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SUPERCONVERGENCE FOR CONTROL-VOLUME MIXED FINITE ELEMENT METHODS ON RECTANGULAR GRIDS*

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Abstract. We consider control-volume mixed finite element methods for the approximation of second-order elliptic problems on rectangular grids. These methods associate control volumes (covolumes) with the vector variable as well as the scalar, obtaining local algebraic representation of the vector equation (e.g., Darcy's law) as well as the scalar equation (e.g., conservation of mass). We establish $O(h^2)$ superconvergence for both the scalar variable in a discrete L^2-norm and the vector variable in a discrete $H(\mathrm{div})$-norm. The analysis exploits a relationship between control-volume mixed finite element methods and the lowest order Raviart–Thomas mixed finite element methods.

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1. Introduction.

We consider the second-order elliptic problem in a domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, written as a first-order system

$$\mathbf{u} = -K\nabla p \text{ in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = f \text{ in } \Omega, \tag{1.2}$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{1.3}$$

The above equations model single-phase flow in porous media, where p is the fluid pressure, the vector $\mathbf{u}$ is the Darcy velocity, K is a symmetric uniformly positive definite and bounded diagonal tensor, representing the rock permeability divided by the fluid viscosity, $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, and f is the source term satisfying the compatibility condition

$$\int_\Omega f\,dx = 0.$$

The choice of homogeneous Neumann boundary condition corresponds to an impermeable boundary, which is the typical physical situation.

In this paper we consider discretizations for (1.1)–(1.3) based on control-volume mixed finite element methods (CVMFEM) and establish $O(h^2)$ superconvergence for the pressure and velocity in a discrete L^2-norm and $H(\mathrm{div})$-norm, respectively. Most of the arguments can be extended to Dirichlet boundary conditions. However, some

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loss of superconvergence occurs on the boundary in that case. Global $O(h)$ convergence has been shown by Chou et al. [9, 10]; here we obtain the $O(h^2)$ rate suggested by various numerical results (e.g., [8, 19, 24, 22]). Superconvergence is proved by $O(h^2)$ estimates of the differences between the scalar and vector discrete solutions and appropriate projections of the exact solutions.

CVMFEM, first introduced in [8], can be viewed as a type of mixed covolume method [9, 10, 11]. CVMFEM are closely related to the Raviart–Thomas mixed finite element methods (MFEM) [27, 7, 28], cell-centered finite difference (CCFD) methods [29, 30, 4], mimetic finite difference (MFD) methods [5, 21, 6], and multipoint flux approximation (MPFA) methods [1, 17]. Some of these relationships are explored in detail in [22].

Like MFEM, CVMFEM are designed to provide simultaneous (accurate) approximations of pressure and velocity, and local mass conservation, $\int_Q \nabla\cdot\mathbf{u}_h = \int_Q f$ on each finite element Q, where $\mathbf{u}_h$ is the computed velocity. These properties can be difficult to obtain when K is heterogeneous (in particular, discontinuous) and/or anisotropic, especially when it incorporates irregular geological features. The methods listed above seek to accomplish this for flow in porous media, among other applications.

Unlike MFEM, CVMFEM have vector control volumes (covolumes) that give rise to a local discrete Darcy law analogous to (1.1). An engineer measuring the permeability of a core sample will typically impose a pressure at each end and observe the flux through the core. The discrete CVMFEM control volume that corresponds to the discrete flux unknown through a face, consisting of the two adjacent halves of the elements on either side of the face (see Figure 1), plays the role of this core, with the element pressures representing the imposed pressures at the ends. The vector test function associated with the control volume is essentially a piecewise-constant vector field, similar to a unit vector in the control volume and a zero vector outside it. The algebraic equation produced by this test function is the local discrete Darcy law. Thus, CVMFEM represent both physical principles in (1.1)–(1.3) locally.

In MFEM, the test vector belongs to the vector trial space and therefore has a continuous normal component. Because the test and trial spaces are the same, the mass matrix is symmetric and positive definite (SPD). In CVMFEM, the normal component of the test vector is discontinuous at the ends of the control volume, and can also be discontinuous at the element face for general distorted grids. If K is elementwise constant and the elements are affine (parallelograms in two dimensions), the mass matrix is SPD, despite the distinct test and trial spaces; in general, it is not symmetric, but symmetry can be restored by appropriate numerical integration [19].

On a uniform grid with constant K, the lowest-order Raviart–Thomas MFEM, denoted RT_0, yields a tridiagonal mass matrix with weights 1/6, 2/3, 1/6, and the basic CCFD results in a diagonal mass matrix. As will be seen below, CVMFEM leads to weights 1/8, 3/4, 1/8. These are all of the form c, $1-2c$, c, where $c = 0$ (CCFD), 1/6 (MFEM), or 1/8 (CVMFEM). In [19], some heuristic reasons to favor $c = 1/8$ are presented: on a uniform grid, the second-order truncation error term is half that of $c = 0$ and $c = 1/6$; on a nonuniform grid, only $c = 1/8$ matches one-sided compact finite differences, avoiding any first-order local truncation error; in terms of Fourier modes, the ratio of the discrete eigenvalue to the continuous eigenvalue is generally closer to 1 for $c = 1/8$. Numerical results in [22] for homogeneous K show second-order convergence for both MFEM and CVMFEM; on orthogonal grids, the flux error for CVMFEM improves on that of MFEM by a factor of approximately 2.6; on the distorted grids used, CVMFEM is worse by a factor of about 1.3.

The rest of the paper is organized as follows. In the next section we recall the Raviart–Thomas MFEM for (1.1)–(1.3). Section 3 describes the CVMFEM and its relation to the Raviart–Thomas MFEM. Superconvergence for the velocity is established in section 4. Section 5 is devoted to superconvergence for the pressure.

2. Mixed finite element methods. We will make use of the following standard notation. For a subdomain $G \subset \mathbb{R}^d$, the $L^2(G)$ inner product (or duality pairing) for scalar and vector valued functions is denoted by $(\cdot,\cdot)_G$. We denote the norm in the Sobolev space $W_p^k(G)$, $k \in \mathbb{R}$, $1 \le p \le \infty$ [2], by $\|\cdot\|_{k,p,G}$. Let $\|\cdot\|_{k,G}$ be the norm of the Hilbert space $H^k(G) = W_2^k(G)$. We omit G in the subscript if $G = \Omega$. For a section of a subdomain boundary $S \subset \mathbb{R}^{d-1}$ we write $\langle\cdot,\cdot\rangle_S$ and $\|\cdot\|_{0,S}$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively.

The mixed variational formulation, which is the basis for the MFEM is as follows. Find $\mathbf{u} \in \mathbf{V}$ and $p \in W$ such that

$$(K^{-1}\mathbf{u}, \mathbf{v}) = (p, \nabla\cdot\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}, \tag{2.1}$$

$$(\nabla\cdot\mathbf{u}, w) = (f, w), \quad w \in W, \tag{2.2}$$

where

$$\mathbf{V} = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \quad W = L_0^2(\Omega) = \left\{ w \in L^2(\Omega) : \int_\Omega w\,dx = 0 \right\},$$

and

$$H(\mathrm{div};\Omega) = \{\mathbf{v} : \mathbf{v} \in (L^2(\Omega))^2,\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\}$$

with a norm

$$\|\mathbf{v}\|_{\mathbf{V}} = (\|\mathbf{v}\|^2 + \|\nabla\cdot\mathbf{v}\|^2)^{1/2}.$$

We assume that Ω can be exactly covered by a rectangular-type finite element partition $\mathcal{T}_h$. Let $\mathbf{V}_h \times W_h \subset \mathbf{V} \times W$ be the lowest-order Raviart–Thomas (RT_0) mixed finite element spaces on $\mathcal{T}_h$ [27]. More precisely, for all $Q \in \mathcal{T}_h$,

$$\mathbf{V}_h(Q) = \{\mathbf{v} = (a_1+b_1x, a_2+b_2y, a_3+b_3z)^T \text{ on } Q\}, \quad W_h(Q) = \{w = \text{constant on } Q\},$$

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{V} : \mathbf{v}|_Q \in \mathbf{V}_h(Q)\ \forall\, Q \in \mathcal{T}_h\}, \quad W_h = \{w \in W : w|_Q \in W_h(Q)\ \forall\, Q \in \mathcal{T}_h\},$$

where the third component of $\mathbf{v}$ should be removed if $d = 2$. The degrees of freedom of $\mathbf{V}_h$ are the constant normal components on the sides. If these are continuous, then $\mathbf{v} \in H(\mathrm{div};\Omega)$. Key properties of the RT_0 spaces are

$$\nabla\cdot\mathbf{V}_h = W_h \tag{2.3}$$

and the existence of an interpolation operator $\Pi : (H^1(\Omega))^d \to \mathbf{V}_h$ (see [27, 7]) such that for $\mathbf{q} \in (H^1(\Omega))^2$

$$(\nabla\cdot(\Pi\mathbf{q} - \mathbf{q}), w) = 0 \quad \forall\, w \in W_h \tag{2.4}$$

and which satisfies the continuity and approximation properties

$$\|\Pi\mathbf{q}\|_{\mathbf{V}} \le C\|\mathbf{q}\|_1, \tag{2.5}$$

$$\|\mathbf{q} - \Pi\mathbf{q}\|_0 \le Ch|\mathbf{q}|_1. \tag{2.6}$$

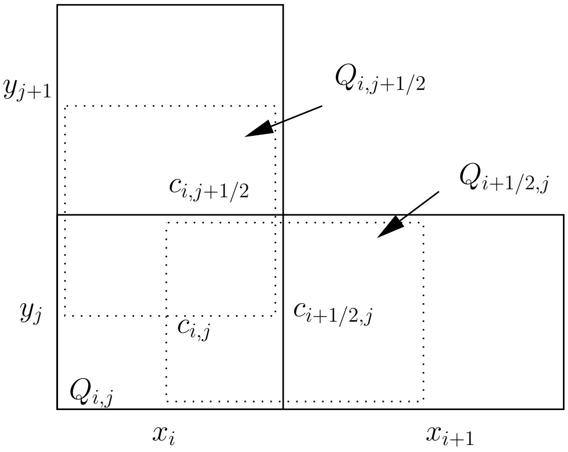

FIG. 1. *Computational grid and control volumes.*

The MFEM for approximating (2.1)–(2.2) is as follows. Find $\tilde{\mathbf{u}}_h \in \mathbf{V}_h$, $\tilde{p}_h \in W_h$ such that

$$(K^{-1}\tilde{\mathbf{u}}_h, \mathbf{v}) = (\tilde{p}_h, \nabla\cdot\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_h, \tag{2.7}$$

$$(\nabla\cdot\tilde{\mathbf{u}}_h, w) = (f, w), \quad w \in W_h. \tag{2.8}$$

It has been shown in [27] that (2.7)–(2.8) has a unique solution and

$$\|p - \tilde{p}_h\|_W + \|\mathbf{u} - \tilde{\mathbf{u}}_h\|_{\mathbf{V}} = O(h).$$

A number of authors have studied superconvergence for the above method or the closely related CCFD method [25, 14, 30, 15, 16, 18, 4] and have shown results of the form

$$|||p - \tilde{p}_h|||_W + |||\mathbf{u} - \tilde{\mathbf{u}}_h|||_{\mathbf{V}} = O(h^2),$$

where $|||\cdot|||_W$ and $|||\cdot|||_{\mathbf{V}}$ are discrete norms defined in (4.8) and (4.9) below (or some variants of them). The goal of this paper is to obtain similar superconvergence results for the CVMFEM.

3. Control volume mixed finite element methods.

Denote the elements of $\mathcal{T}_h$ by $Q_{i,j}$ for $d = 2$ or by $Q_{i,j,k}$ for $d = 3$; see Figure 1 for $d = 2$. For simplicity, in most of the paper we will use the notation and present the arguments for $d = 2$. The case $d = 3$ is a trivial extension.

The center of $Q_{i,j}$ is denoted by $c_{i,j}$. The midpoints of the left and right edges are denoted by $c_{i-1/2,j}$ and $c_{i+1/2,j}$, respectively, with similar notation for the bottom and top edges. With each edge we associate a control volume, where Darcy's law (1.1) is approximated. In particular, letting $c_{i+1/2,j} = (x_{i+1/2}, y_j)$, $c_{i,j} = (x_i, y_j)$, etc., define

$$Q_{i+1/2,j} := (x_i, x_{i+1}) \times (y_{j-1/2}, y_{j+1/2}) \cap \Omega, \tag{3.1}$$

$$Q_{i,j+1/2} := (x_{i-1/2}, x_{i+1/2}) \times (y_i, y_{i+1}) \cap \Omega. \tag{3.2}$$

The control volumes $Q_{i+1/2,j}$ and $Q_{i,j+1/2}$ are referred to as v_1-volumes and v_2-volumes, respectively. The control volumes that have at least one edge on $\partial\Omega$ are called border volumes.

Define the velocity test space

$$\begin{aligned}\mathbf{Y}_h = \{(v_h^1, v_h^2) : v_h^1|_{Q_{i+1/2,j}} &= \text{ constant } \forall\, Q_{i+1/2,j},\ v_h^1 = 0 \text{ on border } v_1\text{-volumes}\\ v_h^2|_{Q_{i,j+1/2}} &= \text{ constant } \forall\, Q_{i,j+1/2},\ v_h^2 = 0 \text{ on border } v_2\text{-volumes}\}.\end{aligned}$$

Thus, for example, the basis function $\mathbf{y}_{i+1/2,j} \in \mathbf{Y}_h$ associated with $c_{i+1/2,j}$ is the vector $(\chi_{i+1/2,j}, 0)$, i.e., $(1,0)$ on $Q_{i+1/2,j}$, $(0,0)$ elsewhere. To see the form of the associated algebraic equation, write (1.1) as $K^{-1}\mathbf{u} + \nabla p = 0$, form the inner product with $\mathbf{y}_{i+1/2,j}$, and integrate

$$\int_{x_i}^{x_{i+1}} \int_{y_{j-1/2}}^{y_{j+1/2}} (K^1)^{-1} u^1 \, dy\, dx + \int_{y_{j-1/2}}^{y_{j+1/2}} (p(x_{i+1}, y) - p(x_i, y))\, dy = 0,$$

where $\mathbf{u} = (u^1, u^2)$ and $K = \operatorname{diag}(K^1, K^2)$. Suppose that K is elementwise constant on $Q_{i,j}$ and $Q_{i+1,j}$. Taking $\mathbf{u} = \mathbf{v}_{i-1/2,j}, \mathbf{v}_{i+1/2,j}, \mathbf{v}_{i+3/2,j} \in \mathbf{V}_h$, the usual RT_0 vector basis functions, we obtain the tridiagonal mass-matrix coefficients

$$1/8\,(K^1_{i,j})^{-1} h_i^x h_j^y,\ 3/8\,(K^1_{i,j})^{-1} h_i^x h_j^y + 3/8\,(K^1_{i+1,j})^{-1} h_{i+1}^x h_j^y,\ 1/8\,(K^1_{i+1,j})^{-1} h_{i+1}^x h_j^y,$$

where h^x and h^y are the element dimensions. For homogeneous K and a uniform grid, this reduces to 1/8, 3/4, 1/8, as noted above.

3.1. Variational formulation for CVMFEM.

Following [9], define the bilinear forms $a(\cdot,\cdot) : (L^2(\Omega))^d \times (L^2(\Omega))^d \to \mathbb{R}$, $b(\cdot,\cdot) : \mathbf{Y}_h \times H^1(\Omega) \to \mathbb{R}$, and $c(\cdot,\cdot) : H(\operatorname{div};\Omega) \times L^2(\Omega) \to \mathbb{R}$ as follows:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &:= (K^{-1}\mathbf{u}, \mathbf{v}),\\ b(\mathbf{v}, p) &:= \sum_{i,j} \langle p, (v^1, 0)^T \cdot \mathbf{n}\rangle_{\partial Q_{i+1/2,j}} + \sum_{i,j} \langle p, (0, v^2)^T \cdot \mathbf{n}\rangle_{\partial Q_{i,j+1/2}},\\ c(\mathbf{u}, w) &:= (\nabla \cdot \mathbf{u}, w).\end{aligned}$$

LEMMA 3.1. *If* $(\mathbf{u}, p) \in H(\operatorname{div};\Omega) \times H^1(\Omega)$ *solves* (1.1)–(1.3)*, then* $(\mathbf{u}, p)$ *satisfies the variational formulation*

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = 0, \quad \mathbf{v} \in \mathbf{Y}_h, \tag{3.3}$$

$$c(\mathbf{u}, w) = (f, w), \quad w \in W_h. \tag{3.4}$$

Proof. Equation (1.1) implies, for $\mathbf{v} \in \mathbf{Y}_h$,

$$\begin{aligned}(K^{-1}\mathbf{u}, \mathbf{v}) = (-\nabla p, \mathbf{v}) &= \sum_{i,j} (-\nabla p, (v^1, 0)^T)_{Q_{i+1/2,j}} + \sum_{i,j} (-\nabla p, (0, v^2)^T)_{Q_{i,j+1/2}}\\ &= -\sum_{i,j} \langle p, (v^1, 0)^T \cdot \mathbf{n}\rangle_{\partial Q_{i+1/2,j}} - \sum_{i,j} \langle p, (0, v^2)^T \cdot \mathbf{n}\rangle_{\partial Q_{i,j+1/2}},\end{aligned}$$

giving (3.3). Equation (3.4) follows trivially from (1.2). □

The CVMFEM may be formulated as follows. Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = 0, \quad \mathbf{v} \in \mathbf{Y}_h, \tag{3.5}$$

$$c(\mathbf{u}_h, w) = (f, w), \quad w \in W_h. \tag{3.6}$$

Note that (3.5) is a Petrov–Galerkin FEM, since the test functions differ from the trial functions. We next recall the transfer operator $\gamma_h : \mathbf{V}_h \to \mathbf{Y}_h$, introduced in [9]. Define, for all $\mathbf{v} \in \mathbf{V}_h$,

$$\gamma_h \mathbf{v} = \left(\sum_{i,j} v^1(c_{i+1/2,j})\chi_{i+1/2,j}, \sum_{i,j} v^2(c_{i,j+1/2})\chi_{i,j+1/2} \right).$$

It has been shown in [9] that for constants $\alpha > 0$ and C independent of h,

$$b(\gamma_h \mathbf{v}, w) = -c(\mathbf{v}, w) \quad \forall\, \mathbf{v} \in \mathbf{V}_h, w \in W_h, \tag{3.7}$$

$$a(\mathbf{v}, \gamma_h \mathbf{v}) \geq \alpha \|\mathbf{v}\|_0^2 \quad \forall\, \mathbf{v} \in \mathbf{V}_h, \tag{3.8}$$

$$\|\gamma_h \mathbf{v}\|_0 \leq C\|\mathbf{v}\|_0. \tag{3.9}$$

4. Velocity superconvergence analysis. In this section we establish superconvergence for the velocity in the CVMFEM. In the treatment of the permeability K we will make use of the following piecewise smooth space. Let $W^{\alpha}_{\mathcal{T}_h}$ consist of functions φ such that $\varphi|_Q \in W^{\alpha}(Q)$ for all $Q \in \mathcal{T}_h$ and $\|\varphi\|_{\alpha,Q}$ is uniformly bounded, independently of h. Let

$$|||\varphi|||_{\alpha} = \max_{Q \in \mathcal{T}_h} \|\varphi\|_{\alpha,Q}.$$

Subtracting (3.5)–(3.6) from (3.3)–(3.4) gives the error equations

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p - p_h) = 0, \quad \mathbf{v} \in \mathbf{Y}_h, \tag{4.1}$$

$$c(\mathbf{u} - \mathbf{u}_h, w) = 0, \quad w \in W_h. \tag{4.2}$$

We first note that (4.2) implies

$$0 = c(\mathbf{u} - \mathbf{u}_h, w) = (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w) = (\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h), w) \quad \forall\, w \in W_h$$

using (2.4). Therefore, using (2.3),

$$\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h) = 0. \tag{4.3}$$

Let P_h be the L^2-orthogonal projection onto W_h, satisfying for any $\varphi \in L^2(\Omega)$

$$(\varphi - P_h\varphi, w) = 0 \quad \forall w \in W_h. \tag{4.4}$$

Taking $\mathbf{v} = \gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)$ and $w = P_h p - p_h$ in (4.1)–(4.2) implies

$$\begin{aligned} &a(\Pi\mathbf{u} - \mathbf{u}_h, \gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)) \\ &\qquad = -a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h(\Pi\mathbf{u} - \mathbf{u}_h)) - b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - p_h), \end{aligned} \tag{4.5}$$

$$c(\Pi\mathbf{u} - \mathbf{u}_h, P_h p - p_h) = 0. \tag{4.6}$$

The second term on the right in (4.5) can be manipulated as follows:

$$\begin{aligned} b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - p_h) &= b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - P_h p) + b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), P_h p - p_h) \\ &= b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - P_h p) - c(\Pi\mathbf{u} - \mathbf{u}_h, P_h p - p_h) \\ &= b(\gamma_h(\Pi\mathbf{u} - \mathbf{u}_h), p - P_h p), \end{aligned}$$

using (3.7) and (4.6) in the last equality. Therefore (4.5) gives

$$a(\Pi\mathbf{u}-\mathbf{u}_h,\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)) = -a(\mathbf{u}-\Pi\mathbf{u},\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)) - b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-P_hp). \tag{4.7}$$

Lemma 4.4 implies that

$$|a(\mathbf{u}-\Pi\mathbf{u},\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h)| \le Ch^2|||K^{-1}|||_{1,\infty}\|\mathbf{u}\|_2\|\Pi\mathbf{u}-\mathbf{u}_h\|_0.$$

Using (4.3), Lemma 4.5 gives

$$|b(\gamma_h(\Pi\mathbf{u}-\mathbf{u}_h),p-P_hp)| \le Ch^2\|p\|_3\|\Pi\mathbf{u}-\mathbf{u}_h\|_0.$$

With the above two bounds and (3.8), (4.7) implies the following superconvergence result.

THEOREM 4.1. *For the CVMFEM approximation* $(\mathbf{u}_h,p_h)$*, there exists a constant* C *independent of* h *such that*

$$\|\Pi\mathbf{u}-\mathbf{u}_h\|_0 \le Ch^2|||K^{-1}|||_{1,\infty}(\|\mathbf{u}\|_2+\|p\|_3).$$

Remark 4.1. The velocity superconvergence result of Theorem 4.1 and the pressure superconvergence bound of Theorem 5.1 require global smoothness of $\mathbf{u}$ and p. There are practical cases when the solution is locally smooth on a given region but possesses reduced regularity globally, such as aquifers with faults or multiple rock layers. Such cases could be treated by establishing interior and negative norm bounds, using techniques developed in [26, 14].

The above result immediately implies superconvergence for the velocity in an L^2 sense along the Gaussian lines. Consider an element $Q=[a_1,b_1]\times[a_2,b_2]$. Following [18, 16], for a vector $\mathbf{q}=(q_1,q_2)$ define

$$|||q_1|||_{1,Q}^2 = (b_2-a_2)\int_{a_1}^{b_1}\left|q_1\left(x_1,\frac{a_2+b_2}{2}\right)\right|^2 dx_1,$$

$$|||q_2|||_{2,Q}^2 = (b_1-a_1)\int_{a_2}^{b_2}\left|q_2\left(\frac{a_1+b_1}{2},x_2\right)\right|^2 dx_2,$$

$$|||\mathbf{q}|||^2 = \sum_{i=1}^{2}\sum_{Q\in\mathcal{T}_h}|||q_i|||_{i,Q}^2.$$

Note that for $\mathbf{q}\in\mathbf{V}_h$, $|||\mathbf{q}|||=\|\mathbf{q}\|_0$.

COROLLARY 4.2. *There exists a constant* C *independent of* h *such that*

$$|||\mathbf{u}-\mathbf{u}_h||| \le Ch^2|||K^{-1}|||_{1,\infty}(\|\mathbf{u}\|_2+\|p\|_3).$$

Proof: It was shown in [16] that

$$|||\mathbf{u}-\Pi\mathbf{u}||| \le Ch^2|\mathbf{u}|_2,$$

where $|\cdot|_2$ denotes the H^2-seminorm. Also, using Theorem 4.1,

$$|||\Pi\mathbf{u}-\mathbf{u}_h||| = \|\Pi\mathbf{u}-\mathbf{u}_h\|_0 \le Ch^2|||K^{-1}|||_{1,\infty}(\|\mathbf{u}\|_2+\|p\|_3).$$

The assertion of the corollary follows from the above two bounds and the triangle inequality. □

It is also easy to see that $\nabla \cdot (\mathbf{u} - \mathbf{u}_h)$ is superconvergent at the midpoints of the elements. Define, for a scalar function g,

$$|||g||| = \left(\sum_{i,j} |Q_{i,j}| g(c_{i,j})^2 \right)^{1/2}. \tag{4.8}$$

Using (4.3) and (2.4),

$$|||\nabla \cdot (\mathbf{u} - \mathbf{u}_h)||| = |||\nabla \cdot (\mathbf{u} - \Pi\mathbf{u})||| = |||\nabla \cdot \mathbf{u} - \widehat{\nabla \cdot \mathbf{u}}||| \le Ch^2 \|\nabla \cdot \mathbf{u}\|_{2,\infty},$$

where the last inequality follows from Lemma 4.6. Defining

$$|||\mathbf{q}|||_{\mathbf{V}}^2 = |||\mathbf{q}|||^2 + |||\nabla \cdot \mathbf{q}|||^2, \tag{4.9}$$

the above results can be summarized as follows.

COROLLARY 4.3. *There exists a constant C independent of h such that*

$$|||\mathbf{u} - \mathbf{u}_h|||_{\mathbf{V}} \le Ch^2 (\|\mathbf{u}\|_2 + \|\nabla \cdot \mathbf{u}\|_{2,\infty} + \|p\|_3). \tag{4.10}$$

We next proceed with the three lemmas needed in the proof of Theorem 4.1.

LEMMA 4.4. *There exists a constant C independent of h such that, for all $\mathbf{v} \in \mathbf{V}_h$,*

$$|a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h \mathbf{v})| \le Ch^2 |||K^{-1}|||_{1,\infty} \|\mathbf{u}\|_2 \|\mathbf{v}\|_0.$$

Proof. We first show that if $\mathbf{q} \in (P_1(Q))^2$, where P_k is the space of polynomials of degree $\le k$, then

$$\int_Q (\mathbf{q} - \Pi\mathbf{q}) \gamma_h \mathbf{v} \, dx \, dy = 0 \quad \forall \mathbf{v} \in \mathbf{V}_h, \ Q \in \mathcal{T}_h. \tag{4.11}$$

The argument follows the proof of Lemma 3.1 in [16]. Let $Q = [a,b] \times [c,d]$ and let $L_1(x)$ and $\tilde{L}_1(y)$ be the linear Legendre polynomials on $[a,b]$ and $[c,d]$, respectively. It is easy to see that any $\mathbf{q} \in (P^1(Q))^2$ can be decomposed into

$$\mathbf{q}(x,y) = \bar{\mathbf{q}}(x,y) + (\alpha \tilde{L}_1(y), \beta L_1(x))^T,$$

where $\bar{\mathbf{q}} \in \mathbf{V}_h(Q)$. Since $\bar{\mathbf{q}} - \Pi\bar{\mathbf{q}} = 0$, it is enough to establish (4.11) for $\mathbf{q}(x,y) = (\alpha \tilde{L}_1(y), \beta L_1(x))^T$. It is shown in [16] that in this case $\Pi\mathbf{q} = 0$. Therefore

$$\begin{aligned} \int_Q (\mathbf{q} - \Pi\mathbf{q}) \gamma_h \mathbf{v} \, dx \, dy &= \int_Q \mathbf{q} \gamma_h \mathbf{v} \, dx \, dy \\ &= \int_Q (\alpha \tilde{L}_1(y) (\gamma_h \mathbf{v})^1(x,y) + \beta L_1(x) (\gamma_h \mathbf{v})^2(x,y)) \, dx \, dy = 0, \end{aligned}$$

using that for any fixed $x_0 \in [a,b]$, $(\gamma_h \mathbf{v})^1(x_0, y) \in P_0[c,d]$, that for any fixed $y_0 \in [c,d]$, $(\gamma_h \mathbf{v})^2(x, y_0) \in P_0[a,b]$, and the orthogonality properties of $L_1(x)$ and $\tilde{L}_1(y)$.

We now have

$$\begin{aligned} a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h \mathbf{v}) &= (K^{-1}(\mathbf{u} - \Pi\mathbf{u}), \gamma_h \mathbf{v}) \\ &= \sum_{Q \in \mathcal{T}_h} [K_Q^{-1}(\mathbf{u} - \Pi\mathbf{u}, \gamma_h \mathbf{v})_Q + ((K^{-1} - K_Q^{-1})(\mathbf{u} - \Pi\mathbf{u}), \gamma_h \mathbf{v})_Q], \end{aligned}$$

where K_Q^{-1} is the value of K^{-1} at the center of Q. Therefore

$$
\begin{aligned}
|a(\mathbf{u}-\Pi\mathbf{u},\gamma_h\mathbf{v})| \le C\|K^{-1}\|_{0,\infty} &\sum_{Q\in\mathcal{T}_h} |(\mathbf{u}-\Pi\mathbf{u},\gamma_h\mathbf{v})_Q| \\
&+ Ch|||K^{-1}|||_{1,\infty}\|\mathbf{u}-\Pi\mathbf{u}\|_0\|\gamma_h\mathbf{v}\|_0.
\end{aligned} \tag{4.12}
$$

Using (4.11), an application of the Bramble–Hilbert lemma [12] implies

$$
|(\mathbf{u}-\Pi\mathbf{u},\gamma_h\mathbf{v})_Q| \le Ch^2|\mathbf{u}|_{2,Q}\|\gamma_h\mathbf{v}\|_{0,Q},
$$

which combined with (4.12), (2.6), and (3.9) completes the proof. $\square$

LEMMA 4.5. *There exists a constant C independent of h such that for all $\mathbf{v}\in\mathbf{V}_h$,*

$$
|b(\gamma_h\mathbf{v}, p-P_hp)| \le Ch^2\|p\|_3\|\mathbf{v}\|_{\mathbf{V}}.
$$

Proof. Let $e_{i+1/2,j} = \partial Q_{i+1/2,j}\cap Q_{i,j}$ and $e_{i,j+1/2} = \partial Q_{i,j+1/2}\cap Q_{i,j}$. Note that in the sums in

$$
\begin{aligned}
&b(\gamma_h\mathbf{v}, p-P_hp) \\
&= \sum_{i,j}\langle p-P_hp, ((\gamma_h\mathbf{v})^1,0)^T\cdot\mathbf{n}\rangle_{\partial Q_{i+1/2,j}} + \sum_{i,j}\langle p-P_hp, (0,(\gamma_h\mathbf{v})^2)^T\cdot\mathbf{n}\rangle_{\partial Q_{i,j+1/2}},
\end{aligned}
$$

every edge $e_{i+1/2,j}$ and $e_{i,j+1/2}$ appears twice (from the two neighboring covolumes). Using that $\frac{\partial v_1}{\partial x}$ and $\frac{\partial v_2}{\partial y}$ are constants on each element $Q_{i,j}$, we have

$$
\begin{aligned}
&b(\gamma_h\mathbf{v}, p-P_hp) \\
&= \sum_{i,j}\left(h_i^x\frac{\partial v_1}{\partial x}\int_{e_{i+1/2,j}}(p-P_hp)\,dy + h_j^y\frac{\partial v_2}{\partial y}\int_{e_{i,j+1/2}}(p-P_hp)\,dx\right) \\
&= \sum_{i,j}\left(\frac{\partial v_1}{\partial x}\left(h_i^x\int_{e_{i+1/2,j}}p\,dy - \int_{Q_{i,j}}p\,dxdy\right)\right. \\
&\qquad\qquad \left.+\frac{\partial v_2}{\partial y}\left(h_j^y\int_{e_{i,j+1/2}}p\,dx - \int_{Q_{i,j}}p\,dxdy\right)\right) \\
&= \sum_{i,j}\left(\left(p,\frac{\partial v_1}{\partial x}\right)_{Q_{i,j},M_x} - \left(p,\frac{\partial v_1}{\partial x}\right)_{Q_{i,j}} + \left(p,\frac{\partial v_2}{\partial y}\right)_{Q_{i,j},M_y} - \left(p,\frac{\partial v_2}{\partial y}\right)_{Q_{i,j}}\right),
\end{aligned} \tag{4.13}
$$

where $(\cdot,\cdot)_{Q,M_x}$ is the quadrature rule on Q which uses the midpoint rule in x and exact integration in y, and $(\cdot,\cdot)_{Q,M_y}$ uses exact integration in x and the midpoint rule in y. Since the midpoint rule is exact for linear polynomials, the Peano kernel theorem [13, Theorem 3.7.1] implies

$$
\begin{aligned}
&\left(p,\frac{\partial v_1}{\partial x}\right)_{Q_{i,j},M_x} - \left(p,\frac{\partial v_1}{\partial x}\right)_{Q_{i,j}} = \int_{Q_{i,j}}\varphi(x)\frac{\partial^2 p}{\partial x^2}(x,y)\frac{\partial v_1}{\partial x}\,dxdy \\
&= \int_{Q_{i,j}}\varphi(x)\frac{\partial^2 p}{\partial x^2}(x,y)\nabla\cdot\mathbf{v}\,dxdy - \int_{Q_{i,j}}\varphi(x)\frac{\partial^2 p}{\partial x^2}(x,y)\frac{\partial v_2}{\partial y}\,dxdy \equiv T_1+T_2,
\end{aligned} \tag{4.14}
$$

where

$$
\varphi(x) = \begin{cases} (x-x_{i-1/2})^2/2, & x_{i-1/2}\le x\le x_i, \\ (x-x_{i+1/2})^2/2, & x_i\le x\le x_{i+1/2}. \end{cases}
$$

For the first term we have

$$|T_1| \le Ch^2 \|p\|_{2,Q_{i,j}} \|\nabla \cdot \mathbf{v}\|_{0,Q_{i,j}}. \tag{4.15}$$

Integrating by parts in T_2 gives

$$\begin{aligned} T_2 = &\int_{Q_{i,j}} \varphi(x) \frac{\partial^3 p}{\partial x^2 \partial y}(x,y) v_2(x,y)\, dxdy \\ &- \left(\int_{e_{i,j,t}} - \int_{e_{i,j,b}} \right) \varphi(x) \frac{\partial^2 p}{\partial x^2}(x,y) v_2(x,y)\, dx \equiv T_{2,1} + T_{2,2}, \end{aligned} \tag{4.16}$$

where $e_{i,j,t}$ and $e_{i,j,b}$ are the top and the bottom edge of $Q_{i,j}$, respectively. For $T_{2,1}$ we have

$$|T_{2,1}| \le Ch^2 \|p\|_{3,Q_{i,j}} \|\mathbf{v}\|_{0,Q_{i,j}}. \tag{4.17}$$

For $T_{2,2}$ we notice that v_2 is continuous across horizontal edges and the assumed regularity of $p(x,y)$ implies that the trace of $\frac{\partial^2 p}{\partial x^2}$ is well defined. When summing over all elements, each edge integral will appear twice from the expressions for the two neighboring elements, with opposite signs. Therefore

$$\sum_{i,j} T_{2,2} = 0. \tag{4.18}$$

Combining (4.14)–(4.18) implies

$$\sum_{i,j} \left(\left(p, \frac{\partial v_1}{\partial x} \right)_{Q_{i,j},M_x} - \left(p, \frac{\partial v_1}{\partial x} \right)_{Q_{i,j}} \right) \le Ch^2 \|p\|_3 \|\mathbf{v}\|_{\mathbf{V}}.$$

The second error term in (4.13) can be bounded in a similar way. Note that for $d = 3$, a similar argument goes through with two terms analogous to T_2. ∎

Lemma 4.6. *For all $g \in W^2_\infty$ there exists a constant C independent of h such that*

$$|||g - P_h g||| \le Ch^2 \|g\|_{2,\infty}.$$

Proof. Let $Q \in \mathcal{T}_h$. The Taylor expansion with integral remainder about the midpoint (x_0, y_0) of Q gives for any $(x,y) \in Q$

$$g(x,y) = g(x_0,y_0) + (x - x_0)\frac{\partial g}{\partial x}(x_0,y_0) + (y - y_0)\frac{\partial g}{\partial y}(x_0,y_0) + R(x,y),$$

where $|R(x,y)| \le Ch^2 \|g\|_{2,\infty,Q}$. Integrating the above equation over Q and using that $\int_Q g = \int_Q P_h g$ gives

$$|Q|(P_h g(x_0,y_0) - g(x_0,y_0)) = \int_Q R(x,y)\, dxdy,$$

which implies

$$|P_h g(x_0,y_0) - g(x_0,y_0)| \le Ch^2 \|g\|_{2,\infty,Q}.$$

The statement of the lemma now follows from the definition (4.8) of $|||\cdot|||$. ∎

5. Pressure superconvergence analysis. In this section we employ a duality argument to derive superconvergence for the pressure at the cell centers. We will make use of the following continuity property of Π [23, 3]. For any $\varepsilon > 0$,

$$\|\Pi \mathbf{q}\|_0 \le C(\|\mathbf{q}\|_\varepsilon + \|\nabla \cdot \mathbf{q}\|_0). \tag{5.1}$$

Consider the auxiliary problem

$$\begin{aligned} -\nabla \cdot K\nabla\varphi &= P_h p - p_h \quad \text{in } \Omega, \\ -K\nabla\varphi \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{5.2}$$

which is well posed since $\int_\Omega P_h p = \int_\Omega p_h = 0$. Elliptic regularity [20] implies that there exists $\varepsilon > 0$ such that

$$\|\varphi\|_{1+\varepsilon} \le C\|P_h p - p_h\|_0. \tag{5.3}$$

Note that (5.3) holds for L-shaped domains. Let $\phi = -K\nabla\varphi$. We have

$$\begin{aligned} \|P_h p - p_h\|_0^2 &= (P_h p - p_h, \nabla \cdot \phi) = (P_h p - p_h, \nabla \cdot \Pi\phi) = c(\Pi\phi, P_h p - p_h) \\ &= -b(\gamma_h \Pi\phi, P_h p - p_h) = -b(\gamma_h \Pi\phi, P_h p - p) - b(\gamma_h \Pi\phi, p - p_h) \\ &= -b(\gamma_h \Pi\phi, P_h p - p) + a(\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi), \end{aligned} \tag{5.4}$$

using (4.1) with $\mathbf{v} = \gamma_h \Pi\phi$. By Lemma 4.5,

$$\begin{aligned} |b(\gamma_h \Pi\phi, P_h p - p)| &\le Ch^2 \|p\|_3 \|\Pi\phi\|_{\mathbf{V}} \\ &\le Ch^2 \|p\|_3 (\|\phi\|_\varepsilon + \|\nabla \cdot \phi\|_0) \le Ch^2 |||K|||_{\epsilon,\infty} \|p\|_3 \|P_h p - p_h\|_0 \end{aligned}$$

using (5.1), (5.3), and that $\|\nabla \cdot \Pi\phi\|_0 \le \|\nabla \cdot \phi\|_0$, which follows from $\nabla \cdot \Pi\phi = P_h \nabla \cdot \phi$. For the last term in (5.4) we write

$$\begin{aligned} |a(\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi)| &= |a(\mathbf{u} - \Pi\mathbf{u}, \gamma_h \Pi\phi) + a(\Pi\mathbf{u} - \mathbf{u}_h, \gamma_h \Pi\phi)| \\ &\le C(h^2 |||K^{-1}|||_{1,\infty} \|\mathbf{u}\|_2 \|\Pi\phi\|_0 + \|K^{-1}\|_{0,\infty} \|\Pi\mathbf{u} - \mathbf{u}_h\|_0 \|\gamma_h \Pi\phi\|_0) \\ &\le Ch^2 |||K^{-1}|||_{1,\infty} (\|\mathbf{u}\|_2 + \|p\|_3) \|\Pi\phi\|_0 \\ &\le Ch^2 |||K|||_{\epsilon,\infty} |||K^{-1}|||_{1,\infty} (\|\mathbf{u}\|_2 + \|p\|_3) \|P_h p - p_h\|_0 \end{aligned}$$

using Lemma 4.4, Theorem 4.1, (3.9), (5.1), (5.3), and (5.2). A combination of (5.4) and the above two bounds gives the following pressure superconvergence result.

THEOREM 5.1. *For the CVMFEM approximation* $(\mathbf{u}_h, p_h)$*, there exists a constant* C *independent of* h *such that*

$$\|P_h p - p_h\|_0 \le Ch^2 |||K|||_{\epsilon,\infty} |||K^{-1}|||_{1,\infty} (\|\mathbf{u}\|_2 + \|p\|_3).$$

It is now easy to obtain superconvergence for the pressure at the midpoints of the elements. Let $|||w|||_W = |||w|||$, where $|||w|||$ is defined in (4.8), and note that $|||w|||_W = \|w\|_0$ for all $w \in W_h$.

COROLLARY 5.2. *There exists a constant* C *independent of* h *such that*

$$|||p - p_h|||_W \le Ch^2 |||K|||_{\epsilon,\infty} |||K^{-1}|||_{1,\infty} (\|\mathbf{u}\|_2 + \|p\|_{2,\infty} + \|p\|_3).$$

Proof. The result follows immediately from the triangle inequality

$$|||p - p_h|||_W \le |||p - P_h p|||_W + |||P_h p - p_h|||_W,$$

Lemma 4.6, and Theorem 5.1. $\square$

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