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# A non-mortar mixed finite element method for elliptic problems on non-matching multiblock grids☆

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**Abstract**

We consider the approximation of second-order elliptic equations on domains that can be described as a union of sub-domains or blocks. We assume that a grid is defined on each block independently, so that the resulting grid over the entire domain need not be conforming (i.e. match) across the block boundaries. Several techniques have been developed to approximate elliptic equations on multiblock grids that utilize a mortar finite element space defined on the block boundary interface itself. We define a mixed finite element method that does not use such a mortar space. The method has an advantage in the case where adaptive local refinement techniques will be used, in that there is no mortar grid to refine. As is typical of mixed methods, our method is locally conservative element-by-element; it is also globally conservative across the block boundaries. Theoretical results show that the approximate solution converges at the optimal rate to the true solution. We present computational results to illustrate and confirm the theory.

## 1. Introduction

It is often advantageous to approximate second-order elliptic and parabolic equations by mixed finite element methods because of their local conservation property and their direct approximation of the vector flux variable. In many applications the complexity of the domain geometry or the solution itself warrants using a multiblock domain structure, wherein the domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, is decomposed into non-overlapping blocks or subdomains $\Omega_i$, $i = 1, \ldots, n$, with grids defined independently on each block. On the $(d-1)$-dimensional interface $\Gamma$ between subdomain blocks, the traces of the grids need not coincide. Two typical examples in subsurface porous medium applications are the modeling of faults, which are natural discontinuities in material properties, and the modeling of wells, the solution's response to which can be resolved often only by using locally refined grids.

We use notation appropriate for applications to porous media, and we consider a model problem. For the unknown pressure scalar function $p(\boldsymbol{x})$ and Darcy velocity vector function $\boldsymbol{u}(\boldsymbol{x})$, we consider the partial differential boundary value problem

$$\boldsymbol{u} = -K(\nabla p - \gamma p) \quad \text{in } \Omega, \tag{1}$$

$$cp + \nabla \cdot \boldsymbol{u} = q \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{u} \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{3}$$

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where $K(\boldsymbol{x})$ is a symmetric, uniformly positive definite tensor representing the permeability divided by the viscosity, $\gamma(\boldsymbol{x})$ is a vector representing gravity effects, $c(\boldsymbol{x}) \geq 0$ represents the compressibility of the medium (after time discretization), and $q(\boldsymbol{x})$ represents the sources (and information from the previous time step). We assume that $c$ and each component of $K$ and $\gamma$ is in $L^\infty(\Omega)$, and that $q \in L^2(\Omega)$. We will assume that $c$ is uniformly positive, and that $\gamma$ is sufficiently small so as to give coercivity (as defined later in (17)). The homogeneous Neumann condition imposed is appropriate for this application; however, other boundary conditions can be handled easily.

A number of papers deal with the analysis and the implementation of mixed methods for our problem on conforming grids (see e.g. [2–4,6–8,10–12,14,17–19,21,22] and the general references [9,20]). Mixed methods on nested locally refined grids are considered in [13,15], but these techniques rely heavily on the fact that the grids are nested and cannot be extended directly to arbitrary non-matching grids.

Techniques have been developed to approximate the problem on non-matching multiblock grids using Galerkin and spectral approximations in the blocks and tying these together through an approximation of the pressure on $\Gamma$. This interface pressure resides in a special finite element space called a mortar space (see [5] and the references cited therein). Recently, using similar ideas, the authors with Cowsar and Wheeler defined and analyzed a mixed finite element approximation to the problem that incorporates a mortar pressure approximation on these interfaces [23,1].

In this work we define and analyze a mixed method that uses no mortar space. One advantage of this approach is that if one desires to adaptively refine grids locally, then there is no need to refine the interface mortar grid on $\Gamma$. Such refinement could be difficult to implement, especially in parallel, since accuracy depends subtly on the relations between the mortar grid and the traces of the grids of the subdomain blocks [1].

We assume that $\Omega$ is a union of non-overlapping polygonal blocks, each covered by a conforming, affine finite element partition. On each block, we employ a standard, partially hybridized mixed method. Lagrange multiplier pressures are introduced on the faces (or edges) of the elements lying on the block interfaces $\Gamma$, as in [4,9,16]. Since the grids are different on the two sides of $\Gamma$, these Lagrange multiplier pressures are doubly valued. Robin type conditions are imposed on $\Gamma$ to unite the subdomain problems. As we show later in the analysis, our method is optimally convergent.

The scheme can be implemented efficiently in parallel using non-overlapping domain decomposition techniques (see e.g. [16]). If $\gamma \equiv 0$, the problem is symmetric; nevertheless, our scheme will be non-symmetric. We comment later on a symmetric variant.

The rest of the paper is organized as follows. An appropriate variational form of the problem is defined in the next section. This form respects the multiblock structure of the domain. The mixed finite element method is presented in Section 3. In Section 4 the scheme is shown to be uniquely solvable and stable. Section 5 is devoted to the error analysis. Finally, in the last section, we define the symmetric variant of our scheme and use it to obtain numerical results that illustrate the performance of our scheme.

## 2. A multiblock variational form

Let $\Omega$ be decomposed into non-overlapping subdomain blocks $\Omega_i$, $i = 1, \ldots, n$; that is, $\Omega$ is the interior of $\bigcup_{i=1}^n \overline{\Omega}_i \subset \mathbb{R}^d$, where $\overline{\Omega}_i$ is the closure of the open set $\Omega_i$. Let $\Gamma_i$ be the interior of $\partial\Omega_i \backslash \partial\Omega$. Then we can define the interface between blocks $i$ and $j$ as $\Gamma_{ij} = \Gamma_i \cap \Gamma_j$ for $i \neq j$ and $\Gamma_{ii} = \emptyset$. The entire interface is $\Gamma = \bigcup_{i,j} \Gamma_{ij}$.

Let $\nu_i$ denote the outer unit normal to $\partial\Omega_i$. Define

$$H_0(\mathrm{div}; \Omega_i) = \{\boldsymbol{v} \in (L^2(\Omega_i))^d : \nabla \cdot \boldsymbol{v} \in L^2(\Omega_i) \text{ and } \boldsymbol{v} \cdot \nu_i = 0 \text{ on } \partial\Omega\}.$$

We denote the $L^2(\Omega_i)$ (or $(L^2(\Omega_i))^d$) inner product by $(\cdot, \cdot)_i$. In these notations, we may omit the subscript $i$ to refer to the entire domain $\Omega$. Let the interface inner product on $L^2(\Gamma_i)$ be denoted by $\langle \cdot, \cdot \rangle_i$, and on $L^2(\Gamma_{ij})$ be denoted by $\langle \cdot, \cdot \rangle_{ij}$. (Note that this latter inner product is zero if $\Gamma_{ij} = \emptyset$. In some cases it will be more generally a duality pairing.)

We first cast our problem (1)–(3) in a multiblock form. We need to choose a parameter $\alpha > 0$. Then we define

$$\boldsymbol{u} = -K(\nabla p - \gamma p) \quad \text{in } \Omega_i, \quad i = 1, \ldots, n, \tag{4}$$

$$cp + \nabla \cdot \boldsymbol{u} = q \qquad \text{in } \Omega_i\,, \quad i = 1, \ldots, n\,, \tag{5}$$

$$\boldsymbol{u} \cdot \nu = 0 \qquad \text{on } \partial\Omega\,, \tag{6}$$

together with the interface conditions

$$\alpha p_i - \boldsymbol{u}_i \cdot \nu_i = \alpha p_j + \boldsymbol{u}_j \cdot \nu_j \quad \text{on } \Gamma_{ij}\,, \quad i, j = 1, \ldots, n\,, \tag{7}$$

where for any function $f$ defined on $\Omega$, we denote both $f|_{\Omega_k}$ and its trace $f|_{\Gamma_k}$ by $f_k$.

The Robin type interface condition (7) is imposed twice on each $\Gamma_{k\ell}$: once for $k = i$ and $\ell = j$ and once for $\ell = i$ and $k = j$. Thus, for each $i$ and $j$, on $\Gamma_{ij}$,

$$\alpha p_i - \boldsymbol{u}_i \cdot \nu_i = \alpha p_j + \boldsymbol{u}_j \cdot \nu_j\,,$$

$$\alpha p_j - \boldsymbol{u}_j \cdot \nu_j = \alpha p_i + \boldsymbol{u}_i \cdot \nu_i\,,$$

so the difference implies that $p_i = p_j$, i.e. $p$ is continuous, and the sum implies that

$$\boldsymbol{u}_i \cdot \nu_i + \boldsymbol{u}_j \cdot \nu_j = 0\,,$$

i.e. flux is continuous (since $\nu_i = -\nu_j$ on $\Gamma_{ij}$).

Clearly the solution to (1)–(3) satisfies (4)–(7). By elliptic regularity, the solution to (4)–(7) also necessarily satisfies (1)–(3), so the two forms are equivalent.

We now define our multiblock variational form. A weak solution of (4)–(7) is a pair of functions $p \in L^2(\Omega)$ and $\boldsymbol{u} \in (L^2(\Omega))^d$ such that for each $i = 1, \ldots, n$, $p_i \in L^2(\Gamma_i)$, $\boldsymbol{u}_i \in H_0(\mathrm{div}; \Omega_i)$ with $\boldsymbol{u}_i \cdot \nu_i \in L^2(\Gamma_i)$, and

$$(K^{-1}\boldsymbol{u}, \boldsymbol{v})_i = (p, \nabla \cdot \boldsymbol{v})_i + (\gamma p, \boldsymbol{v})_i - \langle p_i, \boldsymbol{v} \cdot \nu_i \rangle_i\,, \quad \boldsymbol{v} \in H_0(\mathrm{div}; \Omega_i)\,, \tag{8}$$

$$(cp, w)_i + (\nabla \cdot \boldsymbol{u}, w)_i = (q, w)_i\,, \qquad w \in L^2(\Omega_i)\,, \tag{9}$$

$$\langle \alpha p_i - \boldsymbol{u}_i \cdot \nu_i, \mu_i \rangle_i = \sum_{j=1}^{n} \langle \alpha p_j + \boldsymbol{u}_j \cdot \nu_j, \mu_i \rangle_{ij}\,, \qquad \mu_i \in L^2(\Gamma_i)\,. \tag{10}$$

The derivation of (8) involves multiplying (4) by the test function $K^{-1}\boldsymbol{v}$, integrating over $\Omega_i$, and integrating the term involving $\nabla p$ by parts.

For later purposes, it is important to note that (10) can be replaced by the condition that for each $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} \langle \alpha(p_i - p_j), \mu_i \rangle_{ij} = \sum_{j=1}^{n} \langle \boldsymbol{u}_i \cdot \nu_i + \boldsymbol{u}_j \cdot \nu_j, \mu_i \rangle_{ij}\,, \quad \mu_i \in L^2(\Gamma_i)\,, \tag{11}$$

which relates interface pressure and flux discrepancies on $\Gamma_i$.

## 3. Formulation of the mixed method

For $h > 0$, choose a conforming finite element partition of $\Omega_i$, $1 \leq i \leq n$, where the maximal element diameter is bounded by $h$ and the interior of each element face (or edge) lies entirely in $\Gamma_i$ or in $\partial\Omega \cap \partial\Omega_i$. Let

$$V_{h,i} \times W_{h,i} \times \Lambda_{h,i} \subset H_0(\mathrm{div}; \Omega_i) \times L^2(\Omega_i) \times L^2(\Gamma_i)$$

be any of the usual affine mixed finite element spaces (e.g. the RTN spaces [19,21,18]; BDM spaces [8]; BDDF spaces [6]; BDFM spaces [7]; or CD spaces [10]), with the hybrid Lagrange multiplier spaces $\Lambda_{h,i}$ on $\Gamma_i$ [4,9]. Recall that

$$\nabla \cdot V_{h,i} = W_{h,i}\,, \qquad V_{h,i} \cdot \nu_i = \Lambda_{h,i}\,.$$

Let

$$V_h = \{\boldsymbol{v} \in (L^2(\Omega))^d : \boldsymbol{v}_i = \boldsymbol{v}|_{\Omega_i} \in V_{h,i}\,, \ i = 1, \ldots, n\}\,,$$

$$W_h = \{w \in L^2(\Omega) : w_i = w|_{\Omega_i} \in W_{h,i}\,, \ i = 1, \ldots, n\}\,.$$

We assume that the order of approximation of the spaces is the same on every $\Omega_i$.

In the mixed finite element approximation of (8)–(10), we seek $\boldsymbol{u}_h \in \boldsymbol{V}_h$, $p_h \in W_h$, and, for each $i$, $\lambda_{h,i} \in \Lambda_{h,i}$ such that for $i = 1, \ldots, n$,

$$(K^{-1}\boldsymbol{u}_h, \boldsymbol{v})_i = (p_h, \nabla\cdot\boldsymbol{v})_i + (\gamma p_h, \boldsymbol{v})_i - \langle \lambda_{h,i}, \boldsymbol{v}\cdot\nu_i\rangle_i\,, \quad \boldsymbol{v} \in \boldsymbol{V}_{h,i}\,, \tag{12}$$

$$(cp_h, w)_i + (\nabla\cdot\boldsymbol{u}_h, w)_i = (q, w)_i\,, \quad w \in W_{h,i}\,, \tag{13}$$

$$\langle \alpha\lambda_{h,i} - \boldsymbol{u}_{h,i}\cdot\nu_i, \mu_i\rangle_i = \sum_{j=1}^{n} \langle \alpha\lambda_{h,j} + \boldsymbol{u}_{h,j}\cdot\nu_j, \mu_i\rangle_{ij}\,, \quad \mu_i \in \Lambda_{h,i}\,. \tag{14}$$

We can replace (14) by the condition that for each $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} \langle \alpha(\lambda_{h,i} - \lambda_{h,j}), \mu_i\rangle_{ij} = \sum_{j=1}^{n} \langle \boldsymbol{u}_{h,i}\cdot\nu_i + \boldsymbol{u}_{h,j}\cdot\nu_j, \mu_i\rangle_{ij}\,, \quad \mu_i \in \Lambda_{h,i}\,. \tag{15}$$

We comment on the mass conservation properties of our scheme. Because $W_h$ is discontinuous from element to element, (13) implies local conservation of mass over every element. More generally, there is continuity of the normal component of flux $\boldsymbol{u}\cdot\nu$ across every element face (or edge) that does *not* lie on $\Gamma$. We must pay a price for the non-matching grids on $\Gamma$: We do not necessarily conserve flux across any fixed local portion of $\Gamma$. Since $\mu_i \equiv 1 \in \Lambda_{h,i}$, we continue to conserve mass globally, however.

## 4. Energy estimates and solvability

In this and the next section, we will make repeated use of the following easily verified result.

*LEMMA 1. If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are sequences, then*

$$\sum_{i,j} (a_i + a_j)b_i = \frac{1}{2}\sum_{i,j} (a_i + a_j)(b_i + b_j)\,,$$

$$\sum_{i,j} (a_i - a_j)b_i = \frac{1}{2}\sum_{i,j} (a_i - a_j)(b_i - b_j)\,.$$

In (12), (13) take the test functions $\boldsymbol{v} = \boldsymbol{u}_h$ and $w = p_h$. Sum the two equations, cancel two terms, and sum on $i$ to obtain that

$$(cp_h, p_h) + (K^{-1}\boldsymbol{u}_h, \boldsymbol{u}_h) = (\gamma p_h, \boldsymbol{u}_h) + (q, p_h) - \sum_{i=1}^{n} \langle \lambda_{h,i}, \boldsymbol{u}_{h,i}\cdot\nu_i\rangle_i\,.$$

Next divide (15) by 2, take $\mu_i = \lambda_{h,i}$, sum on $i$, and use Lemma 1 to obtain

$$\frac{1}{4}\alpha\sum_{i,j} \langle \lambda_{h,i} - \lambda_{h,j}, \lambda_{h,i} - \lambda_{h,j}\rangle_{ij} = \frac{1}{2}\sum_{i,j} \langle \boldsymbol{u}_{h,i}\cdot\nu_i + \boldsymbol{u}_{h,j}\cdot\nu_j, \lambda_{h,i}\rangle_{ij}\,.$$

Similarly, divide (15) by $2\alpha$, take $\mu_i = -\boldsymbol{u}_{h,i}\cdot\nu_i$, sum on $i$, and use Lemma 1 three times to obtain that

$$-\frac{1}{2}\sum_{i,j} \langle \lambda_{h,i}, \boldsymbol{u}_{h,i}\cdot\nu_i - \boldsymbol{u}_{h,j}\cdot\nu_j\rangle_{ij} = -\frac{1}{4\alpha}\sum_{i,j} \langle \boldsymbol{u}_{h,i}\cdot\nu_i + \boldsymbol{u}_{h,j}\cdot\nu_j, \boldsymbol{u}_{h,i}\cdot\nu_i + \boldsymbol{u}_{h,j}\cdot\nu_j\rangle_{ij}\,.$$

(The previous two estimates arise from the single Robin type test function $\mu_i = \frac{1}{2}(\lambda_{h,i} - (1/\alpha)\boldsymbol{u}_{h,i}\cdot\nu_i)$.)

If we now sum these three equations and cancel terms, we see that

$$(cp_h, p_h) + (K^{-1}\boldsymbol{u}_h, \boldsymbol{u}_h) + \frac{1}{4}\alpha \sum_{i,j} \langle \lambda_{h,i} - \lambda_{h,j}, \lambda_{h,i} - \lambda_{h,j} \rangle_{ij}$$
$$+ \frac{1}{4\alpha} \sum_{i,j} \langle \boldsymbol{u}_{h,i} \cdot \nu_i + \boldsymbol{u}_{h,j} \cdot \nu_j, \boldsymbol{u}_{h,i} \cdot \nu_i + \boldsymbol{u}_{h,j} \cdot \nu_j \rangle_{ij}$$
$$= (\gamma p_h, \boldsymbol{u}_h) + (q, p_h) . \tag{16}$$

To control the term containing $\gamma$, we make the following assumption of coercivity:

There is some constant $\gamma_* > 0$ such that
for any $w \in L^2(\Omega)$ and $\boldsymbol{v} \in (L^2(\Omega))^d$,
$$\gamma_* \{\|w\|^2 + \|\boldsymbol{v}\|^2\} \leq (cw, w) + (K^{-1}\boldsymbol{v}, \boldsymbol{v}) - (\gamma w, \boldsymbol{v}) . \tag{17}$$

Herein, $\|\cdot\|$ is either the $L^2(\Omega)$-norm or the $(L^2(\Omega))^d$-norm. Since in general, for any $\epsilon > 0$,

$$(f, g) \leq \frac{1}{2}\epsilon \|f\|^2 + \frac{1}{2\epsilon}\|g\|^2 , \tag{18}$$

we can control the term $(q, p_h)$, and so we have shown the following energy estimate giving stability of the scheme, wherein we use the notation

$$|||f|||_i^2 = \langle f, f \rangle_i \quad \text{and} \quad |||f|||_{ij}^2 = \langle f, f \rangle_{ij} .$$

*THEOREM 2. If (17), then there is some constant C, independent of h, such that*

$$\|p_h\| + \|\boldsymbol{u}_h\| + \left\{ \alpha \sum_{i,j} |||\lambda_{h,i} - \lambda_{h,j}|||_{ij}^2 \right\}^{1/2} + \left\{ \frac{1}{\alpha} \sum_{i,j} |||\boldsymbol{u}_{h,i} \cdot \nu_i + \boldsymbol{u}_{h,j} \cdot \nu_j|||_{ij}^2 \right\}^{1/2} \leq C\|q\| .$$

Because the scheme results in a square linear system, existence of a solution is equivalent to uniqueness. For uniqueness, the homogeneous equation (i.e. $q \equiv 0$) has immediately that $p_h$ and $\boldsymbol{u}_h$ are zero. Then (12) implies that $\lambda_{h,i} = 0$, since $\boldsymbol{V}_{h,i} \cdot \nu_i = \Lambda_{h,i}$.

*COROLLARY 3. If (17), then there exists a unique solution to the scheme.*

## 5. Error estimates

Each of the usual mixed spaces that we consider in this paper has a projection operator $\Pi_{h,i}$ onto $\boldsymbol{V}_{h,i}$ satisfying amongst other properties that for any $\boldsymbol{v} \in (H^{1/2+\epsilon}(\Omega_i))^d \cap H_0(\mathrm{div}; \Omega_i)$,

$$(\nabla \cdot (\boldsymbol{v} - \Pi_{h,i}\boldsymbol{v}), w)_i = 0 , \quad w \in W_{h,i} , \tag{19}$$
$$\langle (\boldsymbol{v} - \Pi_{h,i}\boldsymbol{v}) \cdot \nu_i, \mu_i \rangle_i = 0 , \quad \mu_i \in \Lambda_{h,i} . \tag{20}$$

We define the $L^2$-projection of a function $f \in L^2(\Omega)$ by $\hat{f} \in W_h$ satisfying

$$((f - \hat{f}), w)_i = 0 , \quad w \in W_{h,i} = \nabla \cdot \boldsymbol{V}_{h,i} . \tag{21}$$

Similarly, we define the $L^2$-projection of a function $f \in L^2(\Gamma_i)$ by $\bar{f}_i \in \Lambda_{h,i}$ satisfying

$$\langle (f - \bar{f}_i), \mu_i \rangle_i = 0 , \quad \mu_i \in \Lambda_{h,i} = \boldsymbol{V}_{h,i} \cdot \nu_i . \tag{22}$$

In these definitions, we have suppressed the subscripts $h$, as this should cause no confusion below.

For the analysis, let us define the errors as

$$\varphi = p - p_h , \qquad \psi = \boldsymbol{u} - \boldsymbol{u}_h \quad \text{and} \quad \phi_i = p_i - \lambda_{h,i} ,$$

again suppressing the subscripts $h$. Subtract from the variational problem (8), (9) and (11) the mixed method (12), (13) and (15) to obtain for $i = 1, \ldots, n$,

$$(K^{-1}\psi, \boldsymbol{v})_i = (\hat{\varphi}, \nabla\cdot\boldsymbol{v})_i + (\gamma\varphi, \boldsymbol{v})_i - \langle \phi_i, \boldsymbol{v}\cdot\nu_i\rangle_i\,, \quad \boldsymbol{v}\in \boldsymbol{V}_{h,i}\,, \tag{23}$$

$$(c\varphi, w)_i + (\nabla\cdot \Pi_{h,i}\psi, w)_i = 0\,, \qquad w\in W_{h,i}\,, \tag{24}$$

$$\sum_{j=1}^{n} \langle \alpha(\phi_i - \phi_j), \mu_i\rangle_{ij} = \sum_{j=1}^{n} \langle \psi_i\cdot\nu_i + \psi_j\cdot\nu_j, \mu_i\rangle_{ij}\,, \quad \mu_i\in\Lambda_{h,i}\,, \tag{25}$$

wherein we have used properties of the projections to replace two terms.

Take $w=\hat{\varphi}$ and $\boldsymbol{v}=\Pi_{h,i}\psi$ in (23), (24), sum, and sum on $i$ to obtain that

$$(c\varphi,\varphi) + (K^{-1}\psi,\psi) - (\gamma\varphi,\psi) = (c\varphi, p-\hat{p}) + \sum_{i=1}^{n}(K^{-1}\psi, \boldsymbol{u}-\Pi_{h,i}\boldsymbol{u})_i - \sum_{i=1}^{n}(\gamma\varphi, \boldsymbol{u}-\Pi_{h,i}\boldsymbol{u})_i - \sum_{i=1}^{n}\langle \phi_i, \Pi_{h,i}\psi_i\cdot\nu_i\rangle_i\,. \tag{26}$$

Now we need to take $\mu_i = \frac{1}{2}[\bar{\phi}_i - (1/\alpha)\Pi_{h,i}\psi_i\cdot\nu_i]$ in (25). We break this into two steps so that it is easier to describe our manipulation of the expressions. First take $\mu_i = \frac{1}{2}\bar{\phi}_i$. Summing on $i$ and using Lemma 1 results in

$$\frac{1}{4}\alpha\sum_{i,j}\langle \phi_i-\phi_j, \bar{\phi}_i-\bar{\phi}_j\rangle_{ij} = \frac{1}{2}\sum_{i,j}\langle \psi_i\cdot\nu_i, \bar{\phi}_i+\bar{\phi}_j\rangle_{ij}\,. \tag{27}$$

Using (20) and (22),

$$\sum_{i,j}\langle \psi_i\cdot\nu_i, \bar{\phi}_i\rangle_{ij} = \sum_{i,j}\langle \Pi_{h,i}\psi_i\cdot\nu_i, \phi_i\rangle_{ij}\,.$$

Therefore, (27) can be manipulated into

$$\frac{1}{4}\alpha\sum_{i,j}\langle \phi_i-\phi_j, \phi_i-\phi_j\rangle_{ij} = \frac{1}{2}\sum_{i,j}\langle \Pi_{h,i}\psi_i\cdot\nu_i, \phi_i\rangle_{ij} + \frac{1}{4}\alpha\sum_{i,j}\langle \phi_i-\phi_j, (p-\bar{p}_i)-(p-\bar{p}_j)\rangle_{ij} + \frac{1}{2}\sum_{i,j}\langle \psi_i\cdot\nu_i, \bar{\phi}_j\rangle_{ij}\,. \tag{28}$$

Now we take $\mu_i = -(1/2\alpha)\Pi_{h,i}\psi_i\cdot\nu_i$ in (25). Summing on $i$ and using Lemma 1 results in

$$-\frac{1}{2}\sum_{i,j}\langle \phi_i-\phi_j, \Pi_{h,i}\psi_i\cdot\nu_i\rangle_{ij} = -\frac{1}{4\alpha}\sum_{i,j}\langle \psi_i\cdot\nu_i+\psi_j\cdot\nu_j, \Pi_{h,i}\psi_i\cdot\nu_i+\Pi_{h,j}\psi_j\cdot\nu_j\rangle_{ij}\,. \tag{29}$$

Using (20) and (22), we note that

$$\begin{aligned}\langle \phi_j, \Pi_{h,i}\psi_i\cdot\nu_i\rangle_{ij} &= \langle \psi_i\cdot\nu_i, \bar{\phi}_j\rangle_{ij} - \langle (u-\Pi_{h,i}u)\cdot\nu_i, \bar{\phi}_j\rangle_{ij} + \langle \Pi_{h,i}\psi_i\cdot\nu_i, p-\bar{p}_j\rangle_{ij}\\ &= \langle \psi_i\cdot\nu_i, \bar{\phi}_j\rangle_{ij} + \langle (u-\Pi_{h,i}u)\cdot\nu_i, \bar{\phi}_i-\bar{\phi}_j\rangle_{ij} + \langle \Pi_{h,i}\psi_i\cdot\nu_i+\Pi_{h,j}\psi_j\cdot\nu_j, p-\bar{p}_j\rangle_{ij}\,,\end{aligned}$$

so that (29) can be manipulated into

$$\begin{aligned}\frac{1}{4\alpha}\sum_{i,j}\langle \psi_i\cdot\nu_i+\psi_j\cdot\nu_j\cdot, \psi_i\cdot\nu_i+\psi_j\cdot\nu_j\rangle_{ij} &= \frac{1}{2}\sum_{i,j}\langle \Pi_{h,i}\psi_i\cdot\nu_i, \phi_i\rangle_{ij}\\ &+ \frac{1}{4\alpha}\sum_{i,j}\langle \psi_i\cdot\nu_i+\psi_j\cdot\nu_j, (u-\Pi_{h,i}u)\cdot\nu_i+(u-\Pi_{h,j}u)\cdot\nu_j\rangle_{ij}\\ &- \frac{1}{2}\sum_{i,j}\langle \psi_i\cdot\nu_i, \bar{\phi}_j\rangle_{ij} - \frac{1}{2}\sum_{i,j}\langle (u-\Pi_{h,i}u)\cdot\nu_i, \bar{\phi}_i-\bar{\phi}_j\rangle_{ij}\\ &- \frac{1}{2}\sum_{i,j}\langle \Pi_{h,i}\psi_i\cdot\nu_i+\Pi_{h,j}\psi_j\cdot\nu_j, p-\bar{p}_j\rangle_{ij}\,.\end{aligned} \tag{30}$$

We now sum (26), (28) and (30), cancel five terms, manipulate a bit further the last two terms of (30), and invoke (17) and (18) to control the terms. The result is

$$\|\varphi\|^2 + \|\psi\|^2 + \sum_{i,j} \left[ \alpha |||\phi_i - \phi_j|||_{ij}^2 + \frac{1}{\alpha} |||\psi_i \cdot \nu_i + \psi_j \cdot \nu_j|||_{ij}^2 \right]$$
$$\leq C \left\{ \|p - \hat{p}\|^2 + \sum_{i=1}^{n} \left[ \|u - \Pi_{h,i} u\|_i^2 + \alpha |||p - \bar{p}_i|||_i^2 + \frac{1}{\alpha} |||(u - \Pi_{h,i} u) \cdot \nu_i|||_i^2 \right] \right\}. \tag{31}$$

Each of these bounding terms is optimal in its approximation properties.

*THEOREM 4. If (17), then there is some constant C, independent of h, such that*

$$\|p - p_h\| + \|u - u_h\| + \left\{ \alpha \sum_{i,j} |||\lambda_{h,i} - \lambda_{h,j}|||_{i,j}^2 \right\}^{1/2} + \left\{ \frac{1}{\alpha} \sum_{i,j} |||u_{h,i} \cdot \nu_i + u_{h,j} \cdot \nu_j|||_{ij}^2 \right\}^{1/2} \leq Ch^k, \tag{32}$$

*where k is the minimum order of approximation of the scalar and vector variables of the mixed finite element space. Moreover,*

$$\left\{ \sum_{i=1}^{n} \|\nabla \cdot (u - u_{h,i})\|_i^2 \right\}^{1/2} \leq Ch^k. \tag{33}$$

The latter estimate comes from taking $w = \nabla \cdot \Pi_{h,i} \psi$ in (24). We note that for the RTN spaces [19,21,18] and BDFM spaces [7], pressure and velocity are approximated to the same order, so the scheme is indeed optimal in practice. For the BDM spaces [8] and BDDF spaces [6], pressure is approximated to one power of $h$ less than velocity. If $\gamma \equiv 0$, one can easily modify (26) as

$$(c\hat{\varphi}, \hat{\varphi}) + (K^{-1}\psi, \psi)$$
$$= -(c(p - \hat{p}), \hat{\varphi}) + \sum_{i=1}^{n} (K^{-1}\psi, u - \Pi_{h,i} u)_i - \sum_{i=1}^{n} \langle \phi_i, \Pi_{h,i} \psi_i \cdot \nu_i \rangle_i,$$

and then estimate, for $\tilde{c}$ equal to the piece-wise constant projection of $c$,

$$-(c(p - \hat{p}), \hat{\varphi}) = -((c - \tilde{c})(p - \hat{p}), \hat{\varphi}) \leq C \|c\|_{W^{1,\infty}} h \|p - \hat{p}\| \, \|\hat{\varphi}\|$$

to recover the missing power of $h$ for the velocity and interface pressure and flux errors in (32).

## 6. Numerical results

In this section we present some numerical tests to illustrate our method in practice. Actually, we test the symmetric variant of our method. This is defined by replacing (12) with

$$(K^{-1}u_h, v)_i = (p_h, \nabla \cdot v)_i + (\gamma p_h, v)_i - \frac{1}{2} \sum_{j=1}^{n} \langle \lambda_{h,i} + \lambda_{h,j}, v \cdot \nu_i \rangle_{ij}, \quad v \in V_{h,i}. \tag{34}$$

If $\gamma \equiv 0$, then it is easy to verify that (34), (13), and either (14) or (15) is symmetric. An analysis of this modification will be presented elsewhere. Interestingly, a straightforward analysis shows a stability estimate similar to that in Theorem 2, except that there is no bound on the flux discrepancy term. Moreover, a straightforward analysis of the approximation errors fails to show convergence (although a nontrivial modification gives at least some suboptimal convergence).

We choose the symmetric modification because, with $\gamma = 0$, we can then use the Glowinski–Wheeler domain decomposition solution algorithm [16] for symmetric problems. Since this algorithm already uses Lagrange multipliers on the interface, the only additional cost is computing certain projections on $\Gamma$.

We use the lowest order Raviart–Thomas spaces on rectangles [19]. These spaces approximate arbitrary smooth functions as $O(h)$ (i.e. to order 1). We consider three different problems on the unit square, and we take $c \equiv 0$, $\gamma \equiv 0$, and $\alpha = 1$. For each test case, we establish rates of convergence by solving problems on several levels of grid refinement and computing a least squares fit to the error.

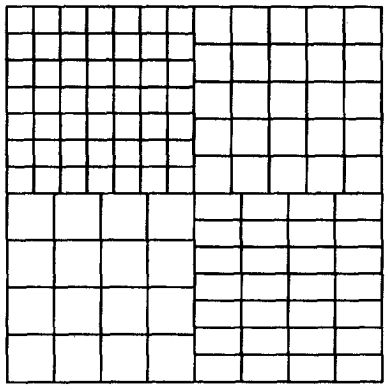


Fig. 1. Initial non-matching grids for test case 1.

In the first example, we solve a problem with the given solution

$$p(x, y) = x^3y^2 + \sin(xy)$$

and coefficient

$$K = \begin{pmatrix} 10 + 5\cos(xy) & 0 \\ 0 & 1 \end{pmatrix}.$$

Dirichlet conditions are imposed on the exterior boundary. The domain is divided into four subdomains covered with grids that do not match on the interface $\Gamma$. The domain decomposition and the grids on the coarsest level are shown in Fig. 1. Numerical errors and convergence rates are presented in Table 1. Herein $\|\cdot\|_M$ is the discrete $L^2$-norm induced by the midpoint quadrature rule on the finite element partition (or its trace on $\Gamma$) and $\|\cdot\|_N$ is the discrete $L^2$-norm in the vector space based on function evaluations at the nodal points of $V_h$. We note that super-convergence $O(h^2)$ is observed for $p_h$ at the midpoints, while only optimal convergence $O(h)$ occurs for $\boldsymbol{u}_h$ and $\lambda_h$. Plots of the computed solution and the numerical error are shown in Fig. 2.

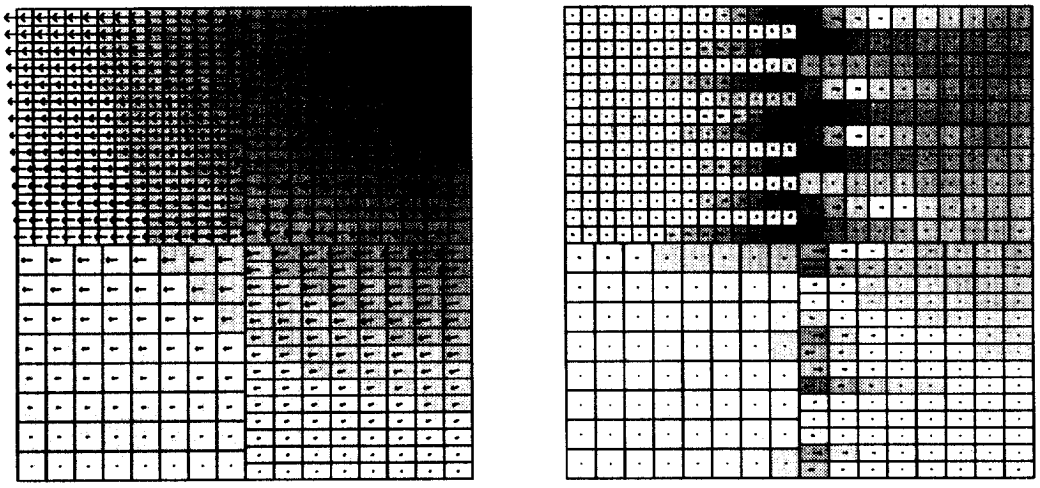
In the second example, we test a problem with a discontinuous coefficient $K = I$ for $0 \leq x < 1/2$ and $K = 10 * I$ for $1/2 < x \leq 1$. The solution

$$p(x, y) = \begin{cases} x^2y^3 + \cos(xy), & 0 \leq x \leq 1/2, \\ \left(\dfrac{2x+9}{20}\right)^2 y^3 + \cos\left(\dfrac{2x+9}{20}\, y\right), & 1/2 \leq x \leq 1, \end{cases}$$

is continuous and has continuous normal flux at $x = 1/2$. The boundary conditions are Dirichlet on the left and right sides and Neumann on the other two sides. The domain is divided into two subdomains with interface $\Gamma$ along $x = 1/2$. The initial grids are $4 \times 7$ on the left and $4 \times 10$ on the right. Convergence rates for the test case are given in Table 2. Note that in this example velocity super-convergence is observed at the nodal points.

Table 1
Discrete norm errors and convergence rates for test case 1

| $1/h$ | $\|p - p_h\|_M$ | $\|\boldsymbol{u} - \boldsymbol{u}_h\|_N$ | $\|p - \lambda_h\|_M$ |
|---|---|---|---|
| 8 | 1.58E-03 | 1.05E-01 | 4.47E-02 |
| 16 | 4.03E-04 | 4.98E-02 | 2.00E-02 |
| 32 | 9.62E-05 | 2.29E-02 | 9.21E-03 |
| 64 | 2.27E-05 | 1.01E-02 | 4.45E-03 |
| 128 | 5.72E-06 | 4.62E-03 | 2.21E-03 |
| rate | $O(h^{2.04})$ | $O(h^{1.13})$ | $O(h^{1.08})$ |



A. *Computed pressure and velocity* B. *Pressure and velocity error*

Fig. 2. Solution and error (magnified) for test case 1.

Table 2
Discrete norm errors and convergence rates for test case 2

| $1/h$ | $\|p - p_h\|_M$ | $\|u - u_h\|_N$ | $\|p - \lambda_h\|_M$ |
|---|---|---|---|
| 8 | 2.89E-04 | 1.77E-02 | 3.93E-03 |
| 16 | 7.60E-05 | 4.48E-03 | 1.79E-03 |
| 32 | 2.00E-05 | 1.19E-03 | 9.00E-04 |
| 64 | 5.24E-06 | 3.78E-04 | 4.80E-04 |
| 128 | 1.40E-06 | 1.73E-04 | 2.58E-04 |
| rate | $O(h^{1.92})$ | $O(h^{1.69})$ | $O(h^{0.98})$ |

Table 3
Discrete norm errors and convergence rates for test case 3

| $1/h_c$ | $\|p - p_h\|_M$ | $\|u - u_h\|_N$ | $\|p - \lambda_h\|_M$ |
|---|---|---|---|
| 8 | 7.30E-03 | 5.90E-02 | 5.27E-03 |
| 16 | 1.84E-03 | 1.73E-02 | 2.57E-02 |
| 32 | 4.61E-04 | 5.01E-03 | 1.28E-02 |
| 64 | 1.15E-04 | 1.48E-03 | 6.38E-03 |
| rate | $O(h^{2.00})$ | $O(h^{1.77})$ | $O(h^{1.01})$ |

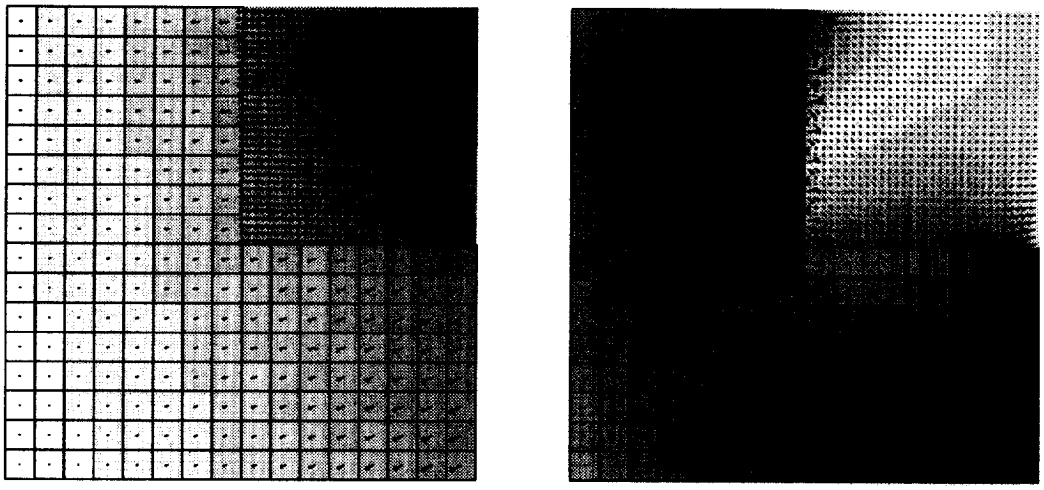
In the third and last example we solve a problem with the solution

$$p(x, y) = x^3y^4 + x^3 + \sin(xy)\cos(y)$$

and coefficient

$$K = \begin{pmatrix} (x+1)^2 + y^2 & 0 \\ 0 & (x+1)^2 \end{pmatrix}$$

on locally refined (i.e. nested) grids. The domain is divided by four, and the grid on the upper-right subdomain is four times finer than the rest of the grids. The convergence rates are given in Table 3, and the numerical solution and error are plotted in Fig. 3.



A. *Computed pressure and velocity* B. *Pressure and velocity error*

Fig. 3. Solution and error (magnified) for test case 3.

## References


[1] T. Arbogast, L.C. Cowsar, M.F. Wheeler and I. Yotov, Mixed finite element methods on non-matching multiblock grids, to appear.

[2] T. Arbogast, C.N. Dawson, P.T. Keenan, M.F. Wheeler and I. Yotov, Enhanced cell-centered finite differences for elliptic equations on general geometry, SIAM J. Sci. Comput. 18 (1997), to appear.

[3] T. Arbogast, M.F. Wheeler and I. Yotov, Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences, SIAM J. Numer. Anal. 34 (1997) 828–852.

[4] D.N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates, RAIRO Modèl, Math. Anal. Numèr. 19 (1985) 7–32.

[5] C. Bernardi, Y. Maday and A.T. Patera, A new nonconforming approach to domain decomposition: the mortar element method, in: H. Brezis and J.L. Lions, eds., Nonlinear Partial Differential Equations and Their Applications (Longman Scientific & Technical, UK, 1994).

[6] F. Brezzi, J. Douglas Jr., R. Duràn and M. Fortin, Mixed finite elements for second order elliptic problems in three variables, Numer. Math. 51 (1987) 237–250.

[7] F. Brezzi, J. Douglas Jr., M. Fortin and L.D. Marini, Efficient rectangular mixed finite elements in two and three space variables, RAIRO Modèl. Math. Anal. Numèr. 21 (1987) 581–604.

[8] F. Brezzi, J. Douglas Jr. and L.D. Marini, Two families of mixed elements for second order elliptic problems, Numer. Math. 88 (1985) 217–235.

[9] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods (Springer-Verlag, New York, 1991).

[10] Z. Chen and J. Douglas Jr., Prismatic mixed finite elements for second order elliptic problems, Calcolo 26 (1989) 135–148.

[11] J. Douglas Jr. and J.E. Roberts, Global estimates for mixed methods for second order elliptic equations, Math. Comput. 44 (1985) 39–52.

[12] R. Durán, Superconvergence for rectangular mixed finite elements, Numer. Math. 58 (1990) 287–298.

[13] R.E. Ewing, R.D. Lazarov, T.F. Russell and P.S. Vassilevski, Analysis of the mixed finite element method for rectangular Raviart–Thomas elements with local refinement, in: Third International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia, 1990, SIAM, 98–114).

[14] R.E. Ewing, R.D. Lazarov and J. Wang, Superconvergence of the velocity along the Gauss lines in mixed finite element methods, SIAM J. Numer. Anal. 28 (1991) 1015–1029.

[15] R.E. Ewing and J. Wang, Analysis of mixed finite element methods on locally refined grids, Numer. Math. 63 (1992) 183–194.

[16] R. Glowinski and M.F. Wheeler, Domain decomposition and mixed finite element methods for elliptic problems, in: R. Glowinski et al., eds., First International Symposium on Domain Decomposition Methods for Partial Differential Equations (Philadelphia, 1988, SIAM, 144–172).

[17] M. Nakata, A. Weiser and M.F. Wheeler, Some superconvergence results for mixed finite element methods for elliptic problems on rectangular domains, in: J.R. Whiteman, ed., The Mathematics of Finite Elements and Applications V (Academic Press, London, 1985, 367–389).

[18] J.C. Nedelec, Mixed finite elements in $R^3$, Numer. Math. 35 (1980) 315–341.

[19] R.A. Raviart and J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, in: Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, Vol. 606 (Springer-Verlag, New York, 1977) 292–315.

[20] J.E. Roberts and J.-M. Thomas, Mixed and hybrid methods, in: P.G. Ciarlet and J. Lions, eds., Handbook of Numerical Analysis, Vol. II (Elsevier Science Publishers B.V., 1991) 523–639.
[21] J.M. Thomas, Sur l'analyse numerique des methodes d'elements finis hybrides et mixtes, These de Doctorat d'etat, Sciences Mathematiques'a l'Universite Pierre et Marie Curie, 1977.
[22] A. Weiser and M.F. Wheeler, On convergence of block-centered finite-differences for elliptic problems, SIAM J. Numer. Anal. 25 (1988) 351–375.
[23] I. Yotov, Mixed finite element methods for flow in porous media, Ph.D. Thesis, Rice University, Houston, Texas, 1996.