

# Mortar Mixed Finite Element Methods on Irregular Multiblock Domains

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## Abstract

We consider an expanded version of the lowest order Raviart-Thomas mixed finite element method for elliptic equations on irregular multiblock domains. The logically rectangular subdomain grids may not match on the interfaces. Continuous or discontinuous piece-wise multilinear mortar finite element spaces are introduced on the interfaces to approximate the scalar variable (pressure) and impose flux-matching conditions. The method is further reduced via quadrature rules to cell-centered finite differences for the subdomain pressures, coupled through the mortars. Under certain subdomain smoothness assumptions, superconvergence for both the pressure and its flux is shown at the cell-centers. A parallel domain decomposition algorithm is used to solve the discrete system by reducing it to a positive definite problem in the mortar spaces.

KEYWORDS: mixed finite element, mortar finite element, error estimates, superconvergence, multiblock, non-matching grids

## 1 Introduction

In this paper we consider mixed finite element methods for second order elliptic problems on multiblock domains. We define a multiblock domain to be a simply connected domain  $\Omega \in \mathbf{R}^d$ ,  $d = 2$  or  $3$ , that is a union of non-overlapping subdomains or blocks  $\Omega_i$ ,  $i = 1, \dots, n$ . Using flow in porous media terms, we solve for the pressure  $p$  and the velocity  $\mathbf{u}$  satisfying

$$\mathbf{u} = -K\nabla p, \quad \nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \quad (1)$$

$$p = 0 \quad \text{on } \partial\Omega, \quad (2)$$

where  $K$  is a symmetric, uniformly positive definite tensor with  $L^\infty(\Omega)$  components related to the permeability. The choice of boundary conditions is merely for simplicity.

Multiblock finite element methods on non-matching grids have become increasingly popular in recent years. They allow for accurate modeling of faults and layers in porous media problems, using sliding grids in structural mechanics applications, and coupling different physics in different parts of the domain. Moreover, they combine the flexibility of modeling fairly irregular geometries with the convenience of constructing the grids locally.

Critical to these methods is to properly impose matching interface conditions. Overconstraining could lead to ill-posed schemes, while underconstraining causes loss of accuracy.

Using an interface (mortar) finite element space as a test space for forcing the matching conditions has been a common tool in standard and spectral finite element methods (see [5] and references therein). Recently mortar techniques have been successfully applied for mixed finite element methods [1, 10, 11]. An alternative non-mortar mixed method has been studied in [4].

In this paper we consider multiblock domains with each block covered by a logically rectangular grid and discretized by the lowest order Raviart-Thomas (RT<sub>0</sub>) mixed spaces [9]. Single block RT discretizations are known to be superconvergent for both pressure and velocity at the nodal points on rectangular [3, 6, 8] and smooth logically rectangular grids [2]. With a proper choice of mortar interface spaces we show convergence and superconvergence results similar to the single block case.

## 2 The expanded mortar mixed finite element method

We assume that each subdomain  $\Omega_i$ ,  $1 \leq i \leq n$ , is the image via a smooth mapping  $F_i$  of a rectangular reference domain  $\hat{\Omega}_i$ . The subdomain  $\Omega_i$  is covered by a logically rectangular curved grid  $\mathcal{T}_{h,i}$  ( $h$  is the maximum grid spacing), which is the image via  $F_i$  of a rectangular grid  $\hat{\mathcal{T}}_{h,i}$  on  $\hat{\Omega}_i$ . Let  $DF = (\partial F_i / \partial \hat{x}_j)$  be the Jacobian matrix of  $F$ , and  $J = |\det(DF)|$  be its Jacobian.

Let  $\hat{\mathbf{V}}_{h,i} \times \hat{W}_{h,i} \subset H(\text{div}; \Omega_i) \times L^2(\Omega_i)$  be the RT<sub>0</sub> mixed finite element spaces on  $\hat{\Omega}_i$  [9]. On each  $d - 1$  dimensional non-matching interface  $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$  we introduce a logically rectangular grid  $\mathcal{T}_{h,ij}$ , which is the image via  $F_i$  and  $F_j$  of rectangular grids  $\hat{\mathcal{T}}_{h,ij}^i$  and  $\hat{\mathcal{T}}_{h,ij}^j$ , respectively. We later impose a condition on  $\mathcal{T}_{h,ij}$  necessary for the stability and the convergence of the method. Let  $\hat{M}_{h,ij}^i$  be the space of either continuous or discontinuous piece-wise (bi)linear functions defined on  $\hat{\mathcal{T}}_{h,ij}^i$ .

If the subdomain grids match on  $\Gamma_{ij}$ , we may use the above choice, or we may take  $\mathcal{T}_{h,ij}$  to be the trace of the local grids and define  $\hat{M}_{h,ij}^i$  to be the standard piece-wise constant Lagrange multiplier space. We note that the latter choice on non-matching interfaces leads to only  $O(h^{1/2})$  velocity approximation ( $O(1)$  on the interfaces).

To complete the definition of the reference finite element spaces, let

$$\hat{\mathbf{V}}_h = \bigoplus_{i=1}^n \hat{\mathbf{V}}_{h,i}, \quad \hat{W}_h = \bigoplus_{i=1}^n \hat{W}_{h,i}, \quad \hat{M}_h = \bigoplus_{1 \leq i < j \leq n} \hat{M}_{h,ij}.$$

We now define the finite element spaces  $\mathbf{V}_h$ ,  $W_h$ , and  $M_h$  on the physical domain  $\Omega$  as follows (see also [2, 9]). For each  $\hat{\mathbf{v}} \in \hat{\mathbf{V}}_h$ ,  $\hat{w} \in \hat{W}_h$ , and  $\hat{\mu} \in \hat{M}_h$ , we define  $\mathbf{v} \in \mathbf{V}_h$ ,  $w \in W_h$ , and  $\mu \in M_h$  for  $x \in \Omega$  by

$$\mathbf{v}(x) = \frac{1}{J(\hat{x})} DF(\hat{x}) \hat{\mathbf{v}}(\hat{x}), \quad w(x) = \hat{w}(\hat{x}), \quad \mu(x) = \hat{\mu}(\hat{x}), \quad (3)$$

where  $x = F(\hat{x})$ ,  $\hat{x} \in \hat{\Omega}$ . The Piola transformation for the vector space preserves the normal component of the velocity across the element boundaries.

Following [2], we introduce the adjusted pressure gradient  $\hat{\mathbf{u}} = -G^{-1} \nabla p$ , where the symmetric and positive definite matrix  $G = J(DF^{-1})^T DF^{-1}$  is chosen to greatly simplify the computational problem on  $\hat{\Omega}$ . Indeed we have

$$G \hat{\mathbf{u}} = -\nabla p \quad \xrightarrow{F^{-1}} \quad \hat{\mathbf{u}} = \hat{\nabla} \hat{p},$$

$$G^T \mathbf{u} = G^T K G \tilde{\mathbf{u}} \xrightarrow{F^{-1}} \hat{\mathbf{u}} = \mathcal{K} \hat{\tilde{\mathbf{u}}}, \quad \mathcal{K} = J D F^{-1} K (D F^{-1})^T.$$

In the expanded mortar mixed method for approximating (1)–(2) we solve for  $\mathbf{u}_h \in \mathbf{V}_h$ ,  $\tilde{\mathbf{u}}_h \in \mathbf{V}_h$ ,  $p_h \in W_h$ , and  $\lambda_h \in M_h$  satisfying after a transformation through (3) for  $1 \leq i \leq n$

$$\int_{\hat{\Omega}_i} \hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}} d\hat{x} = \int_{\hat{\Omega}_i} \hat{p}_h \hat{\nabla} \cdot \hat{\mathbf{v}} d\hat{x} - \int_{\hat{\Gamma}_i} \hat{\lambda}_h \hat{\mathbf{v}} \cdot \hat{\nu}_i d\hat{\sigma}, \quad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_{h,i}, \quad (4)$$

$$\int_{\hat{\Omega}_i} \hat{\mathbf{u}}_h \cdot \hat{\mathbf{v}} d\hat{x} = \int_{\hat{\Omega}_i} \mathcal{K} \hat{\tilde{\mathbf{u}}}_h \cdot \hat{\mathbf{v}} d\hat{x}, \quad \hat{\mathbf{v}} \in \hat{\mathbf{V}}_{h,i}, \quad (5)$$

$$\int_{\hat{\Omega}_i} \hat{\nabla} \cdot \hat{\mathbf{u}}_h \hat{w} d\hat{x} = \int_{\hat{\Omega}_i} J \hat{f} \hat{w} d\hat{x}, \quad \hat{w} \in \hat{W}_{h,i}, \quad (6)$$

$$\sum_{i=1}^n \int_{\hat{\Gamma}_i} \hat{\mathbf{u}}_h \cdot \hat{\nu}_i \hat{\mu} d\hat{\sigma} = 0, \quad \hat{\mu} \in \hat{M}_h, \quad (7)$$

where  $\Gamma_i = \partial\Omega_i \setminus \partial\Omega$ . Continuity of normal flux is weakly imposed across each interface  $\Gamma_{ij}$  by (7).

Existence and uniqueness of a solution is shown in [1, 11], provided that there exists a constant  $C$  independent of  $h$  such that

$$(H1) \quad \|\phi\|_{\Gamma_{ij}} \leq C(\|\mathcal{Q}_{h,i}\phi\|_{\Gamma_{ij}} + \|\mathcal{Q}_{h,j}\phi\|_{\Gamma_{ij}}), \quad \forall \phi \in M_{h,ij},$$

where  $\mathcal{Q}_{h,i}$  is the  $L^2$ -projection onto  $\mathbf{V}_{h,i} \cdot \nu|_{\partial\Omega_i}$  and  $\|\cdot\|_S$  is the  $L^2$ -norm on  $S$ . In fact, solvability holds even if  $C$  depends on  $h$ ; however,  $C$  must be independent of  $h$  for the error estimates in Theorem 2.1.

The hypothesis (H1) is not very restrictive. It requires only that the mortar space be not too fine compared to the traces of the velocity spaces. One choice for the mortar grid that satisfies (H1) for both continuous and discontinuous mortars is to take the trace of either subdomain grid and coarsen it by two.

To further simplify the scheme, we employ the trapezoidal quadrature rule for approximating the three integrals involving a vector-vector product. This allows for a direct elimination of  $\hat{\tilde{\mathbf{u}}}_h$  and  $\hat{\mathbf{u}}_h$  on a sub-domain, leading to a finite difference scheme for  $\hat{p}_h$  at the cell centers and averages of  $\hat{\lambda}_h$  at the centers of the faces on the sub-domain boundary. The stencil in a sub-domain is 9 points if  $d = 2$  and 19 points if  $d = 3$  (see [3] for details).

The following optimal convergence and superconvergence error estimates have been shown in [1, 11], wherein  $|||\cdot|||$  is the discrete  $L^2$ -norm induced by the midpoint quadrature rule, and  $C_{F,l}$  is a constant independent of  $h$ , but dependent on  $\|F_i\|_{l,\infty}$  and  $\|F_i^{-1}\|_{l,\infty}$ .

**Theorem 2.1** *If (H1) holds, then*

$$\|\mathbf{u} - \mathbf{u}_h\| + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\| + \|p - p_h\| \leq C_{F,2} \sum_{i=1}^n (\|p\|_{2,\Omega_i} + \|\mathbf{u}\|_{1,\Omega_i}) h,$$

$$\|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\| \leq C_{F,2} \sum_{i=1}^n \|\nabla \cdot \mathbf{u}\|_{1,\Omega_i} h.$$

*If each subdomain grid is an image of a uniform grid via a  $C^2$  map,  $p \in C^{3,1}(\bar{\Omega}_i) \cap C^0(\bar{\Omega})$ ,  $\mathbf{u} \in (C^1(\bar{\Omega}_i) \cap W^{2,\infty}(\Omega_i))^d \cap H(\text{div}; \Omega)$ , and  $K \in (C^1(\bar{\Omega}_i) \cap W^{2,\infty}(\Omega_i))^{d \times d}$ , then there exists a constant  $C_{F,3}$  dependent on the solution and  $K$  as indicated, such that*

$$|||\mathbf{u} - \mathbf{u}_h||| + |||\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h||| \leq C_{F,3} h^{3/2}, \quad |||p - p_h||| + |||\nabla \cdot (\mathbf{u} - \mathbf{u}_h)||| \leq C_{F,3} h^2.$$

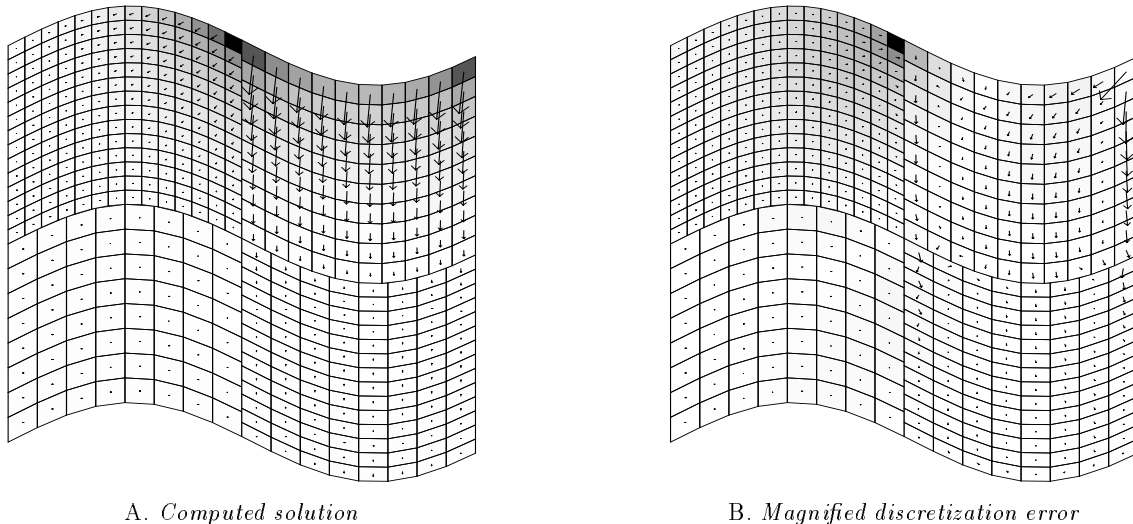


Figure 1: Computed pressure (shade) and velocity (arrows) and discretization error.

### 3 Domain decomposition and a numerical example

The discrete linear system is solved in parallel using a modification of a non-overlapping domain decomposition algorithm by Glowinski and Wheeler [7]. The original system is reduced to a positive definite problem in the mortar spaces, which is then solved using multigrid on the interface with conjugate gradient smoothing. Each conjugate gradient iteration requires subdomain solves and projections between the mortars and the local piece-wise constant spaces.

To illustrate computationally the convergence rates we give an example with known solution, coefficient, and mapping

$$p(x, y) = \begin{cases} x^2 y^3 + \cos(xy), & 0 \leq x \leq 1/2, \\ \left(\frac{2x+9}{20}\right)^2 y^3 + \cos\left(\frac{2x+9}{20}y\right), & 1/2 \leq x \leq 1, \end{cases}$$

$$K = \begin{cases} I, & 0 \leq x < 1/2, \\ 10 * I, & 1/2 < x \leq 1, \end{cases}, \quad \begin{pmatrix} x \\ y \end{pmatrix} = F \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = \begin{pmatrix} \hat{x} \\ \hat{y} + \frac{1}{10} \sin(6\hat{x}) \end{pmatrix}.$$

The discrete  $L^2$ -errors on four levels of refinement and the convergence rates for the two mortar types are given in Table 1. The computed solution and error for the continuous mortars are shown in Figure 1.

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$1/h$	Continuous mortars			Discontinuous mortars		
	$   p - p_h   $	$   \mathbf{u} - \mathbf{u}_h   $	$   \lambda - \lambda_h   $	$   p - p_h   $	$   \mathbf{u} - \mathbf{u}_h   $	$   \lambda - \lambda_h   $
8	6.75E-4	3.05E-2	1.74E-3	7.13E-4	4.60E-2	2.43E-3
16	2.22E-4	1.14E-2	5.43E-4	2.28E-4	1.72E-2	7.01E-4
32	6.42E-5	3.86E-3	1.54E-4	6.50E-5	6.05E-3	1.90E-4
64	1.72E-5	1.29E-3	4.10E-5	1.74E-5	2.10E-3	4.91E-5
1-4	$O(h^{1.77})$	$O(h^{1.53})$	$O(h^{1.80})$	$O(h^{1.79})$	$O(h^{1.49})$	$O(h^{1.88})$
3-4	$O(h^{1.90})$	$O(h^{1.59})$	$O(h^{1.91})$	$O(h^{1.90})$	$O(h^{1.53})$	$O(h^{1.95})$

Table 1: Discrete norm errors and convergence rates

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