu^m = \mu^0 + G_{k,m} r_0,$$

where

$$r_0 = g_k - A_k \mu^0$$

is the initial residual and

$$G_{k,m}r_0 \in \mathscr{K}_{k,m}(r_0),$$

where

$$\mathscr{K}_{k,m}(r_0) = \mathrm{span}(r_0, A_k r_0, \ldots, A_k^{m-1} r_0)$$

is the mth Krylov subspace.

The multigrid algorithm defines an operator $B = B_L$ which can be used in a linear iterative process

$$\mu^{n+1} = \mu^n - B(A\mu^n - g)$$

or as a preconditioner for A. On the coarsest grid we define

$$B_1 = A_1^{-1}.$$

For any $g \in M_k$, $2 \leqslant k \leqslant L$, we define $B_k g$ recursively in terms of B_{k-1}. Let p be any positive integer.

(i) *Initialization*:

$$x^{(0)} = 0, \quad q^{(0)} = 0.$$

(ii) *Presmoothing*:

$$x^{(1)} = x^{(0)} + G_{k,m}\big(g - A_k x^{(0)}\big).$$

(iii) *Coarse grid correction*:

$$x^{(2)} = x^{(1)} + I_k q^{(p)},$$

where $q^{(i)}$, $i = 1, \ldots, p$ is defined by

$$q^{(i)} = q^{(i-1)} + B_{k-1}\big[Q_{k-1}\big(g - A_k x^{(1)}\big) - A_{k-1} q^{(i-1)}\big].$$

(iv) *Postsmoothing*:

$$B_k g = x^{(3)} = x^{(2)} + G_{k,m}\big(g - A_k x^{(2)}\big).$$

In the above algorithm $p = 1$ and $p = 2$ correspond to the V-cycle and the W-cycle, respectively.

4. Theoretical results for the multigrid

The analysis in this section is based on a general framework developed by Bramble et al. [9–11]. Since the bilinear forms $a_k(\cdot,\cdot)$ are defined by solving discrete subdomain problems on different grid levels (see (21)), the standard variational assumption

$$a_k(I_k\mu, I_k\mu) = a_{k-1}(\mu,\mu) \quad \forall \mu \in M_{k-1} \tag{33}$$

is not satisfied. We employ techniques from [11], where analysis is given without assuming (33). The main difficulties here are in proving the ''regularity and approximation'' assumption (see (35) below) and treating the non-stationary CG smoother.

The analysis is based on a recurrence relation for the error propagation operator $I - B_k A_k$, which follows easily from the definition of B_k:

$$I - B_k A_k = K_{k,m}[I - I_k P_{k-1} + I_k (I - B_{k-1} A_{k-1})^p P_{k-1}] K_{k,m}, \tag{34}$$

where $K_{k,m} = I - G_{k,m} A_k$ is the CG error propagation operator. As in [11], we make an assumption on the regularity of the solution and the approximation properties of the discretization. There exists $0 < \alpha \leqslant 1$ such that for $2 \leqslant k \leqslant L$:

$$|A_k((I - I_k P_{k-1})\mu, \mu)| \leqslant C_\alpha \left(\frac{\|A_k \mu\|^2}{\chi_k} \right)^\alpha a_k(\mu,\mu)^{1-\alpha} \quad \forall \mu \in M_k, \tag{35}$$

where χ_k is the largest eigenvalue of A_k. We show in the Appendix A that (35) holds for our application. The property of the CG smoother required for the convergence analysis is given in Lemma 6.

Lemma 6. *Let*

$$K_{k,\omega} = I - \frac{\omega}{\chi_k} A_k, \quad 0 < \omega < 2$$

be the error propagation operator for the weighted Richardson iteration. Then

$$a_k(K_{k,m}\mu, K_{k,m}\mu) \leqslant a_k\left(K^m_{k,\omega}\mu, K^m_{k,\omega}\mu\right) \quad \forall \mu \in M_k. \tag{36}$$

Proof. It is a well known property of CG that (see e.g. [28])

$$a_k(e(\mu^m), e(\mu^m)) = \min_{p\in P^0_m} a_k\big(p(A_k)e(\mu^0), p(A_k)e(\mu^0)\big),$$

where

$$e(\mu) = A_k^{-1} g_k - \mu$$

and

$$P^0_m = \{p : p \text{ is a polynomial of degree } m,\ p(0) = 1\}.$$

The lemma now follows, since $e(\mu^m) = K_{k,m} e(\mu^0)$, $K^m_{k,\omega} \in P^0_m(A_k)$, and $\mu^0 \in M_k$ is an arbitrary vector. □

It follows trivially from the Lemma that the spectrum of $K_{k,m}$ is in the interval $[0, 1)$ and

$$\|A_k K_{k,m}\mu\| \leqslant \|A_k K^m_{k,\omega}\mu\|. \tag{37}$$

We now state and prove a convergence result for the W-cycle. The proof follows closely the proof of Theorem 7 in [11], but differs in incorporating the bound for the smoother (Lemma 6 vs. assumption (A.4) in [11]).

Theorem 7. *Let $p = 2$ in the definition of B_L. If* (35) *holds and m is sufficiently large, then there exists a positive constant M such that*

$$|a_k((I - B_k A_k)\mu, \mu)| \leqslant \delta a_k(\mu, \mu) \quad \forall \mu \in M_k, \tag{38}$$

where

$$\delta = \frac{M}{M + m^\alpha}.$$

Proof. We first show that

$$-a_k((I - B_k A_k)\mu, \mu) \leqslant \delta a_k(\mu, \mu) \quad \forall \mu \in M_k. \tag{39}$$

Let $\tilde{\mu} = K_{k,m}\mu$. Using (34) and (32),

$$\begin{aligned} -a_k((I - B_k A_k)\mu, \mu) &= -a_k((I - I_k P_{k-1} + I_k(I - B_{k-1}A_{k-1})^2 P_{k-1})\tilde{\mu}, \tilde{\mu}) \\ &= -a_k((I - I_k P_{k-1})\tilde{\mu}, \tilde{\mu}) - a_{k-1}((I - B_{k-1}A_{k-1})^2 P_{k-1}\tilde{\mu}, P_{k-1}\tilde{\mu}) \\ &\leqslant -a_k((I - I_k P_{k-1})\tilde{\mu}, \tilde{\mu}), \end{aligned} \tag{40}$$

using in the last inequality that $I - B_{k-1}A_{k-1}$ is symmetric with respect to $a_{k-1}(\cdot,\cdot)$. By (35),

$$-a_k((I - I_k P_{k-1})\tilde{\mu}, \tilde{\mu}) \leqslant C_\alpha \left(\frac{\|A_k\tilde{\mu}\|^2}{\chi_k}\right)^\alpha a_k(\tilde{\mu}, \tilde{\mu})^{1-\alpha}. \tag{41}$$

With (37) we have

$$\frac{\|A_k\tilde{\mu}\|^2}{\chi_k} \leqslant \frac{\|A_k K^m_{k,\omega}\mu\|^2}{\chi_k} = a_k\left(\frac{I-K_{k,\omega}}{\omega}K^m_{k,\omega}\mu, K^m_{k,\omega}\mu\right)$$
$$\leqslant \frac{1}{2m\omega}a_k\left(\left(I-K^{2m}_{k,\omega}\right)\mu,\mu\right) \leqslant \frac{1}{2m\omega}a_k(\mu,\mu), \qquad (42)$$

where the second inequality follows from the spectral bound on $K_{k,\omega}$ and the inequality

$$(1-z)z^{2m} \leqslant \frac{1}{2m}(1-z)\sum_{i=0}^{2m-1} z^i = \frac{1}{2m}\left(1-z^{2m}\right)$$

for $0 \leqslant z \leqslant 1$. Combining together (40)–(42) and using the trivial bound

$$a_k(\tilde{\mu},\tilde{\mu}) \leqslant a_k(\mu,\mu),$$

we obtain

$$-a_k((I-B_kA_k)\mu,\mu) \leqslant C_\alpha(2m\omega)^{-\alpha}a_k(\mu,\mu),$$

which implies (39) if m is sufficiently large. We prove the opposite inequality,

$$a_k((I-B_kA_k)\mu,\mu) \leqslant \delta a_k(\mu,\mu) \quad \forall \mu \in M_k, \qquad (43)$$

by induction on k. For $k=1$ (43) is trivially satisfied. Assume that (43) holds for $k-1$, which implies

$$a_{k-1}((I-B_{k-1}A_{k-1})^2\mu,\mu) \leqslant \delta^2 a_{k-1}(\mu,\mu) \quad \forall \mu \in M_{k-1}.$$

Now, as in (40), we have

$$a_k((I-B_kA_k)\mu,\mu) \leqslant a_k((I-I_kP_{k-1})\tilde{\mu},\tilde{\mu}) + \delta^2 a_{k-1}(P_{k-1}\tilde{\mu},P_{k-1}\tilde{\mu})$$
$$= (1-\delta^2)a_k((I-I_kP_{k-1})\tilde{\mu},\tilde{\mu}) + \delta^2 a_k(\tilde{\mu},\tilde{\mu}). \qquad (44)$$

By (35) and a generalized arithmetic–geometric mean inequality,

$$|a_k((I-I_kP_{k-1})\tilde{\mu},\tilde{\mu})| \leqslant C_\alpha\left\{\alpha\gamma\frac{\|A_k\tilde{\mu}\|^2}{\chi_k} + (1-\alpha)\gamma^{-\alpha/(1-\alpha)}a_k(\tilde{\mu},\tilde{\mu})\right\} \qquad (45)$$

for any $\gamma > 0$. Similarly to (42),

$$\frac{\|A_k\tilde{\mu}\|^2}{\chi_k} \leqslant \frac{1}{2m\omega}\left[a_k(\mu,\mu) - a_k\left(K^m_{k,\omega}\mu, K^m_{k,\omega}\mu\right)\right]. \qquad (46)$$

Combining (44)–(46) and applying (36) gives

$$a_k((I-B_kA_k)\mu,\mu) \leqslant \left(1-\delta^2\right)\frac{C_\alpha\alpha\gamma}{2m\omega}\left[a_k(\mu,\mu) - a_k\left(K^m_{k,\omega}\mu,K^m_{k,\omega}\mu\right)\right]$$
$$+ \left[\left(1-\delta^2\right)C_\alpha(1-\alpha)\gamma^{-\alpha/(1-\alpha)} + \delta^2\right]a_k\left(K^m_{k,\omega}\mu,K^m_{k,\omega}\mu\right). \qquad (47)$$

We finish the argument by choosing γ such that the two coefficients above are equal and bounded above by δ (see Theorem 3 in [10] for details). □

Our next result is for the multigrid V-cycle ($p=1$). It has been shown in [11] that in this case B_L is symmetric and positive definite operator on M_L. Therefore, the V-cycle can be used as a preconditioner for A_L. The next theorem indicates that, under a mild assumption on the number of smoothing iterations, the system B_LA_L is well conditioned independently of L.

Theorem 8. *Assume that* (35) *holds and B is defined with* $p = 1$. *If* $m(k)$ *satisfy*

$$\beta_0 m(k) \leqslant m(k-1) \leqslant \beta_1 m(k) \tag{48}$$

for some constants β_0 *and* β_1 *greater than one, then*

$$\eta_0 a_k(\mu,\mu) \leqslant a_k(B_k A_k \mu,\mu) \leqslant \eta_1 a_k(\mu,\mu) \quad \forall \mu \in M_k, \tag{49}$$

where

$$\eta_0 \geqslant \frac{m(k)^\alpha}{M + m(k)^\alpha} \quad \text{and} \quad \eta_1 \leqslant \frac{M + m(k)^\alpha}{m(k)^\alpha}.$$

Proof. We first show by induction that

$$a_k((I - B_k A_k)\mu,\mu) \leqslant \delta_k a_k(\mu,\mu) \quad \forall \mu \in M_k$$

with $\delta_k = M/(M + m(k)^\alpha)$. This implies the first inequality in (49) with $\eta_o = 1 - \delta_k$. We note that the statement is trivial for $k = 1$ and assume that it holds for $k - 1$. Similarly to (44) we have

$$a_k((I - B_k A_k)\mu,\mu) \leqslant (1 - \delta_{k-1}) a_k((I - I_k P_{k-1})\tilde{\mu},\tilde{\mu}) + \delta_{k-1} a_k(\tilde{\mu},\tilde{\mu}). \tag{50}$$

The remainder of the argument is exactly as in the proof of Theorem 7 ((45)–(47)). To estimate η_1, we recall that (see Theorem 7)

$$-a_k((I - I_k P_{k-1})\tilde{\mu},\tilde{\mu}) \leqslant C_\alpha (2m(k)\omega)^{-\alpha} a_k(\mu,\mu).$$

The rest of the argument follows the proof of Theorem 6 in [11]. It uses the recurrence relation (34) to show that

$$\eta_1 \leqslant \prod_{i=2}^{k} \left(1 + \frac{C}{m(i)^\alpha}\right) \leqslant 1 + \frac{M}{m(k)^\alpha}$$

using (48) for the last inequality. □

5. Computational results

We present several examples that illustrate numerically the convergence of the multigrid V-cycle ($p = 1$). In all cases discontinuous piecewise linear mortar finite elements are used, and one presmoothing and one postsmoothing CG iteration is applied ($m = 1$). The CG is diagonally preconditioned by the harmonic average of the projections of the permeability coefficients onto the mortar spaces.

The first test is on the unit square in $\mathbf{R}^2$. We solve a problem with a discontinuous permeability

$$K = \begin{cases} I, & 0 \leqslant x < 1/2, \\ 10 * I, & 1/2 < x \leqslant 1 \end{cases}$$

and a given analytical solution

$$p(x,y) = \begin{cases} x^2 y^3 + \cos(xy), & 0 \leqslant x \leqslant 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20} y\right), & 1/2 \leqslant x \leqslant 1. \end{cases}$$

In this test we study both the convergence of the mixed method as well as the convergence of the multigrid V-cycle. We run the test case on five levels of grid refinement (both subdomain and mortar grid elements are divided by two in each direction for each refinement). The grids and the computed solution on the first level of refinement are shown on Fig. 1. A mortar grid with 3 elements is used on each interface $\Gamma_{i,j}$. Convergence rates for the mixed method are given in Table 1. Here $\|\cdot\|_M$ is the discrete L^2-norm induced by the midpoint rule on $\mathscr{T}_h$ (or the trace of $\mathscr{T}_h$ on Γ). The rates were established by computing a least squares fit to the error. We observe numerically convergence rates corresponding to those predicted by the theory.

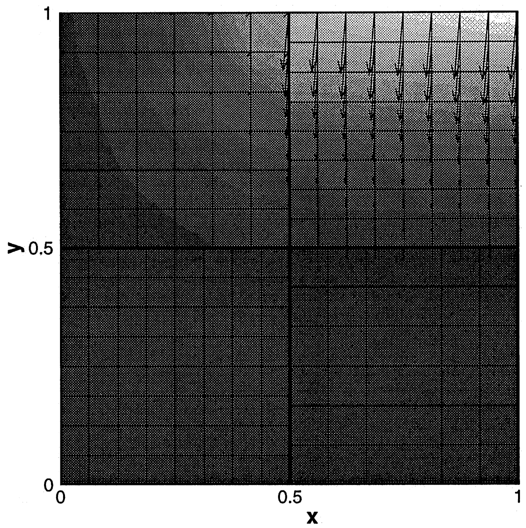

Fig. 1. Computed pressure (shade) and velocity (arrows).

Table 1
Discrete norm errors and convergence rates

$1/h$	$\|p - p_h\|_{\mathrm{M}}$	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|$	$\|\lambda - \mathcal{Q}_h\lambda_h\|_{\mathrm{M}}$
12	2.31E−04	1.98E−02	6.75E−04
24	5.87E−05	6.71E−03	1.72E−04
48	1.48E−05	2.29E−03	4.35E−05
96	3.72E−06	7.89E−04	1.09E−05
192	9.70E−07	2.71E−04	3.62E−06
Rate	$O(h^{1.98})$	$O(h^{1.55})$	$O(h^{1.90})$

Table 2
Number of V-cycles and reduction factors

$1/h$	Levels	V-cycles	δ
12	3	14	0.22
24	4	11	0.16
48	5	12	0.15
96	6	12	0.18
192	7	12	0.20

We next report the reduction factors for the multigrid V-cycle on the interface. On each grid level different number of multigrid levels are used, so that the coarsest mortar grid always consists of a single element on each interface. In Table 2 we give the reduction factor δ and the number of V-cycles needed for the solution of the interface problem. We observe that both δ and the number of V-cycles are insensitive to increasing the number of levels.

Our next example is on three-dimensional irregular domain. The domain is defined to be the image of the unit cube via the mapping:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = F\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} + \frac{1}{10}\sin(6\hat{x}) \\ \hat{z} \end{pmatrix}. \tag{51}$$

Table 3
Multigrid *V*-cycle performance

Domain decomposition	Total mortar elements	Total block elements	CPU time (seconds)	*V*-cycles	δ
Level 1 (2 *multigrid levels*)					
$2 \times 2 \times 1$	16	378	9.49	19	0.41
$2 \times 2 \times 2$	48	756	6.64	14	0.28
$4 \times 2 \times 2$	112	1512	10.06	18	0.36
$4 \times 4 \times 2$	256	3024	18.23	26	0.50
$4 \times 4 \times 4$	576	6048	11.71	15	0.29
Level 2 (3 *multigrid levels*)					
$2 \times 2 \times 1$	64	3024	34.28	23	0.37
$2 \times 2 \times 2$	192	6048	28.02	14	0.29
$4 \times 2 \times 2$	448	12096	20.50	16	0.34
$4 \times 4 \times 2$	1024	24192	31.44	22	0.43
$4 \times 4 \times 4$	2304	48384	26.25	15	0.28
Level 3 (4 *multigrid levels*)					
$2 \times 2 \times 1$	144	10206	80.25	23	0.38
$2 \times 2 \times 2$	432	20412	50.82	14	0.28
$4 \times 2 \times 2$	1008	40824	69.57	16	0.33
$4 \times 4 \times 2$	2304	81648	72.62	22	0.40
$4 \times 4 \times 4$	5184	163296	53.40	15	0.28

The pressure and the permeability are

$$p(x,y,z) = \sin(\pi(x+y+z)), \quad K = \frac{1}{1 + 100 * (x^2 + y^2 + z^2)}.$$

We employ the expanded mixed method on curvilinear grids [3] and transform the problem to a computational problem with a modified full tensor coefficient on a union of rectangular grids. The multigrid algorithm is performed on the rectangular grids which allows for a trivial construction of coarse grids. The subdomain problems are solved efficiently by reducing the mixed method to cell-centered finite differences for the pressure [4]. After convergence the obtained reference solution is transformed back to the curvilinear grids.

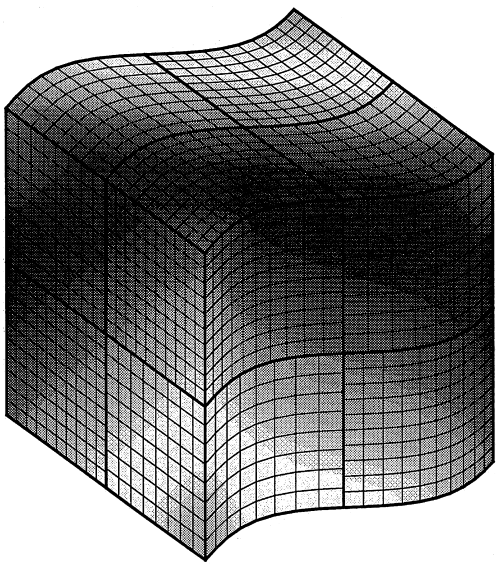

Fig. 2. Computed pressure on the second grid level.

We solve the problem on a series of grid levels and domain decompositions to study the scalability of the multigrid algorithm. We run on three grid levels and five different decompositions (see Table 3). On the first grid level the subdomain grids are taken to be $4 \times 4 \times 4$ and $5 \times 5 \times 5$ in a checkerboard fashion. The mortar grids are 2×2 on each interface. On each subsequent level we refine both subdomain and mortar grids. The computed solution on the second grid level with $2 \times 2 \times 2$ domains is shown in Fig. 2. In Table 3 we report CPU times, number of V-cycles and reduction factors for the three grid levels and five domain decompositions. Note that for a fixed grid level the size of the subdomain problems remains the same when adding new processors. The results indicate that the algorithm scales very well both with increasing the number of processors, as well as the number of unknowns. The timings are performed on an Intel Paragon.

Appendix A. Proof of regularity and approximation assumption

In this section we prove the regularity and approximation assumption (35) for the rectangular RT_0 mixed method from the computational results section. We note that the arguments can be extended to triangular grids, other mixed finite element spaces and higher order approximations.

Theorem 9. *Assume that each subdomain Ω_i is convex, the mixed finite element spaces $\mathbf{V}_{k,i} \times W_{k,i}$ are the lowest order Raviart–Thomas spaces on rectangular-type elements, and the mortar spaces $M_{k,i,j}$ consist of continuous or discontinuous piecewise linears on rectangular grids and satisfy* (15). *Then* (35) *holds.*

Proof. Given any $\lambda_k \in M_k$, recall from (21) that for $1 \leqslant i \leqslant n$

$$a_{k,i}(\lambda_k, \mu) = -\left\langle \mathbf{u}^*_{k,i}(\lambda_k) \cdot \nu_i, \mu \right\rangle_{\Gamma_i} \quad \forall \mu \in L^2(\Gamma_i),$$

where $\mathbf{u}^*_{k,i}(\lambda_k)$ is the velocity solution to the subdomain problem (22) and (23). Let (see (31))

$$\varphi_{k,i} = -\mathscr{P}_i^{M_k} \mathbf{u}^*_{k,i}(\lambda_k) \cdot \nu_i = A_{k,i}\lambda_k,$$

and let $\lambda^i = p_i^*|_{\partial\Omega_i}$, where p_i^* is the solution to

$$-\nabla \cdot K \nabla p_i^* = 0 \quad \text{in } \Omega_i, \tag{A.1}$$

$$-K\nabla p_i^* \cdot \nu_i = \varphi_{k,i} \quad \text{on } \Gamma_i, \tag{A.2}$$

$$p_i^* = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega \tag{A.3}$$

Note that (A.1)–(A.3) is well posed even if $\partial\Omega_i \cap \partial\Omega = \emptyset$, since, using (23),

$$\int_{\partial\Omega_i} \varphi_{k,i} \, \mathrm{d}\sigma = -\int_{\partial\Omega_i} \mathbf{u}^*_{k,i}(\lambda_k) \cdot \nu_i \, \mathrm{d}\sigma = -\int_{\Omega_i} \nabla \cdot \mathbf{u}^*_{k,i}(\lambda_k) \, \mathrm{d}x = 0.$$

By elliptic regularity [27],

$$\|p_i^*\|_{1+\beta,\Omega_i} \leqslant C\|\varphi_{k,i}\|_{-1/2+\beta,\Gamma_i} \tag{A.4}$$

for $0 < \beta \leqslant 1/2$. Let $\mathbf{u}_i^* = -K\nabla p_i^*$. We now have

$$a_{k,i}(\lambda_k - \lambda^i, \mu_k) = \langle \varphi_{k,i}, \mu_k \rangle_{\Gamma_i} - a_{k,i}(\lambda^i, \mu_k) = -\left\langle \left(\mathbf{u}_i^* - \mathbf{u}^*_{k,i}(\lambda^i)\right) \cdot \nu_i, \mu_k \right\rangle_{\Gamma_i}. \tag{A.5}$$

Note that $\mathbf{u}^*_{k,i}(\lambda^i)$ is a mixed finite element approximation of $\mathbf{u}_i^*$; therefore the term on the right in (A.5) can be bounded by standard mixed method error estimates. Following an argument in [36], Theorem 3.1, which employs the $a_{k,i}(\cdot,\cdot)$ characterization (30), we have

$$a_{k,i}\left(\lambda_k - \mathscr{P}_i^{M_{k-1}}\lambda^i,\ \lambda_k - \mathscr{P}_i^{M_{k-1}}\lambda^i\right)^{1/2} \leqslant Ch_{k-1}^{\beta}\|p_i^*\|_{1+\beta,\Omega_i} \leqslant Ch_{k-1}^{\beta}\|\varphi_{k,i}\|_{-1/2+\beta,\Gamma_i}, \tag{A.6}$$

using (A.4) for the second inequality. We now write

$$\|\varphi_{k,i}\|_{-1/2+\beta,\Gamma_i} \leqslant C\|\varphi_{k,i}\|_{-1/2,\Gamma_i}^{1-2\beta}\|\varphi_{k,i}\|_{0,\Gamma_i}^{2\beta} = C\|\varphi_{k,i}\|_{-1/2,\Gamma_i}^{1-2\beta}\|A_{k,i}\lambda_k\|^{2\beta}. \tag{A.7}$$

In addition,

$$\begin{aligned}\|\varphi_{k,i}\|_{-1/2,\Gamma_i} &= \sup_{\|\psi\|_{1/2,\Gamma_i}=1} \langle \varphi_{k,i}, \psi \rangle = \sup_{\|\psi\|_{1/2,\Gamma_i}=1} a_{k,i}\big(\lambda_k, \mathscr{P}_i^{M_k}\psi\big) \\ &\leqslant a_{k,i}(\lambda_k, \lambda_k)^{1/2} \sup_{\|\psi\|_{1/2,\Gamma_i}=1} a_{k,i}\big(\mathscr{P}_i^{M_k}\psi, \mathscr{P}_i^{M_k}\psi\big) \\ &\leqslant C a_{k,i}(\lambda_k, \lambda_k)^{1/2}, \end{aligned} \tag{A.8}$$

using (30) for the last inequality. Combining (A.6)–(A.8),

$$\begin{aligned} a_{k,i}\big(\lambda_k - \mathscr{P}_i^{M_{k-1}}\lambda^i, \lambda_k - \mathscr{P}_i^{M_{k-1}}\lambda^i\big) &\leqslant C h_{k-1}^{2\beta} \|A_{k,i}\lambda_k\|^{4\beta} a_{k,i}(\lambda_k, \lambda_k)^{1-2\beta} \\ &\leqslant C \left(\frac{\|A_{k,i}\lambda_k\|^2}{\chi_k} \right)^{2\beta} a_{k,i}(\lambda_k, \lambda_k)^{1-2\beta}, \end{aligned} \tag{A.9}$$

using that $h_{k-1} \leqslant C h_k \leqslant C \chi_k^{-1}$. We next notice that, with (32),

$$a_{k-1,i}(P_{k-1}\lambda_k, \mu_{k-1}) = a_{k,i}(\lambda_k, I_k\mu_{k-1}) = \langle \varphi_{k,i}, \mu_{k-1} \rangle \quad \forall \mu_{k-1} \in M_{k-1},$$

which allows us to derive, in a way similar to (A.9):

$$a_{k-1,i}\big(\mathscr{P}_i^{M_{k-1}}\lambda^i - P_{k-1}\lambda_k, \mathscr{P}_i^{M_{k-1}}\lambda^i - P_{k-1}\lambda_k\big) \leqslant C \left(\frac{\|A_{k,i}\lambda_k\|^2}{\chi_k} \right)^{2\beta} a_{k,i}(\lambda_k, \lambda_k)^{1-2\beta}. \tag{A.10}$$

The statement of the theorem now follows with $\alpha = 2\beta$, since

$$a_{k,i}(I_k\mu_{k-1}, I_k\mu_{k-1}) \leqslant C a_{k-1,i}(\mu_{k-1}, \mu_{k-1}) \quad \forall \mu_{k-1} \in M_{k-1}. \tag{A.11}$$

Inequality (A.11) can be shown easily using a scaling argument and the characterization (30). □

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