SIAM J. NUMER. ANAL.
Vol. 42, No. 5, pp. 1959–1977

LOCALLY CONSERVATIVE COUPLING OF STOKES AND DARCY FLOWS*

BÉATRICE RIVIÈRE† AND IVAN YOTOV†

Abstract. A locally conservative numerical method for solving the coupled Stokes and Darcy flows problem is formulated and analyzed. The approach employs the mixed finite element method for the Darcy region and the discontinuous Galerkin method for the Stokes region. A discrete inf-sup condition and optimal error estimates are derived.

Key words. multiphysics, porous media flow, incompressible fluid flow, discontinuous Galerkin, mixed finite element, error estimates, inf-sup condition

AMS subject classifications. 35Q35, 65N30, 65N15, 76D07, 76S05

DOI. 10.1137/S0036142903427640

1. Introduction.

The numerical modeling of reactive transport necessitates the use of numerical schemes that do not create artificial mass [14]. Mixed finite element (MFE) and discontinuous Galerkin (DG) methods are two examples of locally mass conservative methods that are used in the geosciences. MFE methods are quite popular for porous media problems [16, 34, 17, 4] and DG methods are attractive for modeling flow on unstructured meshes [33, 31, 30, 32].

Many applications involve different physical processes in different parts of the simulation domain. In this paper we propose a numerical method for approximating the solution to the coupled Darcy–Stokes problem. Such systems arise, for example, in modeling the interaction between surface water (river) and groundwater (aquifer). There are few works in the literature that address the numerical analysis of the coupled Darcy–Stokes problem. In [25], Layton, Schieweck, and Yotov consider a formulation based on the Beavers–Joseph–Saffman interface conditions [5, 35, 24], prove the existence and uniqueness of a weak solution, and analyze a continuous finite element scheme coupled with MFE. A similar formulation is studied by Discacciati, Miglio, and Quarteroni [15], where continuous finite elements are used in both regions. An application of this formulation to vugular porous media is studied in [3]. A singularly perturbed Stokes problem, which models Darcy flow as a limiting case, is considered by Mardal, Tai, and Winther [27]. There, a new finite element is proposed which behaves uniformly in the perturbation parameter. Ewing, Iliev, and Lazarov [18] employ finite difference methods for a similar model involving the Navier–Stokes equations with an added Darcy term.

The model we consider, which is similar to the one in [25], is based on imposing the correct local equations in each region, coupled with appropriate interface conditions. In particular, the fluid region is modeled by the Stokes equations and the porous media region is modeled by the Darcy's law. Continuity of flux, balance of forces, and the Beavers–Joseph–Saffman slip with friction condition (see (2.10) below) are imposed on the interface. In this work we emphasize locally mass conservative discretizations. Conserving mass locally is especially important when the flow equations are coupled

*Received by the editors May 2, 2003; accepted for publication (in revised form) February 27, 2004; published electronically January 20, 2005. This research was partially supported by NSF grants DMS 0107389 and DMS 0112239.
http://www.siam.org/journals/sinum/42-5/42764.html
†Department of Mathematics, 301 Thackeray, University of Pittsburgh, Pittsburgh, PA 15260 (riviere@math.pitt.edu, yotov@math.pitt.edu).

with the reactive transport of chemical species. In the porous media region, the fluid velocity and pressure are obtained by MFE, and in the incompressible flow region, the fluid velocity and pressure are approximated by DG. An advantage of our approach is the possibility of coupling existing highly optimized MFE-based porous media simulators with the flexibility and easy implementation of DG methods for incompressible flows. The meshes at the interface between the two regions may be nonmatching. The estimates are derived for two-dimensional problems. The results are also valid in higher dimension, and depend on the existence of approximation operators (see Remark 4.4 below).

The outline of the paper is as follows. In section 2, the model problem, notation, and scheme are presented. Section 3 contains the derivation of the discrete inf-sup condition. In section 4, approximation results and optimal a priori error estimates are proved. Some concluding remarks follow.

2. Model problem, notation, and scheme. Let Ω be a domain in $\mathbb{R}^d$, $d = 2$, subdivided into two subdomains Ω_1, Ω_2. Let Γ_{12} be the interface $\partial\Omega_1 \cap \partial\Omega_2$. Define $\Gamma_i = \partial\Omega_i \backslash \Gamma_{12}$, $i = 1, 2$. Denote by $\boldsymbol{n}$ the outward normal vector to $\partial\Omega$. Let $\boldsymbol{n}_{12}$ (resp., $\boldsymbol{\tau}_{12}$) be the unit normal (resp., tangential) vector to Γ_{12} outward of Ω_1. Denote by $\boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$ the fluid velocity and by $p = (p_1, p_2)$ the fluid pressure, where $\boldsymbol{u}_i = \boldsymbol{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$. The flow in the domain Ω_1 is assumed to be of Stokes type, and therefore the following equations are satisfied:

$$-\nabla \cdot \boldsymbol{T}(\boldsymbol{u}_1, p_1) = \boldsymbol{f}_1 \quad \text{in } \Omega_1, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0 \quad \text{in } \Omega_1, \tag{2.2}$$

$$\boldsymbol{u}_1 = 0 \quad \text{on } \Gamma_1. \tag{2.3}$$

Here $\boldsymbol{T}$ is the stress tensor

$$\boldsymbol{T}(\boldsymbol{u}_1, p_1) = -p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)$$

which depends on the viscosity $\mu > 0$ and the strain tensor

$$\boldsymbol{D}(\boldsymbol{u}_1) = \frac{1}{2}(\nabla \boldsymbol{u}_1 + \nabla \boldsymbol{u}_1^T).$$

In the region Ω_2, the fluid pressure and velocity satisfy the single phase Darcy flow equations

$$\nabla \cdot \boldsymbol{u}_2 = f_2 \quad \text{in } \Omega_2, \tag{2.4}$$

$$\boldsymbol{u}_2 = -\boldsymbol{K} \nabla p_2 \quad \text{in } \Omega_2, \tag{2.5}$$

$$\boldsymbol{u}_2 \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_2, \tag{2.6}$$

where $\boldsymbol{K}$ is a symmetric and positive definite tensor representing the permeability divided by the viscosity and satisfying, for some $0 < \kappa_0 \leq \kappa_1 < \infty$,

$$\kappa_0 \xi^T \xi \leq \xi^T \boldsymbol{K}(x) \xi \leq \kappa_1 \xi^T \xi \quad \forall x \in \Omega_2, \ \forall \xi \in \mathbb{R}^d. \tag{2.7}$$

The physical quantities are coupled through appropriate interface conditions

$$\boldsymbol{u}_1 \cdot \boldsymbol{n}_{12} = \boldsymbol{u}_2 \cdot \boldsymbol{n}_{12}, \tag{2.8}$$

$$p_1 - 2\mu((\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}) = p_2, \tag{2.9}$$

$$\boldsymbol{u}_1 \cdot \boldsymbol{\tau}_{12} = -2G(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \tag{2.10}$$

Note that condition (2.8) represents the mass conservation across the interface, condition (2.9) imposes balance of forces across the interface, and condition (2.10) is the Beavers–Joseph–Saffman law, where $G > 0$ is a friction constant that can be determined experimentally. The reader should refer to [5, 35, 24, 25] for a detailed description and motivation for the choice of these interface conditions.

For $i = 1, 2$, let $\mathcal{E}_h^i$ be a nondegenerate quasi-uniform subdivision of Ω_i [11] such that the partition $\mathcal{E}_h^1$ consists of triangles and $\mathcal{E}_h^2$ consists of either triangles or rectangles. Let Γ_h^i be the set of interior edges and let h_i denote the maximum diameter of elements in $\mathcal{E}_h^i$. The meshes at the interface between the two domains Ω_i may not match. For $s \geq 0$, $p > 1$, and a domain $E \subset \mathbb{R}^d$, let $W^{s,p}(E)$ be the usual Sobolev spaces [1], let $H^s(E) = W^{s,2}(E)$ be equipped with the usual norm $\|\cdot\|_{s,E}$, and let $L_0^2(E)$ denote the space of L^2 functions with zero average. In the formulation for the Stokes region, we need that both the gradient of $\boldsymbol{u}_1$ and the pressure p_1 have a trace on line segments. For this, it suffices to define the following velocity-pressure spaces for the Stokes region:

$$\boldsymbol{X}^1 = \{\boldsymbol{v}_1 \in (L^2(\Omega_1))^d : \ \forall E \in \mathcal{E}_h^1, \ \boldsymbol{v}_1|_E \in (W^{2,4/3}(E))^d\},$$
$$M^1 = \{q_1 \in L^2(\Omega_1) : \ \forall E \in \mathcal{E}_h^1, \ q_1|_E \in W^{1,4/3}(E)\},$$

with norms

$$|||\boldsymbol{v}_1|||_{s,\Omega_1}^2 = \sum_{E \in \mathcal{E}_h^1} \|\boldsymbol{v}_1\|_{s,E}^2,$$
$$\|\boldsymbol{v}_1\|_{X^1}^2 = |||\nabla \boldsymbol{v}_1|||_{0,\Omega_1}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \|[\boldsymbol{v}_1]\|_{0,e}^2 + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \|\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{0,e}^2,$$
$$\|q_1\|_{M^1} = \|q_1\|_{0,\Omega_1}.$$

Here, the parameter $\sigma_e \geq 0$ takes a constant value over each edge e, and $|e|$ denotes the measure (or length) of e. Given a fixed normal vector $\boldsymbol{n}_e$ on each edge $e = \partial E_e^1 \cap \partial E_e^1$, directed from E_e^1 to E_e^2, the average and jump of functions in $\boldsymbol{X}^1$ and M^1 can be defined as

$$\{w\} = \frac{1}{2}(w|_{E_e^1}) + \frac{1}{2}(w|_{E_e^2}), \quad [w] = (w|_{E_e^1}) - (w|_{E_e^2}) \quad \forall e = \partial E_e^1 \cap \partial E_e^2,$$
$$\{w\} = w|_{E_e^1}, \quad [w] = w|_{E_e^1} \quad \forall e = \partial E_e^1 \cap \partial \Omega_1.$$

The velocity-pressure spaces for the Darcy region are

$$\boldsymbol{X}^2 = \left\{ \boldsymbol{v} \in H(\text{div}; \Omega_2) : \int_{\partial \Omega_2} \boldsymbol{v} \cdot \boldsymbol{n} w = 0 \ \forall w \in H_{0,\Gamma_{12}}^1(\Omega_2) \right\},$$
$$M^2 = L^2(\Omega_2),$$

where $H(\text{div}; \Omega_2)$ is the space of vectors in $(L^2(\Omega_2))^d$ whose divergence lies in $L^2(\Omega_2)$ and

$$H_{0,\Gamma_{12}}^1(\Omega_2) = \{w \in H^1(\Omega_2) : w = 0 \text{ on } \Gamma_{12}\}.$$

The norms associated with $(\boldsymbol{X}^2, M^2)$ are

$$\|\boldsymbol{v}_2\|_{X^2}^2 = \|\boldsymbol{v}_2\|_{0,\Omega_2}^2 + \|\nabla \cdot \boldsymbol{v}_2\|_{0,\Omega_2}^2, \qquad \|q_2\|_{M^2} = \|q_2\|_{0,\Omega_2}. \tag{2.11}$$

We can now define $\boldsymbol{X} = \boldsymbol{X}^1 \times \boldsymbol{X}^2$ and $M = (M^1 \times M^2) \cap L^2_0(\Omega)$, the spaces for the coupled formulation with the usual norms

$$\|\boldsymbol{v}\|^2_X = \|\boldsymbol{v}_1\|^2_{X^1} + \|\boldsymbol{v}_2\|^2_{X^2}, \qquad \|q\|^2_M = \|q_1\|^2_{M^1} + \|q_2\|^2_{M^2}. \tag{2.12}$$

In [25], it was shown that there exists a unique weak solution $(\boldsymbol{u}, p)$ of the coupled problem (2.1)–(2.10), with $\boldsymbol{u}_1 \in (H^1(\Omega_1))^d$, $\boldsymbol{u}_2 \in \boldsymbol{X}^2$, and $p \in M$. We will assume that the solution $(\boldsymbol{u}, p)$ is regular enough, so that it is a strong solution of (2.1)–(2.10). Next, we introduce the bilinear forms $a_1 : \boldsymbol{X}^1 \times \boldsymbol{X}^1 \to \mathbb{R}$ and $b_1 : \boldsymbol{X}^1 \times M^1 \to \mathbb{R}$,

$$\begin{aligned} a_1(\boldsymbol{u}_1, \boldsymbol{v}_1) = {} & 2\mu \sum_{E\in\mathcal{E}^1_h} \int_E \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \sum_{e\in\Gamma^1_h\cup\Gamma_1} \frac{\sigma_e}{|e|} \int_e [\boldsymbol{u}_1]\cdot[\boldsymbol{v}_1] \\ & - 2\mu \sum_{e\in\Gamma^1_h\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + 2\mu\epsilon \sum_{e\in\Gamma^1_h\cup\Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_1] \\ & + \frac{\mu}{G} \sum_{e\in\Gamma_{12}} \int_e \boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12} \boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}, \end{aligned} \tag{2.13}$$

$$b_1(\boldsymbol{v}_1, p_1) = -\sum_{E\in\mathcal{E}_h} \int_E p_1 \nabla\cdot\boldsymbol{v}_1 + \sum_{e\in\Gamma^1_h\cup\Gamma_1} \int_e \{p_1\}[\boldsymbol{v}_1]\cdot\boldsymbol{n}_e. \tag{2.14}$$

Here, ϵ is a constant that takes the value -1 or $+1$, which makes the bilinear form a_1 symmetric or nonsymmetric. The bilinear forms corresponding to the Darcy region are $a_2 : \boldsymbol{X}^2 \times \boldsymbol{X}^2 \to \mathbb{R}$ and $b_2 : \boldsymbol{X}^2 \times M^2 \to \mathbb{R}$:

$$a_2(\boldsymbol{u}_2, \boldsymbol{v}_2) = \int_{\Omega_2} \boldsymbol{K}^{-1}\boldsymbol{u}_2\cdot\boldsymbol{v}_2, \tag{2.15}$$

$$b_2(\boldsymbol{v}_2, q_2) = -\int_{\Omega_2} q_2 \nabla\cdot\boldsymbol{v}_2. \tag{2.16}$$

Let k_1, k_2, and l_2 be positive integers. Let $\boldsymbol{X}_h$ and M_h be finite-dimensional subspaces of $\boldsymbol{X}$ and M, respectively, such that

$$\boldsymbol{X}_h = \boldsymbol{X}^1_h \times \boldsymbol{X}^2_h, \qquad M_h = M^1_h \times M^2_h,$$

where $(\boldsymbol{X}^1_h, M^1_h)$ is the pair of discontinuous finite element spaces

$$\begin{aligned} \boldsymbol{X}^1_h &= \{\boldsymbol{v}_1 \in \boldsymbol{X}^1 : \ \forall E \in \mathcal{E}^1_h,\ \boldsymbol{v}_1 \in (\mathbb{P}_{k_1}(E))^d\}, \\ M^1_h &= \{q_1 \in M^1 : \ \forall E \in \mathcal{E}^1_h,\ q_1 \in \mathbb{P}_{k_1-1}(E)\}. \end{aligned}$$

The discrete spaces corresponding to the Darcy region consist of the standard mixed finite element spaces (such as RT spaces [29], BDM spaces [9], BDFM spaces [8], and BDDF spaces [7]). The mixed spaces $\boldsymbol{X}^2_h$ and M^2_h contain all polynomials of degree at least k_2 and l_2, respectively. Note that for the Raviart–Thomas (RT) spaces, the condition $l_2 = k_2$ holds. We also assume that

$$\forall \boldsymbol{v}_2 \in \boldsymbol{X}^2_h, \quad \boldsymbol{v}_2\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_2.$$

Let E be a mesh element with diameter h_E. Given $p \in L^2_0(\Omega)$, we denote by $\tilde{p}$ the L^2 projection of p in M_h satisfying

$$\forall q \in \mathbb{P}_{k_1-1}(E), \quad \int_E q(\tilde{p} - p) = 0 \quad \forall E \in \mathcal{E}^1_h, \tag{2.17}$$

$$\forall q \in \mathbb{P}_{l_2}(E), \quad \int_E q(\tilde{p} - p) = 0 \quad \forall E \in \mathcal{E}^2_h, \tag{2.18}$$

and, if $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ and $p|_{\Omega_2} \in H^{l_2+1}(\Omega_2)$, then

$$\|p - \tilde{p}\|_{m,E} \leq C h_E^{k_1-m} |p|_{k_1,E}, \quad E \subset \Omega_1,\ m = 0, 1, \tag{2.19}$$

$$\|p - \tilde{p}\|_{m,E} \leq C h_E^{l_2+1-m} |p|_{l_2+1,E}, \quad E \subset \Omega_2,\ m = 0, 1. \tag{2.20}$$

Remark 2.1. One advantage of the DG method is that one can vary the polynomial degrees from element to element. Here we assume that k_1 is the minimum of the polynomial degrees used in the Stokes region.

Here and throughout the paper, C denotes a varying constant that is independent of the diameter of the mesh elements. We also make use of the quasi-local interpolant $\mathbf{\Pi}_h^1 : (H^1(\Omega_1))^d \to \boldsymbol{X}_h^1$ [13, 19, 12, 22] satisfying, for all $\boldsymbol{v}_1 \in (H^1(\Omega_1))^d$,

$$b_1(\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1, q_1) = 0 \quad \forall q_1 \in M_h^1, \tag{2.21}$$

$$\forall e \in \Gamma_h^1 \cup \Gamma_1,\ \int_e [\mathbf{\Pi}_h^1 \boldsymbol{v}_1] \cdot q_1 = 0 \quad \forall \boldsymbol{v}_1 \in (H^1(\Omega_1))^d : \boldsymbol{v}_1 = 0 \text{ on } \Gamma_1,\ \forall q_1 \in M_h^1, \tag{2.22}$$

$$|||\mathbf{\Pi}_h^1 \boldsymbol{v}_1|||_{1,\Omega_1} \leq C \|\boldsymbol{v}_1\|_{1,\Omega_1}. \tag{2.23}$$

The operator $\mathbf{\Pi}_h^1$ has the optimal approximation properties

$$|\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1|_{m,E} \leq C h_E^{s-m} |\boldsymbol{v}_1|_{s,\delta(E)} \quad \forall 1 \leq s \leq k_1 + 1,\ \forall \boldsymbol{v}_1 \in H^s(\Omega_1),\ m = 0, 1, \tag{2.24}$$

where $\delta(E)$ is a suitable macro-element containing E. Moreover, it holds that for at least one edge e of every element $E \in \mathcal{E}_h^1$,

$$\int_e (\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1) = 0 \quad \forall \boldsymbol{v}_1 \in (H^1(\Omega_1))^d. \tag{2.25}$$

We note that (2.25) holds true for all edges in the cases $k = 1$ and $k = 2$. For $k = 3$, we can assume, without loss of generality, that (2.25) is satisfied for all edges in Γ_{12}. We will make use of the following bounds on $\mathbf{\Pi}_h^1$.

LEMMA 2.2. *Let $1 \leq s \leq k_1 + 1$. For all $\boldsymbol{v}_1 \in (H^s(\Omega_1))^d$,*

$$\|\mathbf{\Pi}_h^1 \boldsymbol{v}_1 - \boldsymbol{v}_1\|_{X^1} \leq C h_1^{s-1} |\boldsymbol{v}_1|_{s,\Omega_1}, \tag{2.26}$$

$$\|\mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{X^1} \leq C \|\boldsymbol{v}_1\|_{1,\Omega_1}. \tag{2.27}$$

Proof. From Lemma 3.10 of [22] and from (2.24), we have

$$\|\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1\|_{X^1} \leq C |||\nabla(\boldsymbol{v}_1 - \mathbf{\Pi}_h^1 \boldsymbol{v}_1)|||_{0,\Omega_1} \leq C h_1^{s-1} |\boldsymbol{v}_1|_{s,\Omega_1}. \tag{2.28}$$

The bound (2.27) follows easily from the triangle inequality and (2.26) with $s = 1$, using that $\|\boldsymbol{v}_1\|_{X^1} \leq C \|\boldsymbol{v}_1\|_{1,\Omega_1}$ for $\boldsymbol{v}_1 \in (H^1(\Omega_1))^d$. ☐

We also recall the MFE interpolant $\mathbf{\Pi}_h^2 : \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d \to \boldsymbol{X}_h^2$ for any $\theta > 0$, satisfying [10], for any $\boldsymbol{v}_2 \in \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d$,

$$b_2(\mathbf{\Pi}_h^2 \boldsymbol{v}_2 - \boldsymbol{v}_2, q_2) = 0 \quad \forall q_2 \in M_h^2, \tag{2.29}$$

$$\int_e ((\mathbf{\Pi}_h^2 \boldsymbol{v}_2 - \boldsymbol{v}_2) \cdot \boldsymbol{n}_e) \boldsymbol{w}_2 \cdot \boldsymbol{n}_e = 0 \quad \forall e \in \Gamma_h^2,\ \forall \boldsymbol{w}_2 \in \boldsymbol{X}_h^2. \tag{2.30}$$

Moreover, $\mathbf{\Pi}_h^2$ satisfies the approximation properties

$$\|\boldsymbol{v}_2 - \mathbf{\Pi}_h^2 \boldsymbol{v}_2\|_{0,E} \leq C h_E^s |\boldsymbol{v}_2|_{s,E}, \quad 1 \leq s \leq k_2 + 1, \tag{2.31}$$

$$\|\nabla \cdot (\boldsymbol{v}_2 - \mathbf{\Pi}_h^2 \boldsymbol{v}_2)\|_{0,E} \leq C h_E^s |\nabla \cdot \boldsymbol{v}_2|_{s,E}, \quad 0 \leq s \leq l_2 + 1. \tag{2.32}$$

It has been shown by Mathew in [28] for the Raviart–Thomas elements [29] that

$$\|\mathbf{\Pi}_h^2 \boldsymbol{v}_2\|_{H(\mathrm{div};\Omega_2)} \leq C(\|\boldsymbol{v}_2\|_{\theta,\Omega_2} + \|\nabla\cdot\boldsymbol{v}_2\|_{0,\Omega_2}), \tag{2.33}$$

a result that can be trivially extended to the other families of MFE spaces. Recall the basic trace inequalities on any mesh element E with diameter h_E

$$\forall\phi\in H^1(E),\ \forall e\subset\partial E,\quad \|\phi\|_{0,e}^2 \leq C(h_E^{-1}\|\phi\|_{0,E}^2 + h_E|\phi|_{1,E}^2), \tag{2.34}$$

$$\forall\phi\in H^2(E),\ \forall e\subset\partial E,\quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e}^2 \leq C(h_E^{-1}\|\phi\|_{1,E}^2 + h_E|\phi|_{2,E}^2), \tag{2.35}$$

$$\forall\phi\in\mathbb{P}_k(E),\ \forall e\subset\partial E,\quad \|\nabla\phi\cdot\boldsymbol{n}\|_{0,e} \leq Ch_E^{-1/2}|\phi|_{1,E}, \tag{2.36}$$

Recall also the Korn's inequality proved in [6]

$$\forall\boldsymbol{v}\in\boldsymbol{X}_h^1,\quad C|||\nabla\boldsymbol{v}|||_{0,\Omega_1}^2 \leq |||\boldsymbol{D}(\boldsymbol{v})|||_{0,\Omega_1}^2 + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{1}{|e|}\|[\boldsymbol{v}]\|_{0,e}^2. \tag{2.37}$$

Define the finite-dimensional space of functions on the interface $\Lambda_h = X_h^2\cdot\boldsymbol{n}_{12}$ and let

$$\boldsymbol{V}_h = \left\{\boldsymbol{v} = (\boldsymbol{v}_1,\boldsymbol{v}_2)\in\boldsymbol{X}_h : \sum_{e\in\Gamma_{12}}\int_e \eta(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{n}_{12} = 0\ \forall\eta\in\Lambda_h\right\}.$$

Defining $a = a_1 + a_2$ and $b = b_1 + b_2$, the numerical scheme is, Find $(\boldsymbol{U},P)\in\boldsymbol{V}_h\times M_h$ such that

$$a(\boldsymbol{U},\boldsymbol{v}) + b(\boldsymbol{v},P) = \int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}\quad\forall\boldsymbol{v}\in\boldsymbol{V}_h, \tag{2.38}$$

$$b(\boldsymbol{U},q) = \int_{\Omega_2} f_2 q\quad\forall q\in M_h. \tag{2.39}$$

Remark 2.3. This scheme is locally mass conservative. Indeed, if one chooses the test function in (2.39) such that $q = 1$ on E and $q = 0$ on the rest of the domain, we have

$$\int_{\partial E}\{\boldsymbol{U}\}\cdot\boldsymbol{n}_E = 0\quad\forall E\subset\mathcal{E}_h^1,$$
$$\int_{\partial E}\boldsymbol{U}\cdot\boldsymbol{n}_E = \int_E f_2\quad\forall E\subset\mathcal{E}_h^2.$$

Remark 2.4. The space of weakly-continuous-normal velocities $\boldsymbol{V}_h$ is introduced to facilitate the analysis of the numerical method. A direct construction of this space may, however, be difficult. An equivalent formulation to (2.38)–(2.39) is given in section 5. It is only based on the space $\boldsymbol{X}_h$ and is more suitable for implementation. The space Λ_h plays the role of a Lagrange multiplier or mortar space for imposing continuity of the normal velocities on Γ_{12}. The choice $\Lambda_h = X_h^2\cdot\boldsymbol{n}_{12}$ is critical for the stability and accuracy of the numerical scheme, even in the case of nonmatching grids across Γ_{12}. This choice differs from the mortar space used in [2] in the case of MFE discretizations on nonmatching grids.

In the rest of the section, we show that the solution of the coupled problem satisfies the scheme up to an interface consistency error. We also prove uniqueness and existence of the discrete solution.

LEMMA 2.5. *If* $(\boldsymbol{u}, p) \in \boldsymbol{X} \times M$ *solves the coupled Stokes–Darcy flow problem* (2.1)–(2.10)*, such that* $\boldsymbol{u}_i = \boldsymbol{u}|_{\Omega_i}$ *and* $p_i = p|_{\Omega_i}$*, then* $(\boldsymbol{u}, p)$ *satisfies the variational problem*

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 - \sum_{e \in \Gamma_{12}} \int_e p_2(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{n}_{12} \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.40}$$

$$b(\boldsymbol{u}, q) = \int_{\Omega_2} f_2 q \quad \forall q \in M_h. \tag{2.41}$$

Proof. Multiplying the Stokes equation (2.1) by $\boldsymbol{v}_1 \in \boldsymbol{X}_h^1$ and integrating by parts over one element E,

$$\int_E T(\boldsymbol{u}_1, p_1) : \nabla \boldsymbol{v}_1 - \int_{\partial E} T(\boldsymbol{u}_1, p_1)\boldsymbol{n}_E \cdot \boldsymbol{v}_1 = \int_E \boldsymbol{f}_1 \cdot \boldsymbol{v}_1.$$

Summing over all elements E,

$$\begin{aligned} &\sum_E \int_E (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)) : \nabla \boldsymbol{v}_1 - \sum_{e \in \Gamma_h^1} \int_e [(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))]\boldsymbol{n}_e \cdot \boldsymbol{v}_1 \\ &\quad - \int_{\Gamma_{12}} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 - \int_{\Gamma_1} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n} \cdot \boldsymbol{v}_1 = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1. \end{aligned}$$

It is easy to show that $\boldsymbol{D}(\boldsymbol{u}_1) : \nabla \boldsymbol{v}_1 = \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1)$ and that $\boldsymbol{I} : \nabla \boldsymbol{v}_1 = \nabla \cdot \boldsymbol{v}_1$. Thus, the equation becomes

$$\begin{aligned} &\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla \cdot \boldsymbol{v}_1) \\ &\quad - \sum_{e \in \Gamma_h^1} \int_e \{-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] - \sum_{e \in \Gamma_h^1} \int_e [-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)]\boldsymbol{n}_e \cdot \{\boldsymbol{v}_1\} \\ &\quad - \int_{\Gamma 12} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 - \int_{\Gamma_1} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n} \cdot \boldsymbol{v}_1 = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1. \end{aligned}$$

By regularity of the true solution, we have

$$\begin{aligned} &\sum_E \int_E (2\mu \boldsymbol{D}(\boldsymbol{u}_1) : \boldsymbol{D}(\boldsymbol{v}_1) - p_1 \nabla \cdot \boldsymbol{v}_1) - \int_{\Gamma_{12}} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 \\ &\quad - \sum_{e \in \Gamma_h^1} \int_e \{-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + \epsilon \sum_{e \in \Gamma_h^1} \int_e \{2\mu \boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{u}_1] \\ &\quad - \int_{\Gamma_1} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n} \cdot \boldsymbol{v}_1 + \epsilon \int_{\Gamma_1} 2\mu \boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n} \cdot \boldsymbol{u}_1 = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1. \end{aligned}$$

Let us now consider the interface term

$$(-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} = -p_1 \boldsymbol{n}_{12} + (2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12} + (2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12},$$

which, combined with $\boldsymbol{v}_1 = (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12} + (\boldsymbol{v}_1 \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12}$, gives

$$\begin{aligned} (-p_1 \boldsymbol{I} + 2\mu \boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12} \cdot \boldsymbol{v}_1 = &-p_1(\boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}) + 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}(\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12}) \\ &+ 2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}(\boldsymbol{v}_1 \cdot \boldsymbol{n}_{12}). \end{aligned}$$

Thus,

$$-\int_{\Gamma_{12}}(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1=-\int_{\Gamma_{12}}(-p_1+2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})$$
$$-\int_{\Gamma_{12}}2\mu(\boldsymbol{D}(\boldsymbol{u}_1)\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12}(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}).$$

With the interface conditions (2.9) and (2.10), we obtain

$$-\int_{\Gamma_{12}}(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_{12}\cdot\boldsymbol{v}_1=\int_{\Gamma_{12}}p_2(\boldsymbol{v}_1\cdot\boldsymbol{n}_{12})+\frac{\mu}{G}\int_{\Gamma_{12}}(\boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12})(\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}).$$

Thus,

$$\sum_E\int_E(2\mu\boldsymbol{D}(\boldsymbol{u}_1):\boldsymbol{D}(\boldsymbol{v}_1)-p_1\nabla\cdot\boldsymbol{v}_1)$$
$$-\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{(-p_1\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{u}_1))\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_1]+\epsilon\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{2\mu\boldsymbol{D}(\boldsymbol{v}_1)\boldsymbol{n}_e\}\cdot[\boldsymbol{u}_1]$$
$$+\int_{\Gamma_{12}}p_2\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}+\frac{\mu}{G}\int_{\Gamma_{12}}\boldsymbol{u}_1\cdot\boldsymbol{\tau}_{12}\boldsymbol{v}_1\cdot\boldsymbol{\tau}_{12}=\int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}_1$$

which is equivalent to

$$a_1(\boldsymbol{u}_1,\boldsymbol{v}_1)+b_1(\boldsymbol{v}_1,p_1)+\int_{\Gamma_{12}}p_2\boldsymbol{v}_1\cdot\boldsymbol{n}_{12}=\int_{\Omega_1}\boldsymbol{f}_1\cdot\boldsymbol{v}_1\quad\forall\boldsymbol{v}_1\in\boldsymbol{X}_h^1.\tag{2.42}$$

The Darcy's law (2.5) can be rewritten as $\boldsymbol{K}^{-1}\boldsymbol{u}_2=-\nabla p_2$. As usual, multiplication by $\boldsymbol{v}_2\in\boldsymbol{X}_h^2$ and integration by parts on the Darcy region yields

$$\int_{\Omega_2}\boldsymbol{K}^{-1}\boldsymbol{u}_2\cdot\boldsymbol{v}_2=-\int_{\Omega_2}\nabla p_2\cdot\boldsymbol{v}_2=\int_{\Omega_2}p_2\nabla\cdot\boldsymbol{v}_2-\int_{\partial\Omega_2}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}$$
$$=\int_{\Omega_2}p_2\nabla\cdot\boldsymbol{v}_2-\int_{\Gamma_2}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}+\int_{\Gamma_{12}}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}_{12},$$

or equivalently,

$$a_2(\boldsymbol{u}_2,\boldsymbol{v}_2)+b_2(\boldsymbol{v}_2,p_2)-\int_{\Gamma_{12}}p_2\boldsymbol{v}_2\cdot\boldsymbol{n}_{12}=0\quad\forall\boldsymbol{v}_2\in\boldsymbol{X}_h^2.\tag{2.43}$$

Adding (2.42) and (2.43) yields (2.40). Clearly, (2.2) and the regularity of the solution gives

$$b_1(\boldsymbol{u}_1,q)=0\quad\forall q\in M_h^1.$$

Finally, a simple integration in (2.4) yields

$$b_2(\boldsymbol{u}_2,q)=\int_{\Omega_2}f_2q\quad\forall q\in M_h^2,$$

and adding to the previous equation gives the result. ∎

Next, we prove a coercivity lemma that holds true under the following condition.

Hypothesis A. In the definition of the bilinear form $a_1(\cdot,\cdot)$, let us assume that either the condition (a) or (b) holds true.

(a) $\epsilon = 1$ and $\sigma_e > 1$ for all edges in $\Gamma_h^1 \cup \Gamma_1$. For instance, one may choose $\sigma_e = 2$.

(b) $\epsilon = -1$ and $\sigma_e \geq \sigma_0 > 0$ for σ_0 large enough.

LEMMA 2.6. *Assuming Hypothesis A, there exists a positive constant C_0 such that*

$$C_0 \|\boldsymbol{v}\|_X^2 \leq a(\boldsymbol{v}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{X}_h : \nabla \cdot \boldsymbol{v} = 0 \textit{ a.e. in } \Omega_2.$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{X}_h$. Then $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2)$ with $\boldsymbol{v}_i \in \boldsymbol{X}_h^i$, $i = 1, 2$. Using (2.13) and (2.15),

$$a(\boldsymbol{v}, \boldsymbol{v}) = 2\mu \sum_{E \in \mathcal{E}_h^1} \int_E \boldsymbol{D}(\boldsymbol{v}_1) : \boldsymbol{D}(\boldsymbol{v}_1) + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e}{|e|} \int_e [\boldsymbol{v}_1]^2$$
$$- 2(1-\epsilon)\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})^2 + \int_{\Omega_2} \boldsymbol{K}^{-1} \boldsymbol{v}_2 \cdot \boldsymbol{v}_2.$$

Using Korn's inequality (2.37) and the bound on $\boldsymbol{K}$ (2.7) gives

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq C\mu |||\nabla \boldsymbol{v}|||_{0,\Omega_1}^2 + C \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{\sigma_e - 1}{|e|} \int_e [\boldsymbol{v}_1]^2$$
$$- 2(1-\epsilon)\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] + \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})^2 + \frac{1}{\kappa_1} \|\boldsymbol{v}_2\|_{0,\Omega_2}^2.$$

If $\epsilon = 1$, then the result is straightforward. If $\epsilon = -1$, we have from trace inequality (2.36)

$$2(1-\epsilon)\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e \{\boldsymbol{D}(\boldsymbol{v}_1)\}\boldsymbol{n}_e \cdot [\boldsymbol{v}_1] \leq 4\mu \sum_{e \in \Gamma_h^1 \cup \Gamma_1} h_1^{-1/2} \|\nabla \boldsymbol{v}_1\|_{0,E_e} \left(\frac{|e|}{|e|}\right)^{1/2} \|[\boldsymbol{v}_1]\|_{0,e}$$
$$\leq \frac{C}{2} \mu |||\nabla \boldsymbol{v}_1|||_{0,\Omega_1}^2 + \tilde{C} \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{1}{|e|} \int_e [\boldsymbol{v}_1]^2.$$

Thus, we obtain if $\epsilon = -1$,

$$a(\boldsymbol{v}_1, \boldsymbol{v}_1) \geq \frac{3}{4} \mu |||\nabla \boldsymbol{v}_1|||_{0,\Omega_1}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \frac{C(\sigma_e - 1) - \tilde{C}}{|e|} \int_e [\boldsymbol{v}_1]^2$$
$$+ \frac{\mu}{G} \sum_{e \in \Gamma_{12}} \int_e (\boldsymbol{v}_1 \cdot \boldsymbol{\tau}_{12})^2 + \frac{1}{\kappa_1} \|\boldsymbol{v}_2\|_{0,\Omega_2}^2 \geq C_0 (\|\boldsymbol{v}_1\|_{X^1}^2 + \|\boldsymbol{v}_2\|_{0,\Omega_2}^2)$$

with C_0 positive constant, assuming that σ_e is large enough:

$$(C(\sigma_e - 1) - \tilde{C} \geq C_0 > 0). \qquad \square$$

We are now ready to prove that the discrete scheme (2.38)–(2.39) is solvable.

LEMMA 2.7. *If Hypothesis A holds, then there exists a unique solution to the problem* (2.38)–(2.39).

Proof. Since the problem (2.38)–(2.39) is finite dimensional, it suffices to show that the solution is unique. Set $f_i = 0$ and choose $\boldsymbol{v} = \boldsymbol{U}$ and $q = P$. Then

$$a(\boldsymbol{U}, \boldsymbol{U}) = 0.$$

In addition,

$$b(\boldsymbol{U}, q) = 0 \quad \forall q \in M_h,$$

which implies that $\nabla \cdot \boldsymbol{U} = 0$ in Ω_2, since $\nabla \cdot \boldsymbol{X}_h^2 = M_h^2$. Therefore Lemma 2.6 directly implies that $\boldsymbol{U} = 0$. Thus, the pressure satisfies

$$b(\boldsymbol{v}, P) = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h.$$

The inf-sup condition (3.1) proved below implies that $P = 0$. ∎

3. A discrete inf-sup condition. In this section, a discrete inf-sup condition is proved.

THEOREM 3.1. *There exists a positive constant β such that*

$$\inf_{q_h \in M_h} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_X \|q_h\|_M} \geq \beta. \tag{3.1}$$

Proof. Let $q_h \in M_h$ be given. Then there exists [20, 21] $\boldsymbol{v} \in (H^1(\Omega))^d$ such that

$$\nabla \cdot \boldsymbol{v} = -q_h \quad \text{in } \Omega, \qquad \boldsymbol{v} = 0 \quad \text{on } \partial\Omega,$$

satisfying

$$\|\boldsymbol{v}\|_{1,\Omega} \leq C \|q_h\|_{0,\Omega}.$$

Note that

$$b(v, q_h) = -\int_\Omega (\nabla \cdot \boldsymbol{v}) q_h = \|q_h\|_M^2,$$

which, together with the above a priori bound, implies

$$b(\boldsymbol{v}, q_h) \geq \frac{1}{C} \|\boldsymbol{v}\|_{1,\Omega} \|q_h\|_M.$$

Next, we need to construct an operator $\pi_h : \boldsymbol{X}^1 \times (\boldsymbol{X}^2 \cap (H^1(\Omega_2))^d) \to \boldsymbol{V}_h$ satisfying

$$b(\pi_h \boldsymbol{v} - \boldsymbol{v}, q_h) = 0 \quad \forall q_h \in M_h, \qquad \text{and} \qquad \|\pi_h \boldsymbol{v}\|_X \leq C \|\boldsymbol{v}\|_{1,\Omega}. \tag{3.2}$$

Let $\pi_h \boldsymbol{v} = (\pi_h^1 \boldsymbol{v}, \pi_h^2 \boldsymbol{v}) \in \boldsymbol{X}_h^1 \times \boldsymbol{X}_h^2$. We take $\pi_h^1 \boldsymbol{v} = \boldsymbol{\Pi}_h^1 \boldsymbol{v}_1$ where $\boldsymbol{\Pi}_h^1 : \boldsymbol{X}^1 \to \boldsymbol{X}_h^1$ is the quasi-local interpolant defined in (2.21). Clearly, due to (2.27),

$$\|\pi_h^1 \boldsymbol{v}\|_{X^1} \leq C \|\boldsymbol{v}\|_{1,\Omega_1}. \tag{3.3}$$

To define $\pi_h^2 \boldsymbol{v}$, consider the auxiliary problem

$$\nabla \cdot \nabla \varphi = 0 \quad \text{in } \Omega_2, \tag{3.4}$$

$$\nabla \varphi \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_2, \tag{3.5}$$

$$\nabla \varphi \cdot \boldsymbol{n}_{12} = (\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12} \quad \text{on } \Gamma_{12}. \tag{3.6}$$

The problem is well posed, since

$$\int_{\Gamma_{12}} (\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12} = 0,$$

due to (2.25). Let $\boldsymbol{z} = \nabla\varphi$. We note that the piecewise smooth function $\pi_h^1\boldsymbol{v}\cdot\boldsymbol{n}_{12} \in H^\theta(\Gamma_{12})$ for any $0 < \theta < 1/2$. By elliptic regularity [26],

$$\|\boldsymbol{z}\|_{\theta,\Omega_2} \le C\|(\pi_h^1\boldsymbol{v}-\boldsymbol{v})\cdot\boldsymbol{n}_{12}\|_{\theta-1/2,\Gamma_{12}}, \quad 0 \le \theta \le 1/2. \tag{3.7}$$

Let $\boldsymbol{w} = \boldsymbol{v}+\boldsymbol{z}$. Clearly $\nabla\cdot\boldsymbol{w} = \nabla\cdot\boldsymbol{v}$ in Ω_2 and $\boldsymbol{w}\cdot\boldsymbol{n}_{12} = \pi_h^1\boldsymbol{v}\cdot\boldsymbol{n}_{12}$ on Γ_{12}. We now define $\pi_h^2\boldsymbol{v} := \boldsymbol{\Pi}_h^2\boldsymbol{w}$, where $\boldsymbol{\Pi}_h^2 : \boldsymbol{X}^2 \cap (H^\theta(\Omega_2))^d \to \boldsymbol{X}_h^2$ is the MFE interpolant defined in (2.29). Note that, using (2.29),

$$\begin{aligned} b_2(\pi_h^2\boldsymbol{v}, q_h) &= b_2(\boldsymbol{\Pi}_h^2\boldsymbol{w}, q_h) = b_2(\boldsymbol{w}, q_h)\\ &= -\int_{\Omega_2}(\nabla\cdot\boldsymbol{w})q_h = -\int_{\Omega_2}(\nabla\cdot\boldsymbol{v})q_h = b_2(\boldsymbol{v}, q_h) \quad \forall q_h \in M_h^2, \end{aligned}$$

thus the so-constructed $\pi_h\boldsymbol{v} = (\pi_h^1\boldsymbol{v}, \pi_h^2\boldsymbol{v})$ satisfies

$$b(\pi_h\boldsymbol{v}-\boldsymbol{v}, q_h) = 0 \quad \forall q_h \in M_h.$$

It is easy to see that $\pi_h\boldsymbol{v} \in \boldsymbol{V}_h$. Indeed, for every $e \in \Gamma_h^{12}$ and $\eta \in \Lambda_h$, using (2.30) and the fact that $\Lambda_h = X_h^2\cdot\boldsymbol{n}_{12}$,

$$\int_e \pi_h^2\boldsymbol{v}\cdot\boldsymbol{n}_{12}\eta = \int_e \boldsymbol{\Pi}_h^2\boldsymbol{w}\cdot\boldsymbol{n}_{12}\eta = \int_e \boldsymbol{w}\cdot\boldsymbol{n}_{12}\eta = \int_e \pi_h^1\boldsymbol{v}\cdot\boldsymbol{n}_{12}\eta.$$

It remains to show the bound in (3.2). Using (2.31), (2.32), and (3.7),

$$\begin{aligned} \|\pi_h^2\boldsymbol{v}\|_{X^2} &= \|\boldsymbol{\Pi}_h^2\boldsymbol{w}\|_{X^2}\\ &\le \|\boldsymbol{\Pi}_h^2\boldsymbol{v}\|_{X^2} + \|\boldsymbol{\Pi}_h^2\boldsymbol{z}\|_{X^2}\\ &\le C(\|\boldsymbol{v}\|_{1,\Omega_2} + \|\boldsymbol{z}\|_{\theta,\Omega_2})\\ &\le C(\|\boldsymbol{v}\|_{1,\Omega_1} + \|(\pi_h^1\boldsymbol{v}-\boldsymbol{v})\cdot\boldsymbol{n}\|_{\Gamma_{12}}) \end{aligned}$$

The last term can be bounded as follows. For every $e \in \Gamma_{12}$, and edge (face) of $E \in \mathcal{E}_h^1$, using (2.34) and (2.24),

$$\|(\pi_h^1\boldsymbol{v}-\boldsymbol{v})\cdot\boldsymbol{n}_{12}\|_e \le C(h_E^{-1/2}\|\pi_h^1\boldsymbol{v}-\boldsymbol{v}\|_{0,E} + h_E^{1/2}|\pi_h^1\boldsymbol{v}-\boldsymbol{v}|_{1,E}) \le Ch_E^{1/2}|\boldsymbol{v}|_{1,\delta(E)}. \tag{3.8}$$

Therefore

$$\|\pi_h^2\boldsymbol{v}\|_{X^2} \le C\|\boldsymbol{v}\|_{1,\Omega},$$

which, combined with (3.3), implies the bound in (3.2). Now using (3.2),

$$\frac{1}{C}\|q_h\|_M \le \frac{b(\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{1,\Omega}} = \frac{b(\pi_h\boldsymbol{v}, q_h)}{\|\boldsymbol{v}\|_{1,\Omega}} \le \frac{b(\pi_h\boldsymbol{v}, q_h)}{\frac{1}{C}\|\pi_h\boldsymbol{v}\|_X} \quad \forall q_h \in M_h,$$

which proves (3.1). ∎

4. A priori error estimates. In this section, optimal error estimates in the energy norm are obtained for the velocity field. Also, optimal error estimates in the L^2 norm of the error for the pressure are obtained. We start with an approximation result for the weakly normal-continuous velocity space $\boldsymbol{V}_h$.

LEMMA 4.1. *For* $\boldsymbol{v} \in (H^1(\Omega))^d$ *such that* $\boldsymbol{v}|_{\Omega_1} \in (H^{k_1+1}(\Omega_1))^d$, $\boldsymbol{v}|_{\Omega_2} \in (H^{k_2+1}(\Omega_2))^d$, *and* $\nabla \cdot \boldsymbol{v}|_{\Omega_2} \in (H^{l_2+1}(\Omega_2))^d$, *there exists* $\tilde{\boldsymbol{v}} \in \boldsymbol{V}_h$ *such that*

$$b(\boldsymbol{v} - \tilde{\boldsymbol{v}}, q) = 0 \quad \forall q \in M_h, \tag{4.1}$$

$$\forall e \in \Gamma_h^1 \cup \Gamma_1, \ \int_e [\tilde{\boldsymbol{v}}] \cdot \boldsymbol{q} = 0 \quad \forall \boldsymbol{q} \in (\mathbb{P}_{k_1-1}(e))^d, \tag{4.2}$$

$$\|\boldsymbol{v} - \tilde{\boldsymbol{v}}\|_X \le C\{h_1^{k_1}|\boldsymbol{v}|_{k_1+1,\Omega_1} + h_2^{k_2+1}|\boldsymbol{v}|_{k_2+1,\Omega_2} + h_2^{l_2+1}|\nabla \cdot \boldsymbol{v}|_{l_2+1,\Omega_2}\}. \tag{4.3}$$

Proof. We will show that the interpolant $\pi_h \boldsymbol{v}$ constructed in Theorem 3.1 satisfies the above conditions. Indeed, (4.1) and (4.2) follow directly from the construction of $\pi_h \boldsymbol{v}$. To show (4.3), we first note that (2.26) implies that

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{X^1} \le C h_1^{k_1}|\boldsymbol{v}|_{k_1+1,\Omega_1}. \tag{4.4}$$

Next,

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{X^2} = \|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{w}\|_{X^2} \le \|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{v}\|_{X^2} + \|\boldsymbol{\Pi}_h^2(\boldsymbol{w} - \boldsymbol{v})\|_{X^2}. \tag{4.5}$$

For the first term on the right in (4.5), using (2.31) and (2.32),

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h^2 \boldsymbol{v}\|_{X^2} \le C h_2^{k_2+1}|\boldsymbol{v}|_{k_2+1,\Omega_2} + h_2^{l_2+1}|\nabla \cdot \boldsymbol{v}|_{l_2+1,\Omega_2}. \tag{4.6}$$

The last term in (4.5) can be bounded as follows, using (2.33), (3.7), (3.8), and (2.24):

$$\begin{aligned} \|\boldsymbol{\Pi}_h^2(\boldsymbol{w} - \boldsymbol{v})\|_{X^2} &= \|\boldsymbol{\Pi}_h^2 \boldsymbol{z}\|_{X^2} \le \|\boldsymbol{z}\|_{\theta,\Omega_2} \\ &\le C\|(\pi_h^1 \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_{12}\|_{0,\Gamma_{12}} \le C h_1^{k_1+1/2}|\boldsymbol{v}|_{k_1+1,\Omega_1}. \end{aligned} \tag{4.7}$$

A combination of (4.4)–(4.7) completes the proof. □

THEOREM 4.2. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{X} \times M$ *be the solution of the coupled problem* (2.1)–(2.10). *Assume that* $\boldsymbol{u}|_{\Omega_i} \in H^{k_i+1}(\Omega_i)$ *for* $i = 1, 2$. *Assume that* $p|_{\Omega_1} \in H^{k_1}(\Omega_1)$ *and that* $p|_{\Omega_2} \in H^{l_2+1}(\Omega_2)$. *Assume that Hypothesis A holds. Let* $(\boldsymbol{U}, P)$ *be the discrete solution of* (2.38)–(2.39) *Then, the following estimate holds:*

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{U}\|_X \le{}& C h_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) + C h_2^{k_2+1}|\boldsymbol{u}|_{k_2+1,\Omega_2} \\ &+ C(h_2^{l_2+1} + h_2^{l_2+1/2} h_1^{1/2})|p|_{l_2+1,\Omega_2}. \end{aligned}$$

Proof. Let $\tilde{\boldsymbol{u}}$ be the interpolant of $\boldsymbol{u}$ defined in Lemma 4.1 and let $\tilde{p}$ be the interpolant of p, satisfying (2.17)–(2.20). From (2.40), (2.41), and (2.38)–(2.39), the error equation is

$$\begin{aligned} a(\boldsymbol{U} - \tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, P - \tilde{p}) ={}& a(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{v}) + b(\boldsymbol{v}, p - \tilde{p}) \\ &- \sum_{e \in \Gamma_{12}} \int_e p_2(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{n}_{12} \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \end{aligned} \tag{4.8}$$

$$b(\boldsymbol{U} - \tilde{\boldsymbol{u}}, q) = b(\boldsymbol{u} - \tilde{\boldsymbol{u}}, q) \quad \forall q \in M_h. \tag{4.9}$$

Note that (4.1) implies that $b(\boldsymbol{U}-\tilde{\boldsymbol{u}},q)=0$ for all $q\in M_h$, which implies that

$$\nabla\cdot(\boldsymbol{U}-\tilde{\boldsymbol{u}})=0 \quad \text{in } \Omega_2,$$

since $\nabla\cdot\boldsymbol{X}_h^2=M_h^2$. Define $\boldsymbol{\chi}=\boldsymbol{U}-\tilde{\boldsymbol{u}}$ and $\xi=P-\tilde{p}$. Choose $\boldsymbol{v}=\boldsymbol{\chi}$ and $q=\xi$. Then,

$$a(\boldsymbol{\chi},\boldsymbol{\chi})+b(\boldsymbol{\chi},\xi)=a(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi})+b(\boldsymbol{\chi},p-\tilde{p})-\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12},$$
$$b(\boldsymbol{\chi},\xi)=0.$$

Equivalently,

$$a(\boldsymbol{\chi},\boldsymbol{\chi})=a(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi})+b(\boldsymbol{\chi},p-\tilde{p})-\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12}. \tag{4.10}$$

The first term on the right can be estimated as follows:

$$\begin{aligned}a_1(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi})=2\mu\sum_{E\in\mathcal{E}_h^1}\int_E\boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}}):\boldsymbol{D}(\boldsymbol{\chi})\\
&-2\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{\boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}]+2\mu\epsilon\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{\boldsymbol{D}(\boldsymbol{\chi})\}\boldsymbol{n}_e\cdot[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\\
&+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\cdot[\boldsymbol{\chi}]+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{\tau}_{12}\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\\
&=T_1+\cdots+T_5.\end{aligned}$$

Using Cauchy–Schwarz inequality, and the approximation result (4.3), we have

$$\begin{aligned}T_1&\le 2\mu\sum_{E\in\mathcal{E}_h^1}\|\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{0,E}\|\nabla\boldsymbol{\chi}\|_{0,E}\le\frac{1}{8}|||\nabla\boldsymbol{\chi}|||_{0,\Omega_1}^2+C|||\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})|||_{0,\Omega_1}^2\\
&\le\frac{1}{8}|||\nabla\boldsymbol{\chi}|||_{0,\Omega_1}^2+Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.\end{aligned}$$

Let $L_h(\boldsymbol{u})$ denote the standard Lagrange interpolant of degree k_1 defined in Ω_1 and let us insert it in the second integral term. Note that $L_h(\boldsymbol{u})$ satisfies the optimal error estimates

$$|L_h(\boldsymbol{u})-\boldsymbol{u}|_{m,E}\le Ch_E^{s-m}|\boldsymbol{u}|_{s,E}\quad\forall 2\le s\le k_1+1,\ m=0,1,2. \tag{4.11}$$

For e a segment of $\Gamma_h^1\cup\Gamma_1$, we have

$$\int_e\{\boldsymbol{D}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}]=\int_e\{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}]+\int_e\{\boldsymbol{D}(L_h(\boldsymbol{u})-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}].$$

Expanding the first integral, we obtain from the trace inequality (2.35) and from the fact that the Lagrange interpolant satisfies (4.11)

$$\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}]$$
$$\leq \sum_{e\in\Gamma_h^1\cup\Gamma_1} \frac{\sigma_e^{1/2}}{|e|^{1/2}}\|[\boldsymbol{\chi}]\|_{0,e}\frac{|e|^{1/2}}{\sigma_e^{1/2}}\|\{\boldsymbol{D}(\boldsymbol{u}-L_h(\boldsymbol{u}))\}\boldsymbol{n}_e\|_{0.e}$$
$$\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + C\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{|e|}{\sigma_e}(h_e^{-1}|\boldsymbol{u}-L_h(\boldsymbol{u})|_{1,E_e^{12}}^2 + h_e|\boldsymbol{u}-L_h(\boldsymbol{u})|_{2,E_e^{12}}^2)$$
$$\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Similarly, using the trace inequality (2.36), triangle inequality, and (4.3)

$$\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{\boldsymbol{D}(L_h(\boldsymbol{u})-\tilde{\boldsymbol{u}})\}\boldsymbol{n}_e\cdot[\boldsymbol{\chi}] \leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2$$
$$+ C\sum_{e\in\Gamma_h^1\cup\Gamma_1}|\tilde{\boldsymbol{u}}-L_h(\boldsymbol{u})|_{1,E_e^{12}}^2$$
$$\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Therefore,

$$T_2 \leq \frac{1}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

The third term vanishes because of the continuity of $\boldsymbol{u}$ and property (4.2) of $\tilde{\boldsymbol{u}}$:

$$T_3 = 0. \tag{4.12}$$

Using Cauchy–Schwarz inequality, the jump term is bounded by virtue of (2.24) and (2.34):

$$T_4 \leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + C\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\|_{0,e}^2$$
$$\leq \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k+1,\Omega_1}^2.$$

The last term is bounded as follows, from the trace inequality (2.34):

$$T_5 \leq \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,e}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}$$
$$\leq \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + C\sum_{e\in\Gamma_1}(h_e^{-1}\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,E}^2 + h_e|\boldsymbol{u}-\tilde{\boldsymbol{u}}|_{1,E}^2)$$
$$\leq \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2.$$

Let us now estimate $a_2(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi})$, using the result (4.3),

$$a_2(\boldsymbol{u}-\tilde{\boldsymbol{u}},\boldsymbol{\chi}) = \int_{\Omega_2} \boldsymbol{K}^{-1}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{\chi} \le \frac{1}{8}\|\boldsymbol{K}^{-1/2}\boldsymbol{\chi}\|_{0,\Omega_2}^2 + h_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2}^2.$$

Let us now estimate $b_1(\boldsymbol{\chi}, p-\tilde{p})$. By property (2.17), (2.19), and the trace estimate (2.34),

$$\begin{aligned} b_1(\boldsymbol{\chi}, p-\tilde{p}) &= -\sum_{E\in\mathcal{E}_h}\int_E (p-\tilde{p})\nabla\cdot\boldsymbol{\chi} + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{p-\tilde{p}\}[\boldsymbol{\chi}]\cdot\boldsymbol{n}_e \\ &= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e \{p-\tilde{p}\}[\boldsymbol{\chi}]\cdot\boldsymbol{n}_e \\ &\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e [\boldsymbol{\chi}]^2 + Ch_1^{2k_1}|p|_{k_1,\Omega_1}^2. \end{aligned}$$

Now estimate $b_2(\boldsymbol{\chi}, p-\tilde{p})$ using Cauchy–Schwarz inequality and approximation result (2.20)

$$b_2(\boldsymbol{\chi}, p-\tilde{p}) = -\int_{\Omega_2}(p-\tilde{p})\nabla\cdot\boldsymbol{\chi} \le \frac{1}{8}|||\nabla\boldsymbol{\chi}|||_{0,\Omega_2}^2 + Ch_2^{2l_2+2}|p|_{l_2+1,\Omega_2}^2$$

It remains to bound the last term in (4.10). Since $\boldsymbol{\chi}$ belongs to $\boldsymbol{V}_h$, we have

$$\sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12} = \sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)(\boldsymbol{\chi}_1-\boldsymbol{\chi}_2)\cdot\boldsymbol{n}_{12},$$

where $\tilde{p}_2^e \in \Lambda_h$ is the L^2 projection of p_2 with respect to the L^2 inner product on the edge e. Therefore, by definition of the projection and since $\Lambda_h = \boldsymbol{X}_h^2\cdot\boldsymbol{n}_{12}$, we have

$$\sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_2\cdot\boldsymbol{n}_{12} = 0.$$

We also note that for any edge e and any constant vector $\boldsymbol{c}_e$, we have

$$\begin{aligned} \sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} &= \sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)(\boldsymbol{\chi}_1-\boldsymbol{c}_e)\cdot\boldsymbol{n}_{12} \\ &\le \sum_{e\in\Gamma_{12}}\|p_2-\tilde{p}_2^e\|_{0,e}\|\boldsymbol{\chi}_1-\boldsymbol{c}_e\|_{0,e}. \end{aligned}$$

Assume that each edge e of Γ_{12} is shared by the element $E_e^2 \in \mathcal{E}_h^2$ and parts of the elements $E_{e,i}^1 \in \mathcal{E}_h^1$, $i = 1, n_e$. Then, from the approximation properties and the trace inequality (2.34), we obtain

$$\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} \le Ch_2^{l_2+1/2}\|p_2\|_{l_2+1,E_e^2}\sum_{i=1}^{n_e}(h_1^{-1/2}\|\boldsymbol{\chi}_1-\boldsymbol{c}_e\|_{0,E_{e,i}^1} + h_1^{1/2}\|\nabla\boldsymbol{\chi}_1\|_{0,E_{e,i}^1}),$$

thus

$$\begin{aligned} \sum_{e\in\Gamma_{12}}\int_e (p_2-\tilde{p}_2^e)\boldsymbol{\chi}_1\cdot\boldsymbol{n}_{12} &\le C\sum_{e\in\Gamma_{12}} h_2^{l_2+1/2}|p_2|_{l_2+1,E_e^2}\sum_{i=1}^{n_e} h_1^{1/2}\|\nabla\boldsymbol{\chi}_1\|_{0,E_{e,i}^1} \\ &\le \frac{1}{8}|||\nabla\boldsymbol{\chi}|||_{0,\Omega_1}^2 + Ch_2^{2l_2+1}h_1|p_2|_{l_2+1,\Omega_2}^2. \end{aligned}$$

Combining all bounds above yields

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) \le \frac{1}{4}|||\nabla\boldsymbol{\chi}|||_{0,\Omega_1}^2 + \frac{3}{4}\sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}]\|_{0,e}^2 + \frac{\mu}{2G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2$$
$$+\frac{1}{4}\|\boldsymbol{K}^{-1/2}\boldsymbol{\chi}\|_{0,\Omega_2}^2 + Ch_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2} + C(h_2^{2l_2+2}+h_2^{2l_2+1}h_1)|p|_{l_2+1,\Omega_2}^2$$
$$+Ch_2^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2 + Ch_1^{2k_1}|p|_{k_1,\Omega_1}^2.$$

Equivalently,

$$a(\boldsymbol{\chi},\boldsymbol{\chi}) \le Ch_2^{2k_2+2}|\boldsymbol{u}|_{k_2+1,\Omega_2}^2 + C(h_2^{2l_2+2}+h_2^{2l_2+1}h_1)|p|_{l_2+1,\Omega_2}^2$$
$$+Ch_1^{2k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1}^2 + |p|_{k_1,\Omega_1}^2).$$

Now, since $\nabla\cdot\boldsymbol{\chi}=0$ in Ω_2, the coercivity Lemma 2.6 implies

$$\|\boldsymbol{u}-\boldsymbol{U}\|_X \le \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X + \|\boldsymbol{U}-\tilde{\boldsymbol{u}}\|_X$$
$$\le \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_X + \frac{1}{\sqrt{C_0}}a(\boldsymbol{\chi},\boldsymbol{\chi})^{1/2}$$

which concludes the proof, using (4.3). ∎

THEOREM 4.3. *Under the assumptions and notation of Theorem* 4.2*, we have*

$$\|p-P\|_{0,\Omega} \le Ch_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1}+|p|_{k_1,\Omega_1}) + Ch_2^{k_2+1}|\boldsymbol{u}|_{k_2+1,\Omega_2}$$
$$+C(h_2^{l_2+1}+h_2^{l_2+1/2}h_1^{1/2})|p|_{l_2+1,\Omega_2},$$

where C is a constant independent of h_1, h_2.

Proof. The error equation (4.8) can be written as

$$\forall \boldsymbol{v}\in\boldsymbol{V}_h,\quad a(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{v})+b(\boldsymbol{v},P-\tilde{p}) = b(\boldsymbol{v},p-\tilde{p}) - \sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{v}_1-\boldsymbol{v}_2)\cdot\boldsymbol{n}_{12}. \tag{4.13}$$

From the discrete inf-sup condition (3.1),

$$\|P-\tilde{p}\|_{0,\Omega} \le \frac{1}{\beta}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h}\frac{b(\boldsymbol{v}_h,P-\tilde{p})}{\|\boldsymbol{v}_h\|_X}. \tag{4.14}$$

Using (4.13), for any $\boldsymbol{v}_h\in\boldsymbol{V}_h$,

$$b(\boldsymbol{v}_h,P-\tilde{p}) = -a(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{v}_h) + b(\boldsymbol{v}_h,p-\tilde{p}) - \sum_{e\in\Gamma_{12}}\int_e p_2(\boldsymbol{v}_{h1}-\boldsymbol{v}_{h2})\cdot\boldsymbol{n}_{12}.$$

For the first term on the right,

$$a(\boldsymbol{U}-\boldsymbol{u},\boldsymbol{v}_h) = 2\mu\sum_{E\in\mathcal{E}_h^1}\int_E \boldsymbol{D}(\boldsymbol{U}-\boldsymbol{u}):\boldsymbol{D}(\boldsymbol{v}_h) + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\frac{\sigma_e}{|e|}\int_e[\boldsymbol{U}-\boldsymbol{u}]\cdot[\boldsymbol{v}_h]$$
$$-2\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{\boldsymbol{D}(\boldsymbol{U}-\boldsymbol{u})\boldsymbol{n}_e\}\cdot[\boldsymbol{v}_h] + 2\mu\epsilon\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e\{\boldsymbol{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\}\cdot[\boldsymbol{U}-\boldsymbol{u}]$$
$$+\frac{\mu}{G}\sum_{e\in\Gamma_{12}}\int_e(\boldsymbol{U}-\boldsymbol{u})\cdot\boldsymbol{\tau}_{12}\boldsymbol{v}_h\cdot\boldsymbol{\tau}_{12} + \int_{\Omega_2}\boldsymbol{K}^{-1}(\boldsymbol{U}-\boldsymbol{u})\cdot\boldsymbol{v}_h$$
$$= Q_1+\cdots+Q_6.$$

We now bound each Q_i term. From Cauchy–Schwarz inequality, the terms Q_1, Q_2, Q_5, and Q_6 are easily bounded

$$Q_1 + Q_2 + Q_5 + Q_6 \le C\|\boldsymbol{v}_h\|_X \|\boldsymbol{U} - \boldsymbol{u}\|_X.$$

We now bound Q_3,

$$\begin{aligned}
Q_3 &\le C \sum_{e\in\Gamma_h^1\cup\Gamma_1} \left(\frac{|e|}{\sigma_e}\right)^{1/2} \|\nabla(\boldsymbol{U}-\boldsymbol{u})\|_{0,e} \left(\frac{\sigma_e}{|e|}\right)^{1/2} \|[\boldsymbol{v}_h]\|_{0,e} \\
&\le C\|\boldsymbol{v}_h\|_X \left(\sum_{e\in\Gamma_h^1\cup\Gamma_1} (h_1\|\nabla(\boldsymbol{U}-\tilde{\boldsymbol{u}})\|_{0,e}^2 + h_1\|\nabla(\boldsymbol{u}-\tilde{\boldsymbol{u}})\|_{0,e}^2)\right)^{1/2} \\
&\le C\|\boldsymbol{v}_h\|_X (\|U-\tilde{\boldsymbol{u}}\|_X^2 + Ch_1^{2k_1}|\boldsymbol{u}|_{k_1+1,\Omega_1}^2)^{1/2}.
\end{aligned}$$

Now, Q_4 is bounded similarly, from trace inequality (2.36),

$$\begin{aligned}
Q_4 &\le C \sum_{e\in\Gamma_h^1\cup\Gamma_1} \|\{\boldsymbol{D}(\boldsymbol{v}_h)\boldsymbol{n}_e\}\|_{0,e} \|[\boldsymbol{U}-\boldsymbol{u}]\|_{0,e} \\
&\le C \sum_{e\in\Gamma_h^1\cup\Gamma_1} h^{-1/2}\|\nabla\boldsymbol{v}_h\|_{0,E_e^{12}} \left(\frac{\sigma_e}{|e|}\right)^{1/2-1/2} \|[\boldsymbol{U}-\boldsymbol{u}]\|_{0,e} \\
&\le C\|\boldsymbol{v}_h\|_X \|\boldsymbol{U}-\boldsymbol{u}\|_X.
\end{aligned}$$

Let us now estimate $b(\boldsymbol{v}_h, p-\tilde{p})$. From the property (2.17), it is reduced to

$$\begin{aligned}
b(\boldsymbol{v}_h, p-\tilde{p}) &= \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e \{p-\tilde{p}\}[\boldsymbol{v}_h]\cdot\boldsymbol{n}_e \\
&\le \sum_{e\in\Gamma_h^1\cup\Gamma_1} \left(\frac{\sigma_e}{|e|}\right)^{1/2} \|[\boldsymbol{v}_h]\|_{0,e} \left(\frac{|e|}{\sigma_e}\right)^{1/2} \|\{p-\tilde{p}\}\|_{0,e} \\
&\le \|\boldsymbol{v}_h\|_X Ch_1^{k_1}|p|_{k_1,\Omega_1}.
\end{aligned}$$

Finally, following the same approach as in the proof of Theorem 4.2, we bound the interface integral

$$\begin{aligned}
\sum_{e\in\Gamma_{12}} \int_e p_2(\boldsymbol{v}_{h1}-\boldsymbol{v}_{h2})\cdot\boldsymbol{n}_{12} &= \sum_{e\in\Gamma_{12}} \int_e (p_2-\tilde{p}_2^e)\boldsymbol{v}_{h1}\cdot\boldsymbol{n}_{12} \\
&\le C\|\boldsymbol{v}_h\|_X h_2^{l_2+1/2}h_1^{1/2}|p_2|_{l_2+1,\Omega_2}.
\end{aligned}$$

Combining all the bounds with (4.14) yields

$$\|P-\tilde{p}\|_{0,\Omega} \le C\left(\|\boldsymbol{U}-\boldsymbol{u}\|_X + h_1^{k_1}(|\boldsymbol{u}|_{k_1+1,\Omega_1} + |p|_{k_1,\Omega_1}) + h_2^{l_2+1/2}h_1^{1/2}\|p\|_{l_2+1,\Omega_2}\right).$$

Using Theorem 4.2 concludes the proof. ◻

Remark 4.4. The results proven in this section are valid and unchanged in three-dimensional domains, assuming there exist interpolants $\boldsymbol{\Pi}_h^1$ and $\boldsymbol{\Pi}_h^2$ defined in (2.21) and (2.29). The existence of $\boldsymbol{\Pi}_h^1$ for $k=1$ in three dimensions is given in [13]. The existence of $\boldsymbol{\Pi}_h^2$ in any dimension is a well-known fact [10].

5. Implementation issues and conclusions. In this paper, the convergence of a numerical scheme for solving the coupled Darcy–Stokes problem is proved. In order to parallelize the implementation of the scheme, a Lagrange multiplier $\lambda \in \Lambda_h$ approximating p_2 on Γ_{12} can be introduced. We recall the definition of $\Lambda_h = \boldsymbol{X}_h^2 \cdot \boldsymbol{n}_{12}$ given in section 2. Defining the bilinear form on the interface,

$$\Lambda(\eta, \boldsymbol{v}) = \sum_{e \in \Gamma_{12}} \int_e \eta(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot \boldsymbol{n}_{12} \quad \forall \eta \in \Lambda_h,\ \forall \boldsymbol{v} \in \boldsymbol{X}_h,$$

the scheme can be rewritten as: Find $(\boldsymbol{U}, P, \lambda) \in \boldsymbol{X}_h \times M_h \times \Lambda_h$ such that $\boldsymbol{U}_i = \boldsymbol{U}|_{\Omega_i}$ and $P_i = P|_{\Omega_i}$ satisfy

$$a_1(\boldsymbol{U}_1, \boldsymbol{v}_1) + b_1(\boldsymbol{v}_1, P_1) + \Lambda(\lambda, \boldsymbol{v}_1) = \int_{\Omega_1} \boldsymbol{f}_1 \cdot \boldsymbol{v}_1 \quad \forall \boldsymbol{v}_1 \in \boldsymbol{X}_h^1, \tag{5.1}$$

$$b_1(\boldsymbol{U}_1, q_1) = 0 \quad \forall q_1 \in M_h^1, \tag{5.2}$$

$$a_2(\boldsymbol{U}_2, \boldsymbol{v}_2) + b_2(\boldsymbol{v}_2, P_2) - \Lambda(\lambda, \boldsymbol{v}_2) = 0 \quad \forall \boldsymbol{v}_2 \in \boldsymbol{X}_h^2, \tag{5.3}$$

$$b_2(\boldsymbol{U}_2, q_2) = \int_{\Omega_2} f_2 q_2 \quad \forall q_2 \in M_h^2, \tag{5.4}$$

$$\Lambda(\eta, \boldsymbol{U}) = 0 \quad \forall \eta \in \Lambda_h. \tag{5.5}$$

It can easily be shown that the two discrete formulations are equivalent. Formulation (5.1)–(5.5) is suitable for a parallel implementation. In particular, using an approach from [23], a nonoverlapping domain decomposition algorithm can be formulated that reduces the coupled system to a symmetric and positive definite interface problem for λ. In addition to its parallel efficiency, this approach allows for existing codes solving the Stokes or the Darcy equations to be utilized.

REFERENCES

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] T. Arbogast, L. C. Cowsar, M. F. Wheeler, and I. Yotov, *Mixed finite element methods on nonmatching multiblock grids*, SIAM J. Numer. Anal., 37 (2000), pp. 1295–1315.

[3] T. Arbogast and H. Lehr, *Homogenization of a Darcy-Stokes System Modeling Vuggy Porous Media*, Technical report 02-44, University of Texas at Austin, 2002.

[4] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[5] G. Beavers and D. Joseph, *Boundary conditions at a naturally impermeable wall*, J. Fluid Mech., 30 (1967), pp. 197–207.

[6] S. Brenner, *Korn's inequalities for piecewise H^1 vector fields*, Math. Comp., 73 (2004), pp. 1067–1087.

[7] F. Brezzi, J. Douglas, Jr., R. Duràn, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.

[8] F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modèl. Math. Anal. Numèr., 21 (1987), pp. 581–604.

[9] F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed elements for second order elliptic problems*, Numer. Math., 88 (1985), pp. 217–235.

[10] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[11] P. Ciarlet, *The Finite Element Method for Elliptic Problems*, North–Holland, Amsterdam, 1978.

[12] M. Crouzeix and R. Falk, *Non conforming finite elements for the Stokes problem*, Math. Comp., 52 (1989), pp. 437–456.

[13] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations.* I, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge, 7 (1973), pp. 33–75.
[14] C. Dawson, *Conservative, shock-capturing transport methods with nonconservative velocity approximations*, Comput. Geosci., 3 (1999), pp. 205–227.
[15] M. Discacciati, E. Miglio, and A. Quarteroni, *Mathematical and numerical models for coupling surface and groundwater flows*, Appl. Numer. Math., 43 (2002), pp. 57–74.
[16] J. Douglas, Jr., R. E. Ewing, and M. F. Wheeler, *The approximation of the pressure by a mixed method in the simulation of miscible displacement*, RAIRO Anal. Numér., 17 (1983), pp. 17–34.
[17] L. J. Durlofsky, *Accuracy of mixed and control volume finite element approximations to Darcy velocity and related quantities*, Water Resources Research, 30 (1994), pp. 965–973.
[18] R. E. Ewing, O. P. Iliev, and R. D. Lazarov, *Numerical Simulation of Contamination Transport Due to Flow in Liquid and Porous Media*, Technical report 1992-10, Enhanced Oil Recovery Institute, University of Wyoming, 1992.
[19] M. Fortin and M. Soulie, *A non-conforming piecewise quadratic finite element on triangles*, Internat. J. Numer. Methods Engrg., 19 (1983), pp. 505–520.
[20] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol.* I, Springer-Verlag, New York, 1994.
[21] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations*, Springer-Verlag, Berlin, 1986.
[22] V. Girault, B. Riviere, and M. Wheeler, *A discontinuous Galerkin method with non-overlapping domain decomposition for the Stokes and Navier-Stokes problems*, Math. Comp., 74 (2004), pp. 53–84.
[23] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., SIAM, Philadelphia, 1988, pp. 144–172.
[24] W. Jäger and A. Mikelić, *On the interface boundary condition of Beavers, Joseph, and Saffman*, SIAM J. Appl. Math., 60 (2000), pp. 1111–1127.
[25] W. J. Layton, F. Schieweck, and I. Yotov, *Coupling fluid flow with porous media flow*, SIAM J. Numer. Anal., 40 (2003), pp. 2195–2218.
[26] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Vol. 1, Springer-Verlag, New York, 1972.
[27] K. A. Mardal, X.-C. Tai, and R. Winther, *A robust finite element method for Darcy–Stokes flow*, SIAM J. Numer. Anal., 40 (2002), pp. 1605–1631.
[28] T. P. Mathew, *Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretizations of Elliptic Problems*, Ph.D. thesis, Courant Institute of Mathematical Sciences, New York University, 1989.
[29] R. A. Raviart and J. M. Thomas, *A mixed finite element method for* 2*nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Math. 606, Springer-Verlag, New York, 1977, pp. 292–315.
[30] B. Riviere and M. Wheeler, *Discontinuous Galerkin methods for flow and transport problems in porous media*, Comm. Numer. Methods Engrg., 18 (2002), pp. 63–68.
[31] B. Rivière and M. Wheeler, *Non conforming methods for transport with nonlinear reaction*, in Fluid Flow and Transport in Porous Media: Mathematical and Numerical Treatment, Z. Chen and R. Ewing, eds., Contemp. Math. 295, AMS, Providence, RI, 2002, pp. 421–430.
[32] B. Riviere and M. Wheeler, *A discontinuous Galerkin method for modeling two-phase flow*, Advances in Water Resources, submitted.
[33] B. Rivière, M. Wheeler, and K. Banas, *Part* II. *Discontinuous Galerkin method applied to a single phase flow in porous media*, Comput. Geosci., 4 (2000), pp. 337–349.
[34] T. F. Russell and M. F. Wheeler, *Finite element and finite difference methods for continuous flows in porous media*, in The Mathematics of Reservoir Simulation, Frontiers Appl. Math. 1, R. E. Ewing, ed., SIAM, Philadelphia, 1984, pp. 35–106.
[35] P. Saffman, *On the boundary condition at the surface of a porous media*, Stud. Appl. Math., 50 (1971), pp. 292–315.