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Domain decomposition for coupled Stokes and Darcy flows ☆

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ABSTRACT

A non-overlapping domain decomposition method is presented to solve a coupled Stokes–Darcy flow problem in parallel by partitioning the computational domain into multiple subdomains, upon which families of coupled local problems of lower complexity are formulated. The coupling is based on appropriate interface matching conditions. The global problem is reduced to an interface problem by eliminating the interior subdomain variables. The interface problem is solved by an iterative procedure, which requires solving subdomain problems at each iteration. Finite element techniques appropriate for the type of each subdomain problem are used to discretize it. The condition number of the resulting algebraic system is analyzed and numerical tests verifying the theoretical estimates are provided.

1. Introduction

Partial differential equations modeling various physical phenomena that arise in many fields of science and engineering often need to be coupled, since there are different processes taking place in different parts of the problem domain. Coupling Stokes and Darcy equations is an interesting topic because of its broad scope of possible applications: surface and subsurface water interaction, blood circulation, fuel cells, and filtration problems among others. Furthermore, the Stokes–Darcy flow can be coupled with a transport equation [39], which can be used, for example, to estimate the risk of groundwater contamination from chemicals discharged in rivers or lakes. A variety of numerical methods exist for the coupled Stokes–Darcy problem [24,12,29,35,17]. In this work we propose an approach based on [24] for solving a coupled Stokes–Darcy flow system via domain decomposition (DD). DD methods [25,33,38] naturally lead to designing parallel algorithms and allow different numerical schemes within different subdomains to be employed, which makes them very attractive for multiphysics problems. Another advantage is the possibility to reuse existing computer code libraries for the subdomain problems. We follow the approach from [21] to formulate a non-overlapping DD method that reduces the global problem to an interface problem, which is solved iteratively. Each iteration requires solving in parallel local subdomain problems of lower complexity. The types of the local problems depend on the boundary conditions of the differential problem and the equations used to match the values of the unknowns on the interfaces between the subdomains. In this paper the possibility of using multiple subdomains is considered, which necessitates accounting for interfaces of the following types: Stokes–Darcy, Darcy–Darcy and Stokes–Stokes. On the Stokes–Darcy interfaces the conditions are continuity of the normal velocity and normal stress, as well as the Beavers–Joseph–Saffman [3,36] condition for the tangential Stokes velocity [24]. On the Stokes–Stokes interfaces the velocity vector and the normal stress vector are continuous, while on the Darcy–Darcy interfaces the normal velocity and

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pressure are continuous. For simplicity of notation, we restrict the presentation to two connected regions, one Stokes and one Darcy, each one subdivided into multiple subdomains. However, the formulation and the analysis naturally extend to the case when the Stokes and Darcy regions consist of multiple disconnected components. To the best of our knowledge, this is the first result in the literature that gives analysis of a domain decomposition algorithm for Stokes–Darcy problems with many subdomains. We refer the reader to [13–15,23,18] for previous work on domain decomposition for Stokes–Darcy flows in the two-subdomain case.

The discretization of the Stokes–Darcy flow problem is based on standard conforming Stokes finite elements in the Stokes region and mixed finite elements in the Darcy region. We allow for the grids to be non-matching across the Stokes–Darcy interfaces, but assume that the grids match on Stokes–Stokes and Darcy–Darcy interfaces. The discretization error in the case of non-matching grids on all interfaces is studied in [20]. Our domain decomposition algorithm utilizes a Lagrange multiplier λ on the interfaces to impose the continuity of flux conditions. In particular, on Stokes–Darcy or Darcy–Darcy interfaces λ is the normal stress or pressure, respectively – a scalar quantity, while on Stokes–Stokes interfaces it is the normal stress vector. An interface problem for λ is obtained by eliminating the subdomain velocities and pressures. Computing the action of the interface operator requires solving Stokes subdomain problems of Neumann or Neumann–Robin type and Darcy subdomain problems of Dirichlet type. As a result, the Stokes subdomain problems can be singular. We employ an approach based the FETI methods [16,38], which involves an auxiliary coarse problem to ensure that the local Stokes problems are solvable. We establish that the interface operator S_h (h is the mesh size) is symmetric and positive definite and show that the different interface types have different effect on its condition number. More precisely, $\mathrm{cond}(S_h) = O(h^{-1})$ if there are no Stokes–Stokes interfaces present, and $\mathrm{cond}(S_h) = O(h^{-2})$ otherwise. Furthermore, very small values of the Darcy permeability dominate the discretization effect, in which case we have $\mathrm{cond}(S_h) = O(k^{-1})$ or $\mathrm{cond}(S_h) = O(k^{-1}h^{-1})$ in the cases with or without Stokes–Stokes interfaces, respectively, where k is the characteristic permeability value. We note that our formulation is suitable for the application of optimal interface preconditioners, such as balancing [27,28,10,32,18], which should improve the dependence on h in the condition number, see [18] for the two-subdomain case.

The outline of the paper is the following. In Section 2 we introduce the mathematical model and its variational formulation. The finite element discretization is discussed in Section 3. A non-overlapping DD method is developed in Section 4 and analyzed in Section 5. Results from computational tests are provided in Section 6.

2. The mathematical model and the associated variational problem

The model we consider consists of Stokes flow in the fluid region $\Omega_S \subset \mathbb{R}^d$ and Darcy's law in the porous medium region $\Omega_D \subset \mathbb{R}^d$, where $d = 2, 3$. These are separated by an interface Γ_{SD}. Both Ω_S and Ω_D are bounded domains with outward unit normal vectors $\mathbf{n}_S$ and $\mathbf{n}_D$, respectively. Let $\Gamma_S := \partial\Omega_S \setminus \Gamma_{SD}$ and $\Gamma_D := \partial\Omega_D \setminus \Gamma_{SD}$. The interface Γ_{SD} and the boundaries Γ_S and Γ_D are assumed to be polygonal ($d = 2$) or polyhedral ($d = 3$). Let $\Omega = \Omega_S \cup \Omega_D$ represent the whole domain. The velocity and the pressure in Ω_S, respectively Ω_D, are denoted by $\mathbf{u}_S$ and p_S, respectively $\mathbf{u}_D$ and p_D. The constant viscosity coefficients of the flows in Ω_S and Ω_D are denoted by μ_S and μ_D, respectively. Let $\mathbf{T}(\mathbf{u}_S, p_S)$ and $\mathbf{D}(\mathbf{u}_S)$ denote the stress and the deformation rate tensors characterizing the flow in Ω_S:

$$\mathbf{D}(\mathbf{u}_S) = \frac{1}{2}\left(\nabla\mathbf{u}_S + \nabla\mathbf{u}_S^T\right), \quad \mathbf{T}(\mathbf{u}_S, p_S) = -p_S\mathbf{I} + 2\mu_S\mathbf{D}(\mathbf{u}_S).$$

Assuming Stokes flow and no slip boundary condition, $(\mathbf{u}_S, p_S)$ satisfies in Ω_S

$$-\nabla\cdot\mathbf{T}(\mathbf{u}_S, p_S) \equiv -2\mu_S\nabla\cdot\mathbf{D}(\mathbf{u}_S) + \nabla p_S = \mathbf{f}_S \quad \text{in } \Omega_S, \tag{2.1}$$

$$\nabla\cdot\mathbf{u}_S = 0 \quad \text{in } \Omega_S, \tag{2.2}$$

$$\mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S. \tag{2.3}$$

In the first equation $\mathbf{f}_S$ represents a body force, which has the form $\mathbf{f}_S = \rho\mathbf{b}$, where ρ is the fluid density and $\mathbf{b}$ is the force per unit mass of fluid. Assuming Darcy's law and no flow through Γ_D, $(\mathbf{u}_D, p_D)$ satisfies in Ω_D

$$\mu_D\mathbf{K}^{-1}\mathbf{u}_D + \nabla p_D = \mathbf{f}_D \quad \text{in } \Omega_D, \tag{2.4}$$

$$\nabla\cdot\mathbf{u}_D = q_D \quad \text{in } \Omega_D, \tag{2.5}$$

$$\mathbf{u}_D\cdot\mathbf{n}_D = 0 \quad \text{on } \Gamma_D. \tag{2.6}$$

Here $\mathbf{K}$ is the symmetric and uniformly positive definite rock permeability tensor, $\mathbf{f}_D$ represents the gravity force ($\mathbf{f}_D = \rho\mathbf{g}$, where $\mathbf{g}$ is the gravitational acceleration), and q_D is an external source or sink term. The source q_D is assumed to satisfy the solvability condition

$$\int_{\Omega_D} q_D\,dx = 0 \tag{2.7}$$

The mixed formulation (2.4)–(2.6) in the porous medium region naturally leads to direct approximation of the velocity. The two subdomain models are coupled across Γ_{SD} through the *interface conditions*

$$\mathbf{u}_S \cdot \mathbf{n}_S + \mathbf{u}_D \cdot \mathbf{n}_D = 0 \quad \text{on } \Gamma_{SD}, \tag{2.8}$$

$$-(\mathbf{T}(\mathbf{u}_S, p_S)\mathbf{n}_S) \cdot \mathbf{n}_S \equiv p_S - 2\mu_S(\mathbf{D}(\mathbf{u}_S)\mathbf{n}_S) \cdot \mathbf{n}_S = p_D \quad \text{on } \Gamma_{SD}, \tag{2.9}$$

$$-\frac{\sqrt{K_l}}{\mu_S \alpha_0}(\mathbf{T}(\mathbf{u}_S, p_S)\mathbf{n}_S) \cdot \tau_{SD}^l \equiv -\frac{\sqrt{K_l}}{\alpha_0} 2(\mathbf{D}(\mathbf{u}_S)\mathbf{n}_S) \cdot \tau_{SD}^l = \mathbf{u}_S \cdot \tau_{SD}^l, l = 1, d-1 \quad \text{on } \Gamma_{SD}, \tag{2.10}$$

where $\{\tau_{SD}^l\}_{l=1}^{d-1}$ is an orthogonal system of unit tangent vectors on Γ_{SD}. Conditions (2.8) and (2.9) incorporate continuity of flux and normal stress, respectively. Condition (2.10) is known as the Beavers–Joseph–Saffman law [3,36], where $\sqrt{K_l}/\alpha_0$ is a friction coefficient, $K_l = (\mathbf{K}\tau_{SD}^l) \cdot \tau_{SD}^l$, and $\alpha_0 > 0$ is an experimentally determined slip coefficient.

For a domain G in $\mathbb{R}^d$, the $L^2(G)$ inner product and norm for scalar and vector-valued functions are denoted $(\cdot,\cdot)_G$ and $\|\cdot\|_G$, respectively. The norm and seminorm of the Hilbert spaces $H^k(G)$ are denoted by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$, respectively. We omit G in the subscript if $G = \Omega$. For a section of the domain or element boundary $S \subset \mathbb{R}^{d-1}$ we write $\langle\cdot,\cdot\rangle_S$ and $\|\cdot\|_S$ for the $L^2(S)$ inner product (or duality pairing) and norm, respectively.

We next recall the variational formulation of (2.1)–(2.10) derived in [24]. The velocity–pressure spaces in the fluid region Ω_S are

$$X^S = \{\mathbf{v}_S \in (H^1(\Omega_S))^d, \mathbf{v}_S = 0 \text{ on } \Gamma_S\} \quad \text{and} \quad W^S = L^2(\Omega_S).$$

In the porous medium region Ω_D we introduce the spaces

$$X^D = \{\mathbf{v}_D \in H(\mathrm{div}; \Omega_D) : \langle \mathbf{v}_D \cdot \mathbf{n}_D, \varphi\rangle_{\partial\Omega_D} = 0, \quad \forall \varphi \in H^1_{0,\Gamma_{SD}}(\Omega_D)\}$$

and

$$W^D = L^2(\Omega_D),$$

where

$$H(\mathrm{div}; \Omega_D) = \{\mathbf{v}_D \in (L^2(\Omega_D))^d : \nabla \cdot \mathbf{v}_D \in L^2(\Omega_D)\}$$

and

$$H^1_{0,\Gamma_{SD}}(\Omega_D) = \{\varphi \in H^1(\Omega_D) : \varphi = 0 \text{ on } \Gamma_{SD}\}.$$

The norm on X^D is $\|\mathbf{v}_D\|_{X^D} = \left(\|\mathbf{v}_D\|^2_{\Omega_D} + \|\nabla \cdot \mathbf{v}_D\|^2_{\Omega_D}\right)^{1/2}$. We define $X = X^S \times X^D$ and

$$W = \left\{w = (w_S, w_D) \in W^S \times W^D : \int_\Omega w\, dx = 0\right\}.$$

We also consider the space of continuous-normal-trace velocities

$$V = \{\mathbf{v} = (\mathbf{v}_S, \mathbf{v}_D) \in X : b_{SD}(\mathbf{v}, \mu) = 0, \quad \forall \mu \in \Lambda^{SD}\},$$

where

$$\Lambda^{SD} = H^{1/2}(\Gamma_{SD})$$

and

$$b_{SD}(\mathbf{v}, \mu) = \langle \mathbf{v}_S \cdot \mathbf{n}_S + \mathbf{v}_D \cdot \mathbf{n}_D, \mu\rangle_{\Gamma_{SD}} : V \times \Lambda^{SD} \to \mathbb{R}.$$

A function $\lambda \in \Lambda^{SD}$ can be interpreted physically as the normal stress on the interface separating the two regions:

$$p_S - 2\mu_S(\mathbf{D}(\mathbf{u}_S)\mathbf{n}_S) \cdot \mathbf{n}_S = \lambda = p_D \quad \text{on } \Gamma_{SD}.$$

Remark 2.1. Due to the choice of Λ^{SD} the pairing $b_{SD}(\cdot,\cdot)$ is well-defined. If $\mathbf{v}_D \in H(\mathrm{div}; \Omega_D)$ and $\mathbf{v}_D \cdot \mathbf{n}_D = 0$ on $\partial\Omega_D \setminus \Gamma_{SD}$, then $\mathbf{v}_D \cdot \mathbf{n}_D \in H^{-1/2}(\Gamma_{SD})$, see [17].

The weak form of (2.1)–(2.10) is: find $(\mathbf{u}, p) \in V \times W$ satisfying

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v})_\Omega, \quad \mathbf{v} \in V, \tag{2.11}$$

$$b(\mathbf{u}, w) = -(q_D, w)_{\Omega_D}, \quad w \in W, \tag{2.12}$$

where

$$a(\mathbf{u}, \mathbf{v}) = a_S(\mathbf{u}_S, \mathbf{v}_S) + a_D(\mathbf{u}_D, \mathbf{v}_D) : X \times X \to \mathbb{R},$$

$$b(\mathbf{v}, w) = b_S(\mathbf{v}_S, w_S) + b_D(\mathbf{v}_D, w_D) : X \times W \to \mathbb{R},$$

$$a_S(\mathbf{u}_S,\mathbf{v}_S) = (2\mu_S\mathbf{D}(\mathbf{u}_S):\mathbf{D}(\mathbf{v}_S))_{\Omega_S} + \sum_{l=1}^{d-1}\left\langle \frac{\mu_S\alpha_0}{\sqrt{K_l}}\mathbf{u}_S\cdot\tau_{SD}^l, \mathbf{v}_S\cdot\tau_{SD}^l\right\rangle_{\Gamma_{SD}},$$
$$b_S(\mathbf{v}_S,w_S) = -(w_S,\nabla\cdot\mathbf{v}_S)_{\Omega_S},$$
$$a_D(\mathbf{u}_D,\mathbf{v}_D) = (\mu_D\mathbf{K}^{-1}\mathbf{u}_D,\mathbf{v}_D)_{\Omega_D}, \quad \text{and}$$
$$b_D(\mathbf{v}_D,w_D) = -(w_D,\nabla\cdot\mathbf{v}_D)_{\Omega_D}.$$

We note that the continuity of flux (2.8) is an essential condition for the velocity space, while (2.9) and (2.10) are natural conditions. Existence and uniqueness of a solution to (2.11) and (2.12) is established in [24].

3. Finite element discretization

Let Ω_S, respectively Ω_D, be decomposed into N_S, respectively N_D, non-overlapping Lipschitz polyhedral subdomains:

$$\Omega_S = \cup_{i=1}^{N_S}\Omega_i, \quad \Omega_D = \cup_{i=N_S+1}^{N}\Omega_i, \quad N = N_S + N_D.$$

For $1\leqslant i\leqslant N$, let $\mathbf{n}_i$ be the outward unit normal vector to subdomain Ω_i. The exterior boundary of Ω_i, possibly with zero measure, is denoted by $\Gamma_{i,ext}$:

$$\Gamma_{i,ext} = \partial\Omega_i\cap\partial\Omega, \quad 1\leqslant i\leqslant N.$$

Let Γ_{ij} be the interfaces between the subdomains, again possibly with zero measure:

$$\Gamma_{ij} = \partial\Omega_i\cap\partial\Omega_j, \quad 1\leqslant i<j\leqslant N.$$

We also introduce the following notations to represent the union of the interfaces between the subdomains of the same type:

$$\Gamma_{SS} = \cup_{1\leqslant i<j\leqslant N_s}\left(\partial\Omega_i\cap\partial\Omega_j\right),$$

$$\Gamma_{DD} = \cup_{N_s+1\leqslant i<j\leqslant N}\left(\partial\Omega_i\cap\partial\Omega_j\right).$$

The union of all the interfaces is denoted by Γ:

$$\Gamma = \Gamma_{SD}\cup\Gamma_{DD}\cup\Gamma_{SS}.$$

Let $A_i = \operatorname{diam}(\Omega_i)$. We assume that the subdomain partition is shape-regular, in the sense that there exists a constant $\sigma>0$ independent of N such that

$$\forall 1\leqslant i\leqslant N, \quad \forall j:|\Gamma_{ij}|>0, \quad \frac{A_i}{\rho_i}\leqslant\sigma, \quad \frac{A_i}{\rho_{ij}}\leqslant\sigma, \tag{3.1}$$

where ρ_i, receptively ρ_{ij}, is the diameter of the largest ball contained in Ω_i, respectively Γ_{ij}, and the second condition is required when $\Omega\subset\mathbb{R}^3$.

Let $\mathscr{T}_{h,i}$ be a shape regular finite element partition of $\Omega_i, i=1,N$, where h is the maximum element diameter. The shape regularity of the grids is a standard finite element assumption needed in the error analysis. It is not needed in the analysis of the convergence of the domain decomposition algorithm, other than for the Darcy elements along the interfaces. To simplify the notation, we assume that the characteristic element size is the same in all subdomains. We assume that the traces of the subdomain grids on the interfaces are quasi-uniform partitions:

$$\exists\nu>0:\forall e\in\mathscr{T}_{h,i}|_\Gamma, \quad \operatorname{diam}(e)\geqslant\frac{h}{\nu}. \tag{3.2}$$

We allow for the traces of the grids on Γ_{SD} to be non-matching and assume that no point of the interface boundary $\partial\Gamma_{SD}$ belongs to the interior of a face of an element of $\mathscr{T}_{h,i}$. We assume that the traces of the grids on Γ_{SS} and Γ_{DD} are matching.

For all $1\leqslant i\leqslant N_S$ let $X_i^S = X^S|_{\Omega_i}$, let $W_i^S = W^S|_{\Omega_i}$, and let $X_{h,i}^S\times W_{h,i}^S\subset X_i^S\times W_i^S$, be any Stokes finite element spaces satisfying the inf-sup condition

$$\inf_{0\neq w_{h,i}\in W_{h,i}^S}\sup_{0\neq\mathbf{v}_{h,i}\in X_{h,i}^S}\frac{(w_{h,i},\nabla\cdot\mathbf{v}_{h,i})_{\Omega_i}}{\|\mathbf{v}_{h,i}\|_{H^1(\Omega_i)}\ \|w_{h,i}\|_{L^2(\Omega_i)}}\geqslant\beta_S>0. \tag{3.3}$$

Examples of such spaces include the MINI elements [2], the Taylor–Hood elements [37], and the conforming Crouzeix–Raviart elements [11]. For the analysis we will need a projection operator $\Pi_{S,i}:(H^1(\Omega_i))^d\to X_{h,i}^S$ such that for all $\mathbf{q}_i\in(H^1(\Omega_i))^d$

$$(\nabla\cdot(\mathbf{q}_i-\Pi_{S,i}\mathbf{q}_i),w_{h,i})_{\Omega_i} = 0, \quad \forall w_{h,i}\in W_{h,i}^S. \tag{3.4}$$

The existence of such operator is shown in [7].

Similarly, for all $N_S+1 \leqslant i \leqslant N$, let $X_i^D = X^D|_{\Omega_i}$, let $W_i^D = W^D|_{\Omega_i}$, and let $X_{h,i}^D \times W_{h,i}^D \subset X_i^D \times W_i^D$ be any of the well-known mixed finite element spaces on Ω_i (see [7] section III.3), the RT spaces [34,31], the BDM spaces [6], the BDFM spaces [5], the BDDF spaces [4], or the CD spaces [8]. All of the above spaces satisfy $\nabla \cdot X_{h,i}^D = W_{h,i}^D$ and the inf-sup condition

$$\inf_{0\neq w_{h,i}\in W_{h,i}^D} \sup_{0\neq \mathbf{v}_{h,i}\in X_{h,i}^D} \frac{(w_{h,i}, \nabla\cdot \mathbf{v}_{h,i})_{\Omega_i}}{\|\mathbf{v}_{h,i}\|_{H(\mathrm{div};\Omega_i)}\|w_{h,i}\|_{L^2(\Omega_i)}} \geqslant \beta_D > 0. \tag{3.5}$$

Moreover, there exists a projection operator $\Pi_{D,i} : (H^1(\Omega_i))^d \to X_{h,i}^D$ such that for all $\mathbf{q}_i \in (H^1(\Omega_i))^d$

$$(\nabla\cdot(\mathbf{q}_i - \Pi_{D,i}\mathbf{q}_i), w_{h,i})_{\Omega_i} = 0, \quad \forall w_{h,i} \in W_{h,i}^D \tag{3.6}$$

and, for any element face e,

$$\langle (\mathbf{q}_i - \Pi_{D,i}\mathbf{q}_i)\cdot\mathbf{n}_i, \mu_h\rangle_{\partial\Omega_i} = 0, \quad \forall \mu_h \in X_{h,i}^D\cdot\mathbf{n}_i. \tag{3.7}$$

We also note that, if $\mathbf{q}_i \in (H^\varepsilon(\Omega_i))^d \cap X_i^D, 0<\varepsilon<1$, then $\Pi_{D,i}\mathbf{q}_i$ is well defined and [30,1]

$$\|\Pi_{D,i}\mathbf{q}_i\|_{\Omega_i} \leqslant C(\|\mathbf{q}_i\|_{\varepsilon,\Omega_i} + \|\nabla\cdot\mathbf{q}_i\|_{\Omega_i}). \tag{3.8}$$

The finite element spaces on Ω are

$$X_h^S = \{\mathbf{v}_h \in (H^1(\Omega_S))^d : \mathbf{v}_h|_{\Omega_i} \in X_{h,i}^S,\ 1\leqslant i\leqslant N_S,\ \mathbf{v}_h = 0 \text{ on } \Gamma_S\},$$

$$X_h^D = \{\mathbf{v}_h \in H(\mathrm{div};\Omega_D) : \mathbf{v}_h|_{\Omega_i} \in X_{h,i}^D,\ N_S+1\leqslant i\leqslant N,\ \mathbf{v}_h\cdot\mathbf{n}_D = 0 \text{ on } \Gamma_D\},$$

$$X_h = \{\mathbf{v}_h \in (L^2(\Omega))^d : \mathbf{v}_h|_{\Omega_S} \in X_h^S,\ \mathbf{v}_h|_{\Omega_D} \in X_h^D\},$$

$$W_h^S = \{w_h \in L^2(\Omega_S) : w_h|_{\Omega_i} \in W_{h,i}^S,\ 1\leqslant i\leqslant N_S\},$$

$$W_h^D = \{w_h \in L^2(\Omega_D) : w_h|_{\Omega_i} \in W_{h,i}^D,\ N_S+1\leqslant i\leqslant N\},$$

$$W_h = \{w_h \in L_0^2(\Omega) : w_h|_{\Omega_S} \in W_h^S,\ w_h|_{\Omega_D} \in W_h^D\},$$

$$\Lambda_h^{SD} = \{\mathbf{v}\cdot\mathbf{n}_D : \mathbf{v} \in X_h^D\} \text{ on } \Gamma_{SD}$$

and

$$V_h = \{\mathbf{v}_h \in X_h : b_{SD}(\mathbf{v}_h, \mu_h) = 0\ \forall \mu_h \in \Lambda_h^{SD}\}.$$

In the above, $L_0^2(\Omega)$ denotes the space of $L^2(\Omega)$ functions with zero mean value.

Remark 3.1. Since the function $\mu_h \in \Lambda_h^{SD}$ can be discontinuous, $\Lambda_h^{SD} \not\subset \Lambda^{SD}$. Therefore $V_h \not\subset V$, resulting in a non-conforming and exterior approximation.

The finite element discretization of 2.11,2.12 is the following: find $(\mathbf{u}_h, p_h) \in V_h \times W_h$ satisfying

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega, \quad \mathbf{v} \in V_h, \tag{3.9}$$

$$b(\mathbf{u}_h, w_h) = -(q_D w_h)_{\Omega_D}, \quad w_h \in W_h. \tag{3.10}$$

Existence and uniqueness for (3.9) and (3.10) are proved in [24], along with the optimal error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_X + \|p - p_h\|_W \leqslant C\left(h_S^{k_S} + h_D^{k_D+1} + h_D^{l_D+1}\right), \tag{3.11}$$

where $h_\alpha, \alpha = S, D$, characterizes the mesh used in Ω_α, k_S is the polynomial degree of the velocity space in the fluid region, k_D is the polynomial degree of the velocity space in the porous region, and l_D is the polynomial degree of the pressure space in the porous region.

Remark 3.2. Although the convergence theory in [24] is stated under the assumption that the grids match on the interface Γ_{SD}, it is easy to check that, with the above choice of Λ_h^{SD}, the results in [24] hold for non-matching grids as well.

4. Non-overlapping domain decomposition

In this section we present a domain decomposition algorithm for the solution of the algebraic system arising from (3.9) and (3.10). The goal is to design an algorithm that performs well on distributed parallel computers and can utilize existing and optimized software for solving the Stokes and the Darcy equations.

Using ideas from [21], we use Lagrange multiplier spaces Λ_h^{SD}, respectively Λ_h^{DD}, to impose the continuity of the normal velocity components on Γ_{SD}, respectively Γ_{DD}. The space Λ_h^{DD} is defined analogously to Λ_h^{SD}:

$$\Lambda_h^{DD} = \{\mathbf{v}\cdot\mathbf{n}_D : \mathbf{v} \in X_h^D\} \quad \text{on } \Gamma_{DD}.$$

We also need the Lagrange multiplier space

$$\Lambda_h^{SS} = X_h^S|_{\Gamma_{SS}} \quad \text{on } \Gamma_{SS}.$$

Since the velocity has to be continuous in Ω_S, on these interfaces we need to impose d conditions (constraints). Thus, the functions $\boldsymbol{\lambda}_h \in \Lambda_h^{SS}$ are d-dimensional vectors. For example, if $d = 2$, $\boldsymbol{\lambda}_h = (\lambda_{h,n}, \lambda_{h,\tau})$, where $\lambda_{h,n}$ and $\lambda_{h,\tau}$ are approximations to the normal and the tangential components, respectively, of the stress vector on Γ_{SS}. It is convenient to define the space

$$\tilde{\Lambda}_h = \Lambda_h^{SD} \times \Lambda_h^{DD} \times \Lambda_h^{SS}.$$

To simplify the notations we will omit whenever it is possible the subscript h on the functions from the discrete spaces. We introduce the bilinear forms

$$b_{DD}(\mathbf{v}, \mu_n) = \sum_{\Gamma_{ij}\subset\Gamma_{DD}} \langle \mathbf{v}_i\cdot\mathbf{n}_i + \mathbf{v}_j\cdot\mathbf{n}_j, \mu_n\rangle_{\Gamma_{ij}}, \quad \mathbf{v}\in X_h,\ \ \mu_n \in \Lambda_h^{DD}$$

and

$$b_{SS}(\mathbf{v},\mu) = \sum_{\Gamma_{ij}\subset\Gamma_{SS}} \langle \mathbf{v}_i\cdot\mathbf{n}_i + \mathbf{v}_j\cdot\mathbf{n}_j, \mu_n\rangle_{\Gamma_{ij}} + \sum_{\Gamma_{ij}\subset\Gamma_{SS}}\sum_{l=1}^{d-1}\langle \mathbf{v}_i\cdot\tau_i^l + \mathbf{v}_j\cdot\tau_j^l, \mu_\tau^l\rangle_{\Gamma_{ij}}, \quad \forall\mathbf{v}\in X_h, \quad \forall\mu = (\mu_n,\mu_\tau) \equiv (\mu_n,\mu_\tau^1,\ldots,\mu_\tau^{d-1}) \in \Lambda_h^{SS},$$

where $\{\tau_i^l\}_{l=1}^{d-1}$ is an orthogonal system of unit vectors tangential to $\partial\Omega_i$. We assume that $\{\tau_i^l\}_{l=1}^{d-1}$ are oriented in such a way that $\tau_i^l = -\tau_j^l$; hence $b_{SS}(\cdot,\cdot)$ represents the jumps of the normal and tangential velocity components with respect to the Lagrange multiplier space. Let us also introduce the global interface bilinear form

$$b_I(\mathbf{v},\tilde{\mu}) = b_{SD}(\mathbf{v},\mu_{SD}) + b_{DD}(\mathbf{v},\mu_{DD}) + b_{SS}(\mathbf{v},\mu_{SS}), \qquad \forall\mathbf{v}\in X_h, \quad \forall\tilde{\mu} = (\mu_{SD},\mu_{DD},\mu_{SS}) \in \tilde{\Lambda}_h.$$

Let $b_I^i(\cdot,\cdot)$ be the contribution to $b_I(\cdot,\cdot)$ from Ω_i:

$$b_I^i(\mathbf{v},\tilde{\mu}) = \begin{cases} \langle \mathbf{v}_i\cdot\mathbf{n}_i, \mu_n\rangle_{\partial\Omega_i\setminus\partial\Omega} + \displaystyle\sum_{l=1}^{d-1}\langle \mathbf{v}_i\cdot\tau_i^l, \mu_\tau^l\rangle_{\partial\Omega_i\cap\Gamma_{SS}}, & 1\leqslant i\leqslant N_S, \\ \langle \mathbf{v}_i\cdot\mathbf{n}_i, \mu_n\rangle_{\partial\Omega_i\setminus\partial\Omega}, & N_S+1\leqslant i\leqslant N. \end{cases}$$

For $1\leqslant i\leqslant N_S$, let $a_i(\cdot,\cdot) = a_S(\cdot,\cdot)|_{X_{h,i}^S\times X_{h,i}^S}$ and $b_i(\cdot,\cdot) = b_S(\cdot,\cdot)|_{X_{h,i}^S\times W_{h,i}^S}$. Similarly, for $N_S+1\leqslant i\leqslant N$, let $a_i(\cdot,\cdot) = a_D(\cdot,\cdot)|_{X_{h,i}^D\times X_{h,i}^D}$ and $b_i(\cdot,\cdot) = b_D(\cdot,\cdot)|_{X_{h,i}^D\times W_{h,i}^D}$. The restrictions of the right-hand side functions in (2.1), (2.4) and (2.5) on the subdomains are denoted by

$$\mathbf{f}_i = \begin{cases} \mathbf{f}_S|_{\Omega_i}, & 1\leqslant i\leqslant N_S \\ \mathbf{f}_D|_{\Omega_i}, & N_S+1\leqslant i\leqslant N \end{cases}$$

and

$$q_i = \begin{cases} 0, & 1\leqslant i\leqslant N_S \\ q_D|_{\Omega_i}, & N_S+1\leqslant i\leqslant N \end{cases}.$$

Let $\mathbf{v}_i$ and w_i represent the restrictions of $\mathbf{v}\in X_h$ and $w\in W_h$, respectively, on the subdomain $\Omega_i, 1\leqslant i\leqslant N$. Let $\tilde{X}_h$ be the velocity with no continuity imposed on Γ_{SS}, Γ_{DD}, and Γ_{SD}:

$$\tilde{X}_h = \{\mathbf{v}_h \in (L^2(\Omega))^d : \mathbf{v}_h|_{\Omega_i} \in X_{h,i}^\alpha, \quad 1\leqslant i\leqslant N, \qquad \mathbf{v}_h = \mathbf{0} \ \text{ on } \Gamma_S, \quad \mathbf{v}_h\cdot\mathbf{n}_D = 0 \ \text{ on } \Gamma_D\}.$$

It is easy to see that (3.9) and (3.10) is equivalent to the following discrete formulation: find $(\mathbf{u}_h, p_h, \tilde{\boldsymbol{\lambda}}_h) \in \tilde{X}_h\times W_h\times\tilde{\Lambda}_h$ satisfying

$$\begin{aligned} &\sum_{i=1}^N a_i(\mathbf{u}_{h,i},\mathbf{v}_i) + \sum_{i=1}^N b_i(\mathbf{v}_i,p_{h,i}) + b_I(\mathbf{v},\tilde{\boldsymbol{\lambda}}_h) = \sum_{i=1}^N (\mathbf{f}_i,\mathbf{v}_i)_{\Omega_i}, \quad \forall\mathbf{v}\in\tilde{X}_h \\ &\sum_{i=1}^N b_i(\mathbf{u}_{h,i},w_i) = -\sum_{i=1}^N (w_i,q_i)_{\Omega_i}, \quad \forall w\in W_h \\ &b_I(\mathbf{u}_h,\tilde{\mu}) = 0, \quad \forall\tilde{\mu}\in\tilde{\Lambda}_h. \end{aligned} \tag{4.1}$$

In the above $\tilde{\boldsymbol{\lambda}}_h = (\tilde{\lambda}_{h,n}, \tilde{\lambda}_{h,\tau}^1, \ldots, \tilde{\lambda}_{h,\tau}^{d-1})$ on $\Gamma_{ij}\subset\Gamma_{SS}$, where $\tilde{\lambda}_{h,n}$ approximates $-(\mathbf{T}(\mathbf{u},p)\mathbf{n}_i)\cdot\mathbf{n}_i = -(\mathbf{T}(\mathbf{u},p)\mathbf{n}_j)\cdot\mathbf{n}_j$, and $\tilde{\lambda}_{h,\tau}^l$ approximates $-(\mathbf{T}(\mathbf{u},p)\mathbf{n}_i)\cdot\tau_i^l = -(\mathbf{T}(\mathbf{u},p)\mathbf{n}_j)\cdot\tau_j^l$. On $\Gamma_{ij}\subset\Gamma_{SD}\cup\Gamma_{DD}, \tilde{\boldsymbol{\lambda}}_h = \tilde{\lambda}_{h,n}$, where $\tilde{\lambda}_{h,n}$ approximates p_D.

4.1. Coercivity and continuity of $a_i(\cdot,\cdot)$

Here we study the coercivity and continuity of the subdomain bilinear forms $a_i(\cdot,\cdot)$. This is needed for the well-posedness of the subdomain problems involved in the domain decomposition algorithm, as well as for the analysis of the condition number of the interface operator.

To estimate the bilinear forms in the Darcy region we assume that there exist two constants $K_{min} > 0$ and $K_{max} > 0$ such that

$$\forall \mathbf{x} \in \Omega_D, \quad \forall \xi \in \mathbb{R}^d, \quad K_{min}|\xi|^2 \leqslant \xi^T \mathbf{K}(\mathbf{x})\xi \leqslant K_{max}|\xi|^2, \tag{4.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$.

Recall that the kernel of the operator $\mathbf{D}$ consists of all rigid body motions

$$\mathscr{R}B = \begin{cases} \{\mathbf{a} + b[x_2, -x_1]^T\}, & d = 2, \\ \{\mathbf{a} + \mathbf{b} \times [x_1, x_2, x_3]^T\}, & d = 3, \end{cases}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$. The space spans two translations and one rotation in $\mathbb{R}^2$, and three translations and three rotations in $\mathbb{R}^3$. We will utilize the well known Korn's inequality [38]

$$(\mathbf{D}(\mathbf{v}_i), \mathbf{D}(\mathbf{v}_i))_{\Omega_i} \geqslant C_{K,i}\|\mathbf{v}_i\|^2_{1,\Omega_i} \quad \forall \mathbf{v}_i \in (H^1_{0,\gamma}(\Omega_i))^d \cup (H^1(\Omega_i))^d / \mathscr{R}B, \tag{4.3}$$

where $\gamma \subset \partial\Omega_i$ with $|\gamma| > 0$ and $(H^1_{0,\gamma}(\Omega_i))^d = \{\mathbf{v} \in (H^1(\Omega_i))^d : \mathbf{v} = \mathbf{0} \text{ on } \gamma\}$. Let

$$\ker a_i = \{\mathbf{v} \in X_i^S : a_i(\mathbf{v}, \mathbf{v}) = 0\}.$$

Due to (4.2), $\ker a_i = \mathbf{0}$ in the Darcy region. The following lemma describes the kernels of the subdomain bilinear forms in the Stokes region. We assume that the subdomain boundaries cannot intersect $\partial\Omega$ only along an edge in $\mathbb{R}^3$ and at a point in $\mathbb{R}^2$.

Lemma 4.1. *There are several possible cases for Stokes subdomains, $1 \leqslant i \leqslant N_S$:*

- *if $\partial\Omega_i \subset \Gamma_{SS}$, then $\ker a_i = \mathscr{R}B$,*
- *if $\partial\Omega_i \cap \partial\Omega \neq \varnothing$, then $\ker a_i = \mathbf{0}$,*
- *if $\partial\Omega_i \cap \partial\Omega = \varnothing$ and $\partial\Omega_i$ contains exactly one segment of Γ_{SD}, then in $\mathbb{R}^3$ $\ker a_i$ spans one translation and two rotations and in $\mathbb{R}^2$ it spans one translation and one rotation that are orthogonal to the tangent vectors on the Γ_{SD} segment,*
- *if $\partial\Omega_i \cap \partial\Omega = \varnothing$ and $\partial\Omega_i$ contains exactly two (non-connected) parallel segments of Γ_{SD}, then $\ker a_i$ spans one translation that is orthogonal to the tangent vectors on the segments,*
- *if $\partial\Omega_i \cap \partial\Omega = \varnothing$ and $\partial\Omega_i$ contains exactly two non-parallel segments of Γ_{SD}, then $\ker a_i$ spans one rotation that is orthogonal to the tangent vectors on the segments,*
- *if $\partial\Omega_i$ contains more that two segments of Γ_{SD}, then $\ker a_i = \mathbf{0}$.*

Proof. Recall that

$$a_i(\mathbf{u}_i, \mathbf{v}_i) = (2\mu_S \mathbf{D}(\mathbf{u}_i) : \mathbf{D}(\mathbf{v}_i))_{\Omega_i} + \sum_{l=1}^{d-1} \left\langle \frac{\mu_S \alpha_0}{\sqrt{K_l}} \mathbf{u}_i \cdot \tau^l_{SD}, \mathbf{v}_i \cdot \tau^l_{SD} \right\rangle_{\partial\Omega_i \cap \Gamma_{SD}}.$$

The first case is immediate, since if $\partial\Omega_i \subset \Gamma_{SS}$, then $\ker a_i = \ker \mathbf{D} = \mathscr{R}B$. The second case follows from the Korn's inequality (4.3). Using (4.3), the other cases can be verified by direct calculation by checking which elements of $\mathscr{R}B$ are orthogonal to τ^l_{SD} when restricted to Γ_{SD}. Note that it is sufficient to consider the segments of Γ_{SD} to be the planes $\{x_i = 0\}$ (and $\{x_i = 1\}$ in the case of two parallel segments). □

Lemma 4.2. *There exist positive constants C_1, C_2, C_3, and C_4, independent of h such that*

$$\text{for } 1 \leqslant i \leqslant N_S, C_1\|\mathbf{v}_i\|^2_{1,\Omega_i} \leqslant a_i(\mathbf{v}_i, \mathbf{v}_i) \leqslant C_2\|\mathbf{v}_i\|^2_{1,\Omega_i}, \quad \forall \mathbf{v} \in X^S_{h,i}/\ker a_i, \tag{4.4}$$

$$\text{for } N_S + 1 \leqslant i \leqslant N, C_3\|\mathbf{v}_i\|^2_{\Omega_i} \leqslant a_i(\mathbf{v}_i, \mathbf{v}_i) \leqslant C_4\|\mathbf{v}_i\|^2_{\Omega_i}, \quad \forall \mathbf{v} \in X^D_{h,i}. \tag{4.5}$$

Proof. The upper bound of $a_i(\cdot,\cdot)$ in the Stokes region is straightforward. In particular, if $\partial\Omega_i \cap \Gamma_{SD} \neq \varnothing$,

$$\begin{aligned} a_i(\mathbf{v}_i, \mathbf{v}_i) &\leqslant 2\mu_S|\mathbf{D}(\mathbf{v}_i)|^2_{\Omega_i} + \frac{\mu_S \alpha_0}{\sqrt{K_{min}}} \sum_{l=1}^{d-1} \|\mathbf{v}_i \cdot \tau^l_{SD}\|^2_{\partial\Omega_i \cap \Gamma_{SD}} \\ &\leqslant 2\mu_S\|\mathbf{v}_i\|^2_{1,\Omega_i} + \frac{C_{tr,i}\mu_S \alpha_0}{\sqrt{K_{min}}} \|\mathbf{v}_i\|^2_{1,\Omega_i}, \end{aligned}$$

where $C_{tr,i}$ arises from applying the trace theorem for $H^1(\Omega_i)$ functions. Then the upper bound in (4.4) holds with $C_2 = \mu_S \max\{2, C_{tr,i}\alpha_0/\sqrt{K_{min}}\}$. If $\partial\Omega_i \cap \Gamma_{SD} = \varnothing$, $a_i(\cdot,\cdot)$ is bounded above with $C_2 = 2\mu_S$. If $\partial\Omega_i \subset \Gamma_{SS}$ or $\partial\Omega_i \cap \partial\Omega \neq \varnothing$, the lower bound in the Stokes region follows from Korn's inequality (4.3) with $C_1 = 2\mu_S C_{K,i}$. For the other cases, using (4.3), one needs to check coercivity for all elements of $\mathscr{R}B$ that are not in $\ker a_i$. Let us consider the case $\Omega \subset \mathbb{R}^3, \partial\Omega_i \cap \partial\Omega = \varnothing$, and $\partial\Omega_i$ contains exactly one segment of Γ_{SD}, say γ^i_{SD}, which is in the plane $\{x_3 = 0\}$. Then the vectors in $\mathscr{R}B$ that are not in $\ker a_i$ are $\tau^1_{SD} = (1,0,0)^T$, $\tau^2_{SD} = (0,1,0)^T$, and $\mathbf{r} = (x_2, -x_1, 0)^T$. We have $a_i(\tau^l_{SD}, \tau^l_{SD}) = (\mu_S\alpha_0/\sqrt{K_l})|\gamma^i_{SD}|$ and $\|\tau^l_{SD}\|^2_{1,\Omega_i} = |\Omega_i|$, so (4.4) holds with $C_1 = (\mu_S\alpha_0/\sqrt{K_{max}})|\gamma^i_{SD}|/|\Omega_i|$. For $\mathbf{r} = (x_2, -x_1, 0)^T$ we have

$$\langle \mathbf{r}\cdot\tau^1_{SD}, \mathbf{r}\cdot\tau^1_{SD}\rangle_{\gamma^i_{SD}} + \langle \mathbf{r}\cdot\tau^1_{SD}, \mathbf{r}\cdot\tau^1_{SD}\rangle_{\gamma^i_{SD}} = \|\mathbf{r}\|^2_{\gamma^i_{SD}}$$

and

$$\|\mathbf{r}\|^2_{\gamma^i_{SD}} = \int_{\gamma^i_{SD}} (x_1^2 + x_2^2) \geqslant c_1 A_i^4,$$

where c_1 depends on σ from (3.1). On the other hand, (3.1) implies that

$$\|\mathbf{r}\|^2_{\Omega_i} \leqslant c_2 A_i \|\mathbf{r}\|^2_{\gamma^i_{SD}}$$

and

$$|\mathbf{r}|^2_{1,\Omega_i} = 2|\Omega_i| \leqslant c_3 A_i^3 \leqslant \frac{c_3}{c_1 A_i}\|\mathbf{r}\|^2_{\gamma^i_{SD}}$$

Combining the above inequalities implies (4.4) for $\mathbf{r}$ with $C_1 = c_1(\mu_S\alpha_0/\sqrt{K_{max}})A_i/(c_1c_2A_i^2 + c_3)$. The other cases from Lemma 4.1 can be treated similarly. This completes the proof of (4.4).

The assumption (4.2) directly implies (4.5) with $C_3 = \mu_D/K_{max}$ and $C_4 = \mu_D/K_{min}$. □

4.2. Reduction to an interface problem

In this section we show that the algebraic system (4.1) can be reduced to a symmetric and positive definite interface problem. To do that we introduce families of local problems on each subdomain Ω_i.

Consider the set of Darcy subdomain problems on $\Omega_i, N_S + 1 \leqslant i \leqslant N$, with specified pressure λ_n on Γ_{ij}: find $(\mathbf{u}^*_i(\lambda_n), p^*_i(\lambda_n)) \in X^D_{h,i} \times W^D_{h,i}$ such that

$$a_i(\mathbf{u}^*_i(\lambda_n), \mathbf{v}_i) + b_i(\mathbf{v}_i, p^*_i(\lambda_n)) = -\langle \lambda_n, \mathbf{v}_i\cdot\mathbf{n}_i\rangle_{\partial\Omega_i\setminus\partial\Omega}, \quad \mathbf{v}_i \in X^D_{h,i}, \tag{4.6}$$

$$b_i(\mathbf{u}^*_i(\lambda_n), w_i) = 0, \quad w_i \in W^D_{h,i} \tag{4.7}$$

and the set of Stokes subdomain problems on $\Omega_i, 1 \leqslant i \leqslant N_S$, with specified normal stress λ_n on $\partial\Omega_i \setminus \partial\Omega$ and tangential stress $\boldsymbol{\lambda}_\tau = (\lambda^1_\tau, \ldots, \lambda^{d-1}_\tau)$ on Γ_{SS}, $\boldsymbol{\lambda} = (\lambda_n, \boldsymbol{\lambda}_\tau)$: find $(\mathbf{u}^*_i(\boldsymbol{\lambda}), p^*_i(\boldsymbol{\lambda})) \in X^S_{h,i}/\ker a_i \times W^S_{h,i}$ such that

$$a_i(\mathbf{u}^*_i(\boldsymbol{\lambda}), \mathbf{v}_i) + b_i(\mathbf{v}_i, p^*_i(\boldsymbol{\lambda})) = -\langle \lambda_n, \mathbf{v}_i\cdot\mathbf{n}_i\rangle_{\partial\Omega_i\setminus\partial\Omega} - \sum_{l=1}^{d-1}\langle \lambda^l_\tau, \mathbf{v}_i\cdot\tau^l_i\rangle_{\partial\Omega_i\cap\Gamma_{SS}}, \quad \mathbf{v}_i \in X^S_{h,i}/\ker a_i, \tag{4.8}$$

$$b_i(\mathbf{u}^*_i(\boldsymbol{\lambda}), w_i) = 0, \quad w_i \in W^S_{h,i}. \tag{4.9}$$

Consider also the set of complementary Darcy subdomain problems on $\Omega_i, N_S + 1 \leqslant i \leqslant N$: find $(\bar{\mathbf{u}}_i, \bar{p}_i) \in X^D_{h,i} \times W^D_{h,i}$ such that

$$a_i(\bar{\mathbf{u}}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \bar{p}_i) = (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i}, \quad \mathbf{v}_i \in X^D_{h,i}, \tag{4.10}$$

$$b_i(\bar{\mathbf{u}}_i, w_i) = -(q_i, w_i)_{\Omega_i}, \quad w_i \in W^D_{h,i} \tag{4.11}$$

and the set of complementary Stokes subdomain problems on $\Omega_i, 1 \leqslant i \leqslant N_S$: find $(\bar{\mathbf{u}}_i, \bar{p}_i) \in X^S_{h,i}/\ker a_i \times W^S_{h,i}$ such that

$$a_i(\bar{\mathbf{u}}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \bar{p}_i) = (\mathbf{f}_i, \mathbf{v}_i)_{\Omega_i}, \quad \mathbf{v}_i \in X^S_{h,i}/\ker a_i, \tag{4.12}$$

$$b_i(\bar{\mathbf{u}}_i, w_i) = 0, \quad w_i \in W^S_{h,i}. \tag{4.13}$$

It is straightforward to see that solving (4.1) is equivalent to solving the interface problem: find $\tilde{\boldsymbol{\lambda}} = (\lambda_{SD}, \lambda_{DD}, \boldsymbol{\lambda}_{SS}) \in \tilde{\Lambda}_h$ such that

$$s_h(\tilde{\boldsymbol{\lambda}}, \tilde{\mu}) \equiv -b_I(\mathbf{u}^*(\tilde{\boldsymbol{\lambda}}), \tilde{\mu}) = b_I(\bar{\mathbf{u}}, \tilde{\mu}), \quad \tilde{\mu} \in \tilde{\Lambda}_h \tag{4.14}$$

and recovering global velocity and pressure: $\mathbf{u}_h = \mathbf{u}^*(\tilde{\boldsymbol{\lambda}}) + \bar{\mathbf{u}}, p_h = p^*(\tilde{\boldsymbol{\lambda}}) + \bar{p}$.

Remark 4.1. The subdomain problems (4.6)–(4.9) and (4.10)–(4.13) are well posed due to the coercivity of $a_i(\cdot,\cdot)$ from Lemma 4.2 and the local discrete inf-sup conditions (3.5) and (3.3). The boundary conditions on the interfaces in the Darcy region are of Dirichlet type:

$$p_i = \lambda_n, \quad N_S + 1 \leqslant i \leqslant N \quad \text{on } \Gamma_{SD} \cup \Gamma_{DD}.$$

The boundary conditions on the interfaces for the local Stokes problems are of Neumann type:

$$-(\mathbf{T}\mathbf{n}_i)\cdot\mathbf{n}_i = \lambda_n, \qquad -(\mathbf{T}\mathbf{n}_i)\cdot\tau_i^l = \lambda_\tau^l, \quad 1 \leqslant l \leqslant d-1, \quad 1 \leqslant i \leqslant N_S \quad \text{on } \Gamma_{SS}$$

and of Neumann–Robin type:

$$-(\mathbf{T}\mathbf{n}_i)\cdot\mathbf{n}_i = \lambda_n, \qquad -(\mathbf{T}\mathbf{n}_i)\cdot\tau_i^l - \frac{\mu_S\alpha_0}{\sqrt{K_l}}\mathbf{u}_i\cdot\tau_i^l = 0, \quad 1 \leqslant l \leqslant d-1, \quad 1 \leqslant i \leqslant N_S \quad \text{on } \Gamma_{SD}.$$

Note that the matrices of the Stokes problems can be singular and their solutions are determined up to an element of $\ker a_i$. This can be resolved by introducing an auxiliary coarse problem, as discussed in Section 4.3.

Remark 4.2. Introducing the Steklov–Poincaré type operator $S_h : \tilde{\Lambda}_h \to \tilde{\Lambda}_h$,

$$(S_h\tilde{\boldsymbol{\lambda}}, \tilde{\mu}) = s_h(\tilde{\boldsymbol{\lambda}}, \tilde{\mu}) \quad \forall \tilde{\boldsymbol{\lambda}}, \tilde{\mu} \in \tilde{\Lambda}_h,$$

the interface problem (4.14) can be written as: find $\tilde{\boldsymbol{\lambda}} \in \tilde{\Lambda}_h$ such that

$$S_h\tilde{\boldsymbol{\lambda}} = g_h, \tag{4.15}$$

where $g_h : \tilde{\Lambda}_h \to \mathbb{R}$, $g_h(\tilde{\mu}) = b_l(\bar{\mathbf{u}}_h, \tilde{\mu}), \forall\, \tilde{\mu} \in \tilde{\Lambda}_h$.

The algebraic interpretation of the above method is as follows. Slightly abusing the notations, let u, p, and λ represent the degrees of freedom for velocity, pressure, and Lagrange multipliers, respectively. The discrete analogues of the right hand side functions in the coupled system are denoted by f and q. The linear system arising in (4.1) is of the form

$$\begin{pmatrix} A & B^T & L^T \\ B & 0 & 0 \\ L & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ q \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad \begin{pmatrix} M & L^T \\ L & 0 \end{pmatrix}\begin{pmatrix} \xi \\ \lambda \end{pmatrix} = \begin{pmatrix} r \\ 0 \end{pmatrix},$$

where $\xi = (u, p)^T$ is the vector of subdomain unknowns and $r = (f, q)^T$. The interface problem (4.15) corresponds to the Schur complement system

$$LM^{-1}L^T\lambda = LM^{-1}r. \tag{4.16}$$

If an iterative method is employed for solving (4.16), each iteration will require evaluating the action of

$$M^{-1} = \begin{pmatrix} M_1^{-1} & & \\ & \ddots & \\ & & M_N^{-1} \end{pmatrix},$$

i.e., solving local subdomain problems.

4.3. Floating Stokes subdomains

We refer to Stokes subdomains with non-trivial $\ker a_i$ as floating. In this section we present an approach to handle such floating subdomains based on the FETI methods introduced by Farhat and Roux [16]. The one-level FETI method can be viewed as a preconditioned conjugate gradient (PCG) algorithm incorporating an auxiliary coarse problem; see [38] for implementation details.

In the formulation of the FETI methods the Moore-Penrose pseudoinverses M_i^+ of the local Stokes matrices M_i, $i = 1, \ldots, N_S$, are used if the corresponding subdomain problems are singular. In our approach we avoid computing the Moore-Penrose pseudoinverse by choosing the right hand side vector to be in the range of M_i and setting $M_i^+ = (M_i + \sqrt{\epsilon}D_i)^{-1}$, where ϵ is the machine precision and D_i is the velocity mass matrix. In problems (4.12) and (4.13) we replace the functions $\mathbf{f}_i$ with $\mathbf{f}_i - \bar{\mathbf{f}}_i, i = 1, \ldots, N_S$, where $\bar{\mathbf{f}}_i$ is the orthogonal projection of $\mathbf{f}_i$ onto $\ker a_i$. The global Stokes–Darcy problem can be written as

$$M\xi + L^T\lambda = r, \tag{4.17}$$

subject to the constraint

$$L\xi = 0. \tag{4.18}$$

The solution to (4.17) and (4.18) is of the form

$$\xi = \overline{\xi} + \xi^*(\lambda),$$

where $\overline{\xi}$ solves the local problems with zero stress (or pressure) boundary conditions:

$$M\overline{\xi} = r - \overline{r} \tag{4.19}$$

and $\xi^*(\lambda)$ satisfies the equations

$$M\xi^*(\lambda) + L^T\lambda = \overline{r}, \tag{4.20}$$

$$L\xi^*(\lambda) = -L\overline{\xi}. \tag{4.21}$$

For the solvability of the above we need

$$(\overline{r} - L^T\lambda) \in \text{range}(M),$$

which is equivalent to

$$R^T(\overline{r} - L^T\lambda) = 0, \tag{4.22}$$

where R is a matrix whose columns form a basis for $\ker(M)$. More precisely, R has $k = \sum_i k_i$ columns, with k_i columns for each Stokes subdomain containing the basis for $\ker(M_i)$. If (4.22) holds, we have

$$\xi^*(\lambda) = M^+(\overline{r} - L^T\lambda) + R\alpha, \tag{4.23}$$

where α can be computed after λ is found. Define

$$G = LR.$$

Substituting (4.23) into (4.21), and using (4.19) and the solvability condition (4.22), problem (4.20) and (4.21) is transformed into

$$LM^+(\overline{r} - L^T\lambda) + G\alpha = -L\overline{\xi}, \tag{4.24}$$

$$G^T\lambda = R^T\overline{r}. \tag{4.25}$$

We can write

$$\lambda = \lambda_0 + \lambda_1, \tag{4.26}$$

where $\lambda_0 = G(G^TG)^{-1}R^T\overline{r}$, and $\lambda_1 \in \ker(G^T)$. Next, we introduce the operator $P = I - G(G^TG)^{-1}G^T$, which is the orthogonal projector onto $\ker(G^T)$. Applying P^T on both sides of Eq. (4.24) and using the splitting (4.26) leads us to the interface problem

$$P^TLM^+L^T\lambda_1 = P^TL(M^+(\overline{r} - L^T\lambda_0) + \overline{\xi}). \tag{4.27}$$

Note that for any $\mu \in \ker(G^T)$, $P^TLM^+L^T\mu = P^TLM^+L^TP\mu$, which is symmetric and positive semi-definite, hence the above problem can be solved with the conjugate gradient method. Evaluation of $M^+(\overline{r} - L^T\lambda_0)$ in the right hand side of (4.27) means solving once

$$M\xi_0 = \overline{r} - L^T\lambda_0,$$

which in the Stokes region is a set of compatible Neumann problems since λ_0 satisfies (4.25). Applying at each iterative step the matrix $P^TLM^+L^T$ also involves solving compatible Stokes Neumann problems, because for any $\mu \in \ker(G^T)$, $R^TL^T\mu = 0$, implying that

$$L^T\mu \perp \ker(M).$$

Note that the matrix G^TG is of size $k \times k$. Computing $(G^TG)^{-1}$ requires solving a coarse problem, which reduces to solving local $k_i \times k_i$ problems, due to the block-diagonal structure of G^TG. The coarse problem resembles an element of the balancing preconditioner introduced by Mandel [27]. Once λ is computed, one can recover α using (4.24) and (4.19):

$$\alpha = -(G^TG)^{-1}G^TLM^+(r - L^T\lambda).$$

5. Analysis of the interface operator

Here we derive estimates for the condition number of the interface operator stating explicitly the dependence on the mesh size h, the subdomain size A, and the permeability $\mathbf{K}$. We will omit the subscript h in most places throughout this section. Let

$$\Lambda_h^{SD,DD} = \Lambda_h^{SD} \cup \Lambda_h^{DD} \quad \text{and} \quad \hat{\Lambda}_h^{SD,DD} = \left\{ \mu_n \in \Lambda_h^{SD,DD} : \int_{\partial\Omega_{N_S+1}\setminus\partial\Omega} \mu_n = 0 \right\}.$$

Note that λ_n approximates the pressure on $\Gamma_{SD} \cup \Gamma_{DD}$, which is determined up to a constant. The mean value constraint for $\hat{\Lambda}_h^{SD,DD}$ is one possible way to fix the constant. Also, since the interface solve (4.27) is performed in the subspace $\{\mu : L^T\mu \perp \ker(M)\}$, it is enough to carry out the analysis of the interface operator in

$$\hat{\Lambda}_h^{SS} = \{\mu \in \Lambda_h^{SS} : b_l^i(\mathbf{r}, \mu) = 0 \quad \forall \mathbf{r} \in \ker a_i, 1 \leqslant i \leqslant N_S\}.$$

Let

$$\hat{\Lambda}_h = \hat{\Lambda}_h^{SD,DD} \times \hat{\Lambda}_h^{SS}.$$

In the remainder of the paper we will be using the notation

$$\tilde{\mu} = (\mu_n, \mu) \in \hat{\Lambda}_h, \quad \mu_n \in \hat{\Lambda}_h^{SD,DD}, \quad \mu \in \hat{\Lambda}_h^{SS}.$$

Lemma 5.1. *The bilinear form $s(\cdot,\cdot)$ is symmetric and positive definite on $\hat{\Lambda}_h \times \hat{\Lambda}_h$.*

Proof. The definition (4.14) of $s(\cdot,\cdot)$ gives

$$s(\tilde{\lambda}, \tilde{\mu}) = \sum_{i=1}^{N} s_i(\tilde{\lambda}, \tilde{\mu}),$$

where

$$s_i(\tilde{\lambda}, \tilde{\mu}) = -b_l^i(\mathbf{u}_i^*(\boldsymbol{\lambda}), \tilde{\mu}) = \begin{cases} -\langle \mathbf{u}_i^*(\boldsymbol{\lambda}) \cdot \mathbf{n}_i, \mu_n \rangle_{\partial\Omega_i \setminus \partial\Omega} - \sum_{l=1}^{d-1} \langle \mathbf{u}_i^*(\boldsymbol{\lambda}) \cdot \tau_i^l, \mu_\tau^l \rangle_{\partial\Omega_i \cap \Gamma_{SS}}, & 1 \leqslant i \leqslant N_S, \\ -\langle \mathbf{u}_i^*(\lambda_n) \cdot \mathbf{n}_i, \mu_n \rangle_{\partial\Omega_i \setminus \partial\Omega}, & N_S + 1 \leqslant i \leqslant N. \end{cases} \tag{5.1}$$

Taking $\mathbf{v}_i = \mathbf{u}_i^*(\tilde{\mu})$ in (4.6) and (4.8) gives

$$s_i(\tilde{\mu}, \tilde{\lambda}) = a_i(\mathbf{u}_i^*(\tilde{\lambda}), \mathbf{u}_i^*(\tilde{\mu})),$$

which implies the symmetry of $s(\cdot,\cdot)$. Moreover,

$$s(\tilde{\lambda}, \tilde{\lambda}) = \sum_{i=1}^{N} s_i(\tilde{\lambda}, \tilde{\lambda}) = \sum_{i=1}^{N} a_i(\mathbf{u}_i^*(\tilde{\lambda}), \mathbf{u}_i^*(\tilde{\lambda})) \geqslant 0. \tag{5.2}$$

Let $s(\tilde{\lambda}, \tilde{\lambda}) = 0$. Then $\mathbf{u}_i^*(\tilde{\lambda}) \in \ker a_i$, which implies that $a_i(\mathbf{u}_i^*(\tilde{\lambda}), \mathbf{v}_i) = 0, \forall \mathbf{v}_i \in X_{h,i}$, for $1 \leqslant i \leqslant N$, using the characterization of $\ker a_i$ from Lemma 4.1. Thus, (4.6) and (4.8) become:

$$-(\nabla \cdot \mathbf{v}_i, p_i^*(\lambda_n))_{\Omega_i} + \langle \lambda_n, \mathbf{v}_i \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} = 0, \quad \mathbf{v}_i \in X_{h,i}^D, \quad N_S + 1 \leqslant i \leqslant N \tag{5.3}$$

and

$$-(\nabla \cdot \mathbf{v}_i, p_i^*(\boldsymbol{\lambda}))_{\Omega_i} + \langle \lambda_n, \mathbf{v}_i \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} + \sum_{l=1}^{d-1} \langle \lambda_\tau^l, \mathbf{v}_i \cdot \tau_i^l \rangle_{\partial\Omega_i \cap \Gamma_{SS}} = 0, \quad \mathbf{v}_i \in X_{h,i}^S / \ker a_i, \quad 1 \leqslant i \leqslant N_S, \tag{5.4}$$

respectively. Since also $b_l^i(\mathbf{r}, \boldsymbol{\lambda}) = 0$ and $(\nabla \cdot \mathbf{r}, p_i^*(\boldsymbol{\lambda}))_{\Omega_i} = 0$ for $\mathbf{r} \in \ker a_i$, then (5.4) holds for all $\mathbf{v}_i \in X_{h,i}^S$.

For $i = N_S + 1$, consider the auxiliary problem

$$\begin{aligned} &\psi_i = -\nabla \varphi_i, \quad \nabla \cdot \psi_i = 0 \text{ in } \Omega_i, \\ &\psi_i \cdot \mathbf{n}_i = \lambda_n \text{on} \partial\Omega_i \setminus \partial\Omega, \quad \psi_i \cdot \mathbf{n}_i = 0 \text{ on } \partial\Omega_i \cap \partial\Omega. \end{aligned} \tag{5.5}$$

This problem is well posed since

$$\int_{\partial\Omega_i} \psi_i \cdot \mathbf{n}_i = \int_{\partial\Omega_i \setminus \partial\Omega} \lambda_n = 0.$$

Taking $\mathbf{v}_i = \Pi_{D,i}\psi_i$ in (5.3),

$$0 = -(\nabla \cdot \Pi_{D,i}\psi_i, p_i^*(\lambda_n))_{\Omega_i} + \langle \lambda_n, \Pi_{D,i}\psi_i \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} = \langle \lambda_n, \lambda_n \rangle_{\partial\Omega_i \setminus \partial\Omega} = \|\lambda_n\|^2_{\partial\Omega_i \setminus \partial\Omega},$$

which implies $\lambda_n = 0$ on $\partial\Omega_i \setminus \partial\Omega$. Next, we order the remaining Darcy subdomains as follows: $i = N_S + 2, \ldots, N$ so that $\gamma_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j \neq \varnothing$ for some $j < i$. Then, for $i = N_S + 2, \ldots, N$ we solve the problem

$$\begin{aligned} &\psi_i = -\nabla \varphi_i, \quad \nabla \cdot \psi_i = 0 \quad \text{in } \Omega_i, \\ &\psi_i \cdot \mathbf{n}_i = \lambda_n \text{on} \gamma_i \equiv \partial\Omega_i \setminus (\gamma_{ij} \cup \partial\Omega), \quad \psi_i \cdot \mathbf{n}_i = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega, \quad \varphi_i = 0 \quad \text{on } \gamma_{ij}, \end{aligned} \tag{5.6}$$

which is well posed, since $\gamma_{ij} \neq \varnothing$. Taking $\mathbf{v}_i = \Pi_{D,i}\psi_i$ in (5.3) and using that $\lambda_n = 0$ on γ_{ij}, we obtain

$$0 = -(\nabla \cdot \Pi_{D,i}\psi_i, p_i^*(\lambda_n))_{\Omega_i} + \langle \lambda_n, \Pi_{D,i}\psi_i \cdot \mathbf{n}_i \rangle_{\gamma_i} = \langle \lambda_n, \lambda_n \rangle_{\gamma_i} = \|\lambda_n\|^2_{\gamma_i},$$

hence $\lambda_n = 0$ on γ_i. By induction, $\lambda_n = 0$ on $\Gamma_{SD} \cup \Gamma_{DD}$. It remains to show that $\boldsymbol{\lambda} = \mathbf{0}$ on Γ_{SS}. We may assume the first Stokes subdomain to have a Darcy neighbor: $\partial\Omega_1 \cap \Gamma_{SD} \neq \varnothing$. Let $\psi_1 \in (H(\Omega_1))^d$ and $\varphi_1 \in L^2(\Omega_1)$ solve the problem

$$\begin{aligned} &-\nabla \cdot \mathbf{T}(\psi_1, \varphi_1) = 0 \text{ in } \Omega_1, \quad \nabla \cdot \psi_1 = 0 \text{ in } \Omega_1, \\ &\psi_1 = \mathbf{0} \text{ on } \partial\Omega_1 \cap \partial\Omega, \\ &\psi_1 = \boldsymbol{\lambda} \text{ on } \partial\Omega_1 \cap \Gamma_{SS}, \\ &\mathbf{T}(\psi_1, \varphi_1)\mathbf{n}_1 = 0 \text{ on } \partial\Omega_1 \cap \Gamma_{SD}. \end{aligned} \tag{5.7}$$

Note that there is no compatibility condition for $\boldsymbol{\lambda}$ needed for the well posedness of (5.7), since a Neumann condition is imposed on a non-empty part of the boundary. Furthermore, the Dirichlet boundary data belongs to $H^{1/2}(\partial\Omega_1 \cap (\partial\Omega \cup \Gamma_{SS}))$, since $\boldsymbol{\lambda}|_{\partial\Omega} = \mathbf{0}$.

Let $(\psi_1^h, \varphi_1^h) \in X^S_{h,1} \times W^S_{h,1}$ be the finite element approximation to (ψ_1, φ_1). Taking $\mathbf{v}_i = \psi_1^h$ in (5.4) and using the fact that $\psi_1^h = \boldsymbol{\lambda}$ on $\partial\Omega_1 \cap \Gamma_{SS}$, we obtain

$$\langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle_{\partial\Omega_1 \cap \Gamma_{SS}} = -\langle \lambda_n, \psi_1^h \cdot \mathbf{n}_1 \rangle_{\partial\Omega_1 \cap \Gamma_{SD}} = 0,$$

since $\lambda_n = 0$ on Γ_{SD}. Therefore $\boldsymbol{\lambda} = \mathbf{0}$ on $\partial\Omega_1 \cap \Gamma_{SS}$. Similarly to the Darcy subdomains, we order the remaining Stokes subdomains Ω_i, $i = 2, \ldots, N_S$, so that $\gamma_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j \neq \varnothing$ for some $j < i$. Then, we solve consecutively for $i = 2, \ldots, N_S$ the problem

$$\begin{aligned} &-\nabla \cdot \mathbf{T}(\psi_i, \varphi_i) = 0 \text{ in } \Omega_i, \quad \nabla \cdot \psi_i = 0 \text{ in } \Omega_i, \\ &\psi_i = \mathbf{0} \text{ on } \partial\Omega_i \cap \partial\Omega, \\ &\psi_i = \boldsymbol{\lambda} \text{ on } \gamma_i \equiv \partial\Omega_i \setminus (\gamma_{ij} \cup \partial\Omega), \\ &\mathbf{T}(\psi_i, \varphi_i)\mathbf{n}_i = 0 \text{ on } \gamma_{ij} \cup (\partial\Omega_i \cap \Gamma_{SD}). \end{aligned} \tag{5.8}$$

The above problem is well posed, due to the Neumann condition on γ_{ij}.

If $(\psi_i^h, \varphi_i^h) \in X^S_{h,i} \times W^S_{h,i}$ is the finite element approximation to (ψ_i, φ_i), then taking $\mathbf{v}_i = \psi_i^h$ in (5.4) gives

$$0 = \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle_{\gamma_i} + \langle \boldsymbol{\lambda}, \psi_i^h \rangle_{\gamma_{ij}} + \langle \lambda_n, \psi_i^h \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \cap \Gamma_{SD}} = \langle \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle_{\gamma_i},$$

since $\lambda_n = 0$ on Γ_{SD} and $\boldsymbol{\lambda} = \mathbf{0}$ on γ_{ij}. Therefore, $\boldsymbol{\lambda} = \mathbf{0}$ on γ_i, and by induction we conclude that $\boldsymbol{\lambda} = \mathbf{0}$ on Γ_{SS}. □

As a result of Lemma 5.1, the conjugate gradient (CG) method can be applied for solving the algebraic problem (4.15). We now continue with estimating the condition number of S_h.

Lemma 5.2. *There exist positive constants $C_{D,1}$ and $C_{D,2}$ such that for all $\lambda_n \in \hat{\Lambda}_h^{SD,DD}$*

$$C_{D,1}\frac{K_{\min}^2}{K_{\max}}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}} \leqslant \sum_{i=N_S+1}^{N} s_i(\lambda_n, \lambda_n) \leqslant C_{D,2}\frac{K_{\max}}{h}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}}. \tag{5.9}$$

Proof. The definition (5.1) of $s_i(\cdot,\cdot)$ in the Darcy region gives

$$s_i(\lambda_n, \lambda_n) = -\langle \mathbf{u}_i^*(\lambda_n) \cdot \mathbf{n}_i, \lambda_n \rangle_{\partial\Omega_i \setminus \partial\Omega} \leqslant \|\mathbf{u}_i^*(\lambda_n) \cdot \mathbf{n}_i\|_{\partial\Omega_i \setminus \partial\Omega}\|\lambda_n\|_{\partial\Omega_i \setminus \partial\Omega} \leqslant C(v)h^{-1/2}\|\mathbf{u}_i^*(\lambda_n)\|_{\Omega_i}\|\lambda_n\|_{\partial\Omega_i \setminus \partial\Omega},$$

where we have used a discrete trace inequality, see e.g. Lemma 4.1 in [1]. Note that the above constant depends on the quasi-uniformity constant v from (3.2) as well as the shape regularity of the elements along the interface. The above inequality, combined with (4.5) and (5.2), implies the upper bound in the lemma.

We prove the lower bound by induction. Consider again the auxiliary problem (5.5) with $i = N_s + 1$ and take $\mathbf{v}_i = \Pi_{D,i}\psi_i$ in (4.6) to obtain

$$\begin{aligned} \|\lambda_n\|^2_{\partial\Omega_i \setminus \partial\Omega} &= \langle \lambda_n, \psi_i \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} = \langle \lambda_n, \Pi_{D,i}\psi_i \cdot \mathbf{n}_i \rangle_{\partial\Omega_i \setminus \partial\Omega} = -a_i(\mathbf{u}_i^*(\lambda_n), \Pi_{D,i}\psi_i) + (\nabla \cdot \Pi_{D,i}\psi_i, p_i^*(\lambda_n))_{\Omega_i} = -a_i(\mathbf{u}_i^*(\lambda_n), \Pi_{D,i}\psi_i) \\ &\leqslant CK_{\min}^{-1}\|\mathbf{u}_i^*(\lambda_n)\|_{\Omega_i}\|\psi_i\|_{1/2,\Omega_i} \leqslant CK_{\min}^{-1}\|\mathbf{u}_i^*(\lambda_n)\|_{\Omega_i}\|\lambda_n\|_{\partial\Omega_i \setminus \partial\Omega}, \end{aligned}$$

using (3.7), (3.6), (3.8), and the elliptic regularity [22,26]

$$\|\psi\|_{1/2,\Omega_i} \leqslant C\|\lambda_n\|_{\partial\Omega_i \setminus \partial\Omega}. \tag{5.10}$$

The above bound, in combination with (4.5) and (5.2), implies

$$C\frac{K_{\min}^2}{K_{\max}}\|\lambda_n\|^2_{\partial\Omega_i \setminus \partial\Omega} \leqslant s_i(\lambda_n, \lambda_n), \quad \text{for } i = N_S + 1.$$

Similarly to the procedure in the proof of Lemma 5.1, for $i = N_S + 2, \ldots, N$, we solve the auxiliary problems (5.6) and take $\mathbf{v}_i = \Pi_{D,i}\psi_i$ in (4.6). Then,

$$\begin{aligned}\|\lambda_n\|^2_{\gamma_i} &= \langle \lambda_n, \psi_i \cdot \mathbf{n}_i\rangle_{\gamma_i} = \langle \lambda_n, \Pi_{D,i}\psi_i \cdot \mathbf{n}_i\rangle_{\gamma_i} = -a_i(\mathbf{u}_i^*(\lambda_n), \Pi_{D,i}\psi_i) - \langle \lambda_n, \psi_i \cdot \mathbf{n}_i\rangle_{\gamma_{ij}} \leqslant CK_{\min}^{-1}\|\mathbf{u}_i^*(\lambda_n)\|_{\Omega_i}\|\lambda_n\|_{\gamma_i} + \|\lambda_n\|_{\gamma_{ij}}\|\psi_i \cdot \mathbf{n}_i\|_{\gamma_{ij}} \\ &\leqslant CK_{\min}^{-1}K_{\max}^{1/2}\left(s_i^{1/2}(\lambda_n,\lambda_n)\|\lambda_n\|_{\gamma_i} + \left(\sum_{k=N_S+1}^{i-1} s_k(\lambda_n,\lambda_n)\right)^{1/2}\|\psi_i\|_{1/2,\Omega_i}\right) \\ &\leqslant CK_{\min}^{-1}K_{\max}^{1/2}\left(s_i^{1/2}(\lambda_n,\lambda_n) + \left(\sum_{k=N_S+1}^{i-1} s_k(\lambda_n,\lambda_n)\right)^{1/2}\right)\|\lambda_n\|_{\gamma_i}.\end{aligned}$$

In the second inequality above we used (4.5) and (5.2), the induction hypothesis, and the trace inequality $\|\psi_i \cdot \mathbf{n}_i\|_{\gamma_{ij}} \leqslant C\|\psi_i\|_{1/2,\Omega_i}$, which follows by interpolating $\|\psi_i \cdot \mathbf{n}_i\|_{-1/2,\partial\Omega_i} \leqslant C\|\psi_i\|_{H(\mathrm{div};\Omega_i)} = C\|\psi_i\|_{\Omega_i}$ and $\|\psi_i \cdot \mathbf{n}_i\|_{\epsilon,\partial\Omega_i} \leqslant C\|\psi_i\|_{1/2+\epsilon,\Omega_i}$ for $\epsilon > 0$. In the last inequality we used the elliptic regularity (5.10). After applying Young's inequality we have

$$C\frac{K_{\min}^2}{K_{\max}}\|\lambda_n\|^2_{\gamma_i} \leqslant \sum_{k=N_S+1}^{i} s_k(\lambda_n,\lambda_n), \quad \text{for } i = N_S+2,\ldots,N. \tag{5.11}$$

Summing over the subdomains gives the lower bound for $\sum_{k=N_S+1}^{N} s_k(\lambda_n,\lambda_n)$. □

Next, we consider the contributions to the interface operator due to the Stokes subdomains.

Lemma 5.3. *There exist positive constants $C_{S,1}$ and $C_{S,2}$ such that for all $\tilde{\boldsymbol{\lambda}} \in \hat{\Lambda}_h$*

$$C_{S,1}\frac{K_{\min}^2}{K_{\max}}h\|\boldsymbol{\lambda}\|^2_{\Gamma_{SS}} \leqslant \sum_{i=1}^{N_S} s_i(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \sum_{i=N_S+1}^{N} s_i(\lambda_n,\lambda_n), \tag{5.12}$$

$$\sum_{i=1}^{N_S} s_i(\boldsymbol{\lambda},\boldsymbol{\lambda}) \leqslant C_{S,2}\left(\|\lambda_n\|^2_{\Gamma_{SD}} + \|\boldsymbol{\lambda}\|^2_{\Gamma_{SS}}\right). \tag{5.13}$$

Proof. We begin with establishing the upper bound. The definition (5.1) of $s_i(\cdot,\cdot)$ in the Stokes subdomains gives

$$\begin{aligned}s_i(\boldsymbol{\lambda},\boldsymbol{\lambda}) &= -\langle \mathbf{u}_i^*(\boldsymbol{\lambda}) \cdot \mathbf{n}_i, \lambda_n\rangle_{\partial\Omega_i\cap\Gamma_{SD}} - \langle \mathbf{u}_i^*(\boldsymbol{\lambda}), \boldsymbol{\lambda}\rangle_{\partial\Omega_i\cap\Gamma_{SS}} \leqslant \|\mathbf{u}_i^*(\boldsymbol{\lambda}) \cdot \mathbf{n}_i\|_{\partial\Omega_i\cap\Gamma_{SD}}\|\lambda_n\|_{\partial\Omega_i\cap\Gamma_{SD}} + \|\mathbf{u}_i^*(\boldsymbol{\lambda})\|_{\partial\Omega_i\cap\Gamma_{SS}}\|\boldsymbol{\lambda}\|_{\partial\Omega_i\cap\Gamma_{SS}} \\ &\leqslant C\|\mathbf{u}_i^*(\boldsymbol{\lambda})\|_{1,\Omega_i}(\|\lambda_n\|_{\Gamma_{SD}} + \|\boldsymbol{\lambda}\|_{\Gamma_{SS}}).\end{aligned}$$

Since (4.4) and (5.2) imply

$$\|\mathbf{u}_i^*(\boldsymbol{\lambda})\|_{1,\Omega_i} \leqslant C\max\{1, K_{\max}^{1/4}\}s_i^{1/2}(\boldsymbol{\lambda},\boldsymbol{\lambda}), \tag{5.14}$$

then

$$s_i^{1/2}(\boldsymbol{\lambda},\boldsymbol{\lambda}) \leqslant C\max\{1, K_{\max}^{1/4}\}\left(\|\lambda_n\|_{\Gamma_{SD}} + \|\boldsymbol{\lambda}\|_{\Gamma_{SS}}\right),$$

which yields (5.13).

Next, we derive the lower bound. Consider again problem (5.7) in Ω_1, as well as in any Ω_i such that $\partial\Omega_i \cap \Gamma_{SD} \neq \varnothing$ and its finite element approximation. By regularity of the Stokes solution [19],

$$\|\psi_i^h\|_{1,\Omega_i} \leqslant C\|\boldsymbol{\lambda}\|_{1/2,\partial\Omega_i\cap\Gamma_{SS}} \leqslant C(\nu)h^{-1/2}\|\boldsymbol{\lambda}\|_{\partial\Omega_i\cap\Gamma_{SS}}, \tag{5.15}$$

using also an inverse inequality [9], where ν is the constant from (3.2).

Note that, since $b_i^i(\mathbf{r},\boldsymbol{\lambda}) = 0$ and $b_i(\mathbf{r}, p_i^*(\boldsymbol{\lambda}) = 0$ for $\mathbf{r} \in \ker a_i$, then (4.8) holds for all $\mathbf{v}_i \in X_{h,i}^S$. We now have, taking $\mathbf{v}_i = \psi_i^h$ in (4.8),

$$\begin{aligned}\|\boldsymbol{\lambda}\|^2_{\partial\Omega_i\cap\Gamma_{SS}} &= \langle \boldsymbol{\lambda}, \psi_i^h\rangle_{\partial\Omega_i\cap\Gamma_{SS}} = -a_i(\mathbf{u}_i^*(\boldsymbol{\lambda}), \psi_i^h) - \langle \lambda_n, \psi_i^h \cdot \mathbf{n}_i\rangle_{\partial\Omega_i\cap\Gamma_{SD}} \leqslant C(\max\{1, K_{\min}^{-1/2}\}\|\mathbf{u}_i^*(\boldsymbol{\lambda})\|_{1,\Omega_i} + \|\lambda_n\|_{\partial\Omega_i\cap\Gamma_{SD}})\|\psi_i^h\|_{1,\Omega_i} \\ &\leqslant C\left(\max\left\{1, \frac{K_{\max}^{1/4}}{K_{\min}^{1/2}}\right\}s_i^{1/2}(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \frac{K_{\max}^{1/2}}{K_{\min}}\left(\sum_{k=N_S+1}^{N} s_k(\lambda_n,\lambda_n)\right)^{1/2}\right)h^{-1/2}\|\boldsymbol{\lambda}\|_{\partial\Omega_i\cap\Gamma_{SS}},\end{aligned}$$

using (5.14), (5.11), and (5.15) in the last inequality. The above inequality implies that

$$\frac{K_{\min}^2}{K_{\max}} h\|\boldsymbol{\lambda}\|^2_{\partial\Omega_i\cap\Gamma_{SS}} \leqslant C\left(\max\{1, K_{\min}^{1/2}\} s_i(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \sum_{k=N_S+1}^{N} s_k(\lambda_n,\lambda_n)\right).$$

In the following we will omit $\max\{1, K_{\min}^{1/2}\}$, since $K_{\min}$ is typically small. We continue by induction. Let us order the remaining Stokes subdomains Ω_i, so that $\gamma_{ij} \equiv \partial\Omega_i \cap \partial\Omega_j \neq \varnothing$ for some Ω_j from the set previously considered. Let (ψ_i^h, φ_i^h) be the finite element approximation to the problem

$$-\nabla\cdot\mathbf{T}(\psi_i,\varphi_i) = 0 \text{ in } \Omega_i, \quad \nabla\cdot\psi_i = 0 \quad \text{in } \Omega_i,$$

$$\psi_i = \boldsymbol{\lambda}^\dagger \quad \text{on } \partial\Omega_i,$$

where

$$\boldsymbol{\lambda}^\dagger = \begin{cases} \mathbf{0} \text{ on } \partial\Omega_i \cap \partial\Omega \\ \boldsymbol{\lambda} \text{ on } \gamma_i = \partial\Omega_i \setminus (\partial\Omega \cup \gamma_{ij}), \\ \boldsymbol{\lambda} + g\mathbf{n}_i \text{ on } \gamma_{ij} \end{cases}$$

and $g \in H_{00}^{1/2}(\gamma_{ij}), g \in X_{h,i}^S \cdot \mathbf{n}_i$ is chosen so that

$$\int_{\gamma_{ij}} g = -\int_{\partial\Omega_i\setminus\partial\Omega} \boldsymbol{\lambda}\cdot\mathbf{n}_i,$$

which guarantees that $\int_{\partial\Omega_i} \boldsymbol{\lambda}^\dagger\cdot\mathbf{n}_i = 0$ and the well-posedness of the problem. It is easy to see that

$$\|g\|_{\gamma_{ij}} \leqslant C\|\boldsymbol{\lambda}\|_{\partial\Omega_i\setminus\partial\Omega}. \tag{5.16}$$

By Stokes regularity [19] and inverse inequality,

$$\|\psi_i^h\|_{1,\Omega_i} \leqslant C\|\boldsymbol{\lambda}^\dagger\|_{1/2,\partial\Omega_i} \leqslant C(\nu)h^{-1/2}\|\boldsymbol{\lambda}\|_{\partial\Omega_i\setminus\partial\Omega}. \tag{5.17}$$

Note also that, due to the Dirichlet boundary condition, the pressure is determined up to a constant and it can be fixed by restricting it to $W_{h,i,0}^S$, the space of functions with mean value zero. As a result, we only have that

$$(\nabla\cdot\psi_i^h, w_i)_{\Omega_i} = 0 \quad \forall w_i \in W_{h,i,0}^S.$$

However, we also have that

$$(\nabla\cdot\psi_i^h, 1)_{\Omega_i} = \int_{\partial\Omega_i} \psi_i^h\cdot\mathbf{n}_i = \int_{\partial\Omega_i} \boldsymbol{\lambda}^\dagger\cdot\mathbf{n}_i = 0,$$

implying that $(\nabla\cdot\psi_i^h, w_i)_{\Omega_i} = 0 \quad \forall w_i \in W_{h,i}^S$.

We now have, taking $\mathbf{v}_i = \psi_i^h$ in (4.8) and using (5.14), (5.17), and (5.16),

$$\begin{aligned}\|\boldsymbol{\lambda}\|^2_{\partial\Omega_i\setminus\partial\Omega} &= \langle\boldsymbol{\lambda}, \psi_i^h\rangle_{\gamma_i} + \langle\boldsymbol{\lambda}, \psi_i^h - g\mathbf{n}_i\rangle_{\gamma_{ij}} = -a_i(\mathbf{u}_i^*(\boldsymbol{\lambda}), \psi_i^h) - \langle\boldsymbol{\lambda}, g\mathbf{n}_i\rangle_{\gamma_{ij}} \leqslant C\left(\|\mathbf{u}_i^*(\boldsymbol{\lambda})\|_{1,\Omega_i}\|\psi_i^h\|_{1,\Omega_i} + \|\boldsymbol{\lambda}\|_{\gamma_{ij}}\|g\|_{\gamma_{ij}}\right) \\ &\leqslant C\left(s_i^{1/2}(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \frac{K_{\max}^{1/2}}{K_{\min}}\left(\sum_{k=1}^{i-1} s_k(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \sum_{k=N_S+1}^{N} s_k(\lambda_n,\lambda_n)\right)^{1/2}\right) h^{-1/2}\|\boldsymbol{\lambda}\|_{\partial\Omega_i\setminus\partial\Omega},\end{aligned}$$

where we also used the induction hypothesis in the last inequality. The above inequality implies that

$$\frac{K_{\min}^2}{K_{\max}} h\|\boldsymbol{\lambda}\|^2_{\partial\Omega_i\setminus\partial\Omega} \leqslant C\left(\sum_{k=1}^{i} s_k(\boldsymbol{\lambda},\boldsymbol{\lambda}) + \sum_{k=N_S+1}^{N} s_k(\lambda_n,\lambda_n)\right).$$

Summing over the subdomains implies (5.12). □

Theorem 5.1. *If there are no Stokes–Stokes interfaces, then*

$$C_{D,1}\frac{K_{\min}^2}{K_{\max}} \leqslant \frac{s(\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{\lambda}})}{\|\tilde{\boldsymbol{\lambda}}\|_\Gamma^2} \leqslant 2\max\left\{C_{D,2}\frac{K_{\max}}{h}, C_{S,2}\right\}, \quad \forall\tilde{\boldsymbol{\lambda}} \in \hat{\Lambda}_h. \tag{5.18}$$

In the presence of Stokes–Stokes interfaces, then

$$\frac{K_{\min}^2}{2K_{\max}}\min\{C_{D,1}, C_{S,1}h\} \leqslant \frac{s(\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{\lambda}})}{\|\tilde{\boldsymbol{\lambda}}\|_\Gamma^2} \leqslant 2\max\left\{C_{D,2}\frac{K_{\max}}{h}, C_{S,2}\right\}, \quad \forall\tilde{\boldsymbol{\lambda}} \in \hat{\Lambda}_h. \tag{5.19}$$

Proof. If $\Gamma_{SS} = \varnothing$, from Lemmas 5.2 and 5.3 we conclude that

$$C_{D,1}\frac{K_{\min}^2}{K_{\max}}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}} \leqslant s(\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{\lambda}}) \leqslant C_{D,2}\frac{K_{\max}}{h}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}} + C_{S,2}\|\lambda_n\|^2_{\Gamma_{SD}}, \quad \forall\tilde{\boldsymbol{\lambda}}\in\hat{\Lambda}_h,$$

which implies (5.18). If $\Gamma_{SS}\neq\varnothing$, Lemma 5.2 and Lemma 5.3 imply

$$\frac{K_{\min}^2}{2K_{\max}}(C_{D,1}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}} + C_{S,1}h\|\boldsymbol{\lambda}\|^2_{\Gamma_{SS}}) \leqslant s(\tilde{\boldsymbol{\lambda}},\tilde{\boldsymbol{\lambda}}) \leqslant \quad C_{D,2}\frac{K_{\max}}{h}\|\lambda_n\|^2_{\Gamma_{SD}\cup\Gamma_{DD}} + C_{S,2}(\|\lambda_n\|^2_{\Gamma_{SD}} + \|\boldsymbol{\lambda}\|^2_{\Gamma_{SS}}), \quad \forall\tilde{\boldsymbol{\lambda}}\in\hat{\Lambda}_h,$$

which implies (5.19). □

Corollary 5.1. *The condition number for the algebraic system associated with the coupled Stokes–Darcy flow problem satisfies*

$$\mathrm{cond}(S_h) = O(h^{-1}), \quad \text{if } \Gamma_{SS} = \varnothing,$$
$$\mathrm{cond}(S_h) = O(h^{-2}), \quad \text{if } \Gamma_{SS}\neq\varnothing.$$

We also note that if the permeability in the porous medium has a characteristic length k that is smaller than h, then $C_{S,2}$ dominates $C_{D,2}\frac{K_{\max}}{h}$ on the right hand sides of (5.18) and (5.19), implying that

$$\mathrm{cond}(S_h) = O(k^{-1}), \quad \text{if } \Gamma_{SS} = \varnothing,$$
$$\mathrm{cond}(S_h) = O(k^{-1}h^{-1}), \quad \text{if } \Gamma_{SS}\neq\varnothing.$$

6. Numerical experiments

We present several numerical experiments that illustrate the behavior of the method. In the first test we solve a coupled problem with known analytical solution on different meshes and compute the associated error to verify the convergence of the discretization scheme. In the other tests, which are aimed to examine the convergence of the iterative method, we vary either the mesh size, the permeability, or the number of subdomains. The computational domain is $\Omega = \Omega_S \cup \Omega_D$, where the Stokes domain $\Omega_S = [0,1]\times[\frac{1}{2},1]$ and Darcy domain $\Omega_D = [0,1]\times[0,\frac{1}{2}]$. For simplicity we set

$$\mathbf{T}(\mathbf{u}_S, p_S) = -p_S\mathbf{I} + \mu\nabla\mathbf{u}_S$$

in the Stokes equation in Ω_S, and

$$\mathbf{K} = K\mathbf{I}$$

in the Darcy equation in Ω_D, where K is a positive constant. To discretize the system of equations we use the Taylor–Hood [37] triangular finite elements in Ω_S and the lowest order Raviart–Thomas [34] rectangular finite elements in Ω_D. The grid for the discretization in Ω_S is obtained by first partitioning the domain into rectangles and then dividing each rectangle along its diagonal into two triangles. The grids in Ω_S and Ω_D match on the interface Γ_{SD}.

In our implementation we utilize direct subdomain solvers. This is reasonable, since in practice sufficient number of processors can assure that the subdomain problems are of small to moderate size. As a result the convergence of the interface CG is not affected by inexact subdomain solves. Furthermore, the LU factorization is reused multiple times with different right hand sides at each CG iteration.

For the first test we consider the following analytical solution satisfying the flow equations in Ω_S and Ω_D along with the conditions on the interface Γ_{SD}:

$$\mathbf{u}_S = \begin{bmatrix} (2-x)(1.5-y)(y-\xi) \\ -\frac{y^3}{3} + \frac{y^2}{2}(\xi+1.5) - 1.5\xi y - 0.5 + \sin(\omega x) \end{bmatrix},$$
$$\mathbf{u}_D = \begin{bmatrix} \omega\cos(\omega x)y \\ \chi(y+0.5) + \sin(\omega x) \end{bmatrix},$$
$$p_S = -\frac{\sin(\omega x)+\chi}{2K} + \mu(0.5-\xi) + \cos(\pi y),$$
$$p_D = -\frac{\chi}{K}\frac{(y+0.5)^2}{2} - \frac{\sin(\omega x)y}{K},$$

where

$$\mu = 0.1, \quad K = 1, \quad \alpha_0 = 0.5, \quad G = \frac{\sqrt{\mu K}}{\alpha_0}, \quad \xi = \frac{1-G}{2(1+G)}, \quad \chi = \frac{-30\xi - 17}{48}, \quad \text{and} \quad \omega = 6.0.$$

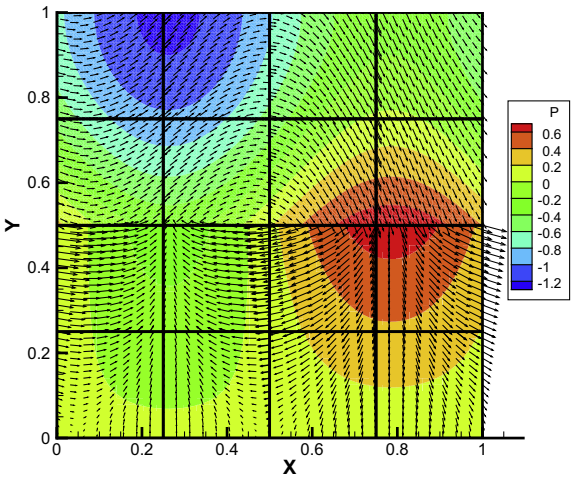

Fig. 1. Computed solution in the first test on a mesh with $h = 1/64$.

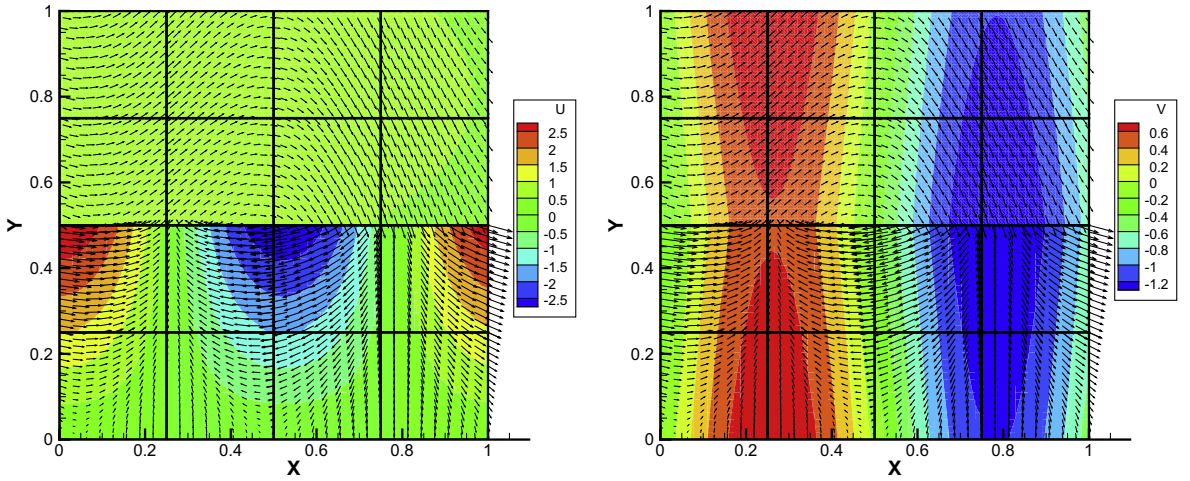

Fig. 2. Computed velocity field in the first test: horizontal velocity (left); vertical velocity (right).

Table 1
Numerical errors and convergence rates in Ω_S.

h	$\|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{1,\Omega_S}$	*rate*	$\|p_S - p_{S,h}\|_{0,\Omega_S}$	*rate*
1/8	1.69e-01		8.83e-03	
1/16	4.23e-02	2.00	2.23e-03	1.98
1/32	1.05e-02	2.00	5.59e-04	2.00
1/64	2.64e-03	2.00	1.41e-04	1.99
1/128	6.68e-04	1.98	3.59e-05	1.97

Table 2
Numerical errors and convergence rates in Ω_D.

h	$\|\|\|\mathbf{u}_D - \mathbf{u}_{D,h}\|\|\|_{\Omega_D}$	*rate*	$\|\|\|p_D - p_{D,h}\|\|\|_{\Omega_D}$	*rate*
1/8	6.04e-02		5.14e-03	
1/16	1.54e-02	1.97	1.29e-03	1.99
1/32	3.88e-03	1.99	3.22e-04	2.00
1/64	9.71e-04	2.00	8.04e-05	2.00
1/128	2.43e-04	2.00	2.02e-05	1.99

Table 3
Convergence of interface CG: $K = 1.0$, 2 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	4.552	14.637	3.2	7
1/16	4.649	23.064	5.0	11
1/32	4.676	45.317	9.7	17
1/64	4.683	90.479	19.3	24
1/128	4.685	180.958	38.6	35

Table 4
Convergence of interface CG: $K = 2.0$, 2 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	8.073	23.930	3.0	8
1/16	8.216	45.477	5.5	12
1/32	8.254	90.490	11.0	17
1/64	8.263	180.918	21.9	25
1/128	8.266	361.902	43.8	36

Table 5
Convergence of interface CG: $K = 0.01$, 2 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.302	8.688	28.8	8
1/16	0.263	8.719	33.1	9
1/32	0.270	8.734	32.3	8
1/64	0.278	8.746	31.5	10
1/128	0.280	8.758	31.3	13

Table 6
Convergence of interface CG: $K = 1.0$, 4 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.404	25.467	63.1	21
1/16	0.236	50.519	213.7	38
1/32	0.134	100.979	754.8	65
1/64	0.092	202.092	2191.0	95
1/128	0.065	404.432	6183.2	146

Table 7
Convergence of interface CG: $K = 2.0$, 4 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.404	50.413	124.8	22
1/16	0.249	100.762	404.8	39
1/32	0.149	201.802	1350.0	66
1/64	0.083	404.096	4874.3	117
1/128	0.084	808.813	9582.9	152

Table 8
Convergence of interface CG: $K = 0.01$, 4 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.118	10.846	91.6	19
1/16	0.127	10.856	85.4	19
1/32	0.087	10.858	124.4	22
1/64	0.056	10.859	195.3	27
1/128	0.035	10.860	308.7	39

Table 9
Convergence of interface CG: $K = 1.0$, 16 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.287	32.000	111.5	42
1/16	0.125	64.000	512.3	78
1/32	0.062	128.000	2060.6	131
1/64	0.032	256.000	7907.6	236
1/128	0.019	512.000	27115.1	410

Table 10
Convergence of interface CG: $K = 1.0$, 64 subdomains.

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.375	31.976	85.3	53
1/16	0.131	64.000	488.2	92
1/32	0.063	128.000	2038.6	152
1/64	0.032	256.000	8035.1	260
1/128	0.019	512.000	27073.1	454

Table 11
Convergence of interface CG: $K = 1.0$, 4 subdomains (Stokes only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.245	8.037	32.8	20
1/16	0.121	8.087	66.6	29
1/32	0.061	8.099	133.4	36
1/64	0.030	8.102	266.6	47
1/128	0.015	8.103	532.1	58

Table 12
Convergence of interface CG: $K = 1.0$, 16 subdomains (Stokes only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.251	5.667	22.6	24
1/16	0.123	5.819	47.4	31
1/32	0.061	5.857	96.5	42
1/64	0.030	5.867	193.0	54
1/128	0.015	5.869	385.5	68

Table 13
Convergence of interface CG: $K = 1.0$, 64 subdomains (Stokes only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	0.222	2.719	12.3	22
1/16	0.124	3.043	24.5	28
1/32	0.061	3.127	51.0	37
1/64	0.030	3.148	103.7	49
1/128	0.015	3.153	207.4	63

Table 14
Convergence of interface CG: $K = 1.0$, 4 subdomains (Darcy only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	4.051	22.623	5.6	10
1/16	4.061	45.193	11.1	15
1/32	4.063	90.433	22.3	22
1/64	4.064	180.929	44.5	32
1/128	4.064	512.000	126.0	46

The right hand sides $\mathbf{f}_S, \mathbf{f}_D$, and q_D for the Stokes–Darcy flow system are obtained by plugging the analytical solution into (2.1), (2.4), and (2.5), respectively. The boundary conditions are as follows: for the Stokes region, the velocity $\mathbf{u}_S$ is specified on the left boundary, and the normal and tangential stresses $(\mathbf{T}\mathbf{n}_S)\cdot\mathbf{n}_S$ and $(\mathbf{T}\mathbf{n}_S)\cdot\tau_S$ are specified on the top and right boundaries; for the Darcy region, the normal velocity $\mathbf{u}_D\cdot\mathbf{n}_D$ is specified on the left boundary and the pressure p_D is specified on the bottom and right boundaries. In this example we use $4\times 4 = 16$ subdomains, 8 in each region. Under these boundary conditions, the six Stokes subdomains on the right are floating: each of the three on the top row has $(u, v) = (1, 0)$ and $(0, 1)$ as a basis of $\ker(M_i)$, while that basis for each of the three on the second row consists only of $(0, 1)$. The computed solution on a mesh with $h = 1/64$ is shown in Fig. 1. The contour plots in Fig. 2 represent the two components of the computed velocity field. We see that the domain decomposition scheme correctly imposes continuity of the normal velocity on Γ_{SD}, but allows for discontinuous tangential velocity across the interface. For our choice of finite element spaces, $k_S = 2$ and $k_D = l_D = 0$, so the error bound (3.11) predicts second order convergence in the Stokes region and first order convergence in the Darcy region. The results reported in Table 1 and Table 2 confirm the expected rates. Note that we report the cell-centered L^2-errors $|||\cdot|||$ in the Darcy region, which are superconvergent for both the pressure and the velocity. This is consistent with the theory for mixed finite element methods for Darcy on rectangular grids.

In the next test, for different permeabilities we vary either the mesh size or the number of subdomains to examine the convergence of the iterative method. In Table 3 and Table 4 we see that on two subdomains, when $h < K$, the minimal eigenvalue of the interface operator does not change much as we refine the mesh, while the maximal eigenvalue changes as $O(h^{-1})$, according to (5.18), which results in condition number of order $O(h^{-1})$. In this case we also see that changing the permeability for a fixed h has no effect on the condition number, which can be explained by the fact that the permeability constants K_{min} and K_{max} appearing in the estimates of the Rayleigh quotient (5.18) cancel one another when we divide the upper bound by the lower bound. Table 5 shows the behavior of the method when $K < h$, in which case both the minimal and the maximal eigenvalues of the interface operator are dominated by constants independent of h, and consequently the condition number does not change significantly as the mesh is refined. This is consistent with (5.18).

In the presence of Stokes–Stokes interfaces, if $h < K$, the bounds in (5.19) imply that the maximal eigenvalue of the interface operator is $O(h^{-1})$ while the minimal is $O(h)$, which means that the condition number is $O(h^{-2})$. This estimate is supported by the results reported in Tables 6 and 7. We also see that the largest eigenvalue is doubled when K is doubled, which is consistent with the upper bound in (5.19). For $K < h$, Table 8 shows that the maximal eigenvalue does not change when the mesh is refined, confirming the upper bound in (5.19).

Finally we test the effect of the subdomain size on the condition number, running the above tests with $K = 1.0$ on $4\times 4 = 16$ and $8\times 8 = 64$ subdomains. Comparing Table 6 on 4 subdomains and Table 9 on 16 subdomains, we see that the minimal eigenvalue is approximately proportional to the subdomain size. This is expected for domain decomposition methods without a coarse solve [38]. However, when comparing Tables 9 and 10, we notice that the minimal eigenvalue does not change. The reason is that the minimal eigenvalue is controlled by the Stokes region, where it is independent of the subdomain size due the coarse solve we have implemented to handle the local Stokes Neumann problems. This can be observed in Tables 11–16, where the results for Stokes only and Darcy only problems are reported. In particular, the minimal eigenvalue is smaller for Stokes compared to Darcy on the same mesh and number of subdomains, while the maximal eigenvalue is larger for Darcy. Furthermore, for Stokes the minimal eigenvalue is $O(h)$ and independent of subdomain size, while the maximal eigenvalue is $O(1)$. For Darcy, the minimal eigenvalue is proportional to the subdomain size and the maximal eigenvalue is $O(h^{-1})$.

Table 15
Convergence of interface CG: $K = 1.0$, 16 subdomains (Darcy only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	2.166	23.601	10.9	16
1/16	2.172	45.239	20.8	22
1/32	2.174	90.477	41.6	31
1/64	2.174	256.000	117.7	45
1/128	2.174	512.000	235.5	63

Table 16
Convergence of interface CG: $K = 1.0$, 64 subdomains (Darcy only).

h	*eig.min.*	*eig.max.*	$cond(S_h)$	*iter.num.*
1/8	1.065	31.361	29.4	26
1/16	1.069	47.773	44.7	31
1/32	1.070	128.000	119.6	44
1/64	1.070	256.000	239.2	61
1/128	1.070	512.000	478.4	87

7. Conclusions

We have developed an iterative substructuring domain decomposition method for coupled Stokes–Darcy flows with multiple subdomains. The discretization uses conforming Stokes finite elements and mixed finite elements in the Darcy region. The coupled system is reduced to an interface problem for a Lagrange multiplier that models the normal stress or pressure on Stokes–Darcy or Darcy–Darcy interfaces and the normal stress vector on Stokes–Stokes interfaces. Computing the action of the interface operator requires solving Stokes subdomain problems with Neumann or Neumann–Robin boundary conditions and Darcy subdomain problems of Dirichlet type. Since the Stokes problems can be singular, a FETI type coarse solve is utilized. No coarse solve is needed to solve the Darcy subdomain problems. The analysis of the condition number for the unpreconditioned algorithm shows possible $O(h^{-2})$ dependence due to the different type of Stokes and Darcy boundary value subdomain problems. We further observe dependence on small permeability values and dependence on the subdomain size in the Darcy region. The algorithm lends itself to the application of optimal interface preconditioners, such as balancing [27,28,10,32,18], which should improve the dependence on h, k, and subdomain size in the condition number. Such preconditioners will be investigated in future work.

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