SUPPLEMENTARY MATERIALS: FLUX-MORTAR MIXED FINITE ELEMENT METHODS ON NONMATCHING GRIDS*

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Abstract. This is a supplementary material for the paper "Flux-mortar mixed finite element methods on non-matching grids" that presents the application of the abstract theory to coupled Stokes-Darcy flow.

Key words. Flux-mortar method, mixed finite element, domain decomposition, non-matching grids, a priori error analysis

AMS subject classifications. 65N12, 65N15, 65N55

SM1. Introduction. In this supplementary material we illustrate the applicability of the general theory to multiphysics problems by formulating and analyzing the flux-mortar mixed finite element method for coupled Stokes-Darcy problems [SM2, SM4, SM6, SM7]. In the presentation we reference sections, equations, and definitions from the main document. The method is presented in section A.1. Well-posedness and error estimates are established in section A.2. Finally, in section A.3 we discuss a non-overlapping domain decomposition method via reduction to an interface problem.

Appendix A. Coupled Stokes-Darcy systems. We combine the concepts introduced in sections 3 and 4. Let Ω_S and Ω_D form a disjoint decomposition of Ω into regions of Stokes and Darcy flow, respectively. For ease of presentation, we assume that both Ω_S and Ω_D are simply connected domains. More general configurations can also be treated, see, e.g. [SM6]. Let the Stokes-Darcy interface be given by $\Gamma_{SD} := \partial\Omega_S \cap \partial\Omega_D$. Let $\Gamma_S = \partial\Omega \cap \partial\Omega_S$ and $\Gamma_D = \partial\Omega \cap \partial\Omega_D$. Denoting the restriction of a function to Ω_S or Ω_D by a subscript S or D, respectively, the governing equations of the coupled Stokes-Darcy problem are [SM7]:

$$\sigma := \tilde{\mu}\epsilon(\boldsymbol{u}_S) - p_S I, \qquad \text{in } \Omega_S, \tag{A.1a}$$
$$-\nabla\cdot\sigma = \boldsymbol{g}_S, \qquad \nabla\cdot\boldsymbol{u}_S = f_S \qquad \text{in } \Omega_S, \tag{A.1b}$$
$$\boldsymbol{u}_D = -K\nabla p_D, \qquad \nabla\cdot\boldsymbol{u}_D = f_D \qquad \text{in } \Omega_D, \tag{A.1c}$$
$$\boldsymbol{\nu}\times(\sigma\boldsymbol{\nu}) = -\boldsymbol{\nu}\times(\beta\boldsymbol{u}_S), \qquad \boldsymbol{\nu}\cdot\boldsymbol{u}_S = \boldsymbol{\nu}\cdot\boldsymbol{u}_D \qquad \text{on } \Gamma_{SD}, \tag{A.1d}$$
$$\boldsymbol{\nu}\cdot(\sigma\boldsymbol{\nu}) = -p_D \qquad \text{on } \Gamma_{SD}, \tag{A.1e}$$
$$\boldsymbol{u}_S = 0 \quad \text{on } \Gamma_S, \qquad p_D = 0 \qquad \text{on } \Gamma_D. \tag{A.1f}$$

Here, β is the Beavers-Joseph-Saffman (BJS) constant, $\boldsymbol{\nu}$ is the unit normal to Γ_{SD} oriented outward with respect to Ω_S, $\boldsymbol{\nu}\times\boldsymbol{v}$ is the cross product if $n = 3$, and $\boldsymbol{\nu}\times\boldsymbol{v} = \boldsymbol{\nu}^\perp\cdot\boldsymbol{v}$ for $n = 2$ with $\perp$ denoting a rotation of $\pi/2$.

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Let us continue by defining the function spaces $V \times W$:

$$V := \left\{ \boldsymbol{v} \in H(\mathrm{div}, \Omega) : \ \boldsymbol{v}_S \in (H^1(\Omega_S))^n, \quad \boldsymbol{v}_S|_{\Gamma_S} = 0 \right\}, \qquad W := L^2(\Omega).$$

Next, we decompose the domain as in section 2.1 such that Γ_{SD} is respected and define the index sets I_S and I_D such that $\Omega_S = \bigcup_{i \in I_S} \Omega_i$, $\Omega_D = \bigcup_{i \in I_D} \Omega_i$, and $I_\Omega = I_S \cup I_D$. The interfaces internal to Ω_S and Ω_D are denoted by Γ_{SS} and Γ_{DD}, respectively.

The variational formulation of problem (A.1) obtains the form (2.1) by defining the bilinear forms a_i and b_i per subdomain as follows [SM6, SM7]:

$$a_i(\boldsymbol{u}_i, \boldsymbol{v}_i) := (K^{-1}\boldsymbol{u}_i, \boldsymbol{v}_i)_{\Omega_i}, \qquad i \in I_D, \tag{A.2a}$$

$$a_i(\boldsymbol{u}_i, \boldsymbol{v}_i) := (\tilde{\mu}\varepsilon(\boldsymbol{u}_i), \varepsilon(\boldsymbol{v}_i))_{\Omega_i} + (\beta \boldsymbol{\nu}_i \times \boldsymbol{u}_i, \boldsymbol{\nu}_i \times \boldsymbol{v}_i)_{\Gamma_i \cap \Gamma_{SD}}, \qquad i \in I_S, \tag{A.2b}$$

$$b_i(\boldsymbol{u}_i, w_i) := (\nabla \cdot \boldsymbol{u}_i, w_i)_{\Omega_i}, \qquad i \in I_\Omega. \tag{A.2c}$$

It is shown in [SM6, SM7] that this variational formulation has a unique solution.

In turn, (2.2) leads us to consider the following norms:

$$\|\boldsymbol{v}\|_V := \sum_{i \in I_S} \|\boldsymbol{v}_i\|_{1,\Omega_i} + \sum_{i \in I_D} \|\boldsymbol{v}_i\|_{\mathrm{div},\Omega_i}, \qquad \|w\|_W := \sum_i \|w_i\|_{\Omega_i}.$$

Next, we define the local trace operators Tr_i. For $i \in I_D$, let $\mathrm{Tr}_i\, \boldsymbol{v}_i = (\boldsymbol{\nu} \cdot \boldsymbol{v}_i)|_{\Gamma_i}$, as in section 3. For $i \in I_S$, let $\mathrm{Tr}_i\, \boldsymbol{v}_i = \boldsymbol{v}_i|_{\Gamma_i}$ as in section 4. Note that this leads to a discrepancy on Γ_{SD} because $\mathrm{Tr}_i\, \boldsymbol{v}_i$ is scalar-valued for $i \in I_D$ but vector-valued for $i \in I_S$. We then define the trace space

$$\Lambda := L^2(\Gamma_{DD}) \oplus (H^{1/2}(\Gamma_{SS} \cup \Gamma_{SD}))^n.$$

Let $\Lambda_i := \{\mu|_{\Gamma_i}, \mu \in \Lambda\}$, where the meaning of the restriction on $\Gamma_i \cap \Gamma_{SD}$ is either the full vector $\boldsymbol{\mu}|_{\Gamma_i \cap \Gamma_{SD}}$ for $i \in I_S$ or the normal component $\boldsymbol{\nu} \cdot \boldsymbol{\mu}|_{\Gamma_i \cap \Gamma_{SD}}$ for $i \in I_D$. The space Λ is endowed with the norm $\|\mu\|_\Lambda := \sum_i \|\mu_i\|_{\Lambda_i}$, in which $\|\mu_i\|_{\Lambda_i}$ is given by $\|\mu_i\|_{\Gamma_i}$ for $i \in I_D$ and by (4.3) for $i \in I_S$.

A.1. Discretization. For each $i \in I_\Omega$, we choose a finite element pair $V_{h,i} \times W_{h,i} \subset V_i \times W_i$ that is stable for the Darcy subproblem if $i \in I_D$ and for the Stokes subproblem if $i \in I_S$.

We next define the discrete flux space $\Lambda_h \subset \Lambda$. On Γ_{DD}, $\Lambda_{h,D} \subset L^2(\Gamma_{DD})$ is defined interface by interface as described in section 3.1. On $\Gamma_{SS} \cup \Gamma_{SD}$, we define $\Lambda_{h,S}$ as the trace of a globally defined Lagrange finite element space, as in section 4.

Due to the boundary condition (A.1f), we redefine $I_{int} := I_S \cup \{i \in I_D : \ \partial\Omega_i \subseteq \Gamma\}$. In turn, the space S_H, defined by (2.6), is given explicitly by (3.12).

We continue with the definition of the operator $\mathcal{Q}_{h,i} : \Lambda \to \mathrm{Tr}_i\, V_{h,i}$. For $i \in I_S$, we define $\mathcal{Q}_{h,i}$ as in section 4.1. For $i \in I_D$, recall that the space Λ has a different number of components on Γ_{DD} and Γ_{SD}. On $\Gamma_i \cap \Gamma_{SD}$, let $\mathcal{Q}_{h,i}\boldsymbol{\lambda}$ be the L^2-projection of $\boldsymbol{\nu} \cdot \boldsymbol{\lambda}$ onto the normal trace space $(\mathrm{Tr}_i\, V_{h,i})|_{\Gamma_i \cap \Gamma_{SD}}$. On $\Gamma_i \cap \Gamma_{DD}$, let $\mathcal{Q}_{h,i}$ be the L^2-projection $\mathcal{Q}_{h,i}^\flat$ from section 3.1.1.

We now define the extension operator $\mathcal{R}_{h,i}$ using (2.7) and the discrete spaces $V_h \times W_h$ according to (2.8). The discrete Stokes-Darcy problem is then defined by (2.10), posed on $V_h \times W_h$, with the bilinear forms from (A.2).

Remark A.1. The choice of a full vector $\boldsymbol{\lambda}_h$ on Γ_{SD} is different from previously developed pressure-mortar methods for the Stokes-Darcy problem [SM4, SM6, SM7, SM8], where λ_h is a scalar on Γ_{SD} modeling $\boldsymbol{\nu} \cdot (\sigma \boldsymbol{\nu}) = -p_D$ and used to impose weakly $\boldsymbol{\nu} \cdot \boldsymbol{u}_S = \boldsymbol{\nu} \cdot \boldsymbol{u}_D$. In a domain decomposition implementation, the BJS boundary condition is incorporated into the subdomain problems [SM6, SM8]. In contrast, in our method, the BJS term $(\beta \boldsymbol{\nu} \times \boldsymbol{u}_i, \boldsymbol{\nu} \times \boldsymbol{v}_i)_{\Gamma_i \cap \Gamma_{SD}}$ is eliminated from the subdomain problems, since $\boldsymbol{v}_{h,i}^0 = 0$ on $\partial\Omega_i$ in (2.7a). The Stokes subdomain problems are of Dirichlet type with data $\mathcal{Q}_{h,i}\boldsymbol{\mu}_h$. In turn, the BJS boundary condition is incorporated into the coupled system (2.10) via the BJS term in the bilinear form $a(\cdot,\cdot)$ in (A.2). In the domain decomposition implementation, the BJS boundary condition is incorporated into the interface operator.

A.2. Well-posedness and error estimates. We next verify the assumptions in Theorems 2.1–2.3.

LEMMA A.1 (A1). *Problem* (2.7) *has a unique solution and the resulting extension operator $\mathcal{R}_h : \Lambda \to V_h$ is continuous, i.e. $\|\mathcal{R}_h \lambda\|_V \lesssim \|\lambda\|_\Lambda \ \forall \lambda \in \Lambda$.*

Proof. This is shown in Lemma 3.3 for $i \in I_D$ and in Lemma 4.1 for $i \in I_S$. ☐

LEMMA A.2 (A2). *The four inequalities* (2.11) *hold for a and b on $V_h \times W_h$.*

Proof. The continuity and coercivity inequalities (2.11a)–(2.11c) have been established for $i \in I_D$ in Lemma 3.4 and for $i \in I_S$ in Lemma 4.2, with a slight addaptation for $i \in I_S$, using that $(\beta \boldsymbol{\nu}_i \times \boldsymbol{u}_i, \boldsymbol{\nu}_i \times \boldsymbol{v}_i)_{\Gamma_i \cap \Gamma_{SD}} \lesssim \|\boldsymbol{u}_{h,i}\|_{1,\Omega_i} \|\boldsymbol{v}_{h,i}\|_{1,\Omega_i}$ and $(\beta \boldsymbol{\nu}_i \times \boldsymbol{v}_i, \boldsymbol{\nu}_i \times \boldsymbol{v}_i)_{\Gamma_i \cap \Gamma_{SD}} \geq 0$. Next, we prove the inf-sup condition (2.11d) by constructing $\boldsymbol{v}_h \in V_h$ for a given $w_h \in W_h$. We follow the approach from Lemmas 3.4 and 4.2 and consider a global divergence problem on Ω, cf. (3.17) to construct $\boldsymbol{v}^w \in (H^1(\Omega))^n$ with the properties

$$\nabla \cdot \boldsymbol{v}^w = w_h \text{ in } \Omega, \quad \boldsymbol{v}^w = 0 \text{ on } \Gamma_S, \quad \|\boldsymbol{v}^w\|_{1,\Omega} \lesssim \|w_h\|_\Omega.$$

Given $\boldsymbol{v}^w$, the construction of $\boldsymbol{\mu}_h$ on $\Gamma_{SS} \cup \Gamma_{SD}$ follows Lemma 4.2 and we define μ_h on Γ_{DD} according to Lemma 3.4. With the interface variable defined, each $\boldsymbol{v}_{h,i}^0 \in V_{h,i}^0$ is then constructed using the stability of the local finite element pairs in Ω_i (see Lemmas 3.4 and 4.2) such that

$$\nabla \cdot \boldsymbol{v}_{h,i}^0 = w_{h,i} - \nabla \cdot \mathcal{R}_{h,i}\mu_h, \qquad \|\boldsymbol{v}_{h,i}^0\|_{V_i} \lesssim \|w_h\|_W.$$

The combination of these constructions gives us $\boldsymbol{v}_h := \boldsymbol{v}_h^0 + \mathcal{R}_h \mu_h \in V_h$ with

$$\sum_i b_i(\boldsymbol{v}_{h,i}, w_{h,i}) = \|w_h\|_\Omega^2, \qquad \sum_i \|\boldsymbol{v}_{h,i}\|_{V_i} \lesssim \|w_h\|_W, \tag{A.3}$$

implying the inf-sup condition (2.11d). ☐

We next note that assumption A4 holds due to Lemma 3.5 and Lemma 4.3.

The interpolants are defined as in sections 3.3 and 4.3. In particular, we define Π^W as the L^2-projection onto W_h, Π^Λ as the L^2-projection onto Λ_h, and $\Pi_i^{V^\Gamma}$ as the L^2-projection onto $V_{h,i}^\Gamma$. The interpolant Π^V is as follows. First, for $i \in I_D$, let Π_i^V be the interpolant introduced in section 3.3 and for $i \in I_S$, we use the interpolant from section 4.3. Finally, Π^V is given as in (4.16). It satisfies

$$\|\boldsymbol{u} - \Pi^V \boldsymbol{u}\|_V \lesssim \sum_{i \in I_S} \|\boldsymbol{u} - \Pi_i^V \boldsymbol{u}\|_{1,\Omega_i} + \sum_{i \in I_D} \|\boldsymbol{u} - \Pi_i^V \boldsymbol{u}\|_{\mathrm{div},\Omega_i} + \|\lambda - \Pi^\Lambda \lambda\|_\Lambda. \tag{A.4}$$

LEMMA A.3 (A5). *The interpolation operator Π^V has the property*

$$b(\boldsymbol{u} - \Pi^V \boldsymbol{u}, w_h) = 0, \quad \forall w_h \in W_h. \tag{A.5}$$

Proof. The statement follows from Lemma 3.7 and Lemma 4.4. ∎

LEMMA A.4. *If* A3 *holds, then the consistency error* $\mathcal{E}_c$ *satisfies*

$$\mathcal{E}_c \lesssim \sum_{i \in I_S} \|\sigma \boldsymbol{\nu} - \Pi_i^{V^\Gamma}(\sigma \boldsymbol{\nu})\|_{\Gamma_i} + h^{-1/2} \sum_{i \in I_D} \|p_D - \mathcal{Q}_{h,i} p_D\|_{\Gamma_i}$$

Proof. We consider the numerator of the definition (2.15) of $\mathcal{E}_c$. We recall the definitions of the bilinear forms in (A.2) and apply integration by parts. Since $(\boldsymbol{u}, p)$ is the solution to (A.1), we substitute the momentum balance (A.1b), Darcy's law (A.1c), the BJS interface condition in (A.1d), and the boundary conditions (A.1f) to derive

$$\begin{aligned}
&\sum_{i \in I_S} \big(a_i(\boldsymbol{u}, \boldsymbol{v}_h) - b_i(\boldsymbol{v}_h, p) - (\boldsymbol{g}_S, \boldsymbol{v}_h)_{\Omega_i}\big) + \sum_{i \in I_D} \big(a_i(\boldsymbol{u}, \boldsymbol{v}_h) - b_i(\boldsymbol{v}_h, p)\big) \\
&\quad = \sum_{i \in I_S} \big((\sigma \boldsymbol{\nu}_i, \boldsymbol{v}_{h,i})_{\Gamma_i} + (\beta \boldsymbol{\nu}_i \times \boldsymbol{u}_i, \boldsymbol{\nu}_i \times \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}}\big) + \sum_{i \in I_D} -(p_D, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i} \\
&\quad = \sum_{i \in I_S} \big((\sigma \boldsymbol{\nu}_i, \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SS}} + (\boldsymbol{\nu}_i \cdot \sigma \boldsymbol{\nu}_i, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}}\big) + \sum_{i \in I_D} -(p_D, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i}.
\end{aligned}$$

The terms on Γ_{DD} and Γ_{SS} are bounded in section 3.4.2, cf. (3.33), and Lemma 4.5, respectively. It remains to bound the terms on Γ_{SD}. Note that there are contributions from Ω_S and Ω_D. For $i \in I_S$, we first note that the locality of the orthogonality (4.9) for each flat face F implies that $(\boldsymbol{\nu}_i \cdot (\mathcal{Q}_{h,i} \boldsymbol{\lambda} - \boldsymbol{\lambda}), \boldsymbol{\nu}_i \cdot \boldsymbol{\chi}_{h,i})_{\Gamma_i \cap \Gamma_{SD}} = 0 \ \forall \boldsymbol{\chi}_{h,i} \in V_{h,i}^F$. Using this, the term $(\boldsymbol{\nu}_i \cdot \sigma \boldsymbol{\nu}_i, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}}$ is manipulated as in Lemma 4.5, while the Darcy term $-(p_D, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}}$ is manipulated as in section 3.4.2. The two expressions are combined using the interface condition (A.1e), resulting in the bound

$$\begin{aligned}
&\sum_{i \in I_S} (\boldsymbol{\nu}_i \cdot \sigma \boldsymbol{\nu}_i, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}} + \sum_{i \in I_D} -(p_D, \boldsymbol{\nu}_i \cdot \boldsymbol{v}_{h,i})_{\Gamma_i \cap \Gamma_{SD}} \\
&\qquad \lesssim \sum_{i \in I_S} \|\sigma \boldsymbol{\nu}_i - \Pi_i^{V^\Gamma}(\sigma \boldsymbol{\nu}_i)\|_{\Gamma_i \cap \Gamma_{SD}} \|\boldsymbol{v}_{h,i}\|_{1,\Omega_i} \\
&\qquad\quad + h^{-1/2} \sum_{i \in I_D} \|p_D - \mathcal{Q}_{h,i} p_D\|_{\Gamma_i \cap \Gamma_{SD}} \|\boldsymbol{v}_{h,i}\|_{\Omega_i}.
\end{aligned}$$

The proof is completed by collecting the bounds on Γ_{DD}, Γ_{SS}, and Γ_{SD}. ∎

THEOREM A.5. *The discrete Stokes-Darcy problem* (2.10) *has a unique solution* $(\boldsymbol{u}_h, p_h) \in V_h \times W_h$. *If* A3 *holds, then there is a unique mortar solution* $\lambda_h \in \Lambda_h$. *Moreover, the following error bound holds with respect to the solution* $(\boldsymbol{u}, p)$ *of* (2.1)*:*

$$\begin{aligned}
&\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|p - p_h\|_W \\
&\quad \lesssim \sum_{i \in I_S} \|\boldsymbol{u} - \Pi_i^V \boldsymbol{u}\|_{1,\Omega_i} + \sum_{i \in I_D} \|\boldsymbol{u} - \Pi_i^V \boldsymbol{u}\|_{\mathrm{div},\Omega_i} + \|\lambda - \Pi^\Lambda \lambda\|_\Lambda + \sum_{i \in I_\Omega} \|p - \Pi_i^W p\|_{W_i} \\
&\qquad + h^{-1/2} \sum_{i \in I_D} \|p_D - \mathcal{Q}_{h,i} p_D\|_{\Gamma_i} + \sum_{i \in I_S} \|\sigma \boldsymbol{\nu} - \Pi_i^{V^\Gamma}(\sigma \boldsymbol{\nu})\|_{\Gamma_i}.
\end{aligned}$$

Proof. With assumptions A1–A5 verified above, the proof is based on Theorems 2.1–2.3. The error estimate follows by combining (2.14), the approximation property (A.4), and the estimate on the consistency error from Lemma A.4. ∎

A.3. Interface problem. The four steps from section 2.5 reduce the coupled Stokes-Darcy problem to a flux-mortar interface problem. For that, we verify the following inf-sup condition.

LEMMA A.6 (A6). *The following inf-sup condition holds for the spaces $\Lambda_h \times S_H$:*

$$\forall s_H \in S_H, \ \exists 0 \neq \mu_h \in \Lambda_h \ \text{such that}\ b(\mathcal{R}_h \mu_h, s_H) \gtrsim \|\mu_h\|_\Lambda \|s_H\|_W.$$

Proof. Setting $w_h := s_H \in S_H \subseteq W_h$ in the proof of the inf-sup condition (2.11d) in Lemma A.2 leads to a pair $(\boldsymbol{v}_h^0, \mu_h)$ with $\boldsymbol{v}_h^0 = 0$ that satisfies (A.3). ∎

Remark A.2. The reduction to interface problem results in a new non-overlapping domain decomposition method for the Stokes-Darcy problem, which satisfies velocity or flux continuity at each iteration. A related approach in the two-subdomain case is studied in [SM1], where the mortar variable on Γ_{SD} is $\lambda = \boldsymbol{\nu} \cdot \boldsymbol{u}_S = \boldsymbol{\nu} \cdot \boldsymbol{u}_D$. In earlier works, flux continuity is either relaxed via the use of Robin transmission conditions [SM3] or it is satisfied only at convergence using pressure and normal stress mortars [SM5, SM8].

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