INTERFACE SOLVERS AND PRECONDITIONERS OF DOMAIN DECOMPOSITION TYPE FOR MULTIPHASE FLOW IN MULTIBLOCK POROUS MEDIA

IVAN YOTOV *

Abstract. A multiblock mortar approach to modeling multiphase flow in porous media decomposes the simulation domain into a series of blocks with possibly different physical and numerical models employed in each block. Matching conditions along the interfaces are imposed through the use of mortar finite elements. A parallel domain decomposition algorithm reduces the algebraic nonlinear system to an interface problem which is solved via a nonlinear multigrid with Newton-GMRES smoothing.

Key words. Multiblock, mortar finite elements, non-matching grids, domain decomposition, Newton-Krylov methods, multiphase flow

AMS subject classifications. 65M55, 65M06, 65H10, 35K65, 76S05, 76T99

1. Introduction. A multiblock mortar methodology has been recently developed for modeling subsurface flow. The simulation domain is decomposed based upon the physics of the model into a series of blocks. The blocks are independently meshed and possibly different physical and numerical models are employed in each block. Interface matching conditions are imposed in a stable and accurate way through the use of mortar finite elements. Mortar finite elements have been successfully applied for standard finite element and spectral finite element discretizations on non-matching grids (see, e.g. [6, 5]). We consider locally conservative mixed finite element (finite volume) methods for subdomain discretizations. Theoretical and numerical results for single phase flow indicate mortar mixed finite element methods are highly accurate (superconvergent) for both pressure and velocity [21, 1, 4, 23]. An extension of the method to a degenerate parabolic equation arising in two phase flow is presented in [22], where optimal convergence is shown. Multiphysics applications can be found in [16].

Critical for the success of this approach is the ability to efficiently solve the resulting discrete nonlinear system. A parallel non-overlapping domain decomposition implementation, based on a method originally proposed by Glowinski and Wheeler [13, 10, 9], provides an efficient scalable solution technique [21, 19]. Some efficient preconditioners have also been developed [15, 20].

In this paper we present an efficient parallel algorithm that reduces the global system to a nonlinear interface problem. One advantage of this approach compared to solving the global system directly is that the subdomains are loosely coupled and it is relatively easy to couple different physical and numerical models. The interface problem is solved via a nonlinear multigrid V-cycle with Newton-GMRES smoothing. A physics based Neumann-Neumann preconditioner is constructed for accelerating the GMRES convergence.

The rest of the paper is organized as follows. In the next section we present a multiblock formulation and discretization for a two-phase flow model. The domain

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260; yotov@math.pitt.edu. This work was supported in part by the DOE grant DE-FG03-99ER25371, the NSF grant DMS 9873326, the University of Pittsburgh CRDF grant, and the ORAU Ralph E. Powe award.

decomposition solvers and preconditioners are described in Section 3. Computational results are given in Section 4.

2. Multiblock formulation and discretization. To illustrate the numerical technique we consider a two-phase flow model. In a multiblock formulation, the domain $\Omega \subset \mathbf{R}^3$, is decomposed into a series of subdomains Ω_k, $k = 1, ..., n_b$. Let $\Gamma_{kl} = \partial\Omega_k \cap \partial\Omega_l$ be the interface between Ω_k and Ω_l.

The governing mass conservation equations [8] are imposed on each subdomain

$$\frac{\partial(\phi\rho_\alpha S_\alpha)}{\partial t} + \nabla \cdot \mathbf{U}_\alpha = q_\alpha, \tag{2.1}$$

where $\alpha = w$ (wetting), n (non-wetting) denotes the phase, S_α is the phase saturation, $\rho_\alpha = \rho_\alpha(P_\alpha)$ is the phase density, ϕ is the porosity, q_α is the source term, and

$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\nabla P_\alpha - \rho_\alpha g \nabla D) \tag{2.2}$$

is the Darcy velocity. Here P_α is the phase pressure, $k_\alpha(S_\alpha)$ is the phase relative permeability, μ_α is the phase viscosity, K is the rock permeability tensor, g is the gravitational constant, and D is the depth. On each interface Γ_{kl} the following physically meaningful continuity conditions are imposed:

$$P_\alpha|_{\Omega_k} = P_\alpha|_{\Omega_l}, \tag{2.3}$$

$$[\mathbf{U}_\alpha \cdot \nu]_{kl} \equiv \mathbf{U}_\alpha|_{\Omega_k} \cdot \nu_k + \mathbf{U}_\alpha|_{\Omega_l} \cdot \nu_l = 0, \tag{2.4}$$

where ν_k denotes the outward unit normal vector on $\partial\Omega_k$. The above equations are coupled via the volume balance equation and the capillary pressure relation

$$S_w + S_n = 1, \quad p_c(S_w) = P_n - P_w, \tag{2.5}$$

which are imposed on each Ω_k and Γ_{kl}. We assume for simplicity that no flow $\mathbf{U}_\alpha \cdot \nu = 0$ is imposed on $\partial\Omega$, although more general types of boundary conditions can also be treated.

2.1. Discretization spaces. The subdomains are discretized using a variant of the mixed finite element method, the expanded mixed method. It has been developed for accurate and efficient treatment of irregular domains (see [3, 2] for single block and [21, 23] for multiblock domains). The original problem is transformed into a problem on a union of regular computational (reference) grids. The permeability after the mapping is usually a full tensor (except in some trivial cases). The mixed method could then be accurately approximated by cell-centered finite differences for the pressure [3].

To simplify the presentation we will only describe here the rectangular reference case. For a definition of the spaces on logically rectangular and triangular grids, we refer to [2] (also see [18, 7]). Let us denote the rectangular partition of Ω_k by $\mathcal{T}_{h_k}$, where h_k is associated with the size of the elements. The lowest order Raviart-Thomas spaces RT_0 [17] are defined on $\mathcal{T}_{h_k}$ by

$$\begin{aligned}
\tilde{\mathbf{V}}_{h_k} = \big\{ & \mathbf{v} = (v_1, v_2, v_3) : \mathbf{v}|_E = (\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)^T : \\
& \alpha_l, \beta_l \in \mathbf{R} \text{ for all } E \in \mathcal{T}_{h_k}, \\
& \text{and each } v_l \text{ is continuous in the } l\text{th coordinate direction} \big\}, \\
\mathbf{V}_{h_k} = \big\{ & \mathbf{v} \in \tilde{\mathbf{V}}_{h_k} : \mathbf{v} \cdot \nu_k = 0 \text{ on } \partial\Omega_k \cap \partial\Omega \big\} \\
W_{h_k} = \big\{ & w : w|_E = \alpha : \alpha \in \mathbf{R} \text{ for all } E \in \mathcal{T}_{h_k} \big\}.
\end{aligned}$$

To impose the interface matching condition (2.3)–(2.4) we introduce a Lagrange multiplier or mortar finite element space $M_{h_{kl}}$ defined on a rectangular grid $\mathcal{T}_{h_{kl}}$ on Γ_{kl}, where h_{kl} is associated with the size of the elements in $\mathcal{T}_{h_{kl}}$. In this space we approximate the interface pressures and saturations, and impose weakly normal continuity of fluxes.

If the subdomain grids adjacent to Γ_{kl} match, we take $\mathcal{T}_{h_{kl}}$ to be the trace of the subdomain grids and define the matching mortar space by

$$M^m_{h_{kl}} = \{\mu : \mu|_e = \alpha : \alpha \in \mathbf{R}, \text{ for all } e \in \mathcal{T}_{h_{kl}}\}.$$

If the grids adjacent to Γ_{kl} are non-matching, the interface grid need not match either of them. A mild condition on $\mathcal{T}_{h_{kl}}$ to guarantee solvability and accuracy of the numerical scheme will be imposed later. We define our non-matching mortar space on an element $e \in \mathcal{T}_{h_{kl}}$ by

$$M^n_h(e) = \big\{\alpha\xi_1\xi_2 + \beta\xi_1 + \gamma\xi_2 + \delta : \alpha, \beta, \gamma, \delta \in \mathbf{R}\big\},$$

where ξ_l are the coordinate variables on e. Then, for each Γ_{kl}, we give two possibilities for the non-matching mortar space, a discontinuous and a continuous version, as

$$M^{n,d}_{h_{kl}} = \big\{\mu : \mu|_e \in M^n_h(e) \text{ for all } e \in \mathcal{T}_{h_{kl}}\big\},$$
$$M^{n,c}_{h_{kl}} = \big\{\mu : \mu|_e \in M^n_h(e) \text{ for all } e \in \mathcal{T}_{h_{kl}},\ \mu \text{ is continuous on } \Gamma_{kl}\big\}.$$

We denote by $M_{h_{kl}}$ any choice of $M^{n,d}_{h_{kl}}$, $M^{n,c}_{h_{kl}}$, or $M^m_{h_{kl}}$ (on matching interfaces).

REMARK 2.1. The usual piece-wise constant Lagrange multiplier space for RT_0 is not a good choice in the case of non-matching grids, since it only provides $O(1)$ approximation on the interfaces and a suboptimal global convergence. With the above choice for mortar space, optimal convergence and, in some cases, superconvergence is recovered for both pressure and velocity (see [21, 1] for single phase flow and [22] for two phase flow).

2.2. The expanded mortar mixed finite element method. Before formulating the method, we note that two of the unknowns in (2.1)–(2.2) can be eliminated using relations (2.5). A common practice is to choose as primary variables one phase pressure and one phase saturation which we denote by P and S.

Following [3], let, for $\alpha = w$, n,

$$\tilde{\mathbf{U}}_\alpha = -\nabla P_\alpha.$$

Then

$$\mathbf{U}_\alpha = -\frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha(\tilde{\mathbf{U}}_\alpha - \rho_\alpha g\nabla D).$$

Let $0 = t_0 < t_1 < t_2 < \ldots$, let $\Delta t^n = t_n - t_{n-1}$, and let $f^n = f(t_n)$.

In the backward Euler multiblock expanded mixed finite element approximation of (2.1)-(2.5) we seek, for $1 \le k < l \le n_b$ and $n = 1, 2, 3\ldots$, $\mathbf{U}^n_{h,\alpha}|_{\Omega_k} \in \mathbf{V}_{h_k}$, $\tilde{\mathbf{U}}^n_{h,\alpha}|_{\Omega_k} \in \tilde{\mathbf{V}}_{h_k}$, $P^n_h|_{\Omega_k} \in W_{h_k}$, $S^n_h|_{\Omega_k} \in W_{h_k}$, and $P^{M,n}_{h,\alpha}|_{\Gamma_{kl}} \in M_{h_{kl}}$ such that, for $\alpha = w$ and n,

$$\int_{\Omega_k} \frac{S^n_{h,\alpha} - S^{n-1}_{h,\alpha}}{\Delta t^n}\, w\, dx + \int_{\Omega_k} \nabla\cdot \mathbf{U}^n_{h,\alpha}\, w\, dx = \int_{\Omega_k} q_\alpha\, w\, dx,\ w \in W_{h_k}, \tag{2.6}$$

$$\int_{\Omega_k} \tilde{\mathbf{U}}^n_{h,\alpha} \cdot \mathbf{v}\, dx = \int_{\Omega_k} P^n_{h,\alpha} \nabla \cdot \mathbf{v}\, dx - \int_{\partial\Omega_k \setminus \partial\Omega} P^{M,n}_{h,\alpha} \mathbf{v} \cdot \nu_k\, d\sigma,\ \ \mathbf{v} \in \mathbf{V}_{h_k}, \tag{2.7}$$

$$\int_{\Omega_k} \mathbf{U}^n_{h,\alpha} \cdot \tilde{\mathbf{v}}\, dx = \int_{\Omega_k} \frac{k^n_{h,\alpha} K}{\mu_{h,\alpha}} \rho^n_{h,\alpha} (\tilde{\mathbf{U}}^n_{h,\alpha} - \rho^n_{h,\alpha} g \nabla D) \cdot \tilde{\mathbf{v}}\, dx,\ \ \tilde{\mathbf{v}} \in \tilde{\mathbf{V}}_{h_k}, \tag{2.8}$$

$$\int_{\Gamma_{kl}} [\mathbf{U}^n_{h,\alpha} \cdot \nu]_{kl}\, \mu\, d\sigma = 0,\ \ \mu \in M_{h_{kl}}. \tag{2.9}$$

Here $k^n_{h,\alpha}$ and $\rho^n_{h,\alpha} \in W_{h_k}$ are given functions of the subdomain primary variables P^n_h and S^n_h. Note that we have chosen $P^{M,n}_{h,\alpha}$ as primary mortar variables. Other choices are possible due to (2.5), but this one leads to the simplest interface operator (described in the next section).

REMARK 2.2. Introducing the pressure gradients $\tilde{\mathbf{U}}_\alpha$ in the expanded mixed method allows for proper handling of the degenerate (for $S_\alpha = 0$) relative permeability $k_\alpha(S_\alpha)$ in (2.7)–(2.8). It also allows, even for a full permeability tensor K, to accurately approximate the mixed method on each subdomain by cell-centered finite differences for P_h and S_h. This is achieved by approximating the vector integrals in (2.7) and (2.8) by a trapezoidal quadrature rule and eliminating $\tilde{\mathbf{U}}_{h,\alpha}$ and $\mathbf{U}_{h,\alpha}$ from the system [3, 2].

REMARK 2.3. A necessary condition for solvability of the scheme is that, for any $\phi \in M_{h_{kl}}$,

$$Q_{h,k}\phi = Q_{h,l}\phi = 0 \Rightarrow \phi = 0, \tag{2.10}$$

where $Q_{h,k}$ is the L^2-projection onto $\mathbf{V}_{h_k} \cdot \nu_k$. This is not a very restrictive condition and requires that the mortar grid is not too fine compared to the subdomain grids. One choice that satisfies this condition for both continuous and discontinuous mortars is to take the trace of either subdomain grid and coarsen it by two in each direction (see [21, 1] for details).

3. Domain decomposition. To solve the discrete system (2.6)–(2.9) on each time step, we reduce it to an interface problem in the mortar space. This approach is based on a domain decomposition algorithm for single phase flow developed originally for conforming grids [13], and later generalized to non-matching grids coupled with mortars [21].

3.1. Interface formulation. Let

$$M_h = \bigoplus_{1 \le k < l \le n_b} M_{h_{kl}}$$

denote the mortar space on $\Gamma = \cup_{1 \le k < l \le n_b} \Gamma_{kl}$ and let $\mathbf{M}_h = M_h \times M_h$. We define a non-linear interface bivariate form $b^n : \mathbf{M}_h \times \mathbf{M}_h \to \mathbf{R}$ as follows. For $\psi = (P^{M,n}_{h,w}, P^{M,n}_{h,n})^T \in \mathbf{M}_h$ and $\mu = (\mu_w, \mu_n) \in \mathbf{M}_h$, let

$$b^n(\psi, \mu) = \sum_{1 \le k < l \le n_b} \int_{\Gamma_{kl}} ([\mathbf{U}^n_{h,w}(\psi) \cdot \nu]_{kl}\, \mu_w + [\mathbf{U}^n_{h,n}(\psi) \cdot \nu]_{kl}\, \mu_n) d\sigma,$$

where $(S^n_h(\psi), \mathbf{U}^n_{h,\alpha}(\psi))$ are solutions to the series of subdomain problems (2.6)–(2.8) with boundary data $P^{M,n}_{h,\alpha}$.

Define a non-linear interface operator $B^n : \mathbf{M}_h \to \mathbf{M}_h$ by

$$\langle B^n \psi, \mu \rangle = b^n(\psi, \mu), \quad \forall \mu \in \mathbf{M}_h,$$

where $\langle \cdot, \cdot \rangle$ is the L^2-inner product in $\mathbf{M}_h$. It is now easy to see that $(\psi, S_h^n(\psi), \mathbf{U}_{h,\alpha}^n(\psi))$ is the solution to (2.6)–(2.9), where $\psi \in \mathbf{M}_h$ solves

$$B^n(\psi) = 0. \tag{3.1}$$

3.2. Iterative solution of the interface problem. We solve the system of nonlinear equations on the interface (3.1) by an inexact Newton method. Each Newton step s is computed by a forward difference GMRES iteration for solving $B'(\psi)s = -B(\psi)$ (we omit superscript n for simplicity). On each GMRES iteration the action of the Jacobian $B'(\psi)$ on a vector μ is approximated by the forward difference

$$D_\delta B(\psi : \mu) = \begin{cases} 0, & \mu = 0, \\ ||\mu|| \frac{B(\psi + \delta ||\psi|| \mu / ||\mu||) - B(\psi)}{\delta ||\psi||}, & \mu \neq 0,\ \psi \neq 0, \\ ||\mu|| \frac{B(\delta \mu / ||\mu||) - B(\psi)}{\delta}, & \mu \neq 0,\ \psi = 0. \end{cases}$$

The choice of δ will be discussed in Section 3.3.2 The inexact Newton-GMRES algorithm is described in [14]. We present here for completeness the forward difference GMRES iteration for approximating $B'(\psi)s = -B(\psi)$.

ALGORITHM 1. `fdgmres`$(s, \psi, B, \delta, \eta, kmax, \rho)$

1. $s = 0$, $r = -B(\psi)$, $v_1 = r/\|r\|_2$, $\rho = \|r\|_2$, $\beta = \rho$, $k = 0$
2. While $\rho > \eta$ and $k < kmax$ do
   (a) $k = k + 1$
   (b) $v_{k+1} = D_\delta B(\psi : v_k)$
   (c) for $j = 1, \ldots k$
   $h_{jk} = (v_j, v_{k+1})$
   $v_{k+1} = v_{k+1} - h_{jk} v_j$
   (d) $h_{k+1,k} = \|v_{k+1}\|_2$
   (e) $v_{k+1} = v_{k+1}/\|v_{k+1}\|_2$
   (f) $e_1 = (1, 0, \ldots, 0)^T \in R^{k+1}$
   Find $y_k \in R^k$ that solves $\min_{R^k} \|\beta e_1 - H_k y_k\|_{R^{k+1}}$
   (g) $\rho = \|\beta e_1 - H_k y_k\|_{R^{k+1}}$
3. $s = V_k y_k$.

Note that each GMRES iteration in Algorithm 1 requires one evaluation of the nonlinear operator B in 2b. Here is the algorithm for evaluating $B(P_w^M, P_n^M)$.

ALGORITHM 2. `feval`(P_w^M, P_n^M)

1. Project (orthogonally) mortar data onto the subdomain grids

$$P_\alpha^M \xrightarrow{Q_k} \bar{P}_{\alpha,k}, \quad \alpha = w, n$$

2. Solve in parallel subdomain problems (2.6)–(2.8) with boundary conditions $\bar{P}_{n,k}$, $\bar{P}_{w,k}$ to compute $\mathbf{U}_{n,k}$, $\mathbf{U}_{w,k}$ on each Ω_k.

3. Project boundary fluxes back to the mortar space

$$\mathbf{U}_{\alpha,k} \cdot \nu_k \xrightarrow{Q_k^T} \mathbf{U}_{\alpha,k}^M$$

4. Compute flux jump in the mortar space. On each Γ_{kl}

$$[\mathbf{U}_\alpha^M]_{kl} = \mathbf{U}_{\alpha,k}^M + \mathbf{U}_{\alpha,l}^M$$

The evaluation of B involves solving subdomain problems (2.6)–(2.8) in parallel and two inexpensive projection steps - from the mortar grid onto the local subdomain grids and from the local grids onto the mortar grid. The subdomain problems are also nonlinear and are solved by a preconditioned Newton-Krylov solver [11].

3.3. Interface GMRES preconditioner. A well known drawback of GMRES is that it may converge very slowly if not preconditioned. A typical behavior of the unpreconditioned GMRES is shown in Fig. 4.2(a). The relative residual is plotted versus the number of iterations for the first four nonlinear Newton steps. Clearly the rate of GMRES convergence deteriorates as the Newton iteration progresses. This results in either a very expensive or a very inexact Newton step. To remedy this we consider a left preconditioned GMRES which is based on solving

$$M^{-1}B'(\psi)s = -M^{-1}B(\psi),$$

where M should be an easily invertible approximation to $B'(\psi)$. The only two changes in Algorithm 1 are in Step 1,

$$r = -M^{-1}B(\psi),$$

and in Step 2b,

$$v_{k+1} = M^{-1}D_\delta B(\psi : v_k).$$

REMARK 3.1. The preconditioned GMRES is consistent with the underlying physical interpretation of the interface operator B while the unpreconditioned GMRES is not. Recall that $B : (P_w^M, P_n^M)^T \to (\mathbf{U}_w \cdot \nu, \mathbf{U}_n \cdot \nu)^T$ is a Dirichlet to Neumann operator. Therefore in Step 2b of Algorithm 1 the two consecutive Krylov vectors v_k and v_{k+1} have different physical meanings. This inconsistency is corrected by the preconditioned step, since

$$M^{-1} : (\mathbf{U}_w \cdot \nu, \mathbf{U}_n \cdot \nu)^T \to (P_w^M, P_n^M)^T$$

and

$$M^{-1}D_\delta B(\psi : v_k) : (P_w^M, P_n^M)^T \to (P_w^M, P_n^M)^T.$$

Therefore the preconditioned GMRES builds a Krylov basis for the space $(P_w^M, P_n^M)^T$.

3.3.1. A Neumann-Neumann preconditioner for $D_\delta B$. We can write the interface operator B and the approximation to its derivative $D_\delta B$ as sums of local subdomain Dirichlet to Neumann operators

$$B = \sum_k^{n_b} B_k, \quad D_\delta B = \sum_k^{n_b} D_\delta B_k.$$

The Neumann-Neumann preconditioner M^{-1} is defined as a sum of (possibly inexact) local Neumann solves (see [12]):

$$M^{-1} = \sum_k^{n_b} \widehat{D_\delta B_k}^{-1},$$

where $\widehat{D_\delta B_k}^{-1}$ is an approximation to $(D_\delta B_k)^{-1}$.

As a first step the Jacobian matrix $B_k'(P_w^M, P_n^M)$ is approximated by its block-diagonal

$$B_k'(P_w^M, P_n^M) = \begin{pmatrix} \frac{\partial \mathbf{U}_{w,k}}{\partial P_w^M} & \frac{\partial \mathbf{U}_{w,k}}{\partial P_n^M} \\ \frac{\partial \mathbf{U}_{n,k}}{\partial P_w^M} & \frac{\partial \mathbf{U}_{n,k}}{\partial P_n^M} \end{pmatrix} \sim \begin{pmatrix} \frac{\partial \mathbf{U}_{w,k}}{\partial P_w^M} & 0 \\ 0 & \frac{\partial \mathbf{U}_{n,k}}{\partial P_n^M} \end{pmatrix}.$$

To compute the diagonal blocks, instead of solving full Dirichlet subdomain problems, we approximate the Darcy's Law (2.2) by a finite difference on a single-cell layer along the interface:

$$\hat{\mathbf{U}}_{\alpha,k} \cdot \nu_k(P_\alpha^M) \sim \lambda_{\alpha,k} \frac{P_{\alpha,k} - P_\alpha^M}{h_k/2},$$

where $\lambda_\alpha = \frac{k_\alpha(S_\alpha)K}{\mu_\alpha}\rho_\alpha$ is the mobility of phase α, $P_{\alpha,k}$ is the pressure in the cell next to the interface, and the gravity term has been omitted for simplicity. This leads to an approximation to $D_\delta B_k(P_\alpha^M : s)$

$$\widehat{D_\delta B_k}(P_\alpha^M : s) = \frac{\hat{B}_k(P_\alpha^M + \delta s) - \hat{B}_k(P_\alpha^M)}{\delta} = -2\frac{\lambda_{\alpha,k}}{h_k} s$$

The preconditioner M^{-1} is now defined for a given mortar flux v_α^M

$$M^{-1} v_\alpha^M = -\sum_k^{n_b} \frac{h_k}{2\lambda_{\alpha,k}} v_\alpha^M.$$

3.3.2. Choosing the forward difference step δ. It is customary to expect that preconditioning a linear system should not change the solution. In our case, however, $D_\delta B$ is a nonlinear approximation to B' and the preconditioned solution differs from the unpreconditioned one

$$s_{precond} = s_{unprecond} + O(\delta).$$

It is therefore desirable to take δ as small as possible. On the other hand, due to inexact evaluation of the operator B, δ should not be too small. It is easy to check that an optimal value is $\delta \sim \sqrt{\eta_B}$, where η_B is the subdomain nonlinear tolerance. Therefore taking η_B as small as possible should lead to better performance of the preconditioned GMRES. This conclusion is confirmed by the numerical experiments presented in Section 4.

REMARK 3.2. It may seem that asking for a very small subdomain tolerance will increase the computational time. Recall however that subdomain problems are repeatedly solved during the GMRES iteration with boundary data very slightly perturbed in the order of $O(\delta)$. If the latest subdomain solution is used as an initial guess, the subdomain solves typically converge in a single nonlinear iteration.

3.4. Multigrid on the interface. A nonlinear mortar multigrid solver solver has been developed to further speed up the interface convergence. The preconditioned GMRES described above is used as a smoother. The algorithm is based on the one presented in [20] for linear elliptic problems. We define a sequence of nested mortar spaces

$$\mathbf{M}_1 \subset \mathbf{M}_2 \subset \cdots \subset \mathbf{M}_J = \mathbf{M}.$$

Each space $\mathbf{M}_j$, $1 \le j \le J$, is associated with a mortar grid $\mathcal{T}_{h_j}^\Gamma$ and an interface operator $B_j : \mathbf{M}_j \to \mathbf{M}_j$ satisfying for any $\psi \in \mathbf{M}_j$

$$\langle B_j\psi, \mu\rangle = \sum_{1\le k<l\le n_b} \int_{\Gamma_{kl}} ([\mathbf{U}_{h_j,w}(\psi)\cdot\nu]_{kl}\,\mu_w + [\mathbf{U}_{h_j,n}(\psi)\cdot\nu]_{kl}\,\mu_n)d\sigma, \quad \forall \mu \in \mathbf{M}_j.$$

The intergrid transfer operators are defined as follows.

Coarse to fine $\mathcal{I}_j : \mathbf{M}_{j-1} \to \mathbf{M}_j$,

$$\mathcal{I}_j \mu_{j-1} = \mu_{j-1}, \quad \mu_{j-1} \in \mathbf{M}_{j-1}.$$

Note that $\mathcal{I}_j$ is the identity operator on $\mathbf{M}_{j-1}$.

Fine to coarse $\mathcal{Q}_{j-1} : \mathbf{M}_j \to \mathbf{M}_{j-1}$,

$$\langle \mathcal{Q}_{j-1}\psi_j, \mu_{j-1} \rangle = \langle \psi_j, \mathcal{I}_j \mu_{j-1} \rangle, \quad \psi_j \in M_j,\ \mu_{j-1} \in M_{j-1}.$$

Note that $\mathcal{Q}_{j-1}$ is the orthogonal projection onto M_{j-1} and the transpose of $\mathcal{I}_j$.

The nonlinear multigrid V-cycle is defined as an iterative process for solving $B(\psi) = r$:

$$\psi_{(n+1)} = MG(\psi_{(n)}, r),$$

where $MG = MG_J$ is the multigrid operator defined by induction.

ALGORITHM 2. $MG_j(g_j, r_j)$ $(2 \le j \le J)$

1. (initialization) $\psi_j^{(0)} = g_j$
2. (pre-smoothing) $\psi_j^{(1)} = \psi_j^{(0)} + s_m(\psi_j^{(0)})$, where $s_m(\psi_j^{(0)})$ is the $m-$th GMRES iterate for solving $B_j'(\psi_j^{(0)})s = r_j - B_j(\psi_j^{(0)})$.
3. (coarse grid correction)
   (a) Initialize level $j-1$: $\psi_{j-1}^{(0)} = \mathcal{Q}_{j-1}(\psi_j^{(1)})$
   (b) Project residual: $r_{j-1} = B_{j-1}(\psi_{j-1}^{(0)}) - \mathcal{Q}_{j-1}(r_j - B_j(\psi_j^{(1)}))$
   (c) Correct: $\psi_j^{(2)} = \psi_j^{(1)} + \mathcal{I}_j[MG_{j-1}(\psi_{j-1}^{(0)}, r_{j-1}) - \psi_{j-1}^{(0)}]$
4. (post-smoothing) $\psi_j^{(3)} = \psi_j^{(2)} + s_m(\psi_j^{(2)})$
   $MG_j(g_j, r_j) = \psi_j^{(3)}$

Note that the smoothing step is equivalent to taking an inexact Newton step for solving $B_j(\lambda) = r_j$.

4. Computational results. In this section we present numerical results illustrating the efficiency of the solvers described in the previous section. In the first test we study the effect of the preconditioner on the convergence of the interface solver. A two phase oil-water displacement is simulated on a three dimensional domain which consists of three blocks. There are three injection wells and one production well. The subdomain grid dimensions are $12 \times 20 \times 20$, $20 \times 20 \times 20$, and $12 \times 20 \times 20$ and the mortar grids (continuous linears) are 10×10. The numerical grids and the oil pressure contours at early time are shown in Figure 4.1 (note that the vertical direction is x). The interface GMRES performance for a typical time step is illustrated in Figure 4.2. It is clear that the preconditioner improves substantially the GMRES convergence as the outer iteration progresses. As a result the Newton steps are more accurate and the interface Newton converges much faster as can be seen in Figure 4.3.

The second test studies the effect of the forward difference step δ and the subdomain nonlinear tolerance η_B on the interface Newton performance which was discussed in Section 3. The results given in Figure 4.4 confirm the conclusion that smaller values of δ and η_B lead to faster convergence.

The last experiment illustrates the convergence of the nonlinear interface multigrid solver. The domain consists of two blocks with grids $8 \times 16 \times 16$ and $8 \times 20 \times 20$ with a discontinuous linears mortar interface on a 4×10 grid. Table 4.1 presents the

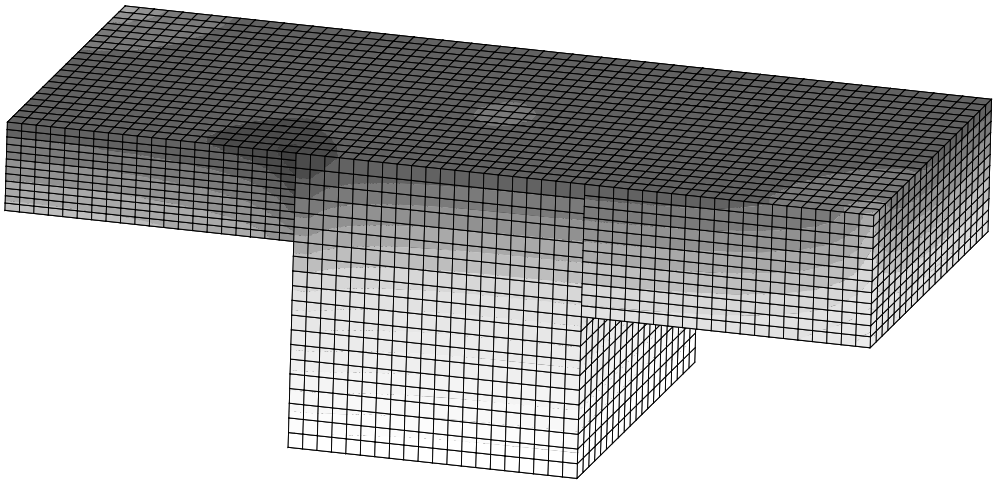

FIG. 4.1. *Numerical grids and oil pressure contours*

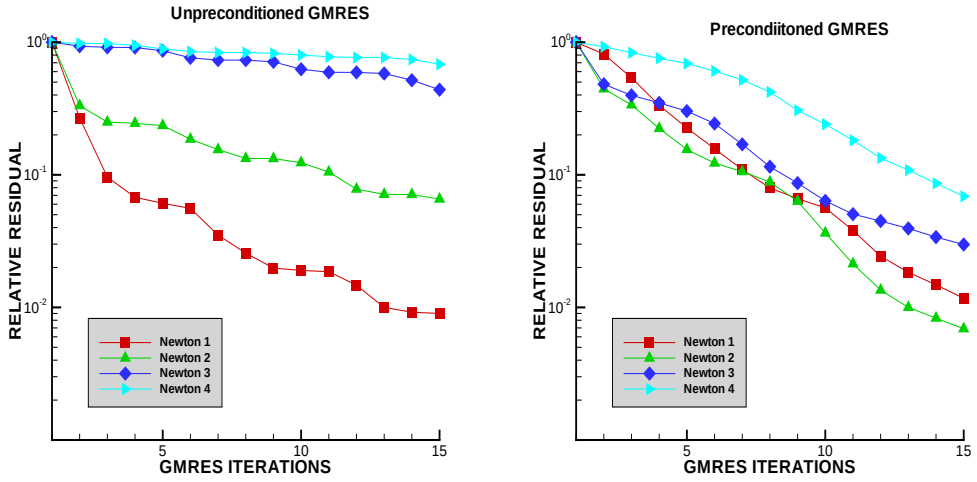

FIG. 4.2. *Effect of preconditioner on interface GMRES convergence*

number of subdomain solves and V-cycles as well as the residual reduction (averaged over all time steps) for various number of levels and smoothings.

Acknowledgments. The computational results in this paper were obtained by running the parallel subsurface simulator IPARS developed in the Center for Subsurface Modeling at University of Texas at Austin. The author thanks Qin Lu, Malgo Peszynska, and Mary Wheeler for their substantial contribution to the code development.

REFERENCES

[1] T. ARBOGAST, L. C. COWSAR, M. F. WHEELER, AND I. YOTOV, *Mixed finite element methods on non-matching multiblock grids*, SIAM J. Numer. Anal., 37 (2000), pp. 1295–1315.

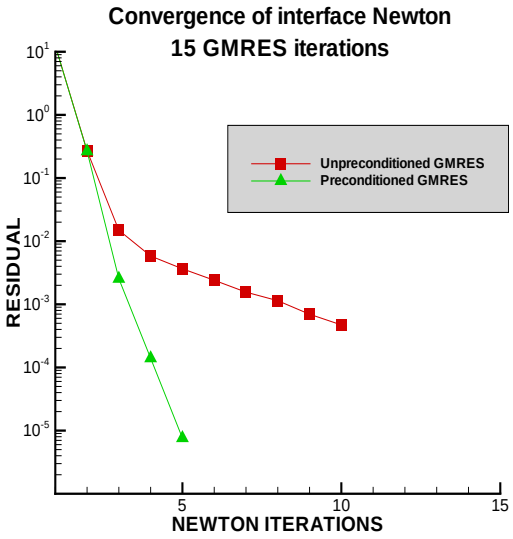

FIG. 4.3. *Effect of preconditioner on interface Newton convergence*

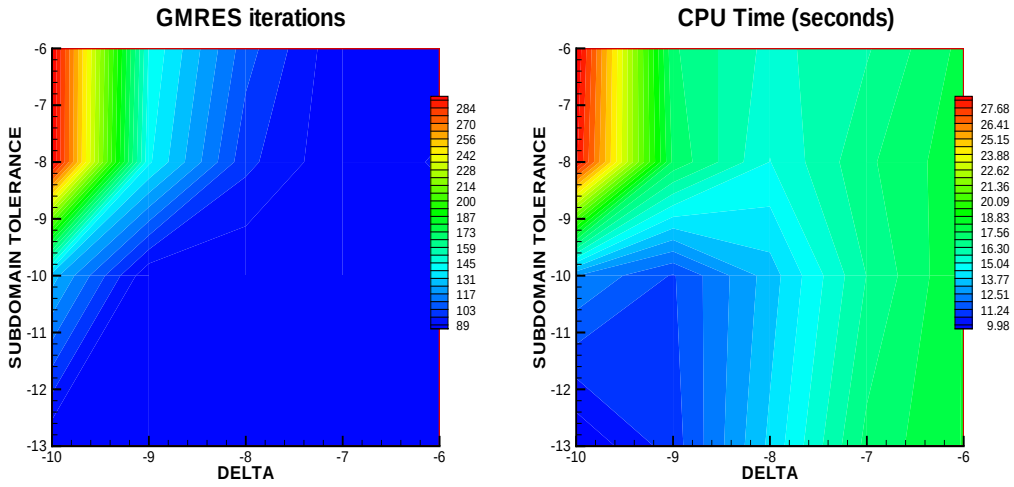

FIG. 4.4. *Effect of the forward difference step δ on the interface Newton performance*

mg levels	smoothings	avg. Fevals	avg. V-cycles	avg. resid. reduction
2	2	61	4	0.07
3	2	81	3.3	0.06
2	4	57	2.3	0.0058
3	4	87	2.3	0.0065

TABLE 4.1
Nonlinear interface multigrid performance

[2] T. Arbogast, C. N. Dawson, P. T. Keenan, M. F. Wheeler, and I. Yotov, *Enhanced cell-centered finite differences for elliptic equations on general geometry*, SIAM J. Sci. Comp., 19 (1998), pp. 404–425.

[3] T. Arbogast, M. F. Wheeler, and I. Yotov, *Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences*, SIAM J. Numer. Anal., 34 (1997), pp. 828–852.

[4] T. Arbogast and I. Yotov, *A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids*, Comput. Meth. Appl. Mech. Eng., 149 (1997), pp. 255–265.

[5] F. Ben Belgacem, *The mortar finite element method with Lagrange multiliers*, Numer. Math., 84 (1999), pp. 173–197.

[6] C. Bernardi, Y. Maday, and A. T. Patera, *A new nonconforming approach to domain decomposition: the mortar element method*, in Nonlinear partial differential equations and their applications, H. Brezis and J. L. Lions, eds., Longman Scientific & Technical, UK, 1994.

[7] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer-Verlag, New York, 1991.

[8] G. Chavent and J. Jaffre, *Mathematical models and finite elements for reservoir simulation*, North-Holland, Amsterdam, 1986.

[9] L. C. Cowsar, J. Mandel, and M. F. Wheeler, *Balancing domain decomposition for mixed finite elements*, Math. Comp., 64 (1995), pp. 989–1015.

[10] L. C. Cowsar and M. F. Wheeler, *Parallel domain decomposition method for mixed finite elements for elliptic partial differential equations*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, Y. Kuznetsov, G. Meurant, J. Periaux, and O. Widlund, eds., SIAM, Philadelphia, 1991.

[11] C. N. Dawson, H. Klie, M. F. Wheeler, and C. Woodward, *A parallel, implicit, cell-centered method for two-phase flow with a preconditioned Newton-Krylov solver*, Comput. Geosci., 1 (1997), pp. 215–249.

[12] Y. H. De Roeck and P. Le Tallec, *Analysis and test of a local domain decomposition preconditioner*, in Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski et al., eds., SIAM, Philadelphia, 1991.

[13] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., SIAM, Philadelphia, 1988, pp. 144–172.

[14] C. T. Kelley, *Iterative methods for linear and nonlinear equations*, SIAM, Philadelphia, 1995.

[15] Y. A. Kuznetsov and M. F. Wheeler, *Optimal order substructuring preconditioners for mixed finite element methods on non-matching grids*, East-West J. Numer. Math., 3 (1995), pp. 127–143.

[16] M. Peszynska, Q. Lu, and M. F. Wheeler, *Multiphysics coupling of codes*, Tech. Rep. 00-02, TICAM, University of Texas at Austin, 2000.

[17] R. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, vol. 606, Springer-Verlag, New York, 1977, pp. 292–315.

[18] J. M. Thomas, *These de Doctorat d'etat*, 'a l'Universite Pierre et Marie Curie, 1977.

[19] M. F. Wheeler and I. Yotov, *Physical and computational domain decompositions for modeling subsurface flows*, in Tenth International Conference on Domain Decomposition Methods, Contemporary Mathematics, vol 218, J. Mandel et al., eds., American Mathematical Society, 1998, pp. 217–228.

[20] ———, *Multigrid on the interface for mortar mixed finite element methods for elliptic problems*, Comput. Meth. Appl. Mech. Eng., 184 (2000), pp. 287–302.

[21] I. Yotov, *Mixed finite element methods for flow in porous media*, PhD thesis, Rice University, Houston, Texas, 1996. TR96-09, Dept. Comp. Appl. Math., Rice University and TICAM report 96-23, University of Texas at Austin.

[22] ———, *A mixed finite element discretization on non-matching multiblock grids for a degenerate parabolic equation arising in porous media flow*, East-West J. Numer. Math., 5 (1997), pp. 211–230.

[23] ———, *Mortar mixed finite element methods on irregular multiblock domains*, in Iterative Methods in Scientific Computation, IMACS series Comp. Appl. Math., J. Wang, M. B. Allen, B. Chen, and T. Mathew, eds., vol. 4, IMACS, 1998, pp. 239–244.