

A mixed finite element discretization on non-matching multiblock grids for a degenerate parabolic equation arising in porous media flow

IVAN YOTOV*

Abstract — Mixed finite element methods on multiblock domains are considered for nonlinear degenerate parabolic equations arising in modeling multiphase flow in porous media. The subdomain grids need not match on the interfaces, where mortar finite element spaces are introduced to properly impose flux-matching conditions. The low regularity of the solution is treated through time integration, and the degeneracy of the diffusion is handled analytically via the Kirchhoff transformation. With an appropriate choice of the mortar spaces, the error for both a semidiscrete (continuous time) scheme and a fully discrete (backward Euler) scheme is bounded entirely by approximation error terms of optimal order.

Keywords. Mixed finite element, degenerate parabolic equation, nonlinear, mortar finite element, multiblock, non-matching grids, error estimates, porous media

AMS subject classifications. 65M60, 65M12, 65M15, 35K65, 76S05

Multiblock finite element techniques on non-matching grids have become increasingly popular in recent years. They combine the flexibility of modeling irregularly shaped domains with the convenience of constructing the grids locally. In porous media applications they also allow accurate modeling of large scale geological features such as faults, layers, and fractures. We define a multiblock domain to be a simply connected domain $\Omega \in \mathbf{R}^d$, $d = 2$ or 3 , that is a union of non-overlapping subdomains or blocks Ω_i , $i = 1, \dots, k$. For the purpose of the analysis we assume that each block Ω_i is convex.

In the numerical modeling of multiphase flow in porous media the coupled nonlinear system of conservation equations is often written as an equation of elliptic or parabolic type for some reference pressure and several saturation equations of advection-diffusion type [7, 13]. The diffusion in the saturation equations is degenerate (is zero at extreme saturation values). This causes a very low regularity of the solution and imposes difficulties for the numerical method.

In this paper we study mixed finite element discretizations for the saturation equation on multiblock domains. The local mass conservation property of the mixed methods is particularly important in modeling porous media flow. We allow the subdomain grids to be non-matching along the interfaces. To properly impose flux-matching conditions we

*Texas Institute for Computational and Applied Mathematics, The University of Texas at Austin, Austin, TX 78712; yotov@ticam.utexas.edu.

introduce mortar finite element spaces along the subdomain boundaries. Mortar spaces have been a common tool in standard and spectral finite element methods (see e.g. [9]). Recently mortar [18, 29, 4] and non-mortar [6] techniques have been successfully applied in the context of mixed finite element methods on multiblock grids.

Due to the degeneracy in the diffusion the solutions can have compact support, thus behaving very differently from the solutions of non-degenerate parabolic problems. In addition, certain error terms cannot be directly bounded in the analysis. Many authors introduce a regularized problem and then approximate it [27, 17, 24, 15, 14]. The numerical error is then a sum of the discretization error for the regularized problem and the difference between the solutions to the regularized and the original problems.

Another common technique is to handle the degeneracy analytically via the Kirchhoff transformation [27, 15, 14, 5, 28], which is the approach we take. We combine the use of mortar finite elements along subdomain interfaces with techniques from [5], where mixed discretizations for similar degenerate parabolic equations on a single block have been studied.

We consider a continuous time scheme and a fully discrete backward Euler scheme and bound the discretization error in the two versions entirely by approximation error terms of optimal order. Critical in the analysis is the choice of mortar spaces along the non-matching interfaces. The mortar elements need to provide one order higher approximation than the traces of the subdomain velocity spaces (see Remark 2.2). At the same time, the mortar spaces should be controlled by the subdomain spaces (see hypothesis (H1) and Remark 3.1).

Our analysis improve the single block results from [5] in several ways. We avoid the assumption from [5] on smoothness of $\partial\Omega$ by requiring a minimal smoothness for the flux (see Lemma 2.2.). Also, we weaken the assumption ([5], A2) on the advective and the source terms and only assume physically reasonable behavior (see (2.7), (2.8)). Finally, by choosing a different flux test function in (3.4), we avoid a non-standard approximation error term involving the divergence of the difference of two discrete projections of the flux, which appears in the analysis in [5].

The rest of the paper is organized as follows. In the next section we illustrate how a degenerate parabolic saturation equation arises in the fractional flow formulation of two phase incompressible flow. In Section 2 we formulate the mortar mixed method for the saturation equation. The analysis of the continuous in time and the fully discrete schemes is presented in Sections 3 and 4, respectively.

1. FRACTIONAL FLOW FORMULATION FOR TWO PHASE FLOW

Two phase immiscible flow in porous media is modeled by the system of conservation equations [7, 13]

$$\frac{\partial(\varphi s_i \rho_i)}{\partial t} + \nabla \cdot \rho_i \mathbf{u}_i = q_i \quad \text{in } \Omega \times (0, T], \quad (1.1)$$

$$\mathbf{u}_i = -\frac{k_i(s_i)K}{\mu_i}(\nabla p_i - \rho_i g \nabla D) \quad \text{in } \Omega \times (0, T], \quad (1.2)$$

$i = w$ (wetting), n (non-wetting), coupled with

$$s_w + s_n = 1, \quad (1.3)$$

$$p_c(s_w) = p_n - p_w, \quad (1.4)$$

where s_i is the phase saturation, ρ_i is the phase density, φ is the porosity, q_i is the source term, \mathbf{u}_i is the Darcy velocity, p_i is the phase pressure, K is the absolute permeability tensor, $k_i(s_i)$ is the phase relative permeability, μ_i is the phase viscosity, g is the gravitational constant, D is the depth, and T is the final time.

We start by reformulating the problem in a standard way by writing it in a fractional flow form (pressure and saturation equation). Let

$$\lambda_i = \frac{k_i}{\mu_i}, \quad i = w, n,$$

denote the phase mobilities, and let

$$\lambda = \lambda_w + \lambda_n$$

be the total mobility. Let

$$\mathbf{u} = \mathbf{u}_w + \mathbf{u}_n$$

be the total velocity. For simplicity of the presentation, we assume incompressible flow and medium (constant ρ_i and φ) and neglect gravity effects. Multiplying equations (1.1) by $1/\rho_i$ and adding them together, we get

$$\nabla \cdot \mathbf{u} = q, \quad (1.5)$$

where $q = q_w/\rho_w + q_n/\rho_n$. Let $s = s_w$, and define the global pressure [13] to be

$$p = p_w + \int_0^{p_c(s)} \left(\frac{\lambda_n}{\lambda} \right) (p_c^{-1}(\zeta)) d\zeta.$$

Thus

$$\mathbf{u} = -\lambda K \nabla p. \quad (1.6)$$

Equation (1.5), coupled with (1.6), is referred to as the pressure equation. Since $\lambda > 0$ and K is a symmetric positive definite tensor, this is an elliptic equation. For compressible flow the pressure equation is parabolic.

To derive the saturation equation, we first observe that

$$\frac{\lambda_w}{\lambda} \mathbf{u} = \mathbf{u}_w - \frac{\lambda_w \lambda_n}{\lambda} K \nabla p_c(s).$$

Substituting this expression into the water conservation equation (1.1), we get the saturation equation

$$\varphi \frac{\partial s}{\partial t} + \nabla \cdot (\beta(s) \mathbf{u} + \alpha(s) K \nabla p_c(s)) = \tilde{q}_w, \quad (1.7)$$

where $\beta(s) = \lambda_w/\lambda$, $\alpha(s) = \lambda_w \lambda_n/\lambda$, and $\tilde{q}_w = q_w/\rho_w$. Note that $p_c(s)$ is a strictly monotone decreasing function; therefore with

$$\sigma(s) = -\alpha(s) \frac{\partial p_c}{\partial s} \quad (1.8)$$

equation (1.7) takes the advection-diffusion form

$$\varphi \frac{\partial s}{\partial t} + \nabla \cdot (\beta(s)\mathbf{u} - \sigma(s)K\nabla s) = \tilde{q}_w. \quad (1.9)$$

The diffusion term vanishes at $s = 0, 1$, the minimum and maximum saturation values. This is due to the behavior of the relative permeability and the capillary pressure functions (see, e.g., [8]). This double degeneracy is the main source of difficulties in the numerical approximation. The solutions of degenerate parabolic equations have very low regularity. It has been shown that (see [25, 16, 19, 2, 1, 3])

$$s \in L^\infty(0, T; L^1(\Omega)), \quad (1.10)$$

$$\frac{\partial s}{\partial t} \in L^2(0, T; H^{-1}(\Omega)). \quad (1.11)$$

Since the saturation s satisfies $0 \leq s(x, t) \leq 1$, $(x, t) \in \Omega \times [0, T]$, we also have

$$s \in L^\infty(0, T; L^\infty(\Omega)). \quad (1.12)$$

We handle the degeneracy in the diffusion analytically using the Kirchhoff transformation [27, 15, 14, 5]. Let

$$D(s) = \int_0^s \sigma(\zeta) d\zeta.$$

Then

$$\nabla D(s) = \sigma(s)\nabla s,$$

and (1.9) becomes

$$\varphi \frac{\partial s}{\partial t} + \nabla \cdot (\beta(s)\mathbf{u} - K\nabla D(s)) = \tilde{q}_w. \quad (1.13)$$

2. A MORTAR MIXED METHOD FOR THE SATURATION EQUATION

In this section we present a multiblock variational formulation of (1.13) that respects the low regularity of the solution, and a mixed finite element discretization using mortar elements on the interfaces. For the rest of the paper we omit the porosity φ , which is assumed constant and does not effect the error analysis. The saturation $s(x, t)$ satisfies

$$\frac{\partial s}{\partial t} + \nabla \cdot (\beta(s)\mathbf{u} - K\nabla D(s)) = \tilde{q}_w(s) \quad \text{in } \Omega \times (0, T], \quad (2.1)$$

$$(\beta(s)\mathbf{u} - K\nabla D(s)) \cdot \nu = 0 \quad \text{on } \partial\Omega \times [0, T], \quad (2.2)$$

$$s(x, 0) = s_0(x) \quad \text{in } \Omega, \quad (2.3)$$

where ν is the outward unit normal vector to $\partial\Omega$. For simplicity we assume that no flow boundary conditions are imposed, although more general boundary conditions can also be

treated. We need several assumptions on the coefficients of the above equation. We first assume that

$$\sigma(s) \geq \begin{cases} \beta_1 |s|^{\nu_1}, & 0 \leq s \leq \alpha_1, \\ \beta_2, & \alpha_1 \leq s \leq \alpha_2, \\ \beta_3 |1-s|^{\nu_2}, & \alpha_2 \leq s \leq 1, \end{cases} \quad (2.4)$$

where β_i , $1 \leq i \leq 3$, are positive constants, and α_i and ν_i , $i = 1, 2$, satisfy

$$0 < \alpha_1 < 1/2 < \alpha_2 < 1, \quad 0 < \nu_i \leq 2.$$

Note that (2.4) controls the rate of degeneracy of the diffusion. We also assume that there exists a positive constant C such that

$$\|D(s_1) - D(s_2)\|_0^2 \leq C(D(s_1) - D(s_2))(s_1 - s_2), \quad \text{for } s_1, s_2 \in L^2(\Omega). \quad (2.5)$$

Here and for the remainder of the paper $(\cdot, \cdot)_S$ and $\langle \cdot, \cdot \rangle_{\partial S}$ denote the L^2 -inner product (or duality pairing) on $S \subset \mathbf{R}^d$ and $\partial S \in \mathbf{R}^{d-1}$, respectively, and $\|\cdot\|_{0,S}$ denotes the L^2 -norm on S . We omit S if $S = \Omega$. A sufficient condition for (2.5) is

$$0 \leq \frac{\partial D}{\partial s}(x, t; s) \leq C \quad \text{for } (x, t) \in \Omega \times [0, T], \quad 0 \leq s \leq 1. \quad (2.6)$$

Bounds (2.4) and (2.6) follow from the physical behavior of the relative permeabilities and the capillary pressure [3, 15]. Typical relations (see, e.g., [8, 20]) are

$$\begin{aligned} \lambda_w(s) &\sim s^2, & p_c(s) &\sim s^{-\delta_1} \text{ as } s \rightarrow 0, \\ \lambda_o(s) &\sim (1-s)^2, & p_c(s) &\sim (1-s)^{\delta_2} \text{ as } s \rightarrow 1, \\ 0 &< \delta_1 < 1, & 0 &< \delta_2 < 1. \end{aligned}$$

Therefore, with (1.8),

$$\sigma(s) \sim s^{1-\delta_1} \text{ as } s \rightarrow 0 \text{ and } \sigma(s) \sim (1-s)^{1+\delta_2} \text{ as } s \rightarrow 1,$$

which implies both (2.4) and (2.6). Finally, we assume that, for $0 \leq s_1, s_2 \leq 1$,

$$|\beta(s_1) - \beta(s_2)|^2 \leq C(D(s_1) - D(s_2))(s_1 - s_2), \quad (2.7)$$

$$|\tilde{q}_w(s_1) - \tilde{q}_w(s_2)|^2 \leq C(D(s_1) - D(s_2))(s_1 - s_2). \quad (2.8)$$

Bounds (2.7) and (2.8) are justified by the following lemma, proven in [15].

LEMMA 2.1. *Suppose σ satisfies (2.4). If $f \in C^1[0, 1]$ and $f'(0) = f'(1) = 0$ with f' Lipschitz at 0 and 1, then there exists a positive constant C such that*

$$|f(a) - f(b)|^2 \leq C(D(a) - D(b))(a - b) \quad \text{for } 0 \leq a, b \leq 1.$$

Note that the fractional flow function $\beta(s)$ satisfies the conditions of Lemma 2.1.. The well term $\tilde{q}_w(s)$ satisfies the conditions of Lemma 2.1. at the injection wells. At the production wells, $\tilde{q}_w(s) \sim k_w(s)$, so $\tilde{q}'_w(0) = 0$. Therefore (2.8) holds, if $s \leq s^* < 1$ at the production well, which covers all cases of physical interest.

REMARK 2.1. *The fractional flow function $\beta(s)$ and the integrated diffusion function $D(s)$ are both S -shaped with zero derivatives at the end points. Bound (2.7) relates the rates of degeneracy of the derivatives of the two functions and indicates, in a sense, that the diffusion dominates the advection.*

In the standard mixed variational formulation, equation (2.1) is multiplied by a test function $w \in L^2(\Omega)$ and integrated in space. In our case however, because of (1.11), the integral $(\frac{\partial s}{\partial t}, w)$ is not well defined. To avoid this problem we integrate (2.1) in time from 0 to t [23, 5] to obtain the equivalent equation

$$s(x, t) + \nabla \cdot \int_0^t \psi \, d\tau = \int_0^t \tilde{q}_w(s) \, d\tau + s_0(x), \quad \text{in } \Omega \times [0, T], \quad (2.9)$$

where

$$\psi = \beta(s)\mathbf{u} - K\nabla D(s). \quad (2.10)$$

Before presenting the variational formulation, we give the following regularity result.

LEMMA 2.2. *Assume that there exists some $0 < \varepsilon < 1/2$ such that, for $1 \leq i \leq k$,*

$$\int_0^t \beta(s)\mathbf{u} \, d\tau \in L^2(0, T; H^\varepsilon(\Omega_i)). \quad (2.11)$$

Then, for every $t \in [0, T]$,

$$\int_0^t \psi \, d\tau \in (H^\varepsilon(\Omega_i))^d \cap H(\text{div}; \Omega_i). \quad (2.12)$$

Proof. The argument is similar to one from [5]. The assumption (2.11) is reasonable, since the fractional flow function $\beta(s)$ has zero derivatives at the degeneracy values $s = 0, 1$. It follows from (2.9) and (1.12) that

$$\int_0^t \psi \, d\tau \in H(\text{div}; \Omega_i).$$

Using (2.10) we have in Ω_i

$$-\nabla \cdot K\nabla \int_0^t D(s) \, d\tau = \nabla \cdot \int_0^t \psi \, d\tau - \nabla \cdot \int_0^t \beta(s)\mathbf{u} \, d\tau.$$

By (2.11) and elliptic regularity, $\int_0^t D(s) \, d\tau \in H^{1+\varepsilon}(\Omega_i)$, which, along with (2.10) and (2.11) implies that

$$\int_0^t \psi \, d\tau \in (H^\varepsilon(\Omega_i))^d. \quad \square$$

Let $\Gamma_{i,j} = \partial\Omega_i \cap \partial\Omega_j$, $\Gamma = \cup_{1 \leq i < j \leq k} \Gamma_{i,j}$, and $\Gamma_i = \partial\Omega_i \cap \Gamma$. Let

$$\mathbf{V}_i = \{\mathbf{v} \in (H^\varepsilon(\Omega_i))^d \cap H(\text{div}; \Omega_i) : \mathbf{v} \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad \mathbf{V} = \bigoplus_{i=1}^k \mathbf{V}_i,$$

$$W_i = L^2(\Omega_i), \quad W = \bigoplus_{i=1}^k W_i = L^2(\Omega), \quad \Lambda = H^{1/2-\varepsilon}(\Gamma).$$

We are now ready to present the multiblock variational formulation of (2.9)–(2.10). With $a = K^{-1}$ and letting γ be the trace of $D(s)$ on Γ , we have, for every time $t \in [0, T]$ and $1 \leq i \leq k$,

$$(a\psi, \mathbf{v})_{\Omega_i} = (D(s), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \gamma, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} + (a\beta(s)\mathbf{u}, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_i, \quad (2.13)$$

$$(s, w)_{\Omega_i} + \left(\nabla \cdot \int_0^t \psi \, d\tau, w \right)_{\Omega_i} = \left(\int_0^t \tilde{q}_w(s) \, d\tau, w \right)_{\Omega_i} + (s_0, w)_{\Omega_i}, \quad w \in W_i, \quad (2.14)$$

$$\sum_{i=1}^k \left\langle \int_0^t \psi \cdot \nu_i \, d\tau, \mu \right\rangle_{\Gamma_i} = 0, \quad \mu \in \Lambda. \quad (2.15)$$

Note that, because of (2.12) and the definitions of \mathbf{V}_i and Λ , the boundary integrals in (2.13) and (2.15) are well defined.

Let $\mathcal{T}_{h,i}$ be a finite element partition of Ω_i with maximal element diameter h . We allow for the possibility that $\mathcal{T}_{h,i}$ and $\mathcal{T}_{h,j}$ need not match on $\Gamma_{i,j}$. Let $\mathbf{V}_{h,i} \times W_{h,i}$ be any of the usual mixed spaces on $\mathcal{T}_{h,i}$ [26, 22, 12, 11, 10]. Let $\mathcal{T}_h^{\Gamma_{i,j}}$ be a finite element partition of $\Gamma_{i,j}$ with maximal element diameter h . Let $\Lambda_{h,i,j} \subset \Lambda_{i,j}$ be the space of continuous or discontinuous piecewise polynomials of degree $k+1$ on $\mathcal{T}_h^{\Gamma_{i,j}}$, where k is associated with the degree of the polynomials in $\mathbf{V}_{h,i} \cdot \nu_i$. More precisely, if $d = 3$, on any boundary element K , $\Lambda_{h,i,j}|_K = P_{k+1}(K)$, if K is a triangle, and $\Lambda_{h,i,j}|_K = Q_{k+1}(K)$, if K is a rectangle. An additional assumption on $\Lambda_{h,i,j}$ and hence on $\mathcal{T}_h^{\Gamma_{i,j}}$ will be made later. The finite element spaces on Ω are now defined as

$$\mathbf{V}_h = \bigoplus_{i=1}^k \mathbf{V}_{h,i}, \quad W_h = \bigoplus_{i=1}^k W_{h,i}, \quad \Lambda_h = \bigoplus_{1 \leq i < j \leq k} \Lambda_{h,i,j}.$$

REMARK 2.2. *We refer to the interface spaces $\Lambda_{h,i,j}$ as “mortar” finite element spaces, following a terminology for similar techniques used with the standard and spectral finite element methods [9]. If the order of approximation for Λ_h is the same as for $\mathbf{V}_h \cdot \nu$, then the mortar mixed methods for elliptic equations lose $O(h^{1/2})$ from the optimal order of convergence [29, 4]. A similar loss is observed for the equations considered here (see the theorems below), which motivates the above choice of mortar spaces.*

In the continuous time mixed finite element method for approximating (2.13)–(2.15) we seek, for each $t \in [0, T]$, $\psi_h(\cdot, t) \in \mathbf{V}_h$, $s_h(\cdot, t) \in W_h$, and $\gamma_h(\cdot, t) \in \Lambda_h$ such that, for $1 \leq i \leq k$,

$$(a\psi_h, \mathbf{v})_{\Omega_i} = (D(s_h), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \gamma_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} + (a\beta(s_h)\mathbf{u}, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (2.16)$$

$$(s_h, w)_{\Omega_i} + \left(\nabla \cdot \int_0^t \psi_h \, d\tau, w \right)_{\Omega_i} = \left(\int_0^t \tilde{q}_w(s_h) \, d\tau, w \right)_{\Omega_i} + (s_{0,h}, w)_{\Omega_i}, \quad w \in W_{h,i}, \quad (2.17)$$

$$\sum_{i=1}^k \left\langle \int_0^t \psi_h \cdot \nu_i \, d\tau, \mu \right\rangle_{\Gamma_i} = 0, \quad \mu \in \Lambda_h, \quad (2.18)$$

where $s_{0,h} \in W_h$ is an approximation of s_0 .

We next consider a backward Euler time discretization. Let $\{t_n\}_{n=0}^N$ be a monotone partition of $[0, T]$ with $t_0 = 0$ and $t_N = T$, let $\Delta t^n = t_n - t_{n-1}$, and let $f^n = f(t_n)$.

In the fully discrete mixed method we seek, for any $0 \leq n \leq N$, $\psi_h^n \in \mathbf{V}_h$, $s_h^n \in W_h$, and $\gamma_h^n \in \Lambda_h$ such that, for $1 \leq i \leq k$,

$$(a\psi_h^n, \mathbf{v})_{\Omega_i} = (D(s_h^n), \nabla \cdot \mathbf{v})_{\Omega_i} - \langle \gamma_h^n, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} + (a\beta(s_h^n) \mathbf{u}^n, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (2.19)$$

$$\begin{aligned} (s_h^n, w)_{\Omega_i} + \left(\nabla \cdot \sum_{j=1}^n \psi_h^j \Delta t^j, w \right)_{\Omega_i} \\ = \left(\sum_{j=1}^n \tilde{q}_w(s_h^j) \Delta t^j, w \right)_{\Omega_i} + (s_{0,h}, w)_{\Omega_i}, \quad w \in W_{h,i}, \end{aligned} \quad (2.20)$$

$$\sum_{i=1}^k \left\langle \sum_{j=1}^n \psi_h^j \cdot \nu_i \Delta t^j, \mu \right\rangle_{\Gamma_i} = 0. \quad \mu \in \Lambda_h, \quad (2.21)$$

As noted in [5], by subtracting equation (2.20) for time levels n and $n-1$, it can be rewritten in the usual backward Euler form

$$\left(\frac{s_h^n - s_h^{n-1}}{\Delta t^n}, w \right)_{\Omega_i} + (\nabla \cdot \psi_h^n, w)_{\Omega_i} = (\tilde{q}_w(s_h^n), w)_{\Omega_i}, \quad w \in W_{h,i}, \quad (2.22)$$

$$(s_h^0, w)_{\Omega_i} = (s_{0,h}, w)_{\Omega_i}, \quad w \in W_{h,i}. \quad (2.23)$$

3. ERROR ANALYSIS OF THE SEMIDISCRETE SCHEME

We start this section with a lemma [29, 4] needed later in the analysis. The proof is included for completeness.

LEMMA 3.1. *For any function $\mathbf{v} \in \mathbf{V}_{h,i}$,*

$$\|\mathbf{v} \cdot \nu_i\|_{0, \partial \Omega_i} \leq Ch^{-1/2} \|\mathbf{v}\|_{0, \Omega_i}.$$

Proof. All spaces under consideration admit nodal bases that include the degrees of freedom of the normal traces on the element boundaries. Since for any element E and any of its faces (edges) e , $|e| \leq Ch^{-1}|E|$, the lemma follows. \square

We need the following projections onto the finite element spaces. The standard mixed projection operator $\Pi : \mathbf{V}_i \rightarrow \mathbf{V}_{h,i}$ satisfies, for $\mathbf{q} \in \mathbf{V}_i$,

$$(\nabla \cdot (\mathbf{q} - \Pi \mathbf{q}), w)_{\Omega_i} = 0, \quad w \in W_{h,i}, \quad (3.1)$$

$$\langle (\mathbf{q} - \Pi \mathbf{q}) \cdot \nu_i, \mathbf{v} \cdot \nu_i \rangle_{\partial \Omega_i} = 0, \quad \mathbf{v} \in \mathbf{V}_{h,i}. \quad (3.2)$$

It is known [21] that

$$\|\Pi \mathbf{q}\|_{H(\text{div}; \Omega_i)} \leq C(\|\mathbf{q}\|_{\varepsilon, \Omega_i} + \|\nabla \cdot \mathbf{q}\|_{0, \Omega_i}). \quad (3.3)$$

In the analysis we apply Π to $\int_0^t \psi$, which is justified by Lemma 2.2.. Let, for any $\varphi \in W$, $\hat{\varphi} \in W_h$ be its L^2 -projection, satisfying

$$(\varphi - \hat{\varphi}, w) = 0, \quad w \in W_h.$$

In a similar way we define the L^2 -projections $\mathcal{P}_h : \Lambda \rightarrow \Lambda_h$, and $\mathcal{Q}_{h,i} : \Lambda \rightarrow \mathbf{V}_{h,i} \cdot \nu_i$. For smooth enough functions, these operators have optimal order approximation properties. As in [29, 4], we make explicit the following assumption on the mortar space Λ_h . There exists a positive constant C independent of h such that, for $1 \leq i < j \leq k$,

$$(H1) \quad \|\mu\|_{0,\Gamma_{i,j}} \leq C(\|\mathcal{Q}_{h,i}\mu\|_{0,\Gamma_{i,j}} + \|\mathcal{Q}_{h,j}\mu\|_{0,\Gamma_{i,j}}), \quad \forall \mu \in \Lambda_h.$$

REMARK 3.1. *Hypothesis (H1) imposes a mild condition on the mortar grids and spaces. It implies that the dimension of the mortar space on a given interface, and the distribution of its degrees of freedom, are controlled by the degrees of freedom of the traces of the velocity spaces on the two sides. This condition prevents overconstraining the matching interface conditions (2.21) and is easily satisfied in practice. See [29] for details.*

We now proceed with the error analysis. Subtracting (2.16)–(2.18) from (2.13)–(2.15), we obtain the error equations

$$\begin{aligned} (a(\psi - \psi_h), \mathbf{v})_{\Omega_i} &= (D(s) - D(s_h), \nabla \cdot \mathbf{v})_{\Omega_i} \\ &\quad - \langle \gamma - \gamma_h, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} + (a(\beta(s) - \beta(s_h))\mathbf{u}, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \end{aligned} \quad (3.4)$$

$$\begin{aligned} (s - s_h, w)_{\Omega_i} &+ \left(\nabla \cdot \int_0^t (\psi - \psi_h) d\tau, w \right)_{\Omega_i} \\ &= \left(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h)) d\tau, w \right)_{\Omega_i} + (s_0 - s_{0,h}, w)_{\Omega_i}, \quad w \in W_{h,i}, \end{aligned} \quad (3.5)$$

$$\sum_{i=1}^k \left\langle \int_0^t (\psi - \psi_h) \cdot \nu_i d\tau, \mu \right\rangle_{\Gamma_i} = 0, \quad \mu \in \Lambda_h. \quad (3.6)$$

To simplify notations, let $\overline{\Phi}(t) = \Pi \int_0^t (\psi - \psi_h) d\tau$. We choose $s_{0,h} = \hat{s}_0$ and take $\mathbf{v} = \overline{\Phi}$, $w = \widehat{D}(s) - \widehat{D}(s_h)$, and $\mu = \mathcal{P}_h \gamma - \gamma_h$ in (3.4)–(3.6). Note that our choice for v differs from this in [5], where v is taken to be an L^2 -projection of the flux error. Consequently, we are able to avoid a non-standard error term involving the divergence of the difference of the Π -projection and the L^2 -projection of $\int_0^t \psi d\tau$, which appears in [5]. We now have

$$\begin{aligned} &(s - s_h, \widehat{D}(s) - \widehat{D}(s_h)) + (a(\psi - \psi_h), \overline{\Phi}) \\ &= \left(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h)) d\tau, \widehat{D}(s) - \widehat{D}(s_h) \right) \\ &\quad + (a(\beta(s) - \beta(s_h))\mathbf{u}, \overline{\Phi}) - \sum_{i=1}^k \langle \gamma - \mathcal{P}_h \gamma, \overline{\Phi} \cdot \nu_i \rangle_{\Gamma_i} \\ &\quad + \sum_{i=1}^k \left\langle \left(\int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right) \cdot \nu_i, \mathcal{P}_h \gamma - \gamma_h \right\rangle_{\Gamma_i}. \end{aligned} \quad (3.7)$$

We integrate (3.7) in time from 0 to t . The first term on the left becomes

$$\int_0^t (s - s_h, \widehat{D}(s) - \widehat{D}(s_h)) d\tau = \int_0^t (s - s_h, D(s) - D(s_h)) d\tau + T_1, \quad (3.8)$$

where

$$T_1 = \int_0^t (\hat{s} - s, D(s) - D(s_h)) d\tau.$$

The second term on the left-hand side of (3.7) becomes

$$\int_0^t (a(\psi - \psi_h), \overline{\Phi}) d\tau = \frac{1}{2} \left\| a^{1/2} \int_0^t (\psi - \psi_h) d\tau \right\|_0^2 + T_2, \quad (3.9)$$

where

$$T_2 = \int_0^t \left(a(\psi - \psi_h), \Pi \int_0^\tau \psi d\xi - \int_0^\tau \psi d\xi \right) d\tau.$$

Combining (3.7)–(3.9), we obtain

$$\int_0^t (s - s_h, D(s) - D(s_h)) d\tau + \frac{1}{2} \left\| a^{1/2} \int_0^t (\psi - \psi_h) d\tau \right\|_0^2 = \sum_{k=1}^6 T_k, \quad (3.10)$$

where

$$\begin{aligned} T_3 &= \int_0^t \left(\int_0^\tau (\tilde{q}_w(s) - \tilde{q}_w(s_h)) d\xi, \widehat{D}(s) - \widehat{D}(s_h) \right) d\tau, \\ T_4 &= \int_0^t (a(\beta(s) - \beta(s_h))\mathbf{u}, \overline{\Phi}) d\tau, \\ T_5 &= - \sum_{i=1}^k \int_0^t \langle \gamma - \mathcal{P}_h \gamma, \overline{\Phi} \cdot \nu_i \rangle_{\Gamma_i} d\tau, \\ T_6 &= \sum_{i=1}^k \int_0^t \left\langle \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \cdot \nu_i d\xi, \mathcal{P}_h \gamma - \gamma_h \right\rangle_{\Gamma_i} d\tau. \end{aligned}$$

We now bound each T_k , $k = 1, \dots, 6$. Note that the terms T_3 and T_4 are bounded differently than in [5], using the weaker and physically reasonable assumptions (2.7) and (2.8), respectively. The terms T_5 and T_6 involve error on the non-matching interfaces and do not appear in the case of a single block. For any $\delta > 0$, we have

$$\begin{aligned} |T_1| &\leq C \int_0^t \|\hat{s} - s\|_0^2 d\tau + \delta \int_0^t \|D(s) - D(s_h)\|_0^2 d\tau, \\ |T_3| &\leq C \int_0^t \int_0^\tau (s - s_h, D(s) - D(s_h)) d\xi d\tau + \delta \int_0^t \|D(s) - D(s_h)\|_0^2 d\tau, \\ |T_4| &\leq \delta \int_0^t (s - s_h, D(s) - D(s_h)) d\tau + C \int_0^t \|\overline{\Phi}\|_0^2 d\tau, \\ |T_5| &\leq C \left\{ h^{-1} \int_0^t \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}^2 d\tau + \int_0^t \|\overline{\Phi}\|_0^2 d\tau \right\}, \end{aligned}$$

using (2.8) for the bound of T_3 , (2.7) for the bound of T_4 , and Lemma 3.1. for the bound of T_5 . To estimate T_2 we integrate by parts in time:

$$\begin{aligned} T_2 = & - \int_0^t \left(\int_0^\tau a(\psi - \psi_h) d\xi, \frac{\partial}{\partial \tau} \left(\Pi \int_0^\tau \psi d\xi - \int_0^\tau \psi d\xi \right) \right) d\tau \\ & + \left(\int_0^t a(\psi - \psi_h) d\tau, \Pi \int_0^t \psi d\tau - \int_0^t \psi d\tau \right); \end{aligned}$$

therefore,

$$\begin{aligned} |T_2| \leq C & \left\{ \int_0^t \left\| \int_0^\tau (\psi - \psi_h) d\xi \right\|_0^2 d\tau + \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\Pi \int_0^\tau \psi d\xi - \int_0^\tau \psi d\xi \right) \right\|_0^2 d\tau \right. \\ & \left. + \left\| \Pi \int_0^t \psi d\tau - \int_0^t \psi d\tau \right\|_0^2 \right\} + \delta \left\| \int_0^t (\psi - \psi_h) d\tau \right\|_0^2 \end{aligned}$$

The term T_6 is the most difficult to bound. Integration by parts in time gives

$$\begin{aligned} T_6 = & - \sum_{i=1}^k \int_0^t \left\langle \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \cdot \nu_i, \int_0^\tau (\mathcal{P}_h \gamma - \gamma_h) d\xi \right\rangle_{\Gamma_i} d\tau \\ & + \sum_{i=1}^k \left\langle \left(\int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right) \cdot \nu_i, \int_0^t (\mathcal{P}_h \gamma - \gamma_h) d\tau \right\rangle_{\Gamma_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} |T_6| \leq C & \left\{ \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \cdot \nu \right\|_{0,\Gamma}^2 d\tau \right. \\ & \left. + \left\| \left(\int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 d\tau \right\} + \delta \int_0^t \left\| \int_0^\tau (\mathcal{P}_h \gamma - \gamma_h) d\xi \right\|_{0,\Gamma}^2 d\tau \\ & + \delta \left\| \int_0^t (\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\Gamma}^2. \end{aligned}$$

To bound the last two terms, we consider, for $1 \leq i \leq n$ and any fixed $t \in (0, T]$, the auxiliary problem

$$\varphi - \Delta \varphi = 0, \quad \text{in } \Omega_i, \quad (3.11)$$

$$\nabla \varphi \cdot \nu_i = \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau, \quad \text{on } \partial \Omega_i. \quad (3.12)$$

Note that the Neumann boundary data is in $H^{1/2-\varepsilon}(\Omega_i)$ for any $\varepsilon > 0$ and by elliptic regularity

$$\|\varphi\|_{m-\varepsilon, \Omega_i} \leq C \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{m-\varepsilon-3/2, \partial \Omega_i}, \quad 1 \leq m \leq 2. \quad (3.13)$$

We now integrate in time from 0 to t and take $\mathbf{v} = \Pi \nabla \varphi$ in (3.4) to obtain

$$\begin{aligned}
& \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\partial\Omega_i}^2 \\
&= - \left(\int_0^t a(\psi - \psi_h) d\tau, \Pi \nabla \varphi \right)_{\Omega_i} + \left(\int_0^t (\widehat{D}(s) - \widehat{D}(s_h)) d\tau, \varphi \right)_{\Omega_i} \\
&+ \left(\int_0^t a(\beta(s) - \beta(s_h)) \mathbf{u} d\tau, \Pi \nabla \varphi \right)_{\Omega_i} + \left\langle \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma) d\tau, \nabla \varphi \cdot \nu_i \right\rangle_{\partial\Omega_i} \\
&= T_7 + T_8 + T_9 + T_{10}.
\end{aligned} \tag{3.14}$$

We bound the terms on the right-hand side of (3.14) as follows. For any $\delta > 0$,

$$\begin{aligned}
|T_7| &\leq C \left\| \int_0^t (\psi - \psi_h) d\tau \right\|_{0,\Omega_i}^2 + \delta (\|\nabla \varphi\|_{\varepsilon,\Omega_i}^2 + \|\nabla \cdot \nabla \varphi\|_{0,\Omega_i}^2) \\
&\leq C \left\| \int_0^t (\psi - \psi_h) d\tau \right\|_{0,\Omega_i}^2 + \delta \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\partial\Omega_i}^2,
\end{aligned}$$

using (3.3) for the first inequality and (3.13) for the last bound. Note that δ in the above inequalities is an arbitrary small generic constant which may be different each time it appears. Similarly,

$$\begin{aligned}
|T_8| &\leq C \int_0^t \|D(s) - D(s_h)\|_{0,\Omega_i}^2 d\tau + \delta \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\partial\Omega_i}^2, \\
|T_9| &\leq C \int_0^t (s - s_h, D(s) - D(s_h))_{\Omega_i} d\tau + \delta \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\partial\Omega_i}^2, \\
|T_{10}| &\leq C \int_0^t \|\mathcal{P}_h \gamma - \gamma\|_{0,\partial\Omega_i}^2 d\tau + \delta \left\| \int_0^t \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\partial\Omega_i}^2.
\end{aligned}$$

Combining together (3.14), the bounds on T_7 – T_{10} , and (H1), we obtain

$$\begin{aligned}
& \left\| \int_0^t (\mathcal{P}_h \gamma - \gamma_h) d\tau \right\|_{0,\Gamma}^2 \\
&\leq C \left\{ \left\| \int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right\|_0^2 + \|\overline{\Phi}(t)\|_0^2 + \int_0^t \|D(s) - D(s_h)\|_0^2 d\tau \right. \\
&\quad \left. + \int_0^t (s - s_h, D(s) - D(s_h)) d\tau + \int_0^t \|\mathcal{P}_h \gamma - \gamma\|_{0,\Gamma}^2 d\tau \right\}.
\end{aligned} \tag{3.15}$$

Combining (3.10), the bounds on T_1 – T_6 , (3.15), and using (2.5) and Gronwall's inequality, we arrive at the following result.

THEOREM 3.1. *Assume that (2.4)–(2.8) and (H1) hold. For the semi-discrete mixed finite element approximation (2.16)–(2.18) of problem (2.1)–(2.3), there exists a positive constant C such that, for every $t \in [0, T]$,*

$$\int_0^t (s - s_h, D(s) - D(s_h)) d\tau + \left\| \int_0^t (\psi - \psi_h) d\tau \right\|_0^2$$

$$\begin{aligned}
 &\leq C \left\{ \int_0^t \|\hat{s} - s\|_0^2 d\tau + \left\| \Pi \int_0^t \psi d\tau - \int_0^t \psi d\tau \right\|_0^2 \right. \\
 &\quad + h^{-1} \int_0^t \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}^2 d\tau + \left\| \left(\int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 \\
 &\quad + \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \right\|_0^2 d\tau \\
 &\quad \left. + \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \cdot \nu \right\|_{0,\Gamma}^2 d\tau \right\}.
 \end{aligned}$$

Theorem 3.1. bounds the size of $\|D(s) - D(s_h)\|_0$ by (2.5). It also allows us to derive a bound on $\|s - s_h\|_{-1}$ (see also [5]), where $\|\cdot\|_{-1}$ is the $H^{-1}(\Omega)$ -norm defined by

$$\|\cdot\|_{-1} = \sup_{\varphi \in H^1(\Omega)} \frac{(\cdot, \varphi)}{\|\varphi\|_1}.$$

THEOREM 3.2. *Assume that (2.4)–(2.8) and (H1) hold. For the semi-discrete mixed finite element approximation (2.16)–(2.18) of problem (2.1)–(2.3), there exists a positive constant C such that, for every $t \in [0, T]$,*

$$\begin{aligned}
 &\|s(\cdot, t) - s_h(\cdot, t)\|_{-1}^2 \\
 &\leq C \left\{ h^2 \|\hat{s} - s\|_0^2 + \int_0^t \|\hat{s} - s\|_0^2 d\tau + \left\| \Pi \int_0^t \psi d\tau - \int_0^t \psi d\tau \right\|_0^2 \right. \\
 &\quad + h^{-1} \int_0^t \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}^2 d\tau + \left\| \left(\int_0^t \psi d\tau - \Pi \int_0^t \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 \\
 &\quad + \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \right\|_0^2 d\tau \\
 &\quad \left. + \int_0^t \left\| \frac{\partial}{\partial \tau} \left(\int_0^\tau \psi d\xi - \Pi \int_0^\tau \psi d\xi \right) \cdot \nu \right\|_{0,\Gamma}^2 d\tau \right\}.
 \end{aligned}$$

Proof. For any $\varphi \in H_0^1(\Omega)$, we have

$$(s - s_h, \varphi) = (s - s_h, \varphi - \hat{\varphi}) + (s - s_h, \hat{\varphi}) = (s - \hat{s}, \varphi - \hat{\varphi}) + (s - s_h, \hat{\varphi}).$$

By (3.5) we have that

$$(s - s_h, \hat{\varphi}) = - \sum_{i=1}^k \left(\nabla \cdot \left(\Pi \int_0^t \psi d\tau - \int_0^t \psi_h d\tau \right), \hat{\varphi} \right)_{\Omega_i} + \left(\int_0^t (\tilde{q}_w(s) - \tilde{q}_w(s_h)) d\tau, \hat{\varphi} \right).$$

For the first term on the right we write

$$- \sum_{i=1}^k \left(\nabla \cdot \left(\Pi \int_0^t \psi d\tau - \int_0^t \psi_h d\tau \right), \hat{\varphi} \right)_{\Omega_i} = - \sum_{i=1}^k \left(\nabla \cdot \left(\Pi \int_0^t \psi d\tau - \int_0^t \psi_h d\tau \right), \varphi \right)_{\Omega_i}$$

$$\begin{aligned}
&= \sum_{i=1}^k \left(\Pi \int_0^t \psi \, d\tau - \int_0^t \psi_h \, d\tau, \nabla \varphi \right)_{\Omega_i} - \sum_{i=1}^k \left\langle \left(\Pi \int_0^t \psi \, d\tau - \int_0^t \psi_h \, d\tau \right) \cdot \nu_i, \varphi \right\rangle_{\Gamma_i} \\
&= \sum_{i=1}^k \left(\Pi \int_0^t \psi \, d\tau - \int_0^t \psi_h \, d\tau, \nabla \varphi \right)_{\Omega_i} - \sum_{i=1}^k \left\langle \left(\Pi \int_0^t \psi \, d\tau - \int_0^t \psi \, d\tau \right) \cdot \nu_i, \varphi \right\rangle_{\Gamma_i} \\
&\quad - \sum_{i=1}^k \left\langle \int_0^t (\psi - \psi_h) \cdot \nu_i \, d\tau, \varphi - \mathcal{P}_h \varphi \right\rangle_{\Gamma_i},
\end{aligned}$$

using (3.6) for the last equality. Therefore

$$\begin{aligned}
&|(s - s_h, \varphi)| \\
&\leq C \left\{ h \|s - \hat{s}\|_0 + \left\| \Pi \int_0^t \psi \, d\tau - \int_0^t \psi_h \, d\tau \right\|_0 + \left\| \left(\Pi \int_0^t \psi \, d\tau - \int_0^t \psi \, d\tau \right) \cdot \nu \right\|_{0,\Gamma} \right. \\
&\quad \left. + \left\| \int_0^t (\psi - \psi_h) \, d\tau \right\|_0 + \left(\int_0^t (s - s_h, D(s) - D(s_h)) \, d\tau \right)^{1/2} \right\} \|\varphi\|_1,
\end{aligned}$$

using that

$$\|\varphi - \hat{\varphi}\|_0 \leq Ch \|\varphi\|_1$$

for the first term on the right, Lemma 3.1. and

$$\|\varphi - \mathcal{P}_h \varphi\|_{0,\Gamma_i} \leq Ch^{1/2} \|\varphi\|_{1,\Omega_i}$$

for the fourth term, and (2.8) for the last term. An application of Theorem 3.1. completes the proof. \square

4. DISCRETE TIME ERROR ANALYSIS

In this section we present the analysis for the fully discrete scheme (2.19)–(2.21). The following error equations are obtained by subtracting (2.19)–(2.21) from (2.13)–(2.15) for $t = t^n$.

$$\begin{aligned}
(a(\psi^n - \psi_h^n), \mathbf{v})_{\Omega_i} &= (D(s^n) - D(s_h^n), \nabla \cdot \mathbf{v})_{\Omega_i} \\
&\quad - \langle \gamma^n - \gamma_h^n, \mathbf{v} \cdot \nu_i \rangle_{\Gamma_i} + (a(\beta(s^n) - \beta(s_h^n)) \mathbf{u}^n, \mathbf{v})_{\Omega_i}, \quad \mathbf{v} \in \mathbf{V}_{h,i}, \quad (4.1)
\end{aligned}$$

$$\begin{aligned}
(s^n - s_h^n, w)_{\Omega_i} &+ \left(\nabla \cdot \left(\int_0^{t^n} \psi \, d\tau - \sum_{j=1}^n \psi_h^j \Delta t^j \right), w \right)_{\Omega_i} \\
&= \left(\int_0^{t^n} \tilde{q}_w(s) \, d\tau - \sum_{j=1}^n \tilde{q}_w(s_h^j) \Delta t^j, w \right)_{\Omega_i} + (s_0 - s_{0,h}, w)_{\Omega_i}, \quad w \in W_{h,i}, \quad (4.2)
\end{aligned}$$

$$\sum_{i=1}^k \left\langle \int_0^{t^n} \psi \cdot \nu_i \, d\tau - \sum_{j=1}^n \psi_h^j \cdot \nu_i \Delta t^j, \mu \right\rangle_{\Gamma_i} = 0, \quad \mu \in M_h. \quad (4.3)$$

We now take $\mathbf{v} = \overline{\Phi}^n \equiv \Pi \left(\int_0^{t^n} \psi \, d\tau - \sum_{j=1}^n \psi_h^j \Delta t^j \right)$, $w = \widehat{D}(s^n) - \widehat{D}(s_h^n)$, and $\mu = \mathcal{P}_h \gamma^n - \gamma_h^n$, to obtain

$$(s^n - s_h^n, \widehat{D}(s^n) - \widehat{D}(s_h^n)) + (a(\psi^n - \psi_h^n), \overline{\Phi}^n)$$

$$\begin{aligned}
 &= \left(\int_0^{t^n} \tilde{q}_w(s) d\tau - \sum_{j=1}^n \tilde{q}_w(s_h^j) \Delta t^j, D(\widehat{s^n}) - D(\widehat{s_h^n}) \right) + (a(\beta(s^n) - \beta(s_h^n)) \mathbf{u}^n, \overline{\Phi}^n) \\
 &- \sum_{i=1}^k \langle \gamma^n - \mathcal{P}_h \gamma^n, \overline{\Phi}^n \cdot \nu_i \rangle_{\Gamma_i} + \sum_{i=1}^k \left\langle \left(\int_0^{t^n} \psi d\tau - \pi \int_0^{t^n} \psi d\tau \right) \cdot \nu_i, \mathcal{P}_h \gamma^n - \gamma_h^n \right\rangle_{\Gamma_i} \quad (4.4)
 \end{aligned}$$

We replace n by j in the above equation, multiply by Δt^j and sum on j . The first term on the left in (4.4) becomes

$$\sum_{j=1}^n (s^j - s_h^j, D(\widehat{s^j}) - D(\widehat{s_h^j})) \Delta t^j = \sum_{j=1}^n (s^j - s_h^j, D(s^j) - D(s_h^j)) \Delta t^j - T_1, \quad (4.5)$$

where

$$T_1 = - \sum_{j=1}^n (\widehat{s^j} - s^j, D(s^j) - D(s_h^j)) \Delta t^j$$

To manipulate the terms involving $\overline{\Phi}^n$ we rewrite it as

$$\overline{\Phi}^n = \Pi \int_0^{t^n} \psi d\tau - \int_0^{t^n} \psi d\tau + \sum_{j=1}^n \left(\int_{t^{j-1}}^{t^j} \psi d\tau - \psi^j \Delta t^j \right) + \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j. \quad (4.6)$$

Note that $\overline{\Phi}^n$ is represented as a sum of an approximation error term, a time discretization error term, and an error term we are trying to bound. Using (4.6), the second term on the left in (4.4) becomes

$$\begin{aligned}
 &\sum_{j=1}^n (a(\psi^j - \psi_h^j), \overline{\Phi}^j) \Delta t^j \\
 &= \sum_{j=1}^n (a(\psi^j - \psi_h^j), \sum_{l=1}^j (\psi^l - \psi_h^l) \Delta t^l) \Delta t^j - T_2 - T_3 \\
 &= \frac{1}{2} \left\| a^{1/2} \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 + \frac{1}{2} \sum_{j=1}^n \left\| a^{1/2} (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 - T_2 - T_3, \quad (4.7)
 \end{aligned}$$

where

$$\begin{aligned}
 T_2 &= - \sum_{j=1}^n \left(a(\psi^j - \psi_h^j), \Pi \int_0^{t^j} \psi d\tau - \int_0^{t^j} \psi d\tau \right) \Delta t^j, \\
 T_3 &= - \sum_{j=1}^n \left(a(\psi^j - \psi_h^j), \sum_{l=1}^j \left(\int_{t^{l-1}}^{t^l} \psi d\tau - \psi^l \Delta t^l \right) \right) \Delta t^j.
 \end{aligned}$$

For the second equality in (4.7) we used the well known identity, for any sequence $\{\alpha_j\}$,

$$\left(\sum_{j=1}^n \alpha_j \right)^2 + \sum_{j=1}^n \alpha_j^2 = 2 \sum_{j=1}^n \left(\alpha_j \sum_{l=1}^j \alpha_l \right).$$

Combining (4.4), (4.5), and (4.7), we arrive at

$$\begin{aligned} & \sum_{j=1}^n (s^j - s_h^j, D(s^j) - D(s_h^j)) \Delta t^j \\ & + \frac{1}{2} \left\| a^{1/2} \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 + \frac{1}{2} \sum_{j=1}^n \left\| a^{1/2} (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 = \sum_{m=1}^7 T_m, \end{aligned} \quad (4.8)$$

where

$$\begin{aligned} T_4 &= \sum_{j=1}^n \left(\int_0^{t^j} \tilde{q}_w(s) d\tau - \sum_{l=1}^j \tilde{q}_w(s_h^l) \Delta t^l, D(\widehat{s}^j) - D(\widehat{s}_h^j) \right) \Delta t^j, \\ T_5 &= \sum_{j=1}^n (a(\beta(s^j) - \beta(s_h^j)) \mathbf{u}^j, \overline{\Phi}^j) \Delta t^j, \\ T_6 &= - \sum_{j=1}^n \sum_{i=1}^k \langle \gamma^j - \mathcal{P}_h \gamma^j, \overline{\Phi}^j \cdot \nu_i \rangle_{\Gamma_i} \Delta t^j, \\ T_7 &= \sum_{j=1}^n \sum_{i=1}^k \left\langle \left(\int_0^{t^j} \psi d\tau - \Pi \int_0^{t^j} \psi d\tau \right) \cdot \nu_i, \mathcal{P}_h \gamma^j - \gamma_h^j \right\rangle_{\Gamma_i} \Delta t^j. \end{aligned}$$

We next bound the terms T_m , $m = 1, \dots, 7$. For any $\delta > 0$ we have

$$\begin{aligned} |T_1| &\leq C \sum_{j=1}^n \|\widehat{s}^j - s^j\|_0^2 \Delta t^j + \delta \sum_{j=1}^n \|D(s^j) - D(s_h^j)\|_0^2 \Delta t^j, \\ |T_4| &\leq C \sum_{j=1}^n \left\{ \left\| \sum_{l=1}^j \left(\int_{t^{l-1}}^{t^l} \tilde{q}_w(s) d\tau - \tilde{q}_w(s_h^l) \Delta t^l \right) \right\|_0^2 + \sum_{l=1}^j (s^l - s_h^l, D(s^l) - D(s_h^l)) \Delta t^l \right\} \Delta t^j \\ &\quad + \delta \sum_{j=1}^n \|D(s^j) - D(s_h^j)\|_0^2 \Delta t^j, \\ |T_5| &\leq \delta \sum_{j=1}^n (s^j - s_h^j, D(s^j) - D(s_h^j)) \Delta t^j + C \sum_{j=1}^n \|\overline{\Phi}^j\|_0^2 \Delta t^j, \\ |T_6| &\leq C \sum_{j=1}^n \left\{ h^{-1} \|\gamma^j - \mathcal{P}_h \gamma^j\|_{0,\Gamma}^2 + \|\overline{\Phi}^j\|_0^2 \right\} \Delta t^j, \end{aligned}$$

where we used (2.8) for the bound of T_4 , (2.7) for the bound of T_5 , and Lemma 3.1. for the bound of T_6 . To bound the rest of the terms we need the following discrete integration by parts identity. For sequences $\{A_j\}_{j=0}^n$ and $\{\beta_j\}_{j=0}^n$,

$$\sum_{j=0}^{n-1} A_j (\beta_{j+1} - \beta_j) = - \sum_{j=1}^n (A_j - A_{j-1}) \beta_j + A_n \beta_n - A_0 \beta_0. \quad (4.9)$$

We will apply (4.9) for $A_j = \sum_{l=1}^j \alpha_l$, where $\{\alpha_j\}_{j=1}^n$ is another sequence. In this case $A_0 = 0$ and (4.9) leads to

$$\sum_{j=1}^n \alpha_j \beta_j = - \sum_{j=1}^{n-1} \sum_{l=1}^j \alpha_l (\beta_{j+1} - \beta_j) + \sum_{j=1}^n \alpha_j \beta_n. \quad (4.10)$$

To estimate T_2 , we apply (4.10) with $\alpha_j = a(\psi^j - \psi_h^j)\Delta t^j$ and $\beta_j = \Pi \int_0^{t^j} \psi d\tau - \int_0^{t^j} \psi d\tau$.

$$\begin{aligned} T_2 &= \sum_{j=1}^{n-1} \left(\sum_{l=1}^j a(\psi^l - \psi_h^l)\Delta t^l, \frac{1}{\Delta t^{j+1}} \left(\Pi \int_{t^j}^{t^{j+1}} \psi d\tau - \int_{t^j}^{t^{j+1}} \psi d\tau \right) \right) \Delta t^{j+1} \\ &\quad - \left(\sum_{j=1}^n a(\psi^j - \psi_h^j)\Delta t^j, \Pi \int_0^{t^n} \psi d\tau - \int_0^{t^n} \psi d\tau \right) \\ &\leq C_1 \sum_{j=1}^{n-1} \left\| \sum_{l=1}^j a(\psi^l - \psi_h^l)\Delta t^l \right\|_0^2 \Delta t^j \\ &\quad + \sum_{j=1}^{n-1} \left\| \frac{1}{\Delta t^{j+1}} \left(\Pi \int_{t^j}^{t^{j+1}} \psi d\tau - \int_{t^j}^{t^{j+1}} \psi d\tau \right) \right\|_0^2 \Delta t^{j+1} \\ &\quad + \delta \left\| \sum_{j=1}^n a(\psi^j - \psi_h^j)\Delta t^j \right\|_0^2 + C \left\| \Pi \int_0^{t^n} \psi d\tau - \int_0^{t^n} \psi d\tau \right\|_0^2, \end{aligned}$$

where we assumed that there exists a constant $C > 0$ such that

$$\Delta t^{j+1} \leq C_1 \Delta t^j, \quad 1 \leq j \leq N-1. \quad (4.11)$$

Similarly,

$$\begin{aligned} |T_3| &\leq C_1 \sum_{j=1}^{n-1} \left\| \sum_{l=1}^j a(\psi^l - \psi_h^l)\Delta t^l \right\|_0^2 \Delta t^j \\ &\quad + \sum_{j=1}^{n-1} \left\| \frac{1}{\Delta t^{j+1}} \left(\int_{t^j}^{t^{j+1}} \psi d\tau - \psi_{j+1}\Delta t^{j+1} \right) \right\|_0^2 \Delta t^{j+1} \\ &\quad + \delta \left\| \sum_{j=1}^n a(\psi^j - \psi_h^j)\Delta t^j \right\|_0^2 + C \left\| \int_0^{t^n} \psi d\tau - \sum_{j=1}^n \psi^j \Delta t^j \right\|_0^2, \end{aligned}$$

and

$$\begin{aligned} |T_7| &\leq \delta \sum_{j=1}^{n-1} \left\| \sum_{l=1}^j (\mathcal{P}_h \gamma^l - \gamma_h^l)\Delta t^l \right\|_{0,\Gamma}^2 \Delta t^j \\ &\quad + C \sum_{j=1}^{n-1} \left\| \frac{1}{\Delta t^{j+1}} \left(\int_{t^j}^{t^{j+1}} \psi d\tau - \Pi \int_{t^j}^{t^{j+1}} \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 \Delta t^{j+1} \\ &\quad + \delta \left\| \sum_{j=1}^n (\mathcal{P}_h \gamma^j - \gamma_h^j)\Delta t^j \right\|_{0,\Gamma}^2 + C \left\| \left(\int_0^{t^n} \psi d\tau - \Pi \int_0^{t^n} \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2. \end{aligned}$$

To further estimate the first and the third terms, we consider, for $1 \leq i \leq k$ and any fixed $1 \leq n \leq N$, the auxiliary problem

$$\varphi - \Delta \varphi = 0, \quad \text{in } \Omega_i, \quad (4.12)$$

$$\nabla \varphi \cdot \nu_i = \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j)\Delta t^j, \quad \text{on } \partial \Omega_i, \quad (4.13)$$

Note that the Neumann boundary data is in $H^{1/2-\varepsilon}(\Omega_i)$ for any $\varepsilon > 0$ and by elliptic regularity

$$\|\varphi\|_{m-\varepsilon,\Omega_i} \leq C \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{m-\varepsilon-3/2,\partial\Omega_i}, \quad 1 \leq m \leq 2. \quad (4.14)$$

We now replace n by j in (4.1), multiply by Δt^j , sum on $1 \leq j \leq n$, and take $\mathbf{v} = \Pi \nabla \varphi$ to obtain

$$\begin{aligned} & \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\partial\Omega_i}^2 \\ &= - \left(\sum_{j=1}^n a(\psi^j - \psi_h^j) \Delta t^j, \Pi \nabla \varphi \right)_{\Omega_i} + \left(\sum_{j=1}^n (D(s) - D(s_h)) \Delta t^j, \varphi \right)_{\Omega_i} \\ &+ \left(\sum_{j=1}^n a(\beta(s^j) - \beta(s_h^j)) \mathbf{u}^j \Delta t^j, \Pi \nabla \varphi \right)_{\Omega_i} + \left\langle \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j, \nabla \varphi \cdot \nu_i \right\rangle_{\partial\Omega_i} \\ &= T_8 + T_9 + T_{10} + T_{11}. \end{aligned} \quad (4.15)$$

We now have, for any small generic constant $\delta > 0$,

$$\begin{aligned} |T_8| &\leq C \left\| \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_{0,\Omega_i}^2 + \delta \|\Pi \nabla \varphi\|_{0,\Omega_i}^2 \\ &\leq C \left\| \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_{0,\Omega_i}^2 + \delta (\|\nabla \varphi\|_{\delta,\Omega_i}^2 + \|\nabla \cdot \nabla \varphi\|_{0,\Omega_i}^2) \\ &\leq C \left\| \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_{0,\Omega_i}^2 + \delta \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\partial\Omega_i}^2, \end{aligned}$$

using (3.3) for the second inequality and (4.14) for the third inequality. Similarly,

$$\begin{aligned} |T_9| &\leq C \sum_{j=1}^n \|D(s) - D(s_h)\|_{0,\Omega_i}^2 \Delta t^j + \delta \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\partial\Omega_i}^2, \\ |T_{10}| &\leq C \sum_{j=1}^n (s - s_h, D(s) - D(s_h))_{\Omega_i} \Delta t^j + \delta \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\partial\Omega_i}^2, \\ |T_{11}| &\leq C \sum_{j=1}^n \|\mathcal{P}_h \gamma^j - \gamma_h^j\|_{0,\partial\Omega_i}^2 \Delta t^j + \delta \left\| \sum_{j=1}^n \mathcal{Q}_{h,i}(\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\partial\Omega_i}^2. \end{aligned}$$

Combining (4.15), the bounds on T_8 – T_{11} , and (H1), we conclude that

$$\left\| \sum_{j=1}^n (\mathcal{P}_h \gamma^j - \gamma_h^j) \Delta t^j \right\|_{0,\Gamma}^2 \leq C \left\{ \left\| \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 + \sum_{j=1}^n \|D(s^j) - D(s_h^j)\|_0^2 \Delta t^j \right.$$

$$+ \left. \sum_{j=1}^n (s^j - s_h^j, D(s^j) - D(s_h^j)) \Delta t^j + \sum_{j=1}^n \|\mathcal{P}_h \gamma^j - \gamma^j\|_{0,\Gamma}^2 \Delta t^j \right\}. \quad (4.16)$$

Putting together (4.8), the bounds on T_1 – T_7 , (4.16), manipulating $\bar{\Phi}^j$ in T_5 and T_6 as in (4.6), applying (2.5) and the discrete Gronwall's inequality, we obtain the following estimate.

THEOREM 4.1. *Assume that (2.4)–(2.8) and (H1) hold, and that there exists a constant $C_1 > 0$ such that*

$$\Delta t^{j+1} \leq C_1 \Delta t^j, \quad 1 \leq j \leq N-1.$$

For the fully discrete mixed finite element approximation (2.19)–(2.21) of (2.1)–(2.3), there exists a positive constant C such that, for any $1 \leq n \leq N$,

$$\begin{aligned} & \sum_{j=1}^n (s^j - s_h^j, D(s^j) - D(s_h^j)) \Delta t^j + \left\| \sum_{j=1}^n (\psi^j - \psi_h^j) \Delta t^j \right\|_0^2 + \sum_{j=1}^n \|(\psi^j - \psi_h^j) \Delta t^j\|_0^2 \\ & \leq C \sum_{j=1}^n \left\{ \|\widehat{s}^j - s^j\|_0^2 + \left\| \sum_{l=1}^j \left(\int_{t^{l-1}}^{t^l} \tilde{q}_w(s) d\tau - \tilde{q}_w(s^l) \Delta t^l \right) \right\|_0^2 \right. \\ & \quad + \left\| \frac{1}{\Delta t^j} \left(\Pi \int_{t^{j-1}}^{t^j} \psi d\tau - \int_{t^{j-1}}^{t^j} \psi d\tau \right) \right\|_0^2 + \left\| \frac{1}{\Delta t^j} \left(\int_{t^{j-1}}^{t^j} \psi d\tau - \psi_j \Delta t^j \right) \right\|_0^2 \\ & \quad \left. + \left\| \frac{1}{\Delta t^j} \left(\Pi \int_{t^{j-1}}^{t^j} \psi d\tau - \int_{t^{j-1}}^{t^j} \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 + h^{-1} \|\mathcal{P}_h \gamma^j - \gamma^j\|_{0,\Gamma}^2 \right\} \Delta t^j. \end{aligned}$$

The following theorem can be shown as in the proof of Theorem 3.2..

THEOREM 4.2. *Under the assumptions of Theorem 4.1., there exists a positive constant C such that, for any $1 \leq n \leq N$,*

$$\begin{aligned} & \|s^n - s_h^n\|_{-1}^2 \\ & \leq C \sum_{j=1}^n \left\{ \|\widehat{s}^j - s^j\|_0^2 + \left\| \sum_{l=1}^j \left(\int_{t^{l-1}}^{t^l} \tilde{q}_w(s) d\tau - \tilde{q}_w(s^l) \Delta t^l \right) \right\|_0^2 + h^{-1} \|\mathcal{P}_h \gamma^j - \gamma^j\|_{0,\Gamma}^2 \right. \\ & \quad + \left\| \frac{1}{\Delta t^j} \left(\Pi \int_{t^{j-1}}^{t^j} \psi d\tau - \int_{t^{j-1}}^{t^j} \psi d\tau \right) \right\|_0^2 + \left\| \frac{1}{\Delta t^j} \left(\int_{t^{j-1}}^{t^j} \psi d\tau - \psi_j \Delta t^j \right) \right\|_0^2 \\ & \quad \left. + \left\| \frac{1}{\Delta t^j} \left(\Pi \int_{t^{j-1}}^{t^j} \psi d\tau - \int_{t^{j-1}}^{t^j} \psi d\tau \right) \cdot \nu \right\|_{0,\Gamma}^2 \right\} \Delta t^j + Ch^2 \|\widehat{s}^n - s^n\|_0^2. \end{aligned}$$

REMARK 4.1. *The estimates from Theorems 3.1.–4.2. bound the discretization error by optimal order approximation terms in time or space. The term $h^{-1/2} \|\gamma - \mathcal{P}_h \gamma\|_{0,\Gamma}$ provides approximation of order $h^{1/2}$ higher than the other terms, assuming enough regularity of γ , since the functions in Λ_h are piece-wise polynomials of one degree higher than these in $\mathbf{V}_h \cdot \nu$.*

Acknowledgments. This work was partially supported by the United States Department of Energy. The author thanks Professors Todd Arbogast and Mary F. Wheeler for useful comments and discussions.

REFERENCES

1. H. W. ALT AND E. DiBENEDETTO, *Nonsteady flow of water and oil through inhomogeneous porous media*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 12 (1985), pp. 335–392.
2. H. W. ALT AND S. LUCKHAUS, *Quasilinear elliptic-parabolic differential equations*, Math. Z., 183 (1983), pp. 311–341.
3. T. ARBOGAST, *The existence of weak solutions to single porosity and simple dual-porosity models of two-phase incompressible flow*, Nonlinear Analysis, Theory, Methods and Applications, 19 (1992), pp. 1009–1031.
4. T. ARBOGAST, L. C. COWSAR, M. F. WHEELER, AND I. YOTOV, *Mixed finite element methods on non-matching multiblock grids*, TICAM Report 96-50, Texas Inst. Comp. Appl. Math., University of Texas at Austin, Nov. 1996.
5. T. ARBOGAST, M. F. WHEELER, AND N. ZHANG, *A nonlinear mixed finite element method for a degenerate parabolic equation arising in flow in porous media*, SIAM J. Numer. Anal., 33 (1996), pp. 1669–1687.
6. T. ARBOGAST AND I. YOTOV, *A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids*, in Symposium on Computational Mechanics Advances, Austin, Texas, 1997. To appear in Comput. Meth. Appl. Mech. Eng.
7. K. AZIZ AND A. SETTARI, *Petroleum reservoir simulation*, Applied Science Publishers, London and New York, 1979.
8. J. BEAR, *Dynamics of fluids in porous media*, Dover, New York, 1972.
9. C. BERNARDI, Y. MADAY, AND A. T. PATERA, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications, H. Brezis and J. L. Lions, eds., Longman Scientific & Technical, UK, 1994.
10. F. BREZZI, J. DOUGLAS, JR., R. DURÀN, AND M. FORTIN, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.
11. F. BREZZI, J. DOUGLAS, JR., M. FORTIN, AND L. D. MARINI, *Efficient rectangular mixed finite elements in two and three space variables*, RAIRO Modél. Math. Anal. Numèr., 21 (1987), pp. 581–604.
12. F. BREZZI, J. DOUGLAS, JR., AND L. D. MARINI, *Two families of mixed elements for second order elliptic problems*, Numer. Math., 88 (1985), pp. 217–235.
13. G. CHAVENT AND J. JAFFRE, *Mathematical models and finite elements for reservoir simulation*, North-Holland, Amsterdam, 1986.
14. K. FADIMBA AND R. SHARPLEY, *Galerkin finite element method for a class of porous medium equations*. Preprint.
15. ———, *A priori estimates and regularization for a class of porous medium equations*, Nonlinear World, 2 (1995), pp. 13–41.
16. A. FRIEDMAN, *The Stefan problem in several space variables*, Trans. Amer. Math. Soc., 133 (1968), pp. 51–87.
17. J. JEROME AND M. ROSE, *Error estimates for the multidimensional two-phase stefan problem*, Math. Comp., 39 (1982), pp. 377–414.
18. Y. A. KUZNETSOV AND M. F. WHEELER, *Optimal order substructuring preconditioners for mixed finite element methods on non-matching grids*, East-West J. Numer. Math., 3 (1995), pp. 127–143.

19. O. LADYZHENSKAYA, V. A. SOLONNIKOV, AND N. N. URAL'CEVA, *Linear and quasilinear equations of parabolic type*, vol. 23 of Transl. Math. Monographs, Amer. Math. Soc., Providence, R.I., 1968.
20. L. W. LAKE, *Fundamentals of enhanced oil recovery*, Prentice Hall, New Jersey, 1989.
21. T. P. MATHEW, *Domain decomposition and iterative refinement methods for mixed finite element discretizations of elliptic problems*, PhD thesis, Courant Institute of Mathematical Sciences, New York University, 1989. Tech. Rep. 463.
22. J. C. NEDELEC, *Mixed finite elements in \mathbf{R}^3* , Numer. Math., 35 (1980), pp. 315–341.
23. R. N. NOCHETTO, *Error estimates for two-phase Stefan problems in several space variables, I: Linear boundary conditions*, Calcolo, 22 (1985), pp. 457–499.
24. R. N. NOCHETTO, *Error estimates for the multidimensional singular parabolic problems*, Japan J. Appl. Math., 4 (1987), pp. 111–138.
25. O. A. OLEINIK, A. S. KALASHNIKOV, AND C. YUI-LIN, *The Cauchy problem and boundary value problems for equations of type of nonstationary filtration*, Izv. Akad. Nauk SSSR Ser Mat, 22 (1958), pp. 667–704. In Russian.
26. R. A. RAVIART AND J. M. THOMAS, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, New York, 1977, pp. 292–315.
27. M. ROSE, *Numerical methods for flows through porous media. I*, Math. Comp., 40 (1983), pp. 435–467.
28. C. S. WOODWARD AND C. N. DAWSON, *Analysis of expanded mixed finite element methods for a nonlinear parabolic equation modeling flow into variably saturated porous media*, TIACM Report 96-51, Texas Inst. Comp. Appl. Math., University of Texas at Austin, Nov. 1996.
29. I. YOTOV, *Mixed finite element methods for flow in porous media*, PhD thesis, Rice University, Houston, Texas, 1996. TR96-09, Dept. Comp. Appl. Math., Rice University and TICAM report 96-23, University of Texas at Austin.