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A multipoint stress mixed finite element method for elasticity on quadrilateral grids

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Abstract

We develop a multipoint stress mixed finite element method for linear elasticity with weak stress symmetry on quadrilateral grids, which can be reduced to a symmetric and positive definite cell centered system. The method utilizes the lowest order Brezzi–Douglas–Marini finite element spaces for the stress and the trapezoidal quadrature rule in order to localize the interaction of degrees of freedom, which allows for local stress elimination around each vertex. We develop two variants of the method. The first uses a piecewise constant rotation and results in a cell-centered system for displacement and rotation. The second uses a continuous piecewise bilinear rotation and trapezoidal quadrature rule for the asymmetry bilinear form. This allows for further elimination of the rotation, resulting in a cell-centered system for the displacement only. Stability and error analysis is performed for both methods. First-order convergence is established for all variables in their natural norms. A duality argument is employed to prove second order superconvergence of the displacement at the cell centers. Numerical results are presented in confirmation of the theory.

KEYWORDS

elasticity, mixed finite element, multipoint stress

1 | INTRODUCTION

Stress–displacement mixed finite element (MFE) elasticity formulations have been studied extensively due to their local momentum conservation with continuous normal stress and locking-free approximation, see [1] and references therein. These methods result in saddle point type algebraic systems, which may be expensive to solve. In this work we develop two stress–displacement MFE methods for elasticity on quadrilateral grids that can be reduced to symmetric and positive definite cell centered systems using a mass lumping technique. We have previously developed such methods on simplicial grids in [2]. Even though the formulation is similar, the stability and error analysis on quadrilaterals differ significantly from those on simplices. The methods are referred to as multipoint stress mixed finite element (MSMFE) methods, adopting the terminology of the multipoint stress approximation (MPSA) method developed in [3–5]. Our approach is motivated by the multipoint flux mixed finite element (MFMFE) method [6–8] for Darcy flow, and its closely related multipoint flux approximation (MPFA) method [9–13]. The MFMFE method utilizes the lowest order Brezzi–Douglas–Marini $\mathcal{BDM}_1$ spaces on simplices and quadrilaterals [14], as well as an enhanced Brezzi–Douglas–Duran–Fortin $\mathcal{BDDF}_1$ space [15] on hexahedral grids. There are two variants of the MFMFE method—symmetric and non-symmetric. The symmetric version is designed for simplices [8], as well as quadrilaterals and hexahedra that are $O(h^2)$-perturbations of parallelograms and parallelepipeds [6, 8]. It is related to the symmetric MPFA method [9, 16, 17]. The symmetric MFMFE method is always well posed, but its convergence may deteriorate on general quadrilaterals or hexahedra. The non-symmetric MFMFE method [7], which is related to the non-symmetric MPFA method [9, 12, 13], exhibits good convergence on general quadrilaterals or hexahedra, but it may become ill-posed due to loss of coercivity if the grids are too distorted.

The MSMFE methods on quadrilaterals we develop in this paper are symmetric and are related to the symmetric MFMFE method. The methods are based on the $\mathcal{BDM}_1$ spaces on quadrilaterals. We consider the formulation with weakly imposed stress symmetry, for which there exist MFE spaces with $\mathcal{BDM}_1$ degrees of freedom for the stress. In this formulation the symmetry of the stress is imposed weakly using a Lagrange multiplier, which is a skew-symmetric matrix and has a physical meaning of rotation. Our first method, referred to as MSMFE-0, is based on the spaces $\mathcal{BDM}_1 \times \mathcal{Q}_0 \times \mathcal{Q}_0$ developed in [18, 19], using $\mathcal{BDM}_1$ stress and piecewise constant displacement and rotation. The $\mathcal{BDM}_1$ space has two normal degrees of freedom per edge, which can be associated with the two vertices. An application of the trapezoidal quadrature rule for the stress bilinear form results in localizing the interaction of stress degrees of freedom around mesh vertices. The stress is then locally eliminated and the method is reduced to a symmetric and positive definite cell centered system for the displacement and rotation. Our second method, MSMFE-1, is based on the spaces $\mathcal{BDM}_1 \times \mathcal{Q}_0 \times \mathcal{Q}_1$, with continuous bilinear rotation, which is proposed in [20]. In this method we employ the trapezoidal quadrature rule both for the stress and the asymmetry bilinear forms. This allows for further local elimination of the rotation after the initial stress elimination, resulting in a symmetric and positive definite cell centered system for the displacement only. To the best of our knowledge, this is the first MFE method for elasticity on quadrilaterals with such property.

We develop stability and error analysis for the two methods. The stability arguments follow the framework established in [18], with modifications to account for the quadrature rules. The argument in [18] explores connections between stable mixed elasticity elements and stable mixed Stokes and Darcy elements. In the case of the MSMFE-0 method, the two stable pairs are $\mathcal{SS}_2 \times \mathcal{Q}_0$ for Stokes and $\mathcal{BDM}_1 \times \mathcal{Q}_0$ for Darcy. Since the only term with quadrature is the stress bilinear form, the stability argument in [18] can be modified in a relatively straightforward way. Proving stability of the MSMFE-1 method is more difficult. In this case the Stokes pair is $\mathcal{SS}_2 \times \mathcal{Q}_1$. The difficulty comes from the fact that the quadrature rule is also applied to the asymmetry bilinear forms, which necessitates establishing an inf-sup condition for $\mathcal{SS}_2 \times \mathcal{Q}_1$ with trapezoidal quadrature in the divergence bilinear form. We do this by a macroelement argument motivated by [21]. It is based on establishing a local macroelement inf-sup condition and combining the locally constructed velocities to obtain the global inf-sup condition. We note that the proof is very different from the argument on simplices in [2]. In particular, on simplices one can establish a local inf-sup condition using vectors that are zero on the boundary of the macroelement, which can be utilized in the global construction. This is not the case on quadrilaterals, which complicates the global construction significantly. The reader is referred to Section 4.1.1, where the global Stokes inf-sup condition is established under a smoothness assumption on the grid given in (M2). We would like to note that this result is important by itself, as it deals with the fundamental issue of inf-sup stability for Stokes finite element approximation with quadrature. We proceed with establishing first order convergence for the stress in the $H(\text{div})$-norm and for the displacement and rotation in the L^2-norm for both methods on elements that are $O(h^2)$-perturbations of parallelograms. This restriction is similar to the one in symmetric MPFA and MFMFE methods [8, 17]. Again, the arguments are very different from the simplicial case, since the map to the reference element is non-affine (bilinear), which complicates the estimation of the quadrature error. We further employ a duality argument to prove second order superconvergence of the displacement at the cell centers.

The rest of the paper is organized as follows. The model problem and its MFE approximation are presented in Section 2. The two methods and their stability are developed in Sections 3 and 4, respectively. The error analysis is performed in Section 5. Numerical results are presented in Section 6.

2 | MODEL PROBLEM AND ITS MFE APPROXIMATION

Let Ω be a simply connected bounded polygonal domain of $\mathbb{R}^2$ occupied by a linearly elastic body. We write $\mathbb{M}$, $\mathbb{S}$, and $\mathbb{N}$ for the spaces of 2×2 matrices, symmetric matrices, and skew-symmetric matrices, all over the field of real numbers, respectively. The material properties are described at each point $\mathbf{x} \in \Omega$ by a compliance tensor $A = A(x)$, which is a symmetric, bounded, and uniformly positive definite linear operator acting from $\mathbb{S}$ to $\mathbb{S}$. We also assume that an extension of A to an operator $\mathbb{M} \to \mathbb{M}$ still possesses the above properties. We will utilize the usual divergence operator div for vector fields. When applied to a matrix field, it produces a vector field by taking the divergence of each row. We will also use the curl operator defined as $\text{curl}\phi = (\partial_2\phi, -\partial_1\phi)$ for a scalar function ϕ. Consequently, for a vector field, the curl operator produces a matrix field, by acting row-wise.

Throughout the paper, C denotes a generic positive constant that is independent of the discretization parameter h. We will also use the following standard notation. For a domain $G \subset \mathbb{R}^2$, the $L^2(G)$ inner product and norm for scalar and vector valued functions are denoted $(\cdot,\cdot)_G$ and $\|\cdot\|_G$, respectively. The norms and seminorms of the Sobolev spaces $W^{k,p}(G)$, $k \in \mathbb{R}$, $p > 0$ are denoted by $\|\cdot\|_{k,p,G}$ and $|\cdot|_{k,p,G}$, respectively. The norms and seminorms of the Hilbert spaces $H^k(G)$ are denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. We omit G in the subscript if $G = \Omega$. For a section of the domain or element boundary S we write $\langle\cdot,\cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively. We will also use the space $H(\text{div};\Omega) = \{v \in L^2(\Omega,\mathbb{R}^2) : \text{div}\, v \in L^2(\Omega)\}$ equipped with the norm $\|v\|_{\text{div}} = (\|v\|^2 + \|\text{div}\, v\|^2)^{1/2}$.

Given a compliance tensor $A \in L^\infty(\Omega,\mathbb{S})$ and a vector field $f \in L^2(\Omega,\mathbb{R}^2)$ representing body forces, the equations of static elasticity in Hellinger–Reissner form determine the stress σ and the displacement u satisfying the constitutive and equilibrium equations respectively:

$$A\sigma = \epsilon(u), \quad \text{div}\,\sigma = f \quad \text{in } \Omega, \tag{2.1}$$

together with the boundary conditions

$$u = g \text{ on } \Gamma_D, \quad \sigma n = 0 \text{ on } \Gamma_N, \tag{2.2}$$

where $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, $\partial\Omega = \Gamma_D \cup \Gamma_N$, and $g \in H^{1/2}(\Gamma_D)$. We assume for simplicity that $\Gamma_D \neq \emptyset$.

We consider a weak formulation for (2.1) and (2.2), in which the stress symmetry is imposed weakly, using the Lagrange multiplier $\gamma = \mathrm{Skew}(\nabla u)$, $\mathrm{Skew}(\tau) = \frac{1}{2}(\tau - \tau^T)$, from the space of skew-symmetric matrices: find $(\sigma, u, \gamma) \in \mathbb{X} \times V \times \mathbb{W}$ such that:

$$(A\sigma, \tau) + (u, \mathrm{div}\ \tau) + (\gamma, \tau) = \langle g, \tau n \rangle_{\Gamma_D}, \quad \forall \tau \in \mathbb{X}, \tag{2.3}$$

$$(\mathrm{div}\ \sigma, v) = (f, v), \quad \forall v \in V, \tag{2.4}$$

$$(\sigma, \xi) = 0, \quad \forall \xi \in \mathbb{W}, \tag{2.5}$$

where the corresponding spaces are

$$\mathbb{X} = \{\tau \in H(\mathrm{div}, \Omega, \mathbb{M}) : \tau n = 0 \ \text{ on } \ \Gamma_N\}, \quad V = L^2(\Omega, \mathbb{R}^2), \quad W = L^2(\Omega, \mathbb{N}).$$

Problem (2.3)–(2.5) has a unique solution [22].

2.1 | MFE method

Let $\mathcal{T}_h$ be a shape-regular and quasi-uniform quadrilateral partition of Ω [23], with $h = \max_{E \in \mathcal{T}_h} \mathrm{diam}(E)$. For any element $E \in \mathcal{T}_h$ there exists a bilinear bijection mapping $F_E : \hat{E} \to E$, where $\hat{E} = [0, 1]^2$ is the reference square. Denote the Jacobian matrix by DF_E and let $J_E = |\det(DF_E)|$. For $\mathbf{x} = F_E(\hat{\mathbf{x}})$ we have

$$DF_E^{-1}(\mathbf{x}) = (DF_E)^{-1}(\hat{\mathbf{x}}), \quad J_{F_E^{-1}}(\mathbf{x}) = \frac{1}{J_E(\hat{\mathbf{x}})}.$$

Let $\hat{E}$ has vertices $\hat{\mathbf{r}}_1 = (0, 0)^T$, $\hat{\mathbf{r}}_2 = (1, 0)^T$, $\hat{\mathbf{r}}_3 = (1, 1)^T$, and $\hat{\mathbf{r}}_4 = (0, 1)^T$ with unit outward normal vectors to the edges denoted by $\hat{n}_i$, $i = 1, \ldots, 4$ (Figure 1). We denote by $\mathbf{r}_i = (x_i, y_i)^T$, $i = 1, \ldots, 4$, the corresponding vertices of the element E, and by n_i, $i = 1, \ldots, 4$, the corresponding unit outward normal vectors. The bilinear mapping F_E and its Jacobian matrix are given by

$$F_E(\hat{\mathbf{r}}) = \mathbf{r}_1 + \mathbf{r}_{21}\hat{x} + \mathbf{r}_{41}\hat{y} + (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}\hat{y}, \tag{2.6}$$

$$DF_E = [\mathbf{r}_{21}, \mathbf{r}_{41}] + [(\mathbf{r}_{34} - \mathbf{r}_{21})\hat{y}, (\mathbf{r}_{34} - \mathbf{r}_{21})\hat{x}], \tag{2.7}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$. It is easy to see that the shape-regularity and quasi-uniformity of the grids imply that $\forall E \in \mathcal{T}_h$,

$$\|DF_E\|_{0,\infty,\hat{E}} \sim h, \quad \|DF_E^{-1}\|_{0,\infty,\hat{E}} \sim h^{-1}, \quad \|J_E\|_{0,\infty,\hat{E}} \sim h^2, \quad \text{and} \quad \|J_{F_E^{-1}}\|_{0,\infty,\hat{E}} \sim h^{-2}, \tag{2.8}$$

where the notation $a \sim b$ means that there exist positive constants c_0, c_1 independent of h such that $c_0 b \le a \le c_1 b$.

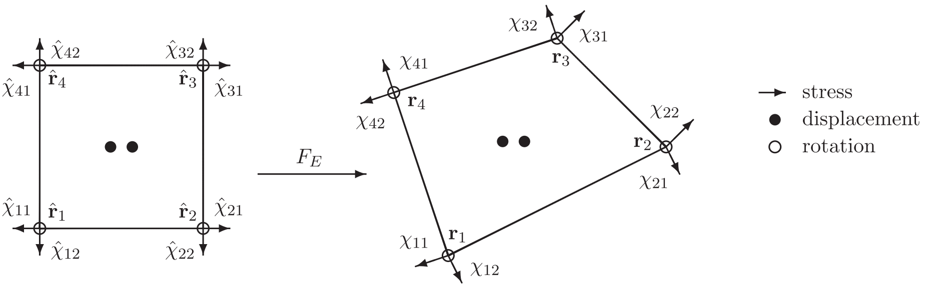


FIGURE 1 Degrees of freedom of $\mathbb{X}_h \times V_h \times \mathbb{W}_h^1$

The finite element spaces $\mathbb{X}_h \times V_h \times \mathbb{W}_h^k \subset \mathbb{X} \times V \times \mathbb{W}$ are the triple $(\mathcal{BDM}_1)^2 \times (\mathcal{Q}_0)^2 \times (\mathcal{Q}_k)^{2\times 2,\text{skew}}$, $k = 0, 1$, where $\mathcal{Q}_k$ denotes the space of polynomials of degree at most k in each variable and each row of an element of $\mathbb{X}_h$ is a vector in $\mathcal{BDM}_1$. On the reference square the spaces are defined as

$$\begin{aligned}\hat{\mathbb{X}}(\hat{E}) &= (\mathcal{P}_1(\hat{E})^2 + r_1\operatorname{curl}(\hat{x}^2\hat{y}) + s_1\operatorname{curl}(\hat{x}\hat{y}^2)) \times (\mathcal{P}_1(\hat{E})^2 + r_2\operatorname{curl}(\hat{x}^2\hat{y}) + s_2\operatorname{curl}(\hat{x}\hat{y}^2)) \\ &= \begin{pmatrix} \alpha_1\hat{x} + \beta_1\hat{y} + \gamma_1 + r_1\hat{x}^2 + 2s_1\hat{x}\hat{y} & \alpha_2\hat{x} + \beta_2\hat{y} + \gamma_2 - 2r_1\hat{x}\hat{y} - s_1\hat{y}^2 \\ \alpha_3\hat{x} + \beta_3\hat{y} + \gamma_3 + r_2\hat{x}^2 + 2s_2\hat{x}\hat{y} & \alpha_4\hat{x} + \beta_4\hat{y} + \gamma_4 - 2r_2\hat{x}\hat{y} - s_2\hat{y}^2 \end{pmatrix},\end{aligned} \tag{2.9}$$

$$\hat{V}(\hat{E}) = (\mathcal{Q}_0(\hat{E}))^2, \quad \hat{\mathbb{W}}^k(\hat{E}) = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix}, \quad p \in \mathcal{Q}_k(\hat{E}) \text{ for } k = 0, 1,$$

where $\alpha_i, \beta_i, \gamma_i, r_i, s_i$ are real constants. Note that $\operatorname{div}\hat{\mathbb{X}}(\hat{E}) = \hat{V}(\hat{E})$ and for all $\hat{\tau} \in \hat{\mathbb{X}}(\hat{E})$, $\hat{\tau} n_{\hat{e}} \in \mathcal{P}_1(\hat{e})^2$ on any edge $\hat{e}$ of $\hat{E}$. It is well known [14, 15] that the degrees of freedom of $\mathcal{BDM}_1(\hat{E})$ can be chosen as the values of the normal components at any two points on each edge $\hat{e} \subset \partial\hat{E}$. In this work we choose these points to be the vertices of $\hat{e}$ (Figure 1). This is motivated by the trapezoidal quadrature rule, introduced in the next section. The spaces on any element $E \in \mathcal{T}_h$ are defined via the transformations

$$\tau \overset{\mathcal{P}}{\leftrightarrow} \hat{\tau} : \tau^T = \frac{1}{J_E} DF_E \hat{\tau}^T \circ F_E^{-1}, \quad v \leftrightarrow \hat{v} : v = \hat{v} \circ F_E^{-1}, \quad \xi \leftrightarrow \hat{\xi} : \xi = \hat{\xi} \circ F_E^{-1}, \tag{2.10}$$

where $\hat{\tau} \in \hat{\mathbb{X}}(\hat{E})$, $\hat{v} \in \hat{V}(\hat{E})$, and $\hat{\xi} \in \hat{\mathbb{W}}^k(\hat{E})$. Note that the Piola transformation (applied row-by-row) is used for $\hat{\mathbb{X}}(\hat{E})$. It satisfies, for all sufficiently smooth $\tau \overset{\mathcal{P}}{\leftrightarrow} \hat{\tau}$, $v \leftrightarrow \hat{v}$, and $\phi \leftrightarrow \hat{\phi}$,

$$(\operatorname{div} \tau, v)_E = (\operatorname{div} \hat{\tau}, \hat{v})_{\hat{E}}, \quad \langle \tau n_e, v \rangle_e = \langle \widehat{\tau n_{\hat{e}}}, \hat{v} \rangle_{\hat{e}}, \quad \text{and} \quad \operatorname{curl} \phi \overset{\mathcal{P}}{\leftrightarrow} \operatorname{curl} \hat{\phi}. \tag{2.11}$$

The spaces on $\mathcal{T}_h$ are defined by

$$\begin{aligned}\mathbb{X}_h &= \left\{\tau \in \mathbb{X} : \tau\big|_E \overset{\mathcal{P}}{\leftrightarrow} \hat{\tau}, \ \hat{\tau} \in \hat{\mathbb{X}}(\hat{E}) \quad \forall E \in \mathcal{T}_h\right\}, \\ V_h &= \left\{v \in V : v\big|_E \leftrightarrow \hat{v}, \ \hat{v} \in \hat{V}(\hat{E}) \quad \forall E \in \mathcal{T}_h\right\}, \\ \mathbb{W}_h^0 &= \left\{\xi \in \mathbb{W} : \xi\big|_E \leftrightarrow \hat{\xi}, \ \hat{\xi} \in \hat{\mathbb{W}}^0(\hat{E}) \quad \forall E \in \mathcal{T}_h\right\}, \\ \mathbb{W}_h^1 &= \left\{\xi \in \mathcal{C}(\Omega, \mathbb{N}) \subset \mathbb{W} : \xi\big|_E \leftrightarrow \hat{\xi}, \ \hat{\xi} \in \hat{\mathbb{W}}^1(\hat{E}) \quad \forall E \in \mathcal{T}_h\right\}.\end{aligned} \tag{2.12}$$

Note that $\mathbb{W}_h^1 \subset H^1(\Omega)$, since it contains continuous piecewise $\mathcal{Q}_1$ functions.

The MFE method for (2.3)–(2.5) is: find $(\sigma_h, u_h, \gamma_h) \in \mathbb{X}_h \times V_h \times \mathbb{W}_h^k$ such that

$$(A\sigma_h, \tau) + (u_h, \operatorname{div}\tau) + (\gamma_h, \tau) = \langle g, \tau n \rangle_{\Gamma_D}, \quad \tau \in \mathbb{X}_h, \tag{2.13}$$

$$(\operatorname{div}\sigma_h, v) = (f, v), \quad v \in V_h, \tag{2.14}$$

$$(\sigma_h, \xi) = 0, \quad \xi \in \mathbb{W}_h^k. \tag{2.15}$$

It is shown in [18] that the method (2.13)–(2.15) in the case $k = 0$ has a unique solution and it is first order accurate for all variables in their corresponding norms. The case $k = 1$ on rectangles is analyzed in [20]. The framework from [18] can be used to analyze the case $k = 1$ on quadrilaterals. A drawback of the method is that the resulting algebraic problem is a coupled stress–displacement–rotation system of a saddle point type. In this paper we develop two methods that utilize a quadrature rule and can be reduced to cell-centered systems for displacement–rotation and displacement only, respectively.

2.2 | A quadrature rule

Let φ and ψ be element-wise continuous functions on Ω. We denote by $(\varphi, \psi)_Q$ the application of the element-wise tensor product trapezoidal quadrature rule for computing (φ, ψ). The integration on any element E is performed by mapping to the reference element $\hat{E}$. For $\tau, \chi \in \mathbb{X}_h$, we have

$$\int_E A\tau : \chi d\mathbf{x} = \int_{\hat{E}} \hat{A}\frac{1}{J_E}\hat{\tau}DF_E^T : \frac{1}{J_E}\hat{\chi}DF_E^T J_E d\hat{\mathbf{x}} = \int_{\hat{E}} \hat{A}\hat{\tau}\frac{1}{J_E}DF_E^T : \hat{\chi}DF_E^T d\hat{\mathbf{x}}.$$

The quadrature rule on an element E is then defined as

$$(A\tau, \chi)_{Q,E} \equiv \left(\hat{A}\hat{\tau}\frac{1}{J_E}DF_E^T, \hat{\chi}DF_E^T\right)_{\hat{Q},\hat{E}} \equiv \frac{|\hat{E}|}{4}\sum_{i=1}^{4}\hat{A}(\hat{\mathbf{r}}_i)\hat{\tau}(\hat{\mathbf{r}}_i)\frac{1}{J_E(\hat{\mathbf{r}}_i)}DF_E^T(\hat{\mathbf{r}}_i) : \hat{\chi}(\hat{\mathbf{r}}_i)DF_E^T(\hat{\mathbf{r}}_i). \tag{2.16}$$

The global quadrature rule is defined as $(A\tau, \chi)_Q \equiv \sum_{E\in\mathcal{T}_h}(A\tau, \chi)_{Q,E}$. We note that the quadrature rule can be defined directly on a physical element E:

$$(A\tau, \chi)_{Q,E} = \frac{1}{2}\sum_{i=1}^{4} | T_i | A(\mathbf{r}_i)\tau(\mathbf{r}_i) : \chi(\mathbf{r}_i), \tag{2.17}$$

where $|T_i|$ is the area of triangle formed by the two edges sharing $\mathbf{r}_i$.

Recall that the stress degrees of freedom are the two normal components per edge evaluated at the vertices (Figure 1). For an element vertex $\mathbf{r}_i$, the tensor $\chi(\mathbf{r}_i)$ is uniquely determined by its normal components to the two edges that share that vertex. Since the basis functions associated with a vertex are zero at all other vertices, the quadrature rule (2.16) decouples the degrees of freedom associated with a vertex from the rest of the degrees of freedom, which allows for local stress elimination.

We also employ the trapezoidal quadrature rule for the stress–rotation bilinear forms in the case of bilinear rotations. For $\tau \in \mathbb{X}_h, \xi \in \mathbb{W}_h^1$, we have

$$(\tau, \xi)_{Q,E} \equiv \left(\frac{1}{J_E}\hat{\tau}F_E^T, \hat{\xi}J_E\right)_{\hat{Q},\hat{E}} \equiv \frac{|\hat{E}|}{4}\sum_{i=1}^{4}\hat{\tau}(\hat{\mathbf{r}}_i)DF_E(\hat{\mathbf{r}}_i)^T : \hat{\xi}(\hat{\mathbf{r}}_i). \tag{2.18}$$

The next lemma shows that the quadrature rule (2.16) produces a coercive bilinear form.

Lemma 2.1 *The bilinear form $(A\tau, \chi)_Q$ is an inner product on $\mathbb{X}_h$ and $(A\tau, \tau)_Q^{1/2}$ is a norm in $\mathbb{X}_h$ equivalent to $\|\cdot\|$, that is, there exist constants $0 < \alpha_0 \le \alpha_1$ independent of h such that*

$$\alpha_0 \| \tau\|^2 \le (A\tau, \tau)_Q \le \alpha_1 \| \tau\|^2 \quad \forall\tau \in \mathbb{X}_h. \tag{2.19}$$

Furthermore, $(\xi, \xi)_Q^{1/2}$ is a norm in $\mathbb{W}_h^1$ equivalent to $\|\cdot\|$, and $\forall\tau \in \mathbb{X}_h, \xi \in \mathbb{W}_h^1$, $(\tau, \xi)_Q \le C \| \tau \| \| \xi\|$.

Proof. The proof follows the argument of Lemma 2.2 in [2], using (2.17). ■

3 | THE MULTIPOINT STRESS MIXED FINITE ELEMENT METHOD WITH CONSTANT ROTATIONS (MSMFE-0)

Let $\mathcal{P}_0$ be the L^2-orthogonal projection onto the space of piecewise constant vector-valued functions on the trace of $\mathcal{T}_h$ on $\partial\Omega$. Our first method, referred to as MSMFE-0, is: find $\sigma_h \in \mathbb{X}_h, u_h \in V_h$, and $\gamma_h \in \mathbb{W}_h^0$ such that

$$(A\sigma_h, \tau)_Q + (u_h, \mathrm{div}\tau) + (\gamma_h, \tau) = \langle\mathcal{P}_0 g, \tau n\rangle_{\Gamma_D}, \quad \tau \in \mathbb{X}_h, \tag{3.1}$$

$$(\mathrm{div}\sigma_h, v) = (f, v), \quad v \in V_h, \tag{3.2}$$

$$(\sigma_h, \xi) = 0, \quad \xi \in \mathbb{W}_h^0. \tag{3.3}$$

The Dirichlet data are incorporated into the scheme as $\mathcal{P}_0 g$, which is necessary for the optimal approximation of the boundary condition term.

Theorem 3.1 *The method* (3.1)–(3.3) *has a unique solution.*

Proof. Using classical stability theory of MFE methods, the required Babuška–Brezzi stability conditions [15] are:

(S1) There exists $c_1 > 0$ such that

$$c_1 \|\tau\|_{\mathrm{div}} \le (A\tau, \tau)_Q^{1/2} \tag{3.4}$$

for $\tau \in \mathbb{X}_h$ satisfying $(\mathrm{div}\tau, v) = 0$ and $(\tau, \xi) = 0$ for all $(v, \xi) \in V_h \times \mathbb{W}_h^0$.

(S2) There exists $c_2 > 0$ such that

$$\inf_{0 \ne (v,\xi) \in V_h \times \mathbb{W}_h^0} \sup_{0 \ne \tau \in \mathbb{X}_h} \frac{(\mathrm{div}\tau, v) + (\tau, \xi)}{\| \tau \|_{\mathrm{div}} (\| v \| + \| \xi \|)} \ge c_2. \tag{3.5}$$

Using (2.11) and $\mathrm{div}\widehat{\mathbb{X}}(\widehat{E}) = \widehat{V}(\widehat{E})$, the condition $(\mathrm{div}\tau, v) = 0,\ \forall v \in V_h$ implies that $\mathrm{div}\tau = 0$. Then (S1) follows from (2.19). The inf-sup condition (S2) has been shown in [18]. ■

3.1 | Reduction to a cell-centered displacement-rotation system

The algebraic system that arises from (3.1)–(3.3) is of the form

$$\begin{pmatrix} A_{\sigma\sigma} & A_{\sigma u}^T & A_{\sigma\gamma}^T \\ -A_{\sigma u} & 0 & 0 \\ -A_{\sigma\gamma} & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma \\ u \\ \gamma \end{pmatrix} = \begin{pmatrix} g \\ -f \\ 0 \end{pmatrix}, \tag{3.6}$$

where $(A_{\sigma\sigma})_{ij} = (A\tau_j, \tau_i)_Q$, $(A_{\sigma u})_{ij} = (\mathrm{div}\tau_j, v_i)$, and $(A_{\sigma\gamma})_{ij} = (\tau_j, \xi_i)$. The method can be reduced to solving a cell-centered displacement-rotation system as follows. Since the quadrature rule $(A\sigma_h, \tau)_Q$ localizes the basis functions interaction around mesh vertices, the matrix $A_{\sigma\sigma}$ is block-diagonal with $2k \times 2k$ blocks associated with vertices, where k is the number of elements that share the vertex, see Figure 2 (left) for an example with $k = 4$. Lemma 2.1 implies that the blocks are symmetric and positive definite. Therefore the stress σ_h can be easily eliminated by solving small local systems, resulting in the cell-centered displacement-rotation system

$$\begin{pmatrix} A_{\sigma u} A_{\sigma\sigma}^{-1} A_{\sigma u}^T & A_{\sigma u} A_{\sigma\sigma}^{-1} A_{\sigma\gamma}^T \\ A_{\sigma\gamma} A_{\sigma\sigma}^{-1} A_{\sigma u}^T & A_{\sigma\gamma} A_{\sigma\sigma}^{-1} A_{\sigma\gamma}^T \end{pmatrix} \begin{pmatrix} u \\ \gamma \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ \tilde{h} \end{pmatrix}. \tag{3.7}$$

The displacement and rotation stencils for an element E include all elements that share a vertex with E, see Figure 2 (right) for an example of the displacement stencil. The matrix in (3.7) is clearly symmetric. Furthermore, for any $(v^T\ \xi^T) \ne 0$,

$$\begin{pmatrix} v^T & \xi^T \end{pmatrix} \begin{pmatrix} A_{\sigma u} A_{\sigma\sigma}^{-1} A_{\sigma u}^T & A_{\sigma u} A_{\sigma\sigma}^{-1} A_{\sigma\gamma}^T \\ A_{\sigma\gamma} A_{\sigma\sigma}^{-1} A_{\sigma u}^T & A_{\sigma\gamma} A_{\sigma\sigma}^{-1} A_{\sigma\gamma}^T \end{pmatrix} \begin{pmatrix} v \\ \xi \end{pmatrix} = (A_{\sigma u}^T v + A_{\sigma\gamma}^T \xi)^T A_{\sigma\sigma}^{-1} (A_{\sigma u}^T v + A_{\sigma\gamma}^T \xi) > 0, \tag{3.8}$$

due to the inf-sup condition (S2), which implies that the matrix is positive definite.

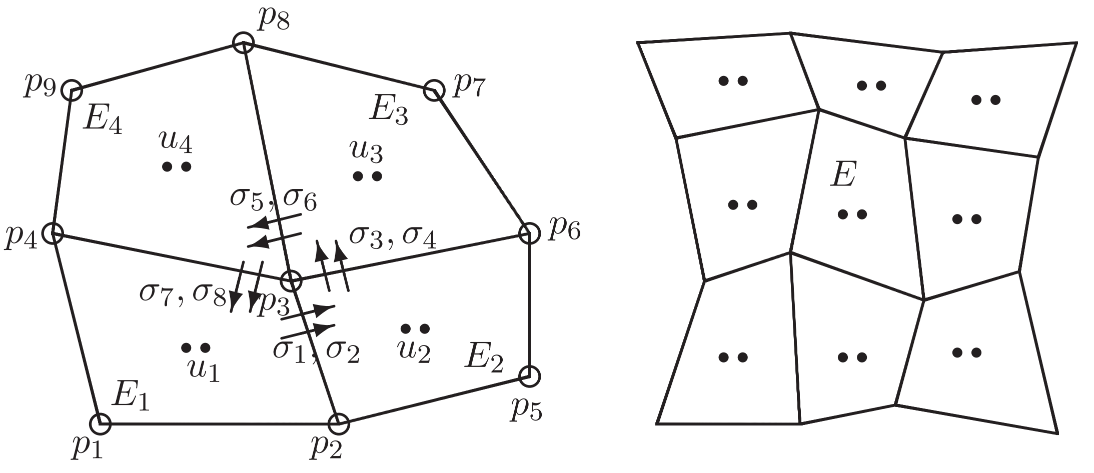


FIGURE 2 Finite elements sharing a vertex (left) and displacement stencil (right)

Remark 3.1 The MSMFE-0 method is more efficient than the original MFE method, since it involves a smaller system, which is symmetric and positive definite. To quantify the computational savings, consider a logically rectangular grid. In this case the number of elements and vertices are approximately the same. Denote this number by m. In the original system (3.6), there are eight stress degrees of freedom per vertex, two displacement degrees of freedom per element, and one rotation degree of freedom per element, resulting in approximately $11m$ unknowns. The reduced system (3.7) has only approximately $3m$ unknowns, a significant reduction. Moreover, the reduced system is symmetric and positive definite, and thus efficient solvers such as the conjugate gradient method or multigrid can be used for its solution. In comparison, the original system (3.6) is indefinite and such fast solvers are not applicable. We further note that the additional cost for solving the local vertex systems required to form (3.7) is $O(m)$, which is negligible for large m compared to the cost for solving the global systems (3.6) or (3.7) using a Krylov space iterative method, which is at least $O(m^2)$.
We remark that further reduction in the system (3.7) is not possible. In the next section we develop a method with continuous bilinear rotations and a trapezoidal quadrature rule applied to the stress-rotation bilinear forms. This allows for further local elimination of the rotation, resulting in a cell-centered system for the displacement only.

4 | THE MULTIPOINT STRESS MIXED FINITE ELEMENT METHOD WITH BILINEAR ROTATIONS (MSMFE-1)

In the second method, referred to as MSMFE-1, we take $k = 1$ in (2.9) and apply the quadrature rule to both the stress bilinear form and the stress-rotation bilinear forms. The method is: find $\sigma_h \in \mathbb{X}_h$, $u_h \in V_h$ and $\gamma_h \in \mathbb{W}_h^1$ such that

$$(A\sigma_h, \tau)_Q + (u_h, \operatorname{div}\tau) + (\gamma_h, \tau)_Q = \langle \mathcal{P}_0 g, \tau n \rangle_{\Gamma_D}, \quad \tau \in \mathbb{X}_h, \tag{4.1}$$

$$(\operatorname{div}\sigma_h, v) = (f, v), \quad v \in V_h, \tag{4.2}$$

$$(\sigma_h, \xi)_Q = 0, \quad \xi \in \mathbb{W}_h^1. \tag{4.3}$$

The stability conditions for the MSMFE-1 method are as follows:

(S3) There exists $c_3 > 0$ such that

$$c_3 \parallel \tau \parallel_{\mathrm{div}}^2 \leq (A\tau, \tau)_Q,$$

for $\tau \in \mathbb{X}_h$ satisfying $(\mathrm{div}\tau, v) = 0$ and $(\tau, \xi)_Q = 0$ for all $(v, \xi) \in V_h \times \mathbb{W}_h^1$.

(S4) There exists $c_4 > 0$ such that

$$\inf_{0\neq(v,\xi)\in V_h\times\mathbb{W}_h^1} \sup_{0\neq\tau\in\mathbb{X}_h} \frac{(\mathrm{div}\tau, v) + (\tau, \xi)_Q}{\parallel \tau \parallel_{\mathrm{div}}(\parallel v \parallel + \parallel \xi \parallel)} \geq c_4. \tag{4.4}$$

4.1 | Well-posedness of the MSMFE-1 method

The stability condition (S3) holds, since the spaces $\mathbb{X}_h$ and V_h are as in the MSMFE-0 method. However, (S4) is different, due to the quadrature rule in $(\tau, \xi)_Q$, and it needs to be verified. The next theorem, proved in [2], provides sufficient conditions for a triple $\mathbb{X}_h \times V_h \times \mathbb{W}_h^1$ to satisfy (S4), where we adopt the notation $b(q, w) = (\mathrm{div}\, q, w)$ and $b(q, w)_Q = (\mathrm{div}\, q, w)_Q$.

Theorem 4.1 *Suppose that* $S_h \subset H(\mathrm{div}; \Omega)$ *and* $U_h \subset L^2(\Omega)$ *satisfy*

$$\inf_{0\neq r\in U_h} \sup_{0\neq z\in S_h} \frac{b(z, r)}{\parallel z \parallel_{\mathrm{div}} \parallel r \parallel} \geq c_5, \tag{4.5}$$

that $Q_h \subset H^1(\Omega, \mathbb{R}^2)$ *and* $W_h \subset L^2(\Omega)$ *are such that* $(w, w)_Q^{1/2}$ *is a norm in* W_h *equivalent to* $\|w\|$ *and*

$$\inf_{0\neq w\in W_h^1} \sup_{0\neq q\in Q_h} \frac{b(q, w)_Q}{\parallel q \parallel_1 \parallel w \parallel} \geq c_6. \tag{4.6}$$

and that

$$\mathrm{curl} Q_h \subset S_h \times S_h. \tag{4.7}$$

Then, $\mathbb{X}_h = S_h \times S_h \subset H(\mathrm{div}; \Omega, \mathbb{M})$, $V_h = U_h \times U_h \subset L^2(\Omega, \mathbb{R}^2)$, *and* $\mathbb{W}_h^1 = \left\{ \xi : \xi = \begin{pmatrix} 0 & w \\ -w & 0 \end{pmatrix}, \ w \in W_h \right\} \subset L^2(\Omega, \mathbb{N})$ *satisfy* (S4).

Remark 4.1 Condition (4.5) states that $S_h \times U_h$ is a stable Darcy pair. Condition (4.6) states that $Q_h \times W_h$ is a stable Stokes pair with quadrature.

Lemma 4.1 *Conditions* (4.5) and (4.7) *hold for* $\mathbb{X}_h \times V_h \times \mathbb{W}_h^1$ *defined in* (2.9) *and* (2.12).

Proof. According to the definition (2.9), we take

$$S_h = \{z \in H(\mathrm{div}; \Omega) : z|_E \overset{\mathcal{P}}{\leftrightarrow} \hat{z} \in \mathcal{BDM}_1(\hat{E}), z \cdot n = 0 \text{ on } \Gamma_N\},$$
$$U_h = \{r \in L^2(\Omega) : r|_E \leftrightarrow \hat{r} \in \mathcal{Q}_0(\hat{E})\}, \quad W_h = \{w \in H^1(\Omega) : w|_E \leftrightarrow \hat{w} \in \mathcal{Q}_1(\hat{E})\}.$$

We note that W_h satisfies the norm equivalence $(w, w)_Q^{1/2} \sim \parallel w \parallel$, see Lemma 2.1. The boundary condition in S_h is needed to guarantee the essential boundary condition in $\mathbb{X}_h$ on Γ_N. Since $\mathcal{BDM}_1 \times \mathcal{Q}_0$ is a stable Darcy pair [15], (4.5) holds. Next, following the construction in [18], we take $\mathcal{SS}_2(\hat{E})$ to be the reduced bi-quadratics (serendipity) space [23],

$$\mathcal{SS}_2(\hat{E}) = \mathcal{P}_2(\hat{E}) + span\{\hat{x}^2\hat{y}, \hat{x}\hat{y}^2\},$$

and define the space Q_h as

$$Q_h = \{q \in H^1(\Omega, \mathbb{R}^2) : q_i|_E \leftrightarrow \hat{q}_i \in \mathcal{SS}_2(\hat{E}), \; i = 1, 2, \; \forall E \in \mathcal{T}_h, \; q = 0 \text{ on } \Gamma_N\}. \tag{4.8}$$

One can verify that $\operatorname{curl}\mathcal{SS}_2(\hat{E}) \subset \mathcal{BDM}_1(\hat{E})$. Due to (2.11), $\operatorname{curl}Q_h \subset S_h \times S_h$, not considering the boundary condition on Γ_N. Finally, since for $q \in Q_h$ we have $q = 0$ on Γ_N, then $(\operatorname{curl}q)\, n = 0$ on Γ_N, see [2, Lemma 4.2]. ∎

To prove (S4), it remains to show that (4.6) holds. It is shown in [21] that $\mathcal{SS}_2 - \mathcal{Q}_1$ is a stable Stokes pair on rectangular grids. We need to verify that it is a stable Stokes pair with quadrature on quadrilaterals, which we do next.

4.1.1 | The inf-sup condition for the Stokes problem

We prove (4.6) using a modification of the macroelement technique presented in [21]. The idea is to establish first a local inf-sup condition and then combine locally constructed velocity vectors to prove the global inf-sup condition. We recall that in [21], it was sufficient to consider velocity functions that vanish on the macroelement boundary in order to control the pressures locally. However, due to the quadrature rule, this is not true in our case. We show how a similar result can be obtained without restricting the velocity basis functions on the macroelement boundary, under a smoothness assumption on the grid $\mathcal{T}_h$.

For a finite element space Y_h and a union of finite elements D, we let $Y_h(D) = Y_h|_D$. For an element E, we consider the span of shape functions associated with all edge degrees of freedom of $Q_h(E)$ and denote it by $Q_h^e(E)$. Let

$$N_E = \{w \in W_h(E) : b(q, w)_{Q,E} = 0, \;\; \forall q \in Q_h^e(E)\}.$$

We make the following assumptions on the mesh.

(M1) Each element E has at most one edge on Γ_N.

(M2) The mesh size h is sufficiently small and there exists a constant C such that for every pair of neighboring elements E and $\tilde{E}$ such that E or $\tilde{E}$ is a non-parallelogram, and every pair of edges $e \subset \partial E \backslash \partial \tilde{E}$, $\tilde{e} \subset \partial \tilde{E} \backslash \partial E$ that share a vertex,

$$\|\mathbf{r}_e - \mathbf{r}_{\tilde{e}}\|_{\mathbb{R}^2} \le Ch^2,$$

where $\mathbf{r}_e$ and $\mathbf{r}_{\tilde{e}}$ are the vectors corresponding to e and $\tilde{e}$, respectively, and $\|\cdot\|_{\mathbb{R}^2}$ is the Euclidean vector norm.

Remark 4.2 Condition (M1) is needed to establish a local inf-sup condition. It requires that corner elements do not have two edges on Γ_N. Condition (M2) is needed to combine the local results and prove the global inf-sup condition (4.6). We note that it is required only for non-parallelogram elements. It is a mesh smoothness condition. For example, it is satisfied if the mesh is generated by a C^2 map of a uniform reference grid. The condition on the mesh size h is given in the proof of Lemma 4.4.

Lemma 4.2 *Let* (M1) *hold. If E is a parallelogram, then N_E is one-dimensional, consisting of functions that are constant on E; otherwise $N_E = 0$.*

Proof. For any $q \in Q_h(E)$, $w \in W_h(E)$, we have

$$b(q,w)_{Q,E} = (\mathrm{tr}(\nabla q), w)_{Q,E} = \frac{1}{4}\sum_{j=1}^{4} \mathrm{tr}\,[DF_E^{-T}(\hat{\mathbf{r}}_j)\hat{\nabla}\hat{q}(\hat{\mathbf{r}}_j)]\hat{w}(\hat{\mathbf{r}}_j)J_E(\hat{\mathbf{r}}_j).$$

Consider first an element with no edges on Γ_N. Denote the basis functions for $Q_h^e(E)$ by $q_i = q_i^n + q_i^t$, $i = 1, \dots, 4$, see Figure 3 (left). Without loss of generality, assume that the edge corresponding to q_1 is horizontal, that is, $y_2 - y_1 = 0$, $x_2 - x_1 \neq 0$, $x_3 - x_4 \neq 0$, $y_4 - y_1 \neq 0$, and $y_3 - y_2 \neq 0$. A direct calculation gives

$$b(q_1^t, w)_{Q,E} = (y_4 - y_1)w(\mathbf{r}_1) + (y_2 - y_3)w(\mathbf{r}_2), \tag{4.9}$$

$$b(q_1^n, w)_{Q,E} = (y_1 - y_2)w(\mathbf{r}_1) + (y_2 - y_1)w(\mathbf{r}_2), \tag{4.10}$$

$$b(q_2^t, w)_{Q,E} = (x_2 - x_1)w(\mathbf{r}_2) + (x_4 - x_3)w(\mathbf{r}_3), \tag{4.11}$$

$$b(q_2^n, w)_{Q,E} = (x_2 - x_3)w(\mathbf{r}_2) + (x_3 - x_2)w(\mathbf{r}_3), \tag{4.12}$$

$$b(q_3^t, w)_{Q,E} = (y_2 - y_3)w(\mathbf{r}_3) + (y_4 - y_1)w(\mathbf{r}_4), \tag{4.13}$$

$$b(q_3^n, w)_{Q,E} = (y_3 - y_4)w(\mathbf{r}_3) + (y_4 - y_3)w(\mathbf{r}_4). \tag{4.14}$$

$$b(q_4^t, w)_{Q,E} = (x_2 - x_1)w(\mathbf{r}_1) + (x_4 - x_3)w(\mathbf{r}_4), \tag{4.15}$$

$$b(q_4^n, w)_{Q,E} = (x_1 - x_4)w(\mathbf{r}_1) + (x_4 - x_1)w(\mathbf{r}_4). \tag{4.16}$$

Let us set the above quantities equal to zero. Consider the vertically oriented edges of E. From (4.11) and (4.15) we get

$$w(\mathbf{r}_2) = w(\mathbf{r}_3)\frac{x_4 - x_3}{x_1 - x_2}, \quad w(\mathbf{r}_1) = w(\mathbf{r}_4)\frac{x_4 - x_3}{x_1 - x_2}. \tag{4.17}$$

If $x_2 \neq x_3$, we also get from (4.12) that $w(\mathbf{r}_2) = w(\mathbf{r}_3)$. This together with (4.17) implies that $w(\mathbf{r}_1) = w(\mathbf{r}_4)$. Similarly, if $x_1 \neq x_4$, it follows from (4.16) that $w(\mathbf{r}_1) = w(\mathbf{r}_4)$, and (4.17) implies that $w(\mathbf{r}_2) = w(\mathbf{r}_3)$. Finally, if $x_2 = x_3$ and $x_1 = x_4$, we arrive to the same conclusion directly from (4.17).

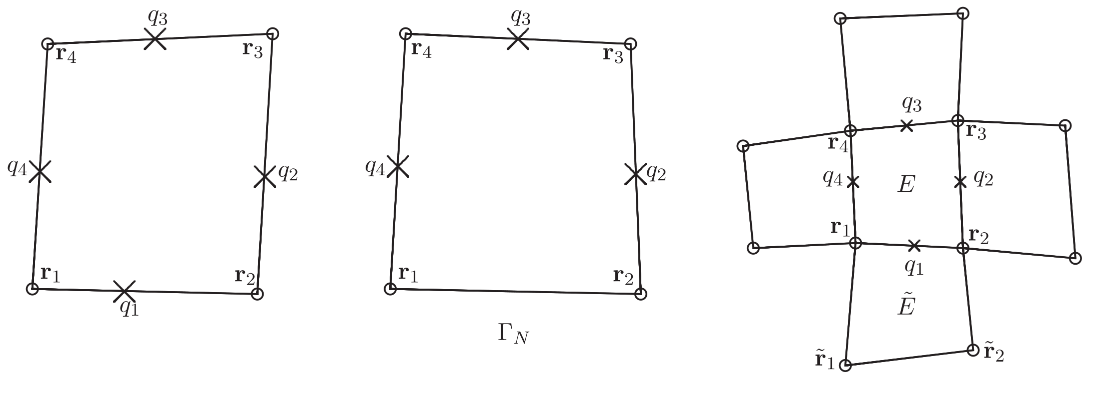


FIGURE 3 Left: interior element; middle: element with bottom edge on Γ_N; right: an interior element, surrounded by four elements

Next, consider the edges corresponding to q_1 and q_3. From (4.13) we get

$$w(\mathbf{r}_3) = w(\mathbf{r}_4)\frac{y_1 - y_4}{y_2 - y_3}. \tag{4.18}$$

If $y_3 \neq y_4$, (4.14) implies that $w(\mathbf{r}_3) = w(\mathbf{r}_4)$. If $y_3 = y_4$, since $y_1 = y_2$, we obtain from (4.18) that $w(\mathbf{r}_3) = w(\mathbf{r}_4)$. Hence, w must be constant on E.

We next consider the case when one of the edges of E is on Γ_N. Let this be the edge associated with q_1, as shown in Figure 3 (middle). Since the above argument above did not use (4.9) or (4.10), the conclusion still applies.

Finally, if w is a nonzero constant in N_E, setting Equations (4.10)–(4.16) to zero implies that E is a parallelogram. ■

Theorem 4.2 *If* (M1)–(M2) *are satisfied, then* (4.6) *holds.*

The proof of Theorem 4.2 is based on several auxiliary lemmas.

Lemmas 4.3 *If* (M1) *holds, then there exists a constant* $\beta > 0$ *independent of h such that,*

$$\forall T \in \mathcal{T}_h, \qquad \sup_{0 \neq q \in Q_h^e(E)} \frac{b(q, w)_{Q,E}}{\|q\|_{1,E}} \geq \beta \|w\|_E, \quad \forall w \in W_h(E)/N_E.$$

Proof. The proof follows from Lemma 4.2 and a scaling argument, see [21, Lemma 3.1]. ■

For $E \in \mathcal{T}_h$, let $\mathbb{P}_h^E$ denote the L^2-projection from $W_h(E)$ onto N_E.

Lemma 4.4 *If* (M1) *and* (M2) *hold, then there exists a constant* $C_1 > 0$, *such that for every* $w \in W_h$ *and for every* $E \in \mathcal{T}_h$ *that is either a non-parallelogram or a parallelogram that neighbors parallelograms, there exists* $q_E \in Q_h^e(E)$ *satisfying*

$$b(q_E, w)_Q \geq C_1 \|(I - \mathbb{P}_h^E)w\|_E^2 \quad \textit{and} \quad \|q_E\|_1 \leq \|(I - \mathbb{P}_h^E)w\|_E. \tag{4.19}$$

Proof. Let $w \in W_h$. Due to Lemma 4.2, if E is not a parallelogram, then $\mathbb{P}_h^E w = 0$ on E. Otherwise, $\mathbb{P}_h^E w$ is the mean value of w on E. Lemma 4.3 implies that for every E there exists $q_E \in Q_h^e(E)$ such that

$$b(q_E, w)_{Q,E} = b(q_E, (I - \mathbb{P}_h^E)w)_{Q,E} \geq C\|(I - \mathbb{P}_h^E)w\|_E^2 \quad \text{and} \quad \|q_E\|_{1,E} \leq \|(I - \mathbb{P}_h^E)w\|_E. \tag{4.20}$$

We note that q_E does not vanish outside of E; however, we will show that under assumption (M2)

$$b(q_E, w)_{Q,\Omega \setminus E} \geq 0. \tag{4.21}$$

In order to prove (4.21) let us consider a neighboring element $\widetilde{E}$, see Figure 3 (right). Let $q_E = \sum_{i=1}^4 \alpha_i q_i$. We first consider a non-parallelogram E. Consider the tangential degree of freedom q_1^t, associated with the edge shared by E and $\widetilde{E}$. Using (4.9), we have

$$b(q_1^t, w)_{Q,E} = (y_4 - y_1)w(\mathbf{r}_1) + (y_2 - y_3)w(\mathbf{r}_2) := \sum_{j=1}^4 \delta_{1,j}^t w(\mathbf{r}_j), \tag{4.22}$$

where $\delta^t_{1,1} = (y_4 - y_1)$, $\delta^t_{1,2} = (y_2 - y_3)$, and $\delta^t_{1,j} = 0$ for $j = 3, 4$. For q^n_1, using (4.10), we have

$$b(q^n_1, w)_{Q,E} = (y_1 - y_2)w(\mathbf{r}_1) + (y_2 - y_1)w(\mathbf{r}_2) := \sum_{j=1}^{4} \delta^n_{1,j} w(\mathbf{r}_j). \tag{4.23}$$

Using a similar expression for the rest of the degrees of freedom, we obtain

$$b(q_E, w)_{Q,E} = \sum_{i=1}^{4} \alpha_i b(q_i, w)_{Q,E} = \sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \delta_{i,j} w(\mathbf{r}_j),$$

where $\delta_{i,j} = \delta^n_{i,j} + \delta^t_{i,j}$. We note that for all i, j, $\delta_{i,j} = 0$ or $|\delta_{i,j}| = O(h)$. Using (4.13), we also compute

$$b(q^t_1, w)_{Q,\widetilde{E}} = (y_1 - \widetilde{y}_1)w(\mathbf{r}_1) + (\widetilde{y}_2 - y_2)w(\mathbf{r}_2) := \sum_{j=1}^{4} \sigma^t_{1,j} w(\mathbf{r}_j), \tag{4.24}$$

where $\sigma^t_{1,1} = (y_1 - \widetilde{y}_1)$, $\sigma^t_{1,2} = (\widetilde{y}_2 - y_2)$, and $\sigma^t_{1,j} = 0$ for $j = 3, 4$. Using (4.14), we have

$$b(q^n_1, w)_{Q,\widetilde{E}} = (y_1 - y_2)w(\mathbf{r}_1) + (y_2 - y_1)w(\mathbf{r}_2) := \sum_{j=1}^{4} \sigma^n_{1,j} w(\mathbf{r}_j). \tag{4.25}$$

Therefore,

$$b(q_E, w)_{Q,\widetilde{E}} = \sum_{i=1}^{4} \alpha_i b(q_i, w)_{Q,\widetilde{E}} = \sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \sigma_{i,j} w(\mathbf{r}_j).$$

Moreover, due to assumption (M2),

$$\sigma_{i,j} = \delta_{i,j} + \theta_{i,j},$$

with $\theta_{i,j} = 0$ if $\delta_{i,j} = 0$ and $|\theta_{i,j}| \le Ch^2$ otherwise. Indeed, consider, for example, $i = j = 1$, then, by (M2),

$$| \sigma_{1,1} - \delta_{1,1} | = | \sigma^t_{1,1} - \delta^t_{1,1} | = | (y_1 - \widetilde{y}_1) - (y_4 - y_1) | \le Ch^2.$$

Therefore, we obtain

$$\begin{aligned} b(q_E, w)_{Q,\widetilde{E}} &= \sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \sigma_{i,j} w(\mathbf{r}_j) = b(q_E, w)_{Q,E} + \sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \theta_{i,j} w(\mathbf{r}_j) \\ &\ge Ch^2 \sum_{j=1}^{4} (w(\mathbf{r}_j))^2 + \sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \theta_{i,j} w(\mathbf{r}_j), \end{aligned} \tag{4.26}$$

using the first inequality in (4.20), that $\mathbb{P}^E_h w = 0$, and that

$$\| w \|^2_E \sim h^2 \sum_{j=1}^{4} (w(\mathbf{r}_j))^2, \tag{4.27}$$

which follows from the norm equivalence $\|w\|_E \sim \|w\|_{Q,E}$ stated in Lemma 2.1 and the shape regularity of the mesh.

Finally, the second inequality in (4.20) and a scaling argument imply that for every $i = 1, \dots, 4$ there exist constants $b_{i,k}$, $k = 1, \dots, 4$, independent of h such that

$$\alpha_i = h\sum_{k=1}^{4} b_{i,k} w(\mathbf{r}_k). \tag{4.28}$$

Then, there exists a constant $\widetilde{C}$ independent of h such that

$$\left|\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_i\theta_{i,j}w(\mathbf{r}_j)\right| = \left|\sum_{i=1}^{4} h\sum_{k=1}^{4} b_{i,k}w(\mathbf{r}_k)\sum_{j=1}^{4}\theta_{i,j}w(\mathbf{r}_j)\right| \le \widetilde{C}h^3\sum_{j=1}^{4}(w(\mathbf{r}_j))^2. \tag{4.29}$$

Combining (4.26)–(4.29) and taking $h \le C/\widetilde{C}$, we obtain (4.21):

$$b(q_E, w)_{Q,\widetilde{E}} \ge Ch^2\sum_{j=1}^{4}(w(\mathbf{r}_j))^2 - \widetilde{C}h^3\sum_{j=1}^{4}(w(\mathbf{r}_j))^2 \ge (C - \widetilde{C}h)h^2\sum_{j=1}^{4}(w(\mathbf{r}_j))^2 \ge 0.$$

Next, consider the case of a parallelogram E with parallelogram neighbors. In this case, (4.22) and (4.24) give

$$b(q_1^t, w)_{Q,E} = (y_4 - y_1)(w(\mathbf{r}_1) - w(\mathbf{r}_2)), \quad b(q_1^t, w)_{Q,\widetilde{E}} = (y_1 - \widetilde{y}_1)(w(\mathbf{r}_1) - w(\mathbf{r}_2)). \tag{4.30}$$

Similarly, (4.23) and (4.25) give

$$b(q_1^n, w)_{Q,E} = (y_1 - y_2)(w(\mathbf{r}_1) - w(\mathbf{r}_2)), \quad b(q_1^n, w)_{Q,\widetilde{E}} = (y_1 - y_2)(w(\mathbf{r}_1) - w(\mathbf{r}_2)).$$

Similar relationships hold for the rest of the basis functions. Therefore there exist positive constants c_i, $i = 1, \dots, 4$, such that $b(q_i, w)_{Q,\widetilde{E}} = c_i b(q_i, w)_{Q,E}$. We can assume that $\alpha_i b(q_i, w)_{Q,E} \ge 0$ for $i = 1, \dots, 4$, since, if $\alpha_i b(q_i, w)_{Q,E} < 0$, it can be omitted from the linear combination $q_E = \sum_{i=1}^{4} \alpha_i q_i$ and the resulting q_E would still satisfy (4.20). Therefore, (4.21) holds:

$$b(q_E, w)_{Q,\widetilde{E}} = \sum_{i=1}^{4}\alpha_i b(q_i, w)_{Q,\widetilde{E}} = \sum_{i=1}^{4} c_i\alpha_i b(q_i, w)_{Q,E} \ge 0.$$

The assertion of the lemma now follows from (4.20) and (4.21), where the second inequality in (4.19) follows from (4.28). ■

We next note that the element norm equivalence (4.27) implies that for $w \in W_h$,

$$\|w\|^2 \sim h^2\sum_{j=1}^{N_W}(w(\mathbf{r}_j))^2, \tag{4.31}$$

where N_W is the number of degrees of freedom of W_h. Therefore, to prove (4.6), it is sufficient to control $h^2(w(\mathbf{r}_j))^2$. We will consider three sets of vertices and show that each set can be controlled. Let

$I_1 = \{j : \mathbf{r}_j$ is a vertex of a non - parallelogram $\}$,
$I_2 = \{j :$ all elements sharing $\mathbf{r}_j$ are parallelograms and at least one has a non - parallelogram neighbor $\}$,
$I_3 = \{j :$ all elements sharing $\mathbf{r}_j$ are parallelograms with parallelogram neighbors $\}$.

Clearly the union of the three sets covers all vertices of the mesh.

Lemma 4.5 *If* (M1)–(M2) *hold, there exists a constant C independent of h such that for every* $w \in W_h$, *there exists* $q \in Q_h$ *such that*

$$b(q, w)_Q \geq Ch^2 \sum_{j \in I_1 \cup I_2} (w(\mathbf{r}_j))^2, \quad \|q\|_1 \leq \| w \| . \tag{4.32}$$

Proof. If $j \in I_1$, Lemma 4.4 and (4.27) imply that there exists $q_j \in Q_h^e(E)$ such that

$$b(q_j, w)_Q \geq Ch^2(w(\mathbf{r}_j))^2, \quad \|q_j\|_1 \leq \|w\|_E, \tag{4.33}$$

where E is the non-parallelogram element with vertex $\mathbf{r}_j$.

Next, consider $j \in I_2$. Let $\mathbf{r}_k$ share an edge with $\mathbf{r}_j$. Note that its two neighboring elements are parallelograms. Denote them by E and $\tilde{E}$ and let q_1^t be the tangential edge basis function. Using (4.30), we can take $\tilde{q}_j = ch(w(\mathbf{r}_j) - w(\mathbf{r}_k))q_1^t$, which satisfies

$$b(\tilde{q}_j, w)_Q \geq Ch^2(w(\mathbf{r}_j) - w(\mathbf{r}_k))^2, \quad \| \tilde{q}_j\|_1 \leq \| w\|_E. \tag{4.34}$$

Let $\mathbf{r}_k$ be the vertex that belongs to a non-parallelogram, denoted by E_k. Then (4.33) implies that there exists $q_k \in Q_h^e(E_k)$ such that

$$b(q_k, w)_Q \geq Ch^2(w(\mathbf{r}_k))^2, \quad \| q_k\|_1 \leq \| w\|_{E_k}. \tag{4.35}$$

Let $q_j = \tilde{q}_j + q_k$. Due to (4.34) and (4.35), q_j satisfies

$$b(q_j, w)_Q \geq Ch^2(w(\mathbf{r}_j))^2, \quad \| q_j\|_1 \leq \| w\|_{E \cup E_k}. \tag{4.36}$$

Finally, $q \in Q_h$ defined as the sum of the functions constructed in (4.33) and (4.36) satisfies (4.32). ∎

We now consider the set of vertices I_3. If $\mathbf{r}_j$ and $\mathbf{r}_k$ are two vertices in the set that share an edge, (4.34) implies that if one of them is controlled, then so is the other. Therefore, it is enough to consider a subset of vertices that do not share an edge, which we denote by $\tilde{I}_3$. For each vertex $\mathbf{r}_j$, let M_j be the union of elements that share $\mathbf{r}_j$. We note that the set $S = \{M_j \,:\, j \in \tilde{I}_3\}$ is non-overlapping. Let $\overline{M} = \cup_{j \in \tilde{I}_3} M_j$. For $M \in S$, let $Q_h^e(M)$ be the span of all edge degrees of freedom of $Q_h(M)$ and let

$$N_M = \{w \in W_h(M) \,:\, b(q, w)_{Q,M} = 0, \ \ \forall q \in Q_h^e(M)\}.$$

Recall that all elements in M are parallelograms. The argument of Lemma 4.2 can be easily extended to show that N_M consists of constant functions. For $M \in S$, let $\mathbb{P}_h^M$ denote the L^2-projection from $W_h(M)$ onto N_M. Since the neighbors of all elements in M are parallelograms, Lemma 4.4 implies that for any $w \in W_h$, there exists $q_M \in Q_h^e(M)$ satisfying

$$b(q_M, w)_Q \geq C_1 \| (I - \mathbb{P}_h^M)w \|_M^2 \quad \text{and} \quad \| q_M\|_1 \leq \| (I - \mathbb{P}_h^M)w\|_M. \tag{4.37}$$

Let

$$M_h = \left\{\mu \in L^2(\Omega) \,:\, \mu\Big|_M \in N_M, \ \ \forall M \in S, \ \ \mu = 0 \ \text{ otherwise}\right\}.$$

Let $\mathbb{P}_h$ be the L^2-projection from W_h onto M_h. Then (4.37) implies that for any $w \in W_h$, there exists $\tilde{q} \in Q_h$ satisfying

$$b(\tilde{q}, w)_Q \geq C_1 \| (I - \mathbb{P}_h)w \|_{\overline{M}}^2 \quad \text{and} \quad \| \tilde{q}\|_1 \leq \| (I - \mathbb{P}_h)w\|_{\overline{M}}. \tag{4.38}$$

The next lemma shows that $\mathbb{P}_h w$ can also be controlled.

Lemma 4.6 *If* (M1) *holds, there exists a constant* $C_2 > 0$ *such that for every* $w \in W_h$ *there exists* $g \in Q_h$ *such that*

$$b(g, \mathbb{P}_h w)_Q = \| \mathbb{P}_h w \|_{\overline{M}}^2 \quad \text{and} \quad \| g \|_1 \le C_2 \| \mathbb{P}_h w \|_{\overline{M}}.$$

Proof. Let $w \in W_h$ be arbitrary. Since $\mathbb{P}_h w \in L^2(\Omega)$, there exists $z \in H^1(\Omega, \mathbb{R}^2)$ such that

$$\text{div } z = \mathbb{P}_h w \quad \text{and} \quad \| z \|_1 \le C \| \mathbb{P}_h w \|_{\overline{M}}.$$

Following [21, Lemma 3.3], there exists an operator $I_h : H^1(\Omega, \mathbb{R}^2) \to \widetilde{Q}_h$ such that

$$(\text{div } z, \mu) = b(I_h z, \mu), \quad \forall \mu \in M_h, \quad \text{and} \quad \| I_h z \|_1 \le C \| z \|_1,$$

where $\widetilde{Q}_h$ is the subspace of Q_h consisting of element-wise mapped bilinear vector functions. We note that the argument in [21, Lemma 3.3] requires that the interfaces between macroelements have at least two edges. Recall that our macroelements consist of all parallelograms sharing a vertex and their neighbors are also parallelograms. We can therefore choose the subset $\widetilde{I}_3$ appropriately to satisfy this requirement. Here we also consider $\Omega \backslash \overline{M}$ as one macroelement.

We next note that for $q \in \widetilde{Q}_h$ and $\mu \in M_h$, on any $E \in \mathcal{T}_h$,

$$b(q, \mu)_E = \int_{\hat{E}} \text{tr}(DF_E^{-T} \hat{\nabla} \hat{q}) \hat{\mu} J_E d\hat{\mathbf{x}}.$$

A direct calculation shows that the integrated quantity on $\hat{E}$ is bilinear, and hence, using that the quadrature rule is exact for bilinears, $b(I_h z, \mu) = b(I_h z, \mu)_Q$. The proof is completed by taking $g = I_h z$. ■

Lemma 4.7 *If* (M1) *holds, there exists a constant C independent of h such that for every* $w \in W_h$*, there exists* $q \in Q_h$ *such that*

$$b(q, w)_Q \ge Ch^2 \sum_{j \in I_3} (w(\mathbf{r}_j))^2, \quad \| q \|_1 \le \| w \| . \tag{4.39}$$

Proof. Let $w \in W_h$ be given, and let $\widetilde{q} \in Q_h, g \in Q_h, C_1$ and C_2 be as in (4.38) and Lemma 4.6. Set $q = \widetilde{q} + \delta g$, where $\delta = 2C_1(1 + C_2^2)^{-1}$. We then have

$$\begin{aligned} b(q, w)_Q &= b(\widetilde{q}, w)_Q + \delta b(g, w)_Q = b(\widetilde{q}, w)_Q + \delta b(g, \mathbb{P}_h w)_Q + \delta b(g, (I - \mathbb{P}_h) w)_Q \\ &\ge C_1 \| (I - \mathbb{P}_h) w \|_{\overline{M}}^2 + \delta \| \mathbb{P}_h w \|_{\overline{M}}^2 - \delta \| g \|_1 \| (I - \mathbb{P}_h) w \|_{\overline{M}} \ge C_1 (1 + C_2^2)^{-1} \| w \|_{\overline{M}}^2, \end{aligned}$$

and $\| q \|_1 \le \| (I - \mathbb{P}_h) w \|_{\overline{M}} + \delta C_2 \| \mathbb{P}_h w \|_{\overline{M}} \le C \| w \|_{\overline{M}}$. The assertion of the lemma follows from (4.27). ■

We are now ready to prove the main result stated in Theorem 4.2:

Proof of Theorem 4.2 The assertion of the theorem follows from Lemmas 4.5, 4.7, and (4.31). ■

We conclude with the solvability result for the MSMFE-1 method (4.1)–(4.3).

Theorem 4.3 *Under the assumptions* (M1)–(M2), *there exists a unique solution of* (4.1)–(4.3).

Proof. The existence and uniqueness of a solution to (4.1)–(4.3) follow from (S3) and (S4). Lemma 2.1 implies the coercivity condition (S3). Assuming (M1)–(M2), the inf-sup condition (S4) follows from a combination of Theorem 4.1, Lemma 4.1, and Theorem 4.2. ∎

4.2 | Reduction to a cell-centered displacement system of the MSMFE-1 method

The algebraic system that arises from (4.1)–(4.3) is of the form (3.6), where the matrix $A_{\sigma\gamma}$ is different from the one in the MSMFE-0 method, due to the quadrature rule, that is, $(A_{\sigma\gamma})_{ij} = (\tau_j, \xi_i)_Q$. As in the MSMFE-0 method, the quadrature rule in $(A\sigma_h, \tau)_Q$ in (4.1) localizes the basis functions interaction around vertices, so the matrix $A_{\sigma\sigma}$ is block diagonal with $2k \times 2k$ blocks, where k is the number of elements that share a vertex. The stress can be eliminated, resulting in the displacement-rotation system (3.7). The matrix in (3.7) is symmetric and positive definite, due to (3.8) and the inf-sup condition (S4).

Furthermore, the quadrature rule in the stress-rotation bilinear forms $(\gamma_h, \tau)_Q$ and $(\sigma_h, \xi)_Q$ also localizes the interaction around vertices, since there is one rotation basis function associated with each vertex. Therefore the matrix $A_{\sigma\gamma}$ is block-diagonal with $1 \times 2k$ blocks, resulting in a diagonal rotation matrix $A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T$. As a result, the rotation γ_h can be trivially eliminated from (3.7), leading to the cell-centered displacement system

$$(A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma u}^T - A_{\sigma u}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T(A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma\gamma}^T)^{-1}A_{\sigma\gamma}A_{\sigma\sigma}^{-1}A_{\sigma u}^T)u = \hat{f}. \tag{4.40}$$

The above matrix is symmetric and positive definite, since it is a Schur complement of the symmetric and positive definite matrix in (3.7), see [24, Theorem 7.7.6].

Remark 4.3 The MSMFE-1 method is more efficient than the MSMFE-0 method and the original MFE method, since it results in a smaller algebraic system. For example, on a logically rectangular grid with approximately m elements and vertices, the MSMFE-1 system (4.40) has approximately $2m$ unknowns compared to $3m$ unknowns in the MSMFE-0 system (3.7) and $11m$ unknowns in the MFE system (3.6).

5 | ERROR ESTIMATES

In this section we establish optimal convergence for all variables, as well as the superconvergence for the displacement. We start by providing several results that will be used in the analysis.

5.1 | Preliminaries

For the rest of the paper we assume that the quadrilateral elements are $O(h^2)$-perturbations of parallelograms known as h^2-parallelograms. In particular, with the notation from Figure 1, we assume that

$$\| \mathbf{r}_{34} - \mathbf{r}_{21} \| \leq Ch^2. \tag{5.1}$$

Elements of this type are obtained by uniform refinements of a general quadrilateral grid or if the mesh is obtained by a smooth map. This is a standard assumption for the symmetric multipoint flux approximation method [8], required due to the reduced approximation properties of the $\mathcal{BDM}_1$ space on general quadrilaterals [25]. If (5.1) holds, it is easy to check that

$$|DF_E|_{1,\infty,\hat{E}} \leq Ch^2 \quad \text{and} \quad \left|\frac{1}{J_E}DF_E\right|_{j,\infty,\hat{E}} \leq Ch^{j-1}, \quad j = 1, 2. \tag{5.2}$$

In the analysis we will utilize several projection operators. It is known [14, 15, 26] that there exists a projection operator $\Pi : \mathbb{X} \cap H^1(\Omega, \mathbb{M}) \to \mathbb{X}_h$ such that

$$(\text{div}(\Pi\tau - \tau), v) = 0, \quad \forall v \in V_h. \tag{5.3}$$

The operator Π is defined locally on an element E by

$$\Pi\tau \overset{\mathcal{P}}{\leftrightarrow} \widehat{\Pi}\widehat{\tau}, \tag{5.4}$$

where $\widehat{\Pi}$ is a reference element interpolant. We will also utilize the lowest order Raviart–Thomas space [15, 27]: $\widehat{\mathbb{X}}^{\mathcal{RT}}(\widehat{E}) = \begin{pmatrix} \alpha_1 + \beta_1\widehat{x} \\ \alpha_2 + \beta_2\widehat{y} \end{pmatrix} \times \begin{pmatrix} \alpha_3 + \beta_3\widehat{x} \\ \alpha_4 + \beta_4\widehat{y} \end{pmatrix}$. The degrees of freedom of $\widehat{\mathbb{X}}^{\mathcal{RT}}(\widehat{E})$ are the values of the normal components at the midpoints of the edges. A projection operator $\Pi^{\mathcal{RT}}$ onto $\mathbb{X}_h^{\mathcal{RT}}$ similar to (5.3) exists [15, 27], which satisfies for any edge e,

$$\langle (\Pi^{\mathcal{RT}}\tau - \tau)n_e, \chi n_e\rangle_e = 0, \quad \forall \chi \in \mathbb{X}_h^{\mathcal{RT}}. \tag{5.5}$$

It is also easy to see that $\Pi^{\mathcal{RT}}$ satisfies

$$\text{div}\,\tau = \text{div}\,\Pi^{\mathcal{RT}}\tau, \quad \forall \tau \in \mathbb{X}_h \tag{5.6}$$

and

$$\| \Pi^{\mathcal{RT}}\tau \| \leq C \| \tau \|, \quad \forall \tau \in \mathbb{X}_h. \tag{5.7}$$

Since the normal trace space $\mathbb{X}_h^{\mathcal{RT}} n$ consists of piecewise constant vector-valued functions on the trace of $\mathcal{T}_h$ on $\partial\Omega$, the L^2-projection operator $\mathcal{P}_0$ utilized in (3.1) can be equivalently characterized as

$$\text{for any } \phi \in L^2(\partial\Omega, \mathbb{R}^2), \quad \mathcal{P}_0\phi \in \mathbb{X}_h^{\mathcal{RT}} n \quad \text{is such that} \quad \langle \phi - \mathcal{P}_0\phi, \tau n\rangle_{\partial\Omega} = 0, \quad \forall \tau \in \mathbb{X}_h^{\mathcal{RT}}. \tag{5.8}$$

Let Q_h^u be a projection operator onto V_h satisfying for any $v \in L^2(\Omega, \mathbb{R}^2)$,

$$(\widehat{Q}^u\widehat{v} - \widehat{v}, \widehat{w})_{\widehat{E}} = 0, \quad \forall \widehat{w} \in \widehat{V}(\widehat{E}), \quad Q_h^u v = \widehat{Q}^u\widehat{v}\circ F_E^{-1} \ \forall E \in \mathcal{T}_h. \tag{5.9}$$

It follows from (2.11) that

$$(Q_h^u v - v, \text{div}\tau) = 0, \quad \forall \tau \in \mathbb{X}_h. \tag{5.10}$$

Let Q_h^γ be the L^2-orthogonal projection operator onto $\mathbb{W}_h^k$ satisfying for any $\xi \in L^2(\Omega, \mathbb{N})$,

$$(Q_h^\gamma\xi - \xi, \zeta) = 0, \quad \forall \zeta \in \mathbb{W}_h^k. \tag{5.11}$$

The next lemma summarizes the well-known approximation properties of the above operators.

Lemma 5.1 *There exists a constant C independent of h such that*

$$\| v - Q_h^u v \| \leq C \| v\|_r h^r, \ \forall v \in H^r(\Omega, \mathbb{R}^2), \ 0 \leq r \leq 1, \tag{5.12}$$

$$\| \xi - Q_h^\gamma\xi \| \leq C \| \xi\|_r h^r, \ \forall \xi \in H^r(\Omega, \mathbb{N}), \ 0 \leq r \leq 1, \tag{5.13}$$

$$\| \tau - \Pi\tau \| \leq C \| \tau\|_r h^r, \ \forall \tau \in H^r(\Omega, \mathbb{M}), \ 1 \leq r \leq 2, \tag{5.14}$$

$$\| \tau - \Pi^{\mathcal{RT}}\tau \| \leq C \| \tau\|_r h^r, \ \forall \tau \in H^r(\Omega, \mathbb{M}), \ 0 \leq r \leq 1, \tag{5.15}$$

$$\| \text{div}(\tau - \Pi\tau) \| + \| \text{div}(\tau - \Pi^{\mathcal{RT}}\tau) \| \leq C \| \text{div}\tau\|_r \ h^r, \forall \tau \in H^{r+1}(\Omega, \mathbb{M}), \quad 0 \leq r \leq 1. \tag{5.16}$$

Proof. Estimates (5.12) and (5.13) can be found in [23]. Estimates (5.14)–(5.16) are proved in [25, 26]. ∎

We note that on general quadrilaterals, (5.12), (5.13), and (5.15) are also valid, while (5.14) and (5.16) hold only with $r = 1$ and $r = 0$, respectively.

Corollary 5.1 *There exists a constant C independent of h such that for all $E \in \mathcal{T}_h$,*

$$\| \Pi\tau\|_{j,E} \le C \| \tau\|_{j,E}, \quad \forall \tau \in H^j(E,\mathbb{M}), \quad j = 1, 2, \tag{5.17}$$

$$\| \Pi^{\mathcal{RT}}\tau\|_{1,E} \le C \| \tau\|_{1,E}, \quad \forall \tau \in H^1(E,\mathbb{M}), \tag{5.18}$$

$$\| Q_h^\gamma \xi\|_{j,E} \le C \| \xi\|_{j,E}, \quad \forall \xi \in H^1(E,\mathbb{N}), \quad j = 1, 2. \tag{5.19}$$

Proof. The proof follows from the approximation properties (5.13)–(5.15) and the use of the inverse inequality, see for example, [2, Lemma 5.1]. ■

We remind the reader that stress tensors are mapped from the reference element via the Piola transformation, while displacements and rotations are mapped using standard change of variables, see (2.10). The following results can be found in [8], where $\tau \overset{\mathcal{P}}{\leftrightarrow} \hat{\tau}$:

$$|\hat{\tau}|_{j,\hat{E}} \le Ch^j \| \tau\|_{j,E}, \quad j \ge 0, \tag{5.20}$$

$$(\hat{\tau} - \hat{\Pi}^{\mathcal{RT}}\hat{\tau}, \hat{\chi}_0)_{\hat{Q},\hat{E}} = 0 \quad \text{for all constant tensors } \hat{\chi}_0, \tag{5.21}$$

$$|(A\Pi\sigma, \tau - \Pi^{\mathcal{RT}}\tau)|_{Q,E} \le Ch \| \sigma\|_{1,E} \| \tau\|_E \quad \forall \tau \in \mathbb{X}_h. \tag{5.22}$$

Also, for $\xi \leftrightarrow \hat{\xi}$, using standard change of variables,

$$|\hat{\xi}|_{j,\hat{E}} \le Ch^{j-1} \| \xi\|_{j,E}, \quad |\hat{\xi}|_{j,\infty,\hat{E}} \le Ch^j \| \xi\|_{j,\infty,E}, \quad j \ge 0. \tag{5.23}$$

For $\tau, \chi \in \mathbb{X}_h$, $\xi \in \mathbb{W}_h^1$, denote the element quadrature errors by

$$\theta(A\tau,\chi)_E \equiv (A\tau,\chi)_E - (A\tau,\chi)_{Q,E}, \quad \delta(\tau,\xi)_E \equiv (\tau,\xi)_E - (\tau,\xi)_{Q,E},$$

and define the global quadrature errors by $\theta(A\tau,\chi)|_E = \theta(A\tau,\chi)_E$, $\delta(\tau,\xi)|_E = \delta(\tau,\xi)_E$. Similarly denote the quadrature errors on the reference element by $\hat{\theta}(\cdot,\cdot)$ and $\hat{\delta}(\cdot,\cdot)$.

Denote $A \in W^{j,\infty}_{\mathcal{T}_h}$ if $A \in W^{j,\infty}(E) \forall E \in \mathcal{T}_h$ and $\|A\|_{j,\infty,E}$ is uniformly bounded independently of h.

Lemma 5.2 *If $A \in W^{1,\infty}_{\mathcal{T}_h}$, there exists a constant C independent of h such that $\forall \tau \in \mathbb{X}_h$, $\chi \in \mathbb{X}_h^{\mathcal{RT}}$,*

$$| \theta(A\tau,\chi) | \le C \sum_{E\in\mathcal{T}_h} h \| A\|_{1,\infty,E} \| \tau\|_{1,E} \| \chi\|_E. \tag{5.24}$$

Moreover, there exist a constant C independent of h such that for all $\tau \in \mathbb{X}_h^{\mathcal{RT}}$ and $\xi \in \mathbb{W}_h^1$,

$$| \delta(\tau,\xi) | \le C \sum_{E\in\mathcal{T}_h} h \| \tau\|_{1,E} \| \xi\|_E, \tag{5.25}$$

$$| \delta(\tau,\xi) | \le C \sum_{E\in\mathcal{T}_h} h \| \tau\|_E \| \xi\|_{1,E}. \tag{5.26}$$

Proof. For a function φ defined on $\hat{E}$, let $\overline{\varphi}$ be its mean value. We have

$$\begin{aligned}
\theta_E(A\tau,\chi) &= \hat{\theta}_{\hat{E}}\left(\hat{A}\hat{\tau}\frac{1}{J_E}DF_E^T, \hat{\chi}DF_E^T\right) \\
&= \hat{\theta}_{\hat{E}}\left((\hat{A} - \bar{\hat{A}})\hat{\tau}\frac{1}{J_E}DF_E^T, \hat{\chi}DF_E^T\right) + \hat{\theta}_{\hat{E}}\left(\bar{\hat{A}}\hat{\tau}\left(\frac{1}{J_E}DF_E^T - \overline{\frac{1}{J_E}DF_E^T}\right), \hat{\chi}DF_E^T\right) \\
&\quad + \hat{\theta}_{\hat{E}}\left(\bar{\hat{A}}\hat{\tau}\overline{\frac{1}{J_E}DF_E^T}, \hat{\chi}(DF_E^T - \overline{DF_E^T})\right) + \hat{\theta}_{\hat{E}}\left(\bar{\hat{A}}\hat{\tau}\overline{\frac{1}{J_E}DF_E^T}, \hat{\chi}\overline{DF_E^T}\right) \equiv \sum_{k=1}^4 I_k.
\end{aligned} \tag{5.27}$$

Using the Bramble–Hilbert lemma [23], (2.8), (5.20), and (5.23), we bound the first term on the right in (5.27) as follows:

$$| I_1 | \leq C|\hat{A}|_{1,\infty,\hat{E}} \| \hat{\tau}\|_{\hat{E}} \| \hat{\chi}\|_{\hat{E}} \leq Ch \| A\|_{1,\infty,E} \| \tau\|_E \| \chi\|_E. \tag{5.28}$$

Similarly, using (2.8), (5.2), (5.20), and (5.23),

$$| I_2 | + | I_3 | \leq Ch \| \hat{A}\|_{0,\infty,\hat{E}} \| \hat{\tau}\|_{\hat{E}} \| \hat{\chi}\|_{\hat{E}} \leq Ch \| A\|_{0,\infty,E} \| \tau\|_E \| \chi\|_E. \tag{5.29}$$

To bound I_4, recall that the trapezoidal quadrature rule is exact for bilinear functions. Since $\hat{\chi} \in \hat{\mathbb{X}}^{\mathcal{RT}}(\hat{E})$ is linear, $I_4 = 0$ for any constant tensor $\hat{\tau}$. Using the Bramble–Hilbert lemma, (2.8), (5.20), and (5.23), we have

$$| I_4 | \leq C \| \hat{A}\|_{0,\infty,\hat{E}}|\hat{\tau}|_{1,\hat{E}} \| \hat{\chi}\|_{\hat{E}} \leq Ch \| A\|_{0,\infty,E} \| \tau\|_{1,E} \| \chi\|_E. \tag{5.30}$$

Combining (5.27)–(5.30) and summing over the elements implies (5.24). Similarly, using the exactness of the quadrature rule for bilinears, the Bramble–Hilbert lemma, (2.8), (5.2), (5.20), and (5.23), we have

$$\begin{aligned} | \delta_E(\tau,\xi) | &= | \hat{\delta}(\hat{\tau}DF_E^T,\hat{\xi}) | \leq | \hat{\delta}(\hat{\tau}(DF_E^T - \overline{DF_E^T}),\hat{\xi}) | + | \hat{\delta}(\tau\overline{DF_E^T},\hat{\xi}) | \\ &\leq C(|DF_E|_{1,\infty,\hat{E}} \| \hat{\tau}\|_{\hat{E}} \| \hat{\xi}\|_{\hat{E}} + \| DF_E\|_{0,\infty,\hat{E}}|\hat{\tau}|_{1,\hat{E}} \| \hat{\xi}\|_{\hat{E}}) \leq Ch \| \tau\|_{1,E} \| \xi\|_E, \end{aligned}$$

which implies (5.25). Bound (5.26) follows in a similar way. ∎

Lemma 5.3 *There exists a constant C independent of h such that for all* $\tau \in \mathbb{X}_h$ *and* $\xi \in \mathbb{W}_h^1$,

$$| (\tau - \Pi^{\mathcal{RT}}\tau,\xi)_Q | \leq Ch \| \tau \| \| \xi\|_1. \tag{5.31}$$

Proof. The proof follows from mapping to the reference element and using (5.21). ∎

5.2 | First order convergence for all variables

The convergence analysis presented below is different from the one on simplices from [2]. In particular, since the quadrature error bounds (5.24)–(5.26) require that one of the arguments is in $\mathbb{X}_h^{\mathcal{RT}}$, rather than $\mathbb{X}_h$, the error equations need to be manipulated in a special way.

Theorem 5.1 *Let* $A \in W^{1,\infty}_{\mathcal{T}_h}$. *If the solution* (σ,u,γ) *of* (2.3)–(2.5) *is sufficiently smooth, for its numerical approximation* (σ_h,u_h,γ_h) *obtained by either the MSMFE-*0 *method* (3.1)–(3.3) *or the MSMFE-*1 *method* (4.1)–(4.3), *there exists a constant C independent of h such that*

$$\| \sigma - \sigma_h\|_{\text{div}} + \| u - u_h \| + \| \gamma - \gamma_h \| \leq Ch(\| \sigma\|_1 + \| \text{div}\sigma\|_1 + \| u\|_1 + \| \gamma\|_1). \tag{5.32}$$

Proof. We present the argument for the MSMFE-1 method, which includes the proof for the MSMFE-0 method. We form the error system by subtracting the MSMFE-1 method (4.1)–(4.3) from (2.3)–(2.5):

$$(A\sigma,\tau) - (A\sigma_h,\tau)_Q + (u - u_h,\text{div}\tau) + (\gamma,\tau) - (\gamma_h,\tau)_Q = \langle g - \mathcal{P}_0 g,\tau n\rangle_{\Gamma_D}, \quad \tau \in \mathbb{X}_h, \tag{5.33}$$

$$(\text{div}(\sigma - \sigma_h),v) = 0, \quad v \in V_h, \tag{5.34}$$

$$(\sigma,\xi) - (\sigma_h,\xi)_Q = 0, \quad \xi \in \mathbb{W}_h^1. \tag{5.35}$$

Using (5.10), (5.5), and (5.8), we rewrite the first error equation as

$$(A(\Pi\sigma - \sigma_h), \tau)_Q + (Q_h^u u - u_h, \operatorname{div}\tau) \\ = -(A\sigma, \tau) + (A\Pi\sigma, \tau)_Q - (\gamma, \tau) + (\gamma_h, \tau)_Q + \langle [g], (\tau - \Pi^{\mathcal{RT}}\tau)n\rangle_{\Gamma_D}. \quad (5.36)$$

For the first two terms on the right above we write

$$-(A\sigma, \tau) + (A\Pi\sigma, \tau)_Q = -(A\sigma, \tau - \Pi^{\mathcal{RT}}\tau) - (A(\sigma - \Pi\sigma), \Pi^{\mathcal{RT}}\tau) \\ - (A\Pi\sigma, \Pi^{\mathcal{RT}}\tau) + (A\Pi\sigma, \Pi^{\mathcal{RT}}\tau)_Q + (A\Pi\sigma, \tau - \Pi^{\mathcal{RT}}\tau)_Q. \quad (5.37)$$

The second two terms on the right in (5.36) can be rewritten as

$$-(\gamma, \tau) + (\gamma_h, \tau)_Q = -(\gamma, \tau - \Pi^{\mathcal{RT}}\tau) - (\gamma - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}}\tau) \\ - (\Pi^{\mathcal{RT}}\tau, Q_h^\gamma \gamma) + (\Pi^{\mathcal{RT}}\tau, Q_h^\gamma \gamma)_Q + (Q_h^\gamma \gamma, \tau - \Pi^{\mathcal{RT}}\tau)_Q + (\gamma_h - Q_h^\gamma \gamma, \tau)_Q. \quad (5.38)$$

Combining the first terms in (5.37) and (5.38) with the last term in (5.36) gives

$$-(A\sigma, \tau - \Pi^{\mathcal{RT}}\tau) - (\gamma, \tau - \Pi^{\mathcal{RT}}\tau) + \langle [g], (\tau - \Pi^{\mathcal{RT}}\tau)n\rangle_{\Gamma_D} = 0, \quad (5.39)$$

which follows from testing (2.3) with $\tau - \Pi^{\mathcal{RT}}\tau$ and using (5.6). The rest of the terms in (5.37) and (5.38) are bounded as follows. Using (5.14) and (5.7), we have

$$| (A(\sigma - \Pi\sigma), \Pi^{\mathcal{RT}}\tau) | \leq Ch \parallel \sigma \parallel_1 \parallel \tau \parallel \leq Ch^2 \parallel \sigma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.40)$$

For the third and fourth terms on the right in (5.37), using (5.24), (5.17), and (5.18), we obtain

$$| \theta(A\Pi\sigma, \Pi^{\mathcal{RT}}\tau) | \leq Ch \parallel \sigma \parallel_1 \parallel \tau \parallel \leq Ch^2 \parallel \sigma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.41)$$

Using (5.22), we write

$$| (A\Pi\sigma, \tau - \Pi^{\mathcal{RT}}\tau)_Q | \leq Ch \parallel \sigma \parallel_1 \parallel \tau \parallel \leq Ch^2 \parallel \sigma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.42)$$

We next bound the terms on the right in (5.38). Due to (5.13) and (5.7), we have

$$| (\gamma - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}}\tau) | \leq Ch \parallel \gamma \parallel_1 \parallel \tau \parallel \leq Ch^2 \parallel \gamma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.43)$$

Using (5.26), (5.7), and (5.19), we have

$$| \delta(\Pi^{\mathcal{RT}}\tau, Q_h^\gamma \gamma) | \leq Ch \parallel \tau \parallel \parallel \gamma \parallel_1 \leq Ch^2 \parallel \gamma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.44)$$

Using Lemma 5.3, we obtain

$$| (Q_h^\gamma \gamma, \tau - \Pi^{\mathcal{RT}}\tau)_Q | \leq Ch \parallel \gamma \parallel_1 \parallel \tau \parallel \leq Ch^2 \parallel \gamma \parallel_1^2 + \epsilon \parallel \tau \parallel^2. \quad (5.45)$$

Combining (5.36)–(5.45), we obtain

$$(A(\Pi\sigma - \sigma_h), \tau)_Q + (Q_h^u u - u_h, \operatorname{div}\tau) \leq Ch^2(\parallel \sigma \parallel_1^2 + \parallel \gamma \parallel_1^2) + \epsilon \parallel \tau \parallel^2 + (\gamma_h - Q_h^\gamma \gamma, \tau)_Q. \quad (5.46)$$

We next note that, using (5.3), the second error equation (5.34) implies that

$$\operatorname{div}(\Pi\sigma - \sigma_h) = 0. \quad (5.47)$$

The third error equation (5.35) implies

$$(\Pi\sigma - \sigma_h, \xi)_Q = (\Pi\sigma - \sigma, \xi)_Q + (\sigma - \Pi^{\mathcal{RT}}\sigma, \xi)_Q - \delta(\Pi^{\mathcal{RT}}\sigma, \xi) + (\Pi^{\mathcal{RT}}\sigma - \sigma, \xi) \\ \leq Ch^2 \parallel \sigma \parallel_1^2 + \epsilon \parallel \xi \parallel^2, \quad (5.48)$$

using (5.14), (5.15), (5.25), and (5.18) for the inequality. We now set $\tau = \Pi\sigma - \sigma_h$ in (5.46), $\xi = \gamma_h - Q_h^{\gamma}\gamma$ in (5.48), use (2.19) and (5.47), and take ϵ small enough to obtain

$$\| \Pi\sigma - \sigma_h\|^2 \leq Ch^2(\| \sigma \|_1^2 + \| \gamma \|_1^2) + \epsilon \| \gamma_h - Q_h^{\gamma}\gamma\|^2. \tag{5.49}$$

We apply the inf-sup condition (4.4) to $(Q_h^u u - u_h, Q_h^{\gamma}\gamma - \gamma_h) \in V_h \times \mathbb{W}_h^1$ and use (5.33) to obtain

$$\begin{aligned}\| Q_h^u u - u_h \| + \| Q_h^{\gamma}\gamma - \gamma_h \| &\leq C \sup_{\tau\in\mathbb{X}_h} \frac{-(A\sigma,\tau) + (A\sigma_h,\tau)_Q - (\gamma,\tau) + (Q_h^{\gamma}\gamma,\tau)_Q + \langle g - \mathcal{P}_0 g, \tau n\rangle_{\Gamma_D}}{\|\tau\|_{\text{div}}} \\ &\leq C(h \| \sigma\|_1 + h \| \gamma\|_1 + \| \Pi\sigma - \sigma_h \|),\end{aligned} \tag{5.50}$$

where the numerator terms have been bounded in a manner similar to the bounds for the terms in the error equation (5.33) presented above. Next, we combine a sufficiently small multiple of (5.50) with (5.49), and choose ϵ in (5.49) small enough to get

$$\| \Pi\sigma - \sigma_h \| + \| Q_h^u u - u_h \| + \| Q_h^{\gamma}\gamma - \gamma_h \| \leq Ch(\| \sigma\|_1 + \| \gamma\|_1). \tag{5.51}$$

The assertion of the theorem follows from (5.51), (5.47), (5.12)–(5.14), and (5.16). The proof for the MSMFE-0 method can be obtained by omitting the quadrature error terms $\delta(\cdot,\cdot)$. ∎

5.3 | Second order convergence for the displacement

We next present superconvergence analysis for the displacement using a duality argument. We need the following improved bounds on the quadrature errors.

Lemma 5.4 *If $A \in W_{\mathcal{T}_h}^{2,\infty}$, there exists a constant C independent of h such that for all $\tau \in \mathbb{X}_h$ and $\chi \in \mathbb{X}_h^{\mathcal{RT}}$*

$$| \theta(A\tau, \chi) | \leq C \sum_{E\in\mathcal{T}_h} h^2 \| \tau\|_{2,E} \| \chi\|_{1,E}. \tag{5.52}$$

For all $\chi \in \mathbb{X}_h^{\mathcal{RT}}$, $\xi \in \mathbb{W}_h^1$ there exists a constant C independent of h such that

$$| \delta(\chi, \xi) | \leq C \sum_{E\in\mathcal{T}_h} h^2 \| \chi\|_{1,E} \| \xi\|_{2,E}. \tag{5.53}$$

Proof. The proof of (5.52) is given in [8, Lemma 4.2]. It uses the Piano kernel theorem [28, Theorem 5.2–3] and the fact that the quadrature rule is exact for bilinear functions. The proof of (5.53) is similar. ∎

We consider the auxiliary elasticity problem: find ϕ and ψ such that

$$\begin{aligned}&\psi = A^{-1}\epsilon(\phi), \quad \text{div}\psi = (Q_h^u u - u_h) \quad \text{in } \Omega, \\ &\phi = 0 \text{ on } \Gamma_D, \quad \psi n = 0 \text{ on } \Gamma_N.\end{aligned} \tag{5.54}$$

We assume that the above problem is H^2-elliptic regular, see [29] for sufficient conditions:

$$\| \phi\|_2 \leq C \| Q_h^u u - u_h \| . \tag{5.55}$$

Theorem 5.2 *If* $A \in W^{2,\infty}_{\mathcal{T}_h}$, (5.55) *holds, and the solution* (σ, u, γ) *of* (2.3)–(2.5) *is sufficiently smooth, there exists a constant C independent of h such that*

$$\| Q_h^u u - u_h \| \leq Ch^2(\| \sigma\|_2 + \| \gamma\|_2). \tag{5.56}$$

Proof. We present the proof for the MSMFE-1 method and note that the proof for the MSMFE-0 method can be obtained by omitting the terms arising due to the quadrature error $\delta(\cdot,\cdot)$. We rewrite the error equation (5.36) as

$$(A(\Pi\sigma - \sigma_h), \tau)_Q + (Q_h^u u - u_h, \operatorname{div}\tau) \\ = (A(\Pi\sigma - \sigma), \tau) - \theta(A\Pi\sigma, \tau) - (\gamma, \tau) + (\gamma_h, \tau)_Q + \langle g - \mathcal{P}_0 g, \tau n\rangle_{\Gamma_D}.$$

and choose $\tau = \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)$ to obtain

$$\| Q_h^u u - u_h\|^2 = -(A(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi))_Q + (A(\Pi\sigma - \sigma), \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)) \\ - \theta(A\Pi\sigma, \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)) - (\gamma, \Pi^{\mathcal{RT}} A^{-1}\varepsilon(\phi)) + (\gamma_h, \Pi^{\mathcal{RT}} A^{-1}\varepsilon(\phi))_Q. \tag{5.57}$$

For the second term on the right in (5.57), using (5.14) and (5.18), we have

$$| (A(\Pi\sigma - \sigma), \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)) | \leq Ch^2 \| \sigma\|_2 \| \phi\|_2. \tag{5.58}$$

The third term on the right in (5.57) is bounded using (5.52), (5.17), and (5.18):

$$| \theta(A\Pi\sigma, \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)) | \leq C \sum_{E\in\mathcal{T}_h} h^2 \| A\Pi\sigma\|_{2,E} \| \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)\|_{1,E} \leq Ch^2 \| \sigma\|_2 \| \phi\|_2. \tag{5.59}$$

The first term on the right in (5.57) can be manipulated as follows:

$$(A(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi))_{Q,E} = ((A - \overline{A})(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi))_{Q,E} \\ +(\overline{A}(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} (A^{-1} - \overline{A}^{-1})\epsilon(\phi))_{Q,E} + (\overline{A}(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} \overline{A}^{-1}(\epsilon(\phi) - \epsilon(\phi_1)))_{Q,E} \\ +(\overline{A}(\Pi\sigma - \sigma_h), \Pi^{\mathcal{RT}} \overline{A}^{-1}\epsilon(\phi_1))_{Q,E} \equiv \sum_{k=1}^{4} I_k, \tag{5.60}$$

where $\overline{A}$ is the mean value of A on E and ϕ_1 is a linear approximation of ϕ such that, see [23],

$$\| \phi - \phi_1\|_E \leq Ch^2 \| \phi\|_{2,E}, \quad \| \phi - \phi_1\|_{1,E} \leq Ch \| \phi\|_{2,E}. \tag{5.61}$$

Using (5.12), (5.61), and (5.18), we have

$$| I_1 | + | I_2 | + | I_3 | \leq Ch \| \Pi\sigma - \sigma_h\|_E \| \phi\|_{2,E}. \tag{5.62}$$

For the last term on the right in (5.60), we first note that for a constant tensor τ_0, $\hat{\tau}_0 = J_E \tau_0 DF_E^{-T} \in \hat{\mathbb{X}}^{\mathcal{RT}}(\hat{E})$, so using (5.4) we have

$$\Pi^{\mathcal{RT}} \tau_0 = \frac{1}{J_E} \hat{\Pi}^{\mathcal{RT}} \hat{\tau}_0 DF_E^T = \frac{1}{J_E} \hat{\tau}_0 DF_E^T = \tau_0. \tag{5.63}$$

Therefore

$$I_4 = (\Pi\sigma - \sigma_h, \epsilon(\phi_1))_{Q,E} = (\Pi\sigma - \sigma_h, \nabla\phi_1)_{Q,E} - (\Pi\sigma - \sigma_h, \operatorname{Skew}(\nabla\phi_1))_{Q,E}. \tag{5.64}$$

For the second term on the right in (5.64) we write

$$(\Pi\sigma - \sigma_h, \operatorname{Skew}(\nabla\phi_1))_{Q,E} = (\Pi\sigma - \sigma_h, \operatorname{Skew}(\nabla\phi_1) - Q_h^\gamma \operatorname{Skew}(\nabla\phi_1))_{Q,E}$$

$$+(\Pi\sigma-\sigma_h, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_{Q,E} \le Ch \parallel \Pi\sigma-\sigma_h\|_E \parallel \phi\|_{2,E} + \mid (\Pi\sigma-\sigma_h, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_{Q,E} \mid, \quad (5.65)$$

using (5.13) for the inequality. For the last term above, using the error equation (5.35), we write

$$(\Pi\sigma-\sigma_h, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_{Q,E} = (\Pi\sigma, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_{Q,E} - (\Pi\sigma, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_E + (\Pi\sigma-\sigma, Q_h^\gamma \mathrm{Skew}(\nabla\phi_1))_E \le Ch^2 \parallel \sigma\|_{2,E} \parallel \phi\|_{2,E}, \quad (5.66)$$

using (5.52) and (5.14) for the inequality.

We next bound the first term on the right in (5.64). Using that $\nabla\phi_1 = \hat{\nabla}\hat{\phi}_1 DF^{-1}$, we write

$$(\Pi\sigma-\sigma_h, \nabla\phi_1)_{Q,E} = (\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}\hat{\phi}_1)_{\hat{Q},\hat{E}}. \quad (5.67)$$

We note that $\hat{\phi}_1$ is bilinear. Let $\tilde{\phi}_1$ be the linear part of $\hat{\phi}_1$. Then we have

$$(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}\hat{\phi}_1)_{\hat{Q},\hat{E}} = (\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}(\hat{\phi}_1-\tilde{\phi}_1))_{\hat{Q},\hat{E}} + (\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}\tilde{\phi}_1)_{\hat{Q},\hat{E}}. \quad (5.68)$$

It follows from (2.6) that $[\hat{\nabla}(\hat{\phi}_1-\tilde{\phi}_1)]_i = ((\mathbf{r}_{34}-\mathbf{r}_{21})\cdot[\nabla\phi_1\circ F_E]_i)\begin{pmatrix}\hat{y}\\ \hat{x}\end{pmatrix}$, $i = 1, 2$. Hence, (5.1) implies

$$\mid (\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}(\hat{\phi}_1-\tilde{\phi}_1))_{\hat{Q},\hat{E}} \mid \le Ch^2 \parallel \hat{\Pi}\hat{\sigma}-\hat{\sigma}_h\|_{\hat{E}} \parallel \nabla\phi\circ F_E\|_{\hat{E}} \le Ch \parallel \Pi\sigma-\sigma_h\|_E \parallel \phi\|_{1,E}, \quad (5.69)$$

where we used (5.20) in the last inequality. For the last term in (5.68), using (5.21) and the exactness of the quadrature rule for linear functions, we obtain

$$\begin{aligned}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h, \hat{\nabla}\tilde{\phi}_1)_{\hat{Q},\hat{E}} &= (\hat{\Pi}^{\mathcal{RT}}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h), \hat{\nabla}\tilde{\phi}_1)_{\hat{Q},\hat{E}} = (\hat{\Pi}^{\mathcal{RT}}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h), \hat{\nabla}\tilde{\phi}_1)_{\hat{E}} \\ &= (\hat{\Pi}^{\mathcal{RT}}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h), \hat{\nabla}(\tilde{\phi}_1-\hat{\phi}_1))_{\hat{E}} + (\hat{\Pi}^{\mathcal{RT}}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h), \hat{\nabla}\hat{\phi}_1)_{\hat{E}}. \quad (5.70)\end{aligned}$$

We bound the first term on the right in (5.70) similarly to (5.69):

$$\mid (\hat{\Pi}^{\mathcal{RT}}(\hat{\Pi}\hat{\sigma}-\hat{\sigma}_h), \hat{\nabla}(\tilde{\phi}_1-\hat{\phi}_1))_{\hat{E}} \mid \le Ch^2 \parallel \hat{\Pi}\hat{\sigma}-\hat{\sigma}_h\|_{\hat{E}} \parallel \nabla\phi\circ F_E\|_{\hat{E}} \le Ch \parallel \Pi\sigma-\sigma_h\|_E \parallel \phi\|_{1,E}. \quad (5.71)$$

Combining (5.60)–(5.71) and summing over the elements, we obtain

$$\begin{aligned}\mid (A(\Pi\sigma-\sigma_h), \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi))_Q \mid &\le C(h \parallel \Pi\sigma-\sigma_h \parallel + h^2 \parallel \sigma\|_2) \parallel \phi\|_2 \\ &+ \mid \sum_{E\in\mathcal{T}_h} (\Pi^{\mathcal{RT}}(\Pi\sigma-\sigma_h), \nabla\phi_1)_E \mid . \quad (5.72)\end{aligned}$$

For the last term above, noting that integration by parts, (5.47), (5.6), $\phi = 0$ on Γ_D, and $(\Pi\sigma-\sigma_h)n = 0$ on Γ_N imply $\sum_{E\in\mathcal{T}_h}(\Pi^{\mathcal{RT}}(\Pi\sigma-\sigma_h), \nabla\phi)_E = 0$, we have

$$\left|\sum_{E\in\mathcal{T}_h}(\Pi^{\mathcal{RT}}(\Pi\sigma-\sigma_h), \nabla\phi_1)_E\right| = \left|\sum_{E\in\mathcal{T}_h}(\Pi^{\mathcal{RT}}(\Pi\sigma-\sigma_h), \nabla(\phi_1-\phi))_E\right| \le Ch \parallel \Pi\sigma-\sigma_h \|\|\| \phi\|_2. \quad (5.73)$$

It is left to bound the last two terms on the right in (5.57). We rewrite them as follows:

$$\begin{aligned}-(\gamma, \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi)) + (\gamma_h, \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi))_Q &= -\delta(\Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi), Q_h^\gamma\gamma) \\ &-(\gamma-Q_h^\gamma\gamma, \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi)) + (\gamma_h-Q_h^\gamma\gamma, \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi))_Q. \quad (5.74)\end{aligned}$$

For the first term on the right-hand side we use (5.53), (5.18), and (5.19):

$$\mid \delta(\Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi), Q_h^\gamma\gamma) \mid \le C\sum_{E\in\mathcal{T}_h} h^2 \parallel \Pi^{\mathcal{RT}}A^{-1}\epsilon(\phi)\|_{1,E} \parallel Q_h^\gamma\gamma\|_{2,E} \le Ch^2 \parallel \phi\|_2 \parallel \gamma\|_2. \quad (5.75)$$

The second term on the right in (5.74) is bounded using the symmetry of $A^{-1}\epsilon(\phi)$, (5.13) and (5.15):

$$| (\gamma - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi)) |=| (\gamma - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi) - A^{-1}\epsilon(\phi)) |\leq Ch^2 \parallel \gamma\|_1 \parallel \phi\|_2. \quad (5.76)$$

For the last term in (5.74) we have

$$\begin{aligned}(\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} A^{-1}\epsilon(\phi))_Q &= (\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} (A^{-1} - \overline{A}^{-1})\epsilon(\phi))_Q \\ &+ (\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} \overline{A}^{-1}(\epsilon(\phi) - \epsilon(\phi_1)))_Q + (\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} \overline{A}^{-1}\epsilon(\phi_1))_Q. \end{aligned} \quad (5.77)$$

We bound the first two terms on the right in (5.77) similarly to I_2 and I_3 in (5.62):

$$\begin{aligned}| (\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} (A^{-1} - \overline{A}^{-1})\epsilon(\phi))_Q \\ + (\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} \overline{A}^{-1}(\epsilon(\phi) - \epsilon(\phi_1)))_Q |\leq Ch \parallel \gamma_h - Q_h^\gamma \gamma \parallel\parallel \phi\|_2. \end{aligned} \quad (5.78)$$

For the last term in (5.77), using (5.63) and the symmetry of $\overline{A}^{-1}\epsilon(\phi_1)$, we have

$$(\gamma_h - Q_h^\gamma \gamma, \Pi^{\mathcal{RT}} \overline{A}^{-1}\epsilon(\phi_1))_Q = (\gamma_h - Q_h^\gamma \gamma, \overline{A}^{-1}\epsilon(\phi_1))_Q = 0. \quad (5.79)$$

The assertion of the theorem follows from combining (5.57)–(5.79) and using (5.51). ■

6 | NUMERICAL RESULTS

In this section we present numerical results that verify the theoretical results from the previous sections. We focus on testing the convergence MSMFE-1 method on different types of quadrilateral grids. We refer the reader to [2] for illustration of the performance of the method on simplicial grids for problems with discontinuous coefficients and for parameters in the nearly incompressible regime. We used deal.II finite element library [30] for the implementation of the method. We consider a homogeneous and isotropic body,

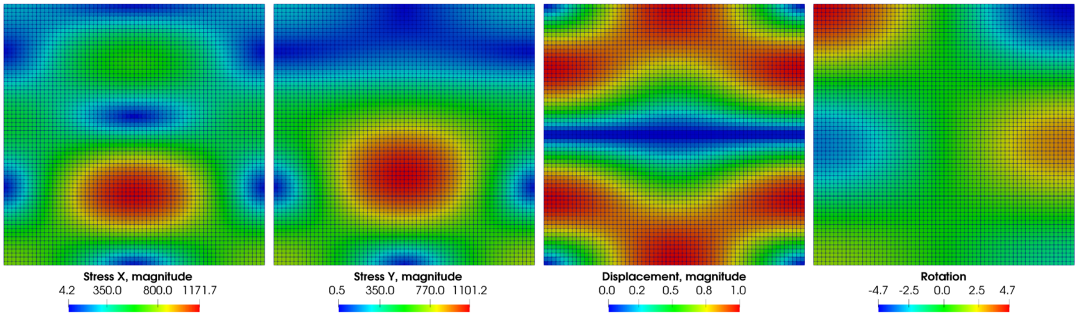
$$A\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{2\mu + 2\lambda}\text{tr}(\sigma)I\right),$$

where I is the 2×2 identity matrix and $\mu > 0$, $\lambda > -\mu$ are the Lamé coefficients. We consider $\Omega = (0, 1)^2$ and the elasticity problem (2.1) and (2.2) with Dirichlet boundary conditions and exact solution [18]

$$u_0 = \begin{pmatrix} \cos(\pi x)\sin(2\pi y) \\ \cos(\pi y)\sin(\pi x) \end{pmatrix}.$$

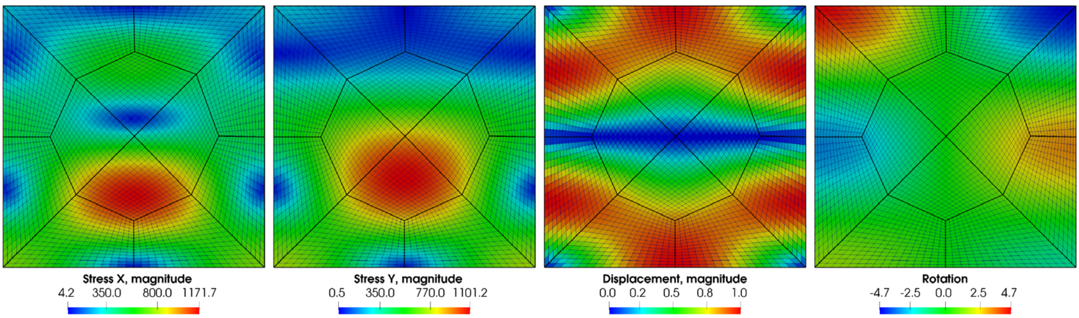
The Lamè coefficients are chosen as $\lambda = 123, \quad \mu = 79.3$.

We study the convergence of the MSMFE-1 method on four different types of grids. For the first test, we use a sequence of square meshes generated by sequential uniform refinement of an initial mesh with characteristic size $h = 1/2$ (Figure 4). For the second test, an initial general quadrilateral grid is used, and a sequence of meshes is obtained by sequential splitting of each element into four. This refinement procedure produces h^2-parallelogram grids (Figure 5), where the initial coarse grid is also shown. For the third test, we consider a sequence of smooth quadrilateral meshes. Each mesh is produced by applying a smooth map $\mathbf{x} = \hat{\mathbf{x}} + 0.1\sin(2\pi\hat{x})\sin(2\pi\hat{y})\begin{pmatrix}1\\1\end{pmatrix}$ to a uniformly refined square mesh, starting with $h = 1/2$ (Figure 6). For the fourth test, we use sequences of quadrilateral grids obtained by random perturbation of square grids. In particular, at each refinement level, the vertices of a square grid a moved randomly within a circle with radius of size $\mathcal{O}(h^\alpha)$ with $\alpha = 1, 3/2, 2$. We note that



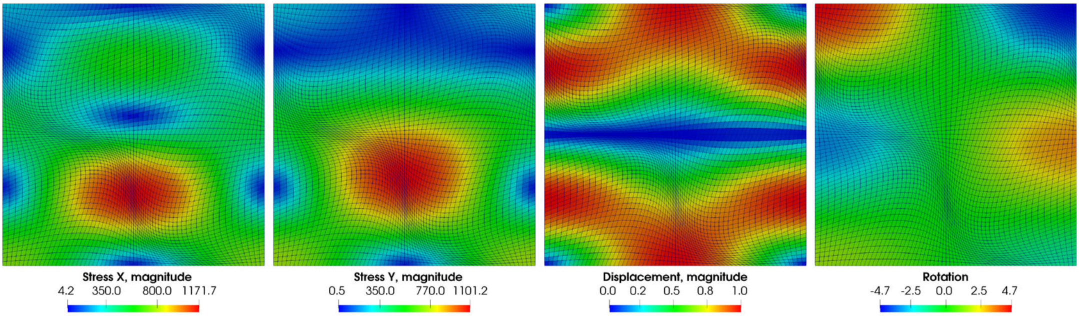
(a) x-stress (b) y-stress (c) Displacement (d) Rotation

FIGURE 4 Computed solution on a square mesh, $h = 1/64$ [Color figure can be viewed at wileyonlinelibrary.com]



(a) x-stress (b) y-stress (c) Displacement (d) Rotation

FIGURE 5 Computed solution on a h^2-parallelogram mesh, $h = 1/32$ [Color figure can be viewed at wileyonlinelibrary.com]



(a) x-stress (b) y-stress (c) Displacement (d) Rotation

FIGURE 6 Computed solution on a smooth quadrilateral mesh, $h = 1/64$ [Color figure can be viewed at wileyonlinelibrary.com]

the grids in the first and third tests satisfy both the stability condition (M2) and the h^2-parallelogram condition (5.1). The grids in the second test satisfy (5.1), but may violate (M2) along the edges of the initial coarse grid. However, we further note that (M2) is not needed on parallelograms and the elements in test two are h^2-parallelograms. Finally, in the fourth test, both (M2) and (5.1) may be violated for $\alpha < 2$.

The computed solutions for tests 1–3 are shown in Figures 4–6, respectively. The solutions are similar despite the different types of grids. The highly distorted elements in the third test do not affect the quality of the solution. The convergence results are presented in Tables 1–3. We observe at least first order of convergence for all variables, as predicted in (5.32), as well as superconvergence of the displacement error evaluated at the cell centers (5.56).

The results from test four on $\mathcal{O}(h^{\alpha})$-perturbed grids illustrate the effect of violating the h^2-parallelogram condition (5.1) and possibly (M2) when $\alpha < 2$. Figure 7 in the $\mathcal{O}(h)$-perturbed grid displays visible effects of the irregularly shaped elements on the computed stress and rotation. The convergence results for $\alpha = 1, 3/2, 2$ are presented in Tables 4–6, respectively. We observe loss of convergence for the stress and rotation in the case $\alpha = 1$. First order convergence in stress is recovered when $\alpha = 3/2$, while the rotation still converges sub-optimally. In the case $\alpha = 2$, which satisfies the h^2-parallelogram condition (5.1), at least first order convergence is observed for all variables, as well as displacement superconvergence, as predicted by the theory.

7 | CONCLUSIONS

We presented two MFE methods for linear elasticity on quadrilateral grids that reduce to symmetric and positive definite cell-centered algebraic systems. The methods utilize the $\mathcal{BDM}_1$ space for the

TABLE 1 Convergence on square grids

	$\|\sigma - \sigma_h\|$		$\|\mathrm{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\| Q_h^u u - u_h \|$		$\|\gamma - \gamma_h\|$	
h	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
1/2	7.61E − 01	—	9.73E − 01	—	7.19E − 01	—	4.76E − 01	—	8.17E − 01	—
1/4	3.74E − 01	1.02	5.42E − 01	0.84	4.56E − 01	0.66	1.06E − 01	2.17	3.91E − 01	1.06
1/8	1.66E − 01	1.17	2.72E − 01	0.99	2.33E − 01	0.97	2.76E − 02	1.93	1.15E − 01	1.77
1/16	7.91E − 02	1.07	1.36E − 01	1.00	1.17E − 01	0.99	7.25E − 03	1.94	3.04E − 02	1.92
1/32	3.90E − 02	1.02	6.79E − 02	1.00	5.86E − 02	1.00	1.84E − 03	1.98	7.75E − 03	1.97
1/64	1.94E − 02	1.01	3.39E − 02	1.00	2.93E − 02	1.00	4.62E − 04	1.99	1.95E − 03	1.99

TABLE 2 Convergence on h^2-parallelogram grids

	$\|\sigma - \sigma_h\|$		$\|\mathrm{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\| Q_h^u u - u_h \|$		$\|\gamma - \gamma_h\|$	
h	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
1/3	5.92E − 01	—	8.00E − 01	—	5.35E − 01	—	1.63E − 01	—	5.98E − 01	—
1/6	2.78E − 01	1.09	4.06E − 01	0.98	3.11E − 01	0.78	1.05E − 01	0.63	3.38E − 01	0.82
1/12	1.37E − 01	1.02	2.03E − 01	1.00	1.58E − 01	0.98	2.95E − 02	1.84	1.38E − 01	1.30
1/24	6.93E − 02	0.98	1.01E − 01	1.00	7.90E − 02	1.00	8.04E − 03	1.87	4.87E − 02	1.50
1/48	3.50E − 02	0.99	5.07E − 02	1.00	3.95E − 02	1.00	2.08E − 03	1.95	1.66E − 02	1.55
1/96	1.76E − 02	0.99	2.53E − 02	1.00	1.97E − 02	1.00	5.26E − 04	1.98	5.67E − 03	1.55

TABLE 3 Convergence on smooth quadrilateral grids

	$\|\sigma - \sigma_h\|$		$\|\mathrm{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\| Q_h^u u - u_h \|$		$\|\gamma - \gamma_h\|$	
h	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
1/4	4.27E − 01	—	6.22E − 01	—	4.71E − 01	—	1.64E − 01	—	4.53E − 01	—
1/8	2.22E − 01	0.94	3.46E − 01	0.85	2.68E − 01	0.81	7.09E − 02	1.21	2.14E − 01	1.08
1/16	1.12E − 01	0.99	1.78E − 01	0.96	1.37E − 01	0.97	2.51E − 02	1.50	9.29E − 02	1.21
1/32	5.61E − 02	1.00	9.00E − 02	0.99	6.84E − 02	1.00	7.35E − 03	1.77	3.21E − 02	1.53
1/64	2.81E − 02	1.00	4.51E − 02	1.00	3.42E − 02	1.00	1.94E − 03	1.92	1.04E − 02	1.63
1/128	1.40E − 02	1.00	2.26E − 02	1.00	1.71E − 02	1.00	4.93E − 04	1.98	3.41E − 03	1.61

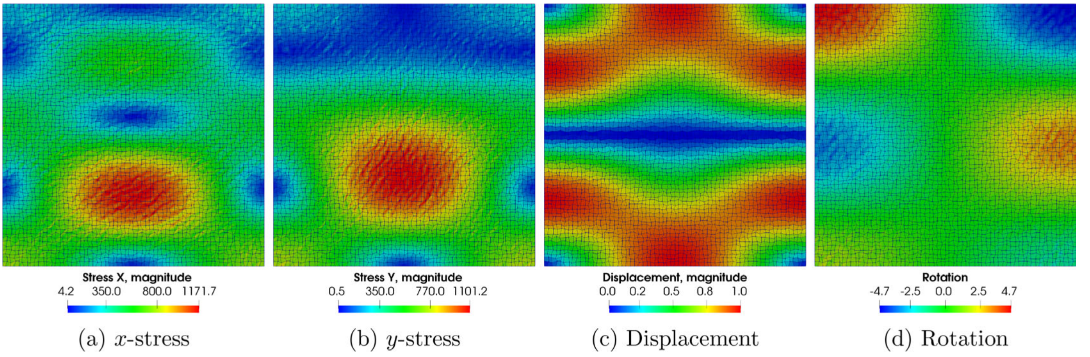


FIGURE 7 Computed solution on a $\mathcal{O}(h)$ randomly perturbed mesh, $h = 1/64$ [Color figure can be viewed at wileyonlinelibrary.com]

weakly-symmetric stress. The MSMFE-0 method employs the trapezoidal rule for the stress bilinear form, which allows for local stress elimination. The MSMFE-1 method employs in addition the trapezoidal rule for the stress-rotation bilinear forms with continuous bilinear rotation. This allows for further local elimination of the rotation variable, resulting in a cell-centered displacement system. To our knowledge, these are the first such MFE methods for elasticity on quadrilaterals in the literature. Well-posedness and error analyses are performed for both methods. The theory is illustrated by numerical experiments. In particular, first order convergence is established and observed for all variables and second order convergence is obtained for the displacement at the cell centers on smooth quadrilateral or h^2-parallelogram grids. A loss of convergence is observed on non-smooth quadrilateral grids. There are several possible extensions of the presented methods. One is to consider a non-symmetric formulation of the MSMFE method, similar to the non-symmetric MFMFE and MPFA methods developed

TABLE 4 Convergence on $\mathcal{O}(h)$ randomly perturbed grids

	$\|\sigma - \sigma_h\|$		$\|\text{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\| Q_h^u u - u_h \|$		$\|\gamma - \gamma_h\|$	
h	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
1/4	4.186e − 01	—	5.894e − 01	—	4.856e − 01	—	1.416e − 01	—	3.968e − 01	—
1/8	1.986e − 01	1.08	3.375e − 01	0.80	2.443e − 01	0.99	4.482e − 02	1.66	1.625e − 01	1.29
1/16	1.173e − 01	0.76	2.239e − 01	0.59	1.254e − 01	0.96	1.557e − 02	1.53	8.360e − 02	0.96
1/32	8.929e − 02	0.39	1.835e − 01	0.29	6.285e − 02	1.00	8.819e − 03	0.82	7.584e − 02	0.14
1/64	8.120e − 02	0.14	1.702e − 01	0.11	3.205e − 02	0.97	7.132e − 03	0.31	7.151e − 02	0.08
1/128	7.915e − 02	0.04	1.660e − 01	0.04	1.696e − 02	0.92	6.582e − 03	0.12	7.193e − 02	−0.01

TABLE 5 Convergence on $\mathcal{O}(h^{3/2})$ randomly perturbed grids

	$\|\sigma - \sigma_h\|$		$\|\text{div}(\sigma - \sigma_h)\|$		$\|u - u_h\|$		$\| Q_h^u u - u_h \|$		$\|\gamma - \gamma_h\|$	
h	Error	Rate	Error	Rate	error	Rate	error	Rate	error	Rate
1/4	3.800e − 01	—	5.492e − 01	—	4.604e − 01	—	1.084e − 01	—	3.883e − 01	—
1/8	1.680e − 01	1.18	2.884e − 01	0.93	2.335e − 01	0.98	2.883e − 02	1.91	1.191e − 01	1.71
1/16	8.008e − 02	1.07	1.475e − 01	0.97	1.174e − 01	0.99	7.422e − 03	1.96	3.233e − 02	1.88
1/32	3.981e − 02	1.01	7.499e − 02	0.98	5.865e − 02	1.00	1.916e − 03	1.95	1.083e − 02	1.58
1/64	2.021e − 02	0.98	3.832e − 02	0.97	2.932e − 02	1.00	4.981e − 04	1.94	5.407e − 03	1.00
1/128	1.048e − 02	0.95	1.998e − 02	0.94	1.466e − 02	1.00	1.335e − 04	1.90	3.610e − 03	0.58

TABLE 6 Convergence on $\mathcal{O}(h^2)$ randomly perturbed grids

	$\|\sigma-\sigma_h\|$		$\|\mathrm{div}(\sigma-\sigma_h)\|$		$\|u-u_h\|$		$\|Q_h^u u-u_h\|$		$\|\gamma-\gamma_h\|$	
h	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**	**Error**	**Rate**
1/4	3.750e − 01	—	5.457e − 01	—	4.568e − 01	—	1.056e − 01	—	3.899e − 01	—
1/8	1.664e − 01	1.17	2.866e − 01	0.93	2.333e − 01	0.97	2.786e − 02	1.92	1.154e − 01	1.76
1/16	7.913e − 02	1.07	1.456e − 01	0.98	1.171e − 01	0.99	7.256e − 03	1.94	3.046e − 02	1.92
1/32	3.898e − 02	1.02	7.328e − 02	0.99	5.860e − 02	1.00	1.842e − 03	1.98	7.784e − 03	1.97
1/64	1.941e − 02	1.01	3.671e − 02	1.00	2.931e − 02	1.00	4.625e − 04	1.99	1.976e − 03	1.98
1/128	9.697e − 03	1.00	1.837e − 02	1.00	1.465e − 02	1.00	1.157e − 04	2.00	5.146e − 04	1.94

in [7, 9, 12, 13], which provide first order convergence on general quadrilateral grids. A second extension is to develop the methods on hexahedral grids, following the approach in [6] for the MFMFE method in the symmetric case and [7] in the non-symmetric case. This would require developing stable weakly-symmetric elasticity finite element spaces $\mathbb{X}_h \times V_h \times \mathbb{W}_h$ that utilize an enhanced $\mathcal{BDDF}_1$ space for the stress. A third extension is to develop higher order methods, which requires constructing suitable general order family of stress spaces and quadrature rules. Such construction has been done for the MFMFE method on quadrilaterals and hexahedra in [31], see also [32] for second-order MFMFE methods on hybrid grids. A fourth possible extension is to develop these methods on polytopal grids using mimetic finite differences (MFD) or mixed virtual element methods (VEM). Relevant works include the local flux MFD method [33], the MFD method for elasticity from [34], and the mixed VEM method for elasticity developed in [35].

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