SIAM J. NUMER. ANAL.
Vol. 34, No. 2, pp. 828–852, April 1997

MIXED FINITE ELEMENTS FOR ELLIPTIC PROBLEMS WITH TENSOR COEFFICIENTS AS CELL-CENTERED FINITE DIFFERENCES*

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Abstract. We present an expanded mixed finite element approximation of second-order elliptic problems containing a tensor coefficient. The mixed method is expanded in the sense that three variables are explicitly approximated, namely, the scalar unknown, the negative of its gradient, and its flux (the tensor coefficient times the negative gradient). The resulting linear system is a saddle point problem. In the case of the lowest order Raviart–Thomas elements on rectangular parallelepipeds, we approximate this expanded mixed method by incorporating certain quadrature rules. This enables us to write the system as a simple, cell-centered finite difference method requiring the solution of a sparse, positive semidefinite linear system for the scalar unknown. For a general tensor coefficient, the sparsity pattern for the scalar unknown is a 9-point stencil in two dimensions and 19 points in three dimensions. Existing theory shows that the expanded mixed method gives optimal order approximations in the L^2- and H^{-s}-norms (and superconvergence is obtained between the L^2-projection of the scalar variable and its approximation). We show that these rates of convergence are retained for the finite difference method. If h denotes the maximal mesh spacing, then the optimal rate is $O(h)$. The superconvergence rate $O(h^2)$ is obtained for the scalar unknown and rate $O(h^{3/2})$ for its gradient and flux in certain discrete norms; moreover, the full $O(h^2)$ is obtained in the strict interior of the domain. Computational results illustrate these theoretical results.

Key words. mixed finite element, finite difference, tensor coefficient, error estimates, superconvergence

AMS subject classifications. 65N06, 65N12, 65N15, 65N22, 65N30

PII. S0036142994262585

1. Introduction.

We consider mixed finite element approximations of second-order elliptic problems with Dirichlet, Neumann, and Robin boundary conditions. In mixed form, the problem is to find $(\mathbf{u}, p)$ such that

$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u} = -K\nabla p \quad \text{in } \Omega, \tag{1.1b}$$

$$p = g^D \quad \text{on } \Gamma^D, \tag{1.1c}$$

$$\mathbf{u} \cdot \nu = g^N \quad \text{on } \Gamma^N, \tag{1.1d}$$

$$\mathbf{u} \cdot \nu - g_1^R p = g_2^R \quad \text{on } \Gamma^R, \tag{1.1e}$$

where Ω is a bounded domain in $\mathbf{R}^d$ ($d = 2$ or 3) with boundary $\partial\Omega = \overline{\Gamma}^D \cup \overline{\Gamma}^N \cup \overline{\Gamma}^R$ ($\Gamma^D \cap \Gamma^N = \Gamma^D \cap \Gamma^R = \Gamma^N \cap \Gamma^R = \emptyset$); $K(\mathbf{x})$ is a symmetric, positive definite second-order tensor with components in $L^\infty(\Omega)$; ν is the outward, unit, normal vector on $\partial\Omega$; and $g_1^R(\mathbf{x}) \geq 0$. In applications to flow in porous media, p is the pressure, $\mathbf{u}$ is

*Received by the editors January 31, 1994; accepted for publication (in revised form) July 11, 1995. This work was supported in part by the Department of Energy, the National Science Foundation, and the State of Texas Governor's Energy Office.

http://www.siam.org/journals/sinum/34-2/26258.html

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the Darcy velocity, and K is the conductivity tensor (permeability divided by fluid viscosity); moreover, such boundary conditions arise naturally [5]. Generally, K is a full tensor, either as measured directly or as homogenized from microscale data to the grid-scale [6], [18], [21].

For simplicity, we assume that the problem has a unique solution; however, our results carry over to the semidefinite, pure Neumann problem, where $\Gamma^N = \partial\Omega$ and $f(\mathbf{x})$ and $g^N(\mathbf{x})$ satisfy the compatibility condition $\int_\Omega f(\mathbf{x})\,d\mathbf{x} = \int_{\partial\Omega} g^N(\mathbf{x})\,ds(\mathbf{x})$.

We use affine mixed finite elements for approximating the solution of (1.1). We concentrate on the case of the lowest order Raviart–Thomas spaces [28], [26], since these are widely used in practice, though many of our results will be more general.

The usual mixed formulation requires the solution of a linear system in the form of a saddle point problem, which can be expensive to solve. An alternate approach was suggested by Arnold and Brezzi [4] (see also [10]). They used the hybrid (or Lagrange multiplier) form of the equations. In their method, one eliminates the velocity unknowns in terms of the pressures and the Lagrange multiplier pressures that live on the element edges if $d = 2$ or faces if $d = 3$; furthermore, one can easily eliminate the pressures to leave a system for the Lagrange multipliers alone. Although there are more overall unknowns, it is simple to implement and requires the solution of a sparse, positive definite linear system; that is, it is a face-centered finite difference method. The lowest order Raviart–Thomas spaces have one Lagrange multiplier unknown per edge or face.

In the case that K is a diagonal tensor and one uses the lowest order Raviart–Thomas spaces defined over a rectangular grid, Russell and Wheeler [29] showed that the system could be simplified by an appropriate use of quadrature rules. They were able to approximate the usual mixed formulation so as to require the solution of a sparse, positive definite linear system for the pressure unknowns. There is only one such unknown per element, so the system is substantially smaller and therefore easier to solve than in the hybrid method. Moreover, Russell and Wheeler showed that in fact their quadrature rules turned the mixed method into a cell-centered finite difference scheme with a five-point stencil (or "computational molecule") if $d = 2$, or seven if $d = 3$. Weiser and Wheeler [32] showed that the modified scheme converges at the rate of the unmodified scheme. This is true also of the superconvergence that is obtained for the velocity $\mathbf{u}$ and pressure p in certain discrete norms [25], [16], [17], [19]. If h denotes the maximal mesh spacing, then the optimal convergence rate is $O(h)$, but $O(h^2)$ superconvergence is obtained in these discrete norms for the pure Neumann problem.

The main goal of this paper is to derive and exploit a connection between the expanded mixed method and a certain cell-centered finite difference method. Using approximate integration, a cell-centered finite difference stencil for the pressure will be obtained after eliminating the velocity when K is not diagonal, without any loss in the rate of convergence, and retaining the superconvergence phenomenon. This has two advantages: first, a sparse, positive definite linear system results, and, second, the method can be relatively easily incorporated into existing standard cell-centered finite difference reservoir or groundwater simulators that handle diagonal K [27]. We also address another computational difficulty, namely, that in practice the conductivity K can be zero in a subdomain of Ω. The standard mixed variational formulation requires inverting K, an impossibility in this degenerate case. Although our theory does not extend to the degenerate case, our scheme is at least computationally well defined.

We consider the following expanded mixed formulation of (1.1) by explicitly introducing the negative pressure gradient. We find $(\mathbf{u}, \tilde{\mathbf{u}}, p)$ satisfying (1.1) with (1.1b) replaced by

$$\tilde{\mathbf{u}} = -\nabla p \quad \text{in } \Omega, \tag{1.2a}$$

$$\mathbf{u} = K\tilde{\mathbf{u}} \quad \text{in } \Omega. \tag{1.2b}$$

Let V^0 and V^N be the subspaces of $V = H(\Omega;\ \text{div})$ consisting of functions with normal trace on Γ^N (weakly) equal to zero and g^N, respectively; let $\tilde{V} = (L^2(\Omega))^d$; let $W = L^2(\Omega)$; and let $\Lambda = H^{1/2}(\partial\Omega)$. Let $(\cdot,\cdot)_S$ denote the $L^2(S)$-inner product (i.e., integration over the set S) or the duality pairing, where we omit S if $S = \Omega$. We have the following equivalent variational formulation of the expanded system (1.1a), (1.2), (1.1c)–(1.1e): find $\mathbf{u} \in V^N$, $\tilde{\mathbf{u}} \in \tilde{V}$, $p \in W$, and $\lambda \in \Lambda$ such that

$$(\nabla \cdot \mathbf{u}, w) = (f, w), \qquad w \in W, \tag{1.3a}$$

$$(\tilde{\mathbf{u}}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} - (\lambda, \mathbf{v}\cdot\nu)_{\Gamma^R}, \qquad \mathbf{v} \in V^0, \tag{1.3b}$$

$$(\mathbf{u}, \tilde{\mathbf{v}}) = (K\tilde{\mathbf{u}}, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{V}, \tag{1.3c}$$

$$(\mathbf{u}\cdot\nu, \mu)_{\Gamma^R} = (g_2^R + g_1^R\lambda, \mu)_{\Gamma^R}, \qquad \mu \in \Lambda. \tag{1.3d}$$

One of the authors described the use of the expanded mixed formulation briefly in a practical setting in [33]. A similar formulation was considered by Chen [11] to approximate a nonlinear problem, using only the Brezzi–Douglas–Marini (BDM) spaces [9]. He also presented a convergence analysis (see also [12]) but did not discuss implementation. Koebbe [22] used the expanded mixed formulation to solve problems with a tensor coefficient. He was concerned with implementation, but he did not attempt to obtain a finite difference stencil. Rather, he solved a saddle point problem. We consider a more general set of test and trial functions than either [11] or [22], since they both took $\tilde{V} = V$.

The rest of the paper is organized as follows. In section 2 we formulate the discrete approximation of (1.3). Stability and solvability are shown, and a convergence theorem is given in section 3. The cell-centered finite difference stencil for the pressure on rectangles is derived in section 4 and analyzed in section 5, showing convergence at the optimal convergence rate $O(h)$; moreover, our superconvergence results for both $\mathbf{u}$ and p are presented in this section. It turns out that superconvergence for p and λ is of rate $O(h^2)$, but for $\mathbf{u}$ and $\tilde{\mathbf{u}}$ it is only $O(h^{3/2})$, being degraded somewhat by the treatment of the boundary conditions; however, we show that full superconvergence of rate $O(h^2)$ is obtained for $\mathbf{u}$ in the strict interior of the domain. Finally, in section 6 we discuss some numerical results demonstrating our convergence results. We also present a modification of the finite difference scheme that apparently achieves the full $O(h^2)$ superconvergence even up to the domain boundary.

2. The expanded mixed finite element method. Let $\{\mathcal{T}_h\}_{h>0}$ be a quasi-uniform family of finite element partitions of Ω such that no element crosses the boundaries of Γ^D, Γ^N, or Γ^R, where h is the maximal element diameter. Let $V_h \times W_h$ be any of the usual mixed finite element approximating subspaces of $H(\Omega;\ \text{div}) \times W$; that is, the Raviart–Thomas–Nedelec (RTN) spaces [30], [28], [26]; BDM spaces [9]; Brezzi–Douglas–Fortin–Marini (BDFM) spaces [8]; Brezzi–Douglas–Duràn–Fortin (BDDF) spaces [7]; or Chen–Douglas (CD) spaces [13]. Let $\Lambda_h \subset L^2(\partial\Omega)$ be the corresponding hybrid space of Lagrange multipliers for the pressure [4], [10] restricted to $\partial\Omega$. Define $V_h^0 = V_h \cap V^0$, $V_h^N = \{\mathbf{v} \in V_h : (\mathbf{v}\cdot\nu - g^N, \mu)_{\Gamma^N} = 0 \text{ for all } \mu \in \Lambda_h\}$, and $\Lambda_h^R = \Lambda_h|_{\Gamma^R}$.

Let $\tilde{V}_h$ be a finite element subspace of $\tilde{V}$ satisfying $V_h^N \subseteq \tilde{V}_h$. Generally speaking, $\tilde{V}_h$ should have full flexibility on Γ^N. Note that we do not require $\tilde{V}_h \subseteq H(\Omega;\ \mathrm{div})$, since continuity of the normal component across the edges is not needed. Thus we can think of $\tilde{V}_h$ as a possibly discontinuous version of V_h with full degrees of freedom near $\partial\Omega$ (although later we will simply take $\tilde{V}_h = V_h$).

In the mixed finite element approximation of (1.3), we seek $\mathbf{u}_h \in V_h^N$, $\tilde{\mathbf{u}}_h \in \tilde{V}_h$, $p_h \in W_h$, $\lambda_h \in \Lambda_h^R$ such that

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w), \qquad w \in W_h, \tag{2.1a}$$

$$(\tilde{\mathbf{u}}_h, \mathbf{v}) = (p_h, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} - (\lambda_h, \mathbf{v}\cdot\nu)_{\Gamma^R}, \qquad \mathbf{v} \in V_h^0, \tag{2.1b}$$

$$(\mathbf{u}_h, \tilde{\mathbf{v}}) = (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{V}_h, \tag{2.1c}$$

$$(\mathbf{u}_h\cdot\nu, \mu)_{\Gamma^R} = (g_2^R + g_1^R\lambda_h, \mu)_{\Gamma^R}, \qquad \mu \in \Lambda_h^R. \tag{2.1d}$$

We have many families of methods for various choices of $\tilde{V}_h$.

We find it convenient for the analysis and for finite differences below to formulate the scheme without explicit reference to the Neumann boundary condition. Both Neumann and Robin conditions affect the flow or flux, so let Γ^F denote the interior of $\bar{\Gamma}^N \cup \bar{\Gamma}^R$ and define

$$g_1^F = \begin{cases} 0 & \text{on } \Gamma^N, \\ g_1^R & \text{on } \Gamma^R, \end{cases} \qquad g_2^F = \begin{cases} g^N, & \text{on } \Gamma^N, \\ g_2^R & \text{on } \Gamma^R, \end{cases}$$

and $\Lambda_h^F = \Lambda_h|_{\Gamma^F}$. An equivalent formulation is to find $\mathbf{u}_h \in V_h$, $\tilde{\mathbf{u}}_h \in \tilde{V}_h$, $p_h \in W_h$, $\lambda_h \in \Lambda_h^F$ such that

$$(\nabla \cdot \mathbf{u}_h, w) = (f, w), \qquad w \in W_h, \tag{2.2a}$$

$$(\tilde{\mathbf{u}}_h, \mathbf{v}) = (p_h, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} - (\lambda_h, \mathbf{v}\cdot\nu)_{\Gamma^F}, \qquad \mathbf{v} \in V_h, \tag{2.2b}$$

$$(\mathbf{u}_h, \tilde{\mathbf{v}}) = (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in \tilde{V}_h, \tag{2.2c}$$

$$(\mathbf{u}_h\cdot\nu, \mu)_{\Gamma^F} = (g_2^F + g_1^F\lambda_h, \mu)_{\Gamma^F}, \qquad \mu \in \Lambda_h^F. \tag{2.2d}$$

It is easy to see that this modification does not change the scheme (2.1).

3. Convergence of the expanded mixed method. In this section we present some error estimates for the approximate solution. For a domain S, let $\|\cdot\|_{j,q,S}$ denote the norm of $W^{j,q}(S)$, the Sobolev space of j-times differentiable functions in $L^q(S)$, and let $H^j(S) = W^{j,2}(S)$, $\|\cdot\|_{j,S}$ denote its norm and $\|\cdot\|_{-j,S}$ denote the norm of its dual space $H^{-j}(S) = (H^j(S))'$. We may omit S if $S = \Omega$. Our error will be measured in the norms of L^2 and H^{-s}. We let C denote a generic positive constant that is independent of the discretization parameter h.

We make explicit the following five hypotheses:

(H1) Problem (1.1) is 2-regular; i.e., given f, g^D, and g_2^F, there exists a unique solution $p \in H^2(\Omega)$ such that

$$\|p\|_2 \le C\{\|f\|_0 + \|g^D\|_{3/2,\Gamma^D} + \|g_2^F\|_{1/2,\Gamma^F}\},$$

where C depends only on Ω, K, and g_1^F;

(H2) $\nabla \cdot V_h = W_h$;

(H3) $V_h \cdot \nu|_{\partial\Omega} = \Lambda_h$;

(H4) $V_h^N \subset \tilde{V}_h$;

(H5) K is uniformly positive definite in Ω, and $g_1^R \ge 0$.

Sufficient conditions for (H1) can be found in standard references on the theory of elliptic partial differential equations (such as [23] and [20]; e.g., Grisvard [20] gives the result for a two-dimensional, convex domain with a pure Dirichlet or Neumann boundary condition). Note that in general (H1) essentially requires that the portions $\bar{\Gamma}^D$ and $\bar{\Gamma}^F$ of $\partial\Omega$ be separated and so, if the boundary is connected, that one of these sets vanish. The second and third hypotheses hold for the usual spaces defined over triangles, tetrahedra, prisms, and rectangular parallelepipeds.

We need four projection operators and their approximation properties. Let $\mathcal{P}_h$ denote L^2-projection of W onto W_h: for $\varphi \in W$, $\mathcal{P}_h\varphi \in W_h$ is defined by

$$(\mathcal{P}_h\varphi - \varphi, w) = 0, \quad w \in W_h.$$

For $\varphi \in W$,

$$\|\mathcal{P}_h\varphi - \varphi\|_{-s} \le C\|\varphi\|_j\, h^{j+s}, \quad 0 \le s \le l,\ 0 \le j \le l, \tag{3.1}$$

where l is associated with the degree of the polynomials in W_h. Similarly, let $\tilde{\Pi}$ denote the L^2-projection of $\tilde{V}$ onto $\tilde{V}_h$ and $\mathcal{Q}_h$ denote $L^2(\partial\Omega)$-projection onto Λ_h (or more often $L^2(\Gamma^F)$-projection onto Λ_h^F). For $\mathbf{q} \in H^j(\Omega)$ and $\psi \in H^j(\partial\Omega)$,

$$\|\mathbf{q} - \tilde{\Pi}\mathbf{q}\|_{-s} \le C\|\mathbf{q}\|_j\, h^{j+s}, \quad 0 \le s \le k,\ 0 \le j \le k, \tag{3.2}$$

$$\|\mathcal{Q}_h\psi - \psi\|_{-s,\partial\Omega} \le C\|\psi\|_{j,\partial\Omega}\, h^{j+s}, \quad 0 \le s \le m,\ 0 \le j \le m, \tag{3.3}$$

where k and m are associated with the degree of the polynomials in V_h and Λ_h, respectively, and where in (3.3) we can restrict to Γ^F.

Each of the mixed spaces we consider has a projection operator $\Pi : (H^1(\Omega))^d \to V_h$ with the four properties

$$(\nabla\cdot\Pi\mathbf{q}, w) = (\nabla\cdot\mathbf{q}, w), \quad w \in W_h \quad (\text{i.e., } \nabla\cdot\Pi\mathbf{q} = \mathcal{P}_h\nabla\cdot\mathbf{q}), \tag{3.4}$$

$$\|\Pi\mathbf{q} - \mathbf{q}\|_0 \le C\|\mathbf{q}\|_j\, h^j, \quad 1 \le j \le k, \tag{3.5}$$

$$(\Pi\mathbf{q}\cdot\nu, \mu)_e = (\mathbf{q}\cdot\nu, \mu)_e, \quad \mu \in \Lambda_h \quad (\text{i.e., } \Pi\mathbf{q}\cdot\nu = \mathcal{Q}_h\mathbf{q}\cdot\nu), \tag{3.6}$$

where e is any element edge or face. The divergence and normal fluxes are well approximated by (3.1) and (3.3).

Remark. For all our mixed spaces, $l \le k$ and $m = k$. For the RTN and BDFM spaces, $l = k$, and for the BDM and BDDF spaces, $l = k-1$. The CD spaces generalize these spaces on prisms. The lowest-order RTN spaces have $k = l = m = 1$.

Before considering convergence, we show that the solution exists and is both unique and stable.

THEOREM 3.1. *Assume* (H1)–(H5). *If* $(\mathbf{u}_h, \tilde{\mathbf{u}}_h, p_h, \lambda_h)$ *is a solution to* (2.2)*, then*

$$\|\nabla\cdot\mathbf{u}_h\|_0 \le \|f\|_0, \tag{3.7a}$$

$$\begin{aligned} &\|\mathbf{u}_h\|_0 + \|\tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u}_h\cdot\nu\|_{0,\Gamma^F} + \|p_h\|_0 + \|\sqrt{g_1^F}\lambda_h\|_{0,\Gamma^F} + \|\lambda_h\|_{-1/2,\Gamma^F} \\ &\quad \le C\{\|f\|_0 + \|g^D\|_{1/2,\Gamma^D} + \|g_2^F\|_{1/2,\Gamma^F}\}, \end{aligned} \tag{3.7b}$$

where C depends on Ω, $\|K\|_{1,\infty}$, and $\|g_1^F\|_{0,\infty,\Gamma^F}$.

Proof. In (2.2), take $w = \nabla\cdot\mathbf{u}_h \in W_h$ to see (3.7a) and $\tilde{\mathbf{v}} = \mathbf{u}_h \in \tilde{V}_h$ and $\mu = \mathbf{u}_h\cdot\nu \in \Lambda_h^F$ to see that

$$\|\mathbf{u}_h\|_0 \le C\|\tilde{\mathbf{u}}_h\|_0, \tag{3.8}$$

$$\|\mathbf{u}_h\cdot\nu\|_{0,\Gamma^F} \le C\{\|g_2^F\|_{0,\Gamma^F} + \|\sqrt{g_1^F}\lambda_h\|_{0,\Gamma^F}\}. \tag{3.9}$$

Now let $\mathbf{v} = \mathbf{u}_h$, $\tilde{\mathbf{v}} = \tilde{\mathbf{u}}_h$, $w = p_h$, and $\mu = \lambda_h$. Together, the four equations of (2.2) imply that for any $\epsilon > 0$,

$$\begin{aligned}(3.10)\qquad &(K\tilde{\mathbf{u}}_h, \tilde{\mathbf{u}}_h) + (g_1^F \lambda_h, \lambda_h)_{\Gamma^F} \\ &\quad = (f, p_h) - (g^D, \mathbf{u}_h \cdot \nu)_{\Gamma^D} - (g_2^F, \lambda_h)_{\Gamma^F} \\ &\quad \le C\{\|f\|_0^2 + \|g^D\|_{1/2,\Gamma^D}^2 + \|g_2^F\|_{1/2,\Gamma^F}^2\} \\ &\qquad + \epsilon\{\|p_h\|_0^2 + \|\mathbf{u}_h\|_0^2 + \|\nabla\cdot\mathbf{u}_h\|_0^2 + \|\mathbf{u}_h\cdot\nu\|_{0,\Gamma^F}^2 + \|\lambda_h\|_{-1/2,\Gamma^F}^2\},\end{aligned}$$

where we use the following argument to handle Γ^D. Let $Eg^D \in H^1(\Omega)$ denote an extension of g^D such that

$$(3.11)\qquad \|Eg^D\|_1 + \|Eg^D\|_{1/2,\partial\Omega} \le C\|g^D\|_{1/2,\Gamma^D}$$

(first extend onto $\partial\Omega$, and then extend into Ω). Then for any $\mathbf{v} \in V_h$,

$$(3.12)\qquad (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} = (Eg^D, \nabla\cdot\mathbf{v}) + (\nabla Eg^D, \mathbf{v}) - (Eg^D, \mathbf{v}\cdot\nu)_{\Gamma^F}.$$

We use a duality argument to control the pressures. Given $\rho \in L^2(\Omega)$ and $\psi \in H^{1/2}(\Gamma^F)$, solve the problem

$$(3.13a)\qquad -\nabla\cdot K\nabla\varphi = \rho \quad \text{in } \Omega,$$

$$(3.13b)\qquad \varphi = 0 \quad \text{on } \Gamma^D,$$

$$(3.13c)\qquad -K\nabla\varphi\cdot\nu - g_1^F\varphi = \psi \quad \text{on } \Gamma^F.$$

By the regularity assumption (H1),

$$(3.14)\qquad \|\varphi\|_2 \le C\{\|\rho\|_0 + \|\psi\|_{1/2,\Gamma^F}\}.$$

Let $\mathbf{v} = -\Pi K\nabla\varphi \in V_h$. Then (3.4), (3.6), and (3.5) show that $\nabla\cdot\mathbf{v} = \mathcal{P}_h\rho \in W_h$, $\mathbf{v}\cdot\nu = \mathcal{Q}_h(\psi + g_1^F\varphi)$ on Γ^F, and

$$(3.15)\qquad \|\mathbf{v}\|_0 + \|\nabla\cdot\mathbf{v}\|_0 + \|\mathbf{v}\cdot\nu\|_{0,\Gamma^F} \le C\{\|\rho\|_0 + \|\psi\|_{1/2,\Gamma^F}\}.$$

Now (2.2b) implies that

$$\begin{aligned}(p_h, \rho) - (\lambda_h, \psi)_{\Gamma^F} &= (\tilde{\mathbf{u}}_h, \mathbf{v}) + (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} + (\lambda_h, g_1^F\varphi)_{\Gamma^F} \\ &\le C\{\|\tilde{\mathbf{u}}_h\|_0 + \|g^D\|_{1/2,\Gamma^D} + \|\sqrt{g_1^F}\lambda_h\|_{0,\Gamma^F}\}\{\|\rho\|_0 + \|\psi\|_{1/2,\Gamma^F}\},\end{aligned}$$

using the argument of (3.11)–(3.12) above. Therefore, first with $\rho = p_h$ and $\psi = 0$ to obtain an estimate of $\|p_h\|_0$ and then with $\rho = 0$ and a supremum on ψ having unit norm to obtain an estimate of $\|\lambda_h\|_{-1/2,\Gamma^F}$,

$$(3.16)\qquad \|p_h\|_0 + \|\lambda_h\|_{-1/2,\Gamma^F} \le C\{\|\tilde{\mathbf{u}}_h\|_0 + \|g^D\|_{1/2,\Gamma^D} + \|\sqrt{g_1^F}\lambda_h\|_{0,\Gamma^F}\}.$$

For ϵ small enough, the theorem follows from (3.10), (3.7a), (3.8)–(3.9), and (3.16). ∎

Since (2.2) is a finite dimensional, square, linear system, uniqueness is equivalent to existence. To see uniqueness, let $f = g^D = g_2^F = 0$.

COROLLARY 3.2. *Assume* (H1)–(H5). *There exists a unique solution to* (2.2).

The next theorem expresses the error in approximating (1.3) by (2.2).

THEOREM 3.3. *Assume* (H1)–(H5). *There exists a constant* C*, independent of* h *and dependent on* Ω*,* p*,* $\mathbf{u}$*,* $\|K\|_{0,\infty}$*, and* $\|g_1^F\|_{0,\infty,\Gamma^F}$ *such that*

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_0 + \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_0 + \|\sqrt{g_1^F}(\lambda-\lambda_h)\|_{0,\Gamma^F} \\
&\quad \le C\{\|\mathbf{u}-\Pi\mathbf{u}\|_0 + \|\tilde{\mathbf{u}}-\tilde{\Pi}\tilde{\mathbf{u}}\|_0 + \|\lambda-\mathcal{Q}_h\lambda\|_{0,\Gamma^F}\} \\
&\quad \le Ch^j, \quad 1 \le j \le \min(k,m),
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{-s} = \|\nabla\cdot(\mathbf{u}-\Pi\mathbf{u})\|_{-s} \\
&\quad \le Ch^{j+s}, \quad 0 \le s \le l,\ 0 \le j \le l,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\|(\mathbf{u}-\mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \le C\{\|\sqrt{g_1^F}(\lambda-\lambda_h)\|_{0,\Gamma^F} + \|(\mathbf{u}-\Pi\mathbf{u})\cdot\nu\|_{0,\Gamma^F}\} \\
&\quad \le Ch^j, \quad 0 \le j \le \min(k,m).
\end{aligned} \tag{3.19}$$

Moreover, if $0 \le s \le \min(k,l,m)-1$*,* Ω *is* $(s+2)$ *regular, and* C *depends also on* $\|K\|_{s+1,\infty}$ *and* $\|g_1^F\|_{s+1,\infty,\Gamma^F}$*, then for any* $0 \le j \le \min(k,l,m)$*,*

$$\begin{aligned}
&\|\mathcal{P}_h p - p_h\|_{-s} + \|\mathcal{Q}_h\lambda-\lambda_h\|_{-s-1/2,\Gamma^F} \\
&\quad \le C\{\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u}-\mathbf{u}_h\|_0 + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 + \|(\mathbf{u}-\mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \\
&\qquad + \|\sqrt{g_1^F}(\lambda-\lambda_h)\|_{0,\Gamma^F} + \|\mathcal{Q}_h\lambda-\lambda\|_{0,\Gamma^F}\}h^{s+1} \le Ch^{j+s+1},
\end{aligned} \tag{3.20}$$

$$\|p-p_h\|_{-s} \le Ch^{j+s}, \tag{3.21}$$

$$\|\lambda-\lambda_h\|_{-s-1/2,\Gamma^F} \le Ch^{j+s+1/2}, \tag{3.22}$$

$$\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{-s} + \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{-s} \\
&\quad \le C\{[\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u}-\mathbf{u}_h\|_0]h^s \\
&\qquad + [\|\mathcal{P}_h p - p_h\|_0 + \|\mathcal{Q}_h\lambda-\lambda_h\|_{0,\Gamma^F}]h^{s-1} \\
&\qquad + \|\mathcal{P}_h p - p_h\|_{-s+1} + \|\mathcal{Q}_h\lambda-\lambda_h\|_{-s+1/2,\Gamma^F}\} \le Ch^{j+s}.
\end{aligned} \tag{3.23}$$

Remark. This theorem implies optimal order convergence for $\|p-p_h\|_{-s}$, $\|\mathbf{u}-\mathbf{u}_h\|_{-s}$, $\|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{-s}$, and $\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{-s}$. Moreover, $\mathcal{P}_h p$ and p_h are superclose in the H^{-s}-norm.

Proof. The theorem can be shown using techniques that have been developed for the analysis of the usual mixed method. For example, using the four projection operators, the proof is a relatively simple extension of that presented by Douglas and Roberts [15]. Chen [11], [12] also analyzed a similar expanded mixed method. We present briefly the proof here for completeness and for later analysis of the finite difference scheme (see also [1], where a somewhat more general expanded mixed finite element method is studied).

From (1.3) (with (1.3b) and (1.3d) extended to Γ^F) and (2.2) we get the error equations

$$(\nabla\cdot(\Pi\mathbf{u}-\mathbf{u}_h), w) = 0, \qquad w \in W_h, \tag{3.24a}$$

$$(\tilde{\Pi}\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h, \mathbf{v}) = (\mathcal{P}_h p - p_h, \nabla\cdot\mathbf{v}) - (\mathcal{Q}_h\lambda-\lambda_h, \mathbf{v}\cdot\nu)_{\Gamma^F}, \qquad \mathbf{v} \in V_h, \tag{3.24b}$$

$$(\mathbf{u}-\mathbf{u}_h, \tilde{\mathbf{v}}) = (K(\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h), \tilde{\mathbf{v}}), \qquad \tilde{\mathbf{v}} \in V_h, \tag{3.24c}$$

$$((\Pi\mathbf{u}-\mathbf{u}_h)\cdot\nu, \mu)_{\Gamma^F} = (g_1^F(\lambda-\lambda_h), \mu)_{\Gamma^F}, \qquad \mu \in \Lambda_h^F. \tag{3.24d}$$

Take $w = \mathcal{P}_h p - p_h$, $\mathbf{v} = \Pi\mathbf{u} - \mathbf{u}_h$, $\tilde{\mathbf{v}} = \tilde{\Pi}\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h$, and $\mu = \mathcal{Q}_h\lambda - \lambda_h$ to obtain the error estimate

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\sqrt{g_1^F}(\lambda - \lambda_h)\|_{0,\Gamma^F} \le C\{\|\mathbf{u} - \Pi\mathbf{u}\|_0 + \|\tilde{\mathbf{u}} - \tilde{\Pi}\tilde{\mathbf{u}}\|_0 + \|\lambda - \mathcal{Q}_h\lambda\|_{0,\Gamma^F}\},$$

and take $\tilde{\mathbf{v}} = \Pi\mathbf{u} - \mathbf{u}_h$ to see that

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \le C\{\|\mathbf{u} - \Pi\mathbf{u}\|_0 + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0\}.$$

Together with the approximation error estimates, these give (3.17). Now take $w = \nabla\cdot(\Pi\mathbf{u} - \mathbf{u}_h)$ to see that in fact

$$\nabla\cdot(\Pi\mathbf{u} - \mathbf{u}_h) = 0; \tag{3.25}$$

thus, (3.18) follows. Finally, take $\mu = (\Pi\mathbf{u} - \mathbf{u}_h)\cdot\nu$ to see that

$$\|(\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \le C\{\|\sqrt{g_1^F}(\lambda - \lambda_h)\|_{0,\Gamma^F} + \|(\mathbf{u} - \Pi\mathbf{u})\cdot\nu\|_{0,\Gamma^F}\},$$

giving (3.19).

The estimates (3.20)–(3.22) of the pressure errors are more involved. Let $\rho \in H^s(\Omega)$, $\psi \in H^{s+1/2}(\Gamma^F)$, and $\varphi \in H^{s+2}(\Omega)$ solve (3.13), and set $\mathbf{v} = -\Pi K\nabla\varphi \in V_h$. Then $\nabla\cdot\mathbf{v} = \mathcal{P}_h\rho$, $\mathbf{v}\cdot\nu = \mathcal{Q}_h(g_1^F\varphi + \psi)$ on Γ^F, and (3.14)–(3.15) hold. In fact, by the $(s+2)$-regularity,

$$\|\varphi\|_{s+2} \le C\{\|\rho\|_s + \|\psi\|_{s+1/2,\Gamma^F}\}. \tag{3.26}$$

Now (3.24b) gives

$$\begin{aligned}
&(\mathcal{P}_h p - p_h, \rho) - (\mathcal{Q}_h\lambda - \lambda_h, \psi)_{\Gamma^F} \\
&\quad = -(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \Pi K\nabla\varphi - K\nabla\varphi) - (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \nabla\varphi - \Pi\nabla\varphi) \\
&\qquad -(K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \Pi\nabla\varphi) + (\mathcal{Q}_h\lambda - \lambda_h, g_1^F\varphi)_{\Gamma^F},
\end{aligned} \tag{3.27}$$

and, using (3.24c), integration by parts, (3.24a), and (3.24d), the last two terms are

$$\begin{aligned}
&-(K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \Pi\nabla\varphi) + (\mathcal{Q}_h\lambda - \lambda_h, g_1^F\varphi)_{\Gamma^F} \\
&\quad = -(\mathbf{u} - \mathbf{u}_h, \Pi\nabla\varphi - \nabla\varphi) - (\mathbf{u} - \mathbf{u}_h, \nabla\varphi) + (g_1^F(\mathcal{Q}_h\lambda - \lambda_h), \varphi)_{\Gamma^F} \\
&\quad = -(\mathbf{u} - \mathbf{u}_h, \Pi\nabla\varphi - \nabla\varphi) + (\nabla\cdot(\mathbf{u} - \mathbf{u}_h), \varphi) \\
&\qquad -((\mathbf{u} - \mathbf{u}_h)\cdot\nu, \varphi)_{\Gamma^F} + (g_1^F(\mathcal{Q}_h\lambda - \lambda_h), \varphi)_{\Gamma^F} \\
&\quad = -(\mathbf{u} - \mathbf{u}_h, \Pi\nabla\varphi - \nabla\varphi) + (\nabla\cdot(\mathbf{u} - \mathbf{u}_h), \varphi - \mathcal{P}_h\varphi) \\
&\qquad -((\mathbf{u} - \mathbf{u}_h)\cdot\nu, \varphi - \mathcal{Q}_h\varphi)_{\Gamma^F} + (g_1^F(\lambda - \lambda_h), \varphi - \mathcal{Q}_h\varphi)_{\Gamma^F} \\
&\qquad +(\mathcal{Q}_h\lambda - \lambda, g_1^F\varphi - \mathcal{Q}_h(g_1^F\varphi))_{\Gamma^F}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\mathcal{P}_h p - p_h, \rho) - (\mathcal{Q}_h\lambda - \lambda_h, \psi)_{\Gamma^F} \\
&\quad \le C\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 + \|(\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \\
&\qquad\quad +\|\sqrt{g_1^F}(\lambda - \lambda_h)\|_{0,\Gamma^F} + \|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^F}\} \\
&\qquad \cdot\{\|K\nabla\varphi - \Pi K\nabla\varphi\|_0 + \|\nabla\varphi - \Pi\nabla\varphi\|_0 \\
&\qquad\quad +\|\varphi - \mathcal{P}_h\varphi\|_0 + \|\varphi - \mathcal{Q}_h\varphi\|_{0,\Gamma^F} + \|g_1^F\varphi - \mathcal{Q}_h(g_1^F\varphi)\|_{0,\Gamma^F}\} \\
&\quad \le C\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 + \|(\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \\
&\qquad +\|\sqrt{g_1^F}(\lambda - \lambda_h)\|_{0,\Gamma^F} + \|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^F}\}\|\varphi\|_{s+2}h^{s+1}.
\end{aligned} \tag{3.28}$$

Take $\psi = 0$ and a supremum on $\rho \in H^s(\Omega)$ having unit norm to obtain an estimate of $\|\mathcal{P}_h p - p_h\|_{-s}$, and then take $\rho = 0$ and a supremum on $\psi \in H^{s+1/2}(\Gamma^F)$ having unit norm to obtain an estimate of $\|\mathcal{Q}_h\lambda - \lambda_h\|_{-s-1/2,\Gamma^F}$.

We finally obtain the estimates (3.23). Take $\psi \in (H^s(\Omega))^d$ and $\mathbf{v} = \Pi\psi$ in (3.24b) to see that

$$\begin{aligned}
(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \psi) &= (\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \psi - \Pi\psi) + (\mathcal{P}_h p - p_h, \nabla\cdot(\Pi\psi - \psi)) \\
&\quad - (\mathcal{Q}_h\lambda - \lambda_h, (\Pi\psi - \psi)\cdot\nu)_{\Gamma^F} \\
&\quad + (\mathcal{P}_h p - p_h, \nabla\cdot\psi) - (\mathcal{Q}_h\lambda - \lambda_h, \psi\cdot\nu)_{\Gamma^F} \\
&\le \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0\, \|\psi - \Pi\psi\|_0 + \|\mathcal{P}_h p - p_h\|_0\, \|\nabla\cdot(\Pi\psi - \psi)\|_0 \\
&\quad + \|\mathcal{Q}_h\lambda - \lambda_h\|_{0,\Gamma^F}\, \|(\Pi\psi - \psi)\cdot\nu\|_{0,\Gamma^F} \\
&\quad + [\|\mathcal{P}_h p - p_h\|_{-s+1} + \|\mathcal{Q}_h\lambda - \lambda_h\|_{-s+1/2,\Gamma^F}]\|\psi\|_s \\
&\le C\{[\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0\, h + \|\mathcal{P}_h p - p_h\|_0 + \|\mathcal{Q}_h\lambda - \lambda_h\|_{0,\Gamma^F}]h^{s-1} \\
&\quad + \|\mathcal{P}_h p - p_h\|_{-s+1} + \|\mathcal{Q}_h\lambda - \lambda_h\|_{-s+1/2,\Gamma^F}\}\|\psi\|_s.
\end{aligned}$$

This gives the second part of (3.23); the rest is similar, starting from (3.24c). ∎

4. A cell-centered finite difference method. In this section we derive a finite difference stencil for the pressure in the case of the lowest-order RTN spaces on rectangles [28], [26]. Recall that on an element $E \in \mathcal{T}_h$,

$$\begin{aligned}
V_h(E) &= \{(\alpha_1 x_1 + \beta_1, \alpha_2 x_2 + \beta_2, \alpha_3 x_3 + \beta_3)^T : \alpha_i, \beta_i \in \mathbf{R}\}, \\
W_h(E) &= \{\alpha : \alpha \in \mathbf{R}\},
\end{aligned}$$

and on an edge or face e,

$$\Lambda_h(e) = \{\alpha : \alpha \in \mathbf{R}\},$$

where the last component in V_h should be deleted if $d = 2$. We use the standard nodal basis, where for V_h and Λ_h the nodes are at the midpoints of the edges or faces of the elements and for W_h the nodes are at the midpoints of the elements (cell centers). We choose $\tilde{V}_h = V_h$ in (2.2).

Our goal is to express approximately $\mathbf{u}_h$ and $\tilde{\mathbf{u}}_h$ in terms of p_h and λ_h from (2.2b)–(2.2d); then (2.2a) will give us an equation for the pressures. To do this, we use numerical quadrature rules for evaluating some of the integrals in (2.2). The approximate problem is to solve for $\mathbf{u}_h \in V_h$, $\tilde{\mathbf{u}}_h \in V_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h^F$ such that

(4.1a) $\quad (\nabla\cdot\mathbf{u}_h, w) = (f, w), \qquad w \in W_h,$

(4.1b) $\quad (\tilde{\mathbf{u}}_h, \mathbf{v})_{\mathbf{TM}} = (p_h, \nabla\cdot\mathbf{v}) - (g^D, \mathbf{v}\cdot\nu)_{\Gamma^D} - (\lambda_h, \mathbf{v}\cdot\nu)_{\Gamma^F}, \qquad \mathbf{v} \in V_h,$

(4.1c) $\quad (\mathbf{u}_h, \tilde{\mathbf{v}})_{\mathbf{TM}} = (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\mathbf{T}}, \qquad \tilde{\mathbf{v}} \in V_h,$

(4.1d) $\quad (\mathbf{u}_h\cdot\nu, \mu)_{\Gamma^F} = (g_2^F + g_1^F\lambda_h, \mu)_{\Gamma^F}, \qquad \mu \in \Lambda_h^F.$

In this paper, $(\cdot,\cdot)_{\mathbf{M}}$ and $(\cdot,\cdot)_{\mathbf{T}}$ mean an application of the midpoint and the trapezoidal rule, respectively (in each coordinate direction), and for $\mathbf{v}, \mathbf{q} \in \mathbf{R}^d$,

$$(\mathbf{v}, \mathbf{q})_{\mathbf{TM}} = \begin{cases} (v_1, q_1)_{\mathbf{T}\times\mathbf{M}} + (v_2, q_2)_{\mathbf{M}\times\mathbf{T}} & \text{if } d = 2, \\ (v_1, q_1)_{\mathbf{T}\times\mathbf{M}\times\mathbf{M}} + (v_2, q_2)_{\mathbf{M}\times\mathbf{T}\times\mathbf{M}} + (v_3, q_3)_{\mathbf{M}\times\mathbf{M}\times\mathbf{T}} & \text{if } d = 3. \end{cases}$$

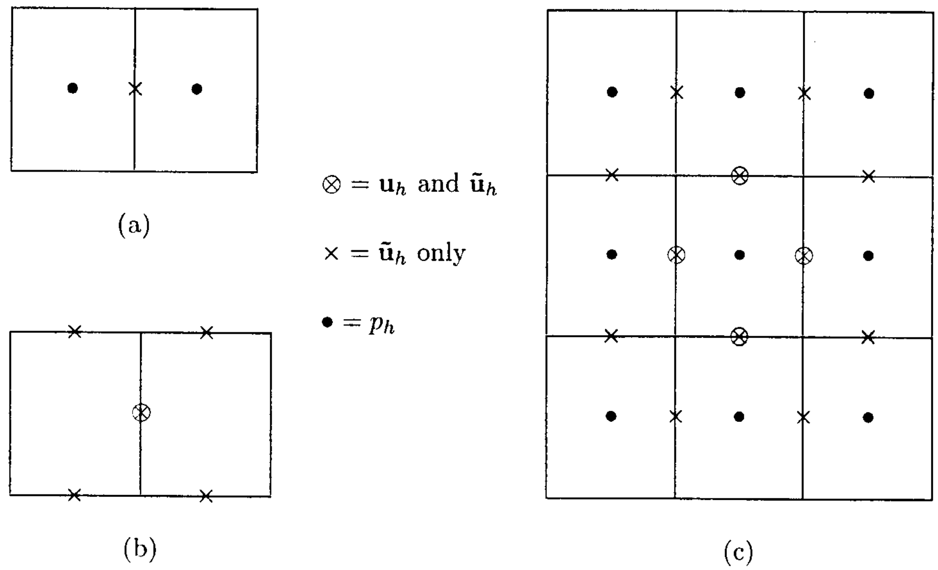


FIG. 4.1. *Nodal points for the standard basis functions* ($d = 2$). (a) *Stencil for the dependence of* $\tilde{\mathbf{u}}_h$ *on* p_h. (b) *Stencil for the dependence of* $\mathbf{u}_h$ *on* $\tilde{\mathbf{u}}_h$. (c) *Stencil for the pressure* p_h.

In other words, for computing an integral of the ith component of the vectors, $i = 1, 2(, 3)$, we apply the trapezoidal rule in the ith direction and the midpoint rule in the other directions. This choice of quadrature rules is compatible with the nodal basis functions for V_h; it gives diagonal coefficient matrices for $\tilde{\mathbf{u}}_h$ in (4.1b) and for $\mathbf{u}_h$ in (4.1c). This technique is sometimes referred to as a lumped mass approximation. It happens that for $\mathbf{v}, \mathbf{q} \in V_h$,

$$(\mathbf{v}, \mathbf{q})_{\mathbf{TM}} = (\mathbf{v}, \mathbf{q})_{\mathbf{T}}.$$

Also, the matrix given by $(\mathbf{v}_i, \mathbf{v}_j)_{\mathbf{TM}}$, where i and j run over a nodal basis of V_h, is diagonal, independently of whether K is diagonal or not. This explains why the expanded mixed method was used.

With the aid of Fig. 4.1, we now describe the meaning of (4.1). Equation (4.1b) expresses the normal component of $\tilde{\mathbf{u}}_h$ at any nodal point as a difference of the pressure at the midpoints of the two adjacent elements (see Fig. 4.1 (a)) or, near the boundary, as a difference of a pressure and either a Lagrange multiplier pressure or a Dirichlet pressure. This corresponds to a finite difference approximation of the equation $\tilde{\mathbf{u}} = -\nabla p$.

Equation (4.1c) expresses the normal component of $\mathbf{u}_h$ at any node by the normal components of $\tilde{\mathbf{u}}_h$ at the nodes of the adjacent elements as in Fig. 4.1 (b). Note that $\mathbf{u}_h$ does not depend on the components of $\tilde{\mathbf{u}}_h$ on the far left and right edges. Thus we get a relatively compact finite difference approximation of the equation $\mathbf{u} = K\tilde{\mathbf{u}}$.

Finally, substituting (4.1b)–(4.1d) in (4.1a), we obtain a finite difference stencil for the pressure, an approximation of the elliptic equation $-\nabla \cdot K \nabla p = f$ (see Fig. 4.1 (c)). We get a 9-point stencil in two dimensions and a 19-point stencil in three dimensions.

If K is a diagonal tensor, the stencil is reduced to five or seven points, and we recover the scheme of Russell and Wheeler [29], except that K is evaluated at somewhat different points. The difference is that in [29] the **TM** rule is used for the last integral in (4.1c); whereas, here we use the trapezoidal rule to maintain symmetry in the case that K is not diagonal.

If a uniform mesh and a constant K are used, we obtain a standard finite difference procedure. In the strict interior

$$\begin{aligned} f_{ij}\, h^2 &= 2(K_{11}+K_{22})p_{h,ij} - K_{11}(p_{h,i-1,j}+p_{h,i+1,j}) - K_{22}(p_{h,i,j-1}+p_{h,i,j+1}) \\ &\quad + \tfrac{1}{2}K_{12}(p_{h,i+1,j-1}+p_{h,i-1,j+1}-p_{h,i-1,j-1}-p_{h,i+1,j+1}), \end{aligned}$$

and the local truncation error is $O(h^2)$, except near the boundary. Many other $O(h^2)$ finite difference schemes can be constructed that vary mainly in how K is treated and the second-order derivatives are approximated (see standard texts of finite difference methods, e.g., [31] or [24]). We show in the next section that our scheme has global convergence properties. Moreover, it is symmetric and locally conservative, and it has a compact 9- or 19-point stencil and connections to mixed finite element methods. Moreover, it can be extended easily to nonrectangular grids (see [1], [3], and [2]).

5. An error analysis of the finite difference method. For either quadrature rule $\mathbf{Q}$, let χ_S denote the characteristic function of the set S and extend the definition of the discrete inner products to

$$(w,w)_{\mathbf{Q},S} = (w, w\chi_S)_{\mathbf{Q}}.$$

For $w \in W \cap C^0(\bar\Omega)$, $\mathbf{v} \in \tilde V \cap (C^0(\bar\Omega))^d$, and h implicitly fixed, let

$$\|w\|^2_{\mathbf{M},S} = (w,w)_{\mathbf{M},S}, \quad \|\mathbf{v}\|^2_{\mathbf{TM},S} = (\mathbf{v},\mathbf{v})_{\mathbf{TM},S}, \quad \text{and} \quad \|\mathbf{v}\|^2_{\mathbf{T},S} = (\mathbf{v},\mathbf{v})_{\mathbf{T},S},$$

where again we omit S if $S = \Omega$; these can also be defined on W_h or $\tilde V_h$, where they are norms. Clearly for $\mathbf{v} \in V_h$,

$$\frac{1}{C}\|\mathbf{v}\|_0 \le \|\mathbf{v}\|_{\mathbf{TM}} = \|\mathbf{v}\|_{\mathbf{T}} \le C\|\mathbf{v}\|_0; \tag{5.1}$$

that is, these three norms are equivalent.

THEOREM 5.1. *Assume* (H1)–(H5). *There exists a unique solution to* (4.1). *If* $(\mathbf{u}_h, \tilde{\mathbf{u}}_h, p_h, \lambda_h)$ *is the solution to* (4.1), *then* (3.7a) *and* (3.7b) *hold.*

The proof is analogous to that given for Theorem 3.1 and its corollary, using (5.1).

Let us proceed with our error analysis of the finite difference scheme. We present proofs only for the case $d = 2$; the generalization to $d = 3$ is straightforward.

We use relatively standard cell-centered finite difference notation. Let the grid points be denoted by

$$(x_{i+1/2}, y_{j+1/2}), \quad i = 0,\dots,N_x,\ j = 0,\dots,N_y,$$

and then define

$$\begin{aligned} x_i &= \tfrac{1}{2}(x_{i+1/2}+x_{i-1/2}), \quad && i = 1, N_x, \\ y_j &= \tfrac{1}{2}(y_{j+1/2}+y_{j-1/2}), \quad && j = 1, N_y, \\ h_i^x &= x_{i+1/2}-x_{i-1/2}, && i = 1, N_x, \\ h_j^y &= y_{j+1/2}-y_{j-1/2}, && j = 1, N_y, \\ h &= \max_{i,j}(h_i^x, h_j^y). \end{aligned}$$

We write $\mathbf{q} = (q^x, q^y)$ for $\mathbf{q} \in \mathbf{R}^2$, and for any function $g(x, y)$, let g_{ij} denote $g(x_i, y_j)$, let $g_{i+1/2,j}$ denote $g(x_{i+1/2}, y_j)$, etc.

Before stating our results, we need the following definition.

DEFINITION 5.2. *An asymptotic family of grids is said to be generated by a C^2 map if each grid is the image by a fixed map of a grid that is uniform in each coordinate direction, where each component of the map is strictly monotone and in $C^2(\bar{\Omega})$.*

Denote this map by $\mathbf{F}(x, y) = (F^x(x), F^y(y))$ and note that in this case

$$h^x_{i+1} - h^x_i = F^x_{i+3/2} - 2F^x_{i+1/2} + F^x_{i-1/2} = \frac{d^2 F^x(\bar{x})}{dx^2} (\bar{h}^x)^2,$$

where $\bar{x}$ is between $x_{i-1/2}$ and $x_{i+3/2}$ and $\bar{h}^x$ is the uniform grid spacing. This, together with the smoothness of $\mathbf{F}$, implies

$$|h^x_{i+1} - h^x_i| \le Ch^2 \quad \text{and, similarly,} \quad |h^y_{j+1} - h^y_j| \le Ch^2. \tag{5.2}$$

5.1. An auxiliary estimate.

LEMMA 5.3. *Assume that $p \in C^{3,1}(\bar{\Omega})$, $\mathbf{u} \in \left(C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega)\right)^d$, and $K \in \left(C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega)\right)^{d\times d}$. There exist $\mathbf{U} \in V_h$, $\tilde{\mathbf{U}} \in V_h$, $P \in W_h$, and $\lambda^* \in \Lambda^F_h$ such that*

$$(\tilde{\mathbf{U}}, \mathbf{v})_{\mathbf{TM}} = (P, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v} \cdot \nu)_{\Gamma^D} - (\lambda^*, \mathbf{v} \cdot \nu)_{\Gamma^F}, \quad \mathbf{v} \in V_h, \tag{5.1.1a}$$

$$(\mathbf{U}, \tilde{\mathbf{v}})_{\mathbf{TM}} = (K\tilde{\mathbf{U}}, \tilde{\mathbf{v}})_{\mathbf{T}}, \quad \tilde{\mathbf{v}} \in V_h, \tag{5.1.1b}$$

and there exists a constant C independent of h such that, for all i, j,

$$|P_{ij} - p_{ij}| \le Ch^2, \tag{5.1.2}$$

$$|\tilde{U}^x_{i+1/2,j} - \tilde{u}^x_{i+1/2,j}| + |\tilde{U}^y_{i,j+1/2} - \tilde{u}^y_{i,j+1/2}| \le Ch^{\tilde{r}}, \tag{5.1.3}$$

$$|U^x_{i+1/2,j} - u^x_{i+1/2,j}| + |U^y_{i,j+1/2} - u^y_{i,j+1/2}| \le Ch^r, \tag{5.1.4}$$

$$|\lambda^* - \mathcal{Q}_h \lambda| \le Ch^2, \tag{5.1.5}$$

where λ^* is given by (5.1.6) *below, $\tilde{r} = 2$ for all points not on Γ^D and otherwise $\tilde{r} = 1$, and $r = 1$ in general but $r = 2$ in special circumstances. If K is diagonal, then $r = 2$ for all points not on Γ^D. If the grids are generated by a C^2 map, then $r = 2$ for points strictly in the interior of Ω that lie on an edge or face e such that $\bar{e} \cap \bar{\Gamma}^D = \emptyset$.*

Proof. We apply a result due to Weiser and Wheeler [32, Lemma 4.1 and appendix] to $(\tilde{\mathbf{u}}, p)$ satisfying the elliptic problem

$$\begin{aligned} \nabla \cdot \tilde{\mathbf{u}} &= F \equiv \nabla \cdot \nabla p && \text{in } \Omega, \\ \tilde{\mathbf{u}} &= -\nabla p && \text{in } \Omega, \\ p &= G \equiv p|_{\partial\Omega} && \text{on } \partial\Omega. \end{aligned}$$

This gives a P satisfying (5.1.2), and, through (5.1.1a), $\tilde{\mathbf{U}}$ satisfies (5.1.3) for $\tilde{r} = 2$ strictly in the interior of Ω.

We define λ^* by the requirement that on the boundary

$$\tilde{U}^x_{i+1/2,j} = \tilde{u}^x_{i+1/2,j} \quad \text{and} \quad \tilde{U}^y_{i,j+1/2} = \tilde{u}^y_{i,j+1/2}, \tag{5.1.6}$$

and then (5.1.3) holds for $\tilde{r} = 2$ on Γ^F as well. Moreover, (5.1.5) holds easily by Taylor's theorem and (5.1.2). Finally, $\tilde{r} = 1$ on Γ^D.

It remains to show (5.1.4). Choosing $\tilde{\mathbf{v}}$ in (5.1.1b) to be the basis function associated with node $(i+1/2, j)$, we have in the strict interior of Ω

$$\begin{aligned} U^x_{i+1/2,j} &= \tfrac{1}{2}\left[(K_{11})_{i+1/2,j-1/2} + (K_{11})_{i+1/2,j+1/2}\right]\tilde{U}^x_{i+1/2,j} \\ &\quad + \frac{1}{2(h^x_i + h^x_{i+1})}\Big\{\Big[(K_{12})_{i+1/2,j-1/2}\tilde{U}^y_{i+1,j-1/2} \\ &\qquad + (K_{12})_{i+1/2,j+1/2}\tilde{U}^y_{i+1,j+1/2}\Big] h^x_{i+1} \\ &\qquad + \Big[(K_{12})_{i+1/2,j-1/2}\tilde{U}^y_{i,j-1/2} + (K_{12})_{i+1/2,j+1/2}\tilde{U}^y_{i,j+1/2}\Big] h^x_i\Big\}, \\ &\qquad\qquad i = 1, \dots, N_x - 1, \end{aligned} \tag{5.1.7a}$$

and on $\partial\Omega$

$$\begin{aligned} U^x_{i+1/2,j} &= \tfrac{1}{2}\Big\{\left[(K_{11})_{i+1/2,j-1/2} + (K_{11})_{i+1/2,j+1/2}\right]\tilde{U}^x_{i+1/2,j} \\ &\qquad + \Big[(K_{12})_{i+1/2,j-1/2}\tilde{U}^y_{\hat{\imath}+1,j-1/2} + (K_{12})_{i+1/2,j+1/2}\tilde{U}^y_{\hat{\imath}+1,j+1/2}\Big]\Big\}, \\ &\qquad i = \hat{\imath} = 0 \text{ or } i = \hat{\imath} + 1 = N_x. \end{aligned} \tag{5.1.7b}$$

Since $\mathbf{u} = K\tilde{\mathbf{u}}$, Taylor's theorem gives

$$\begin{aligned} u^x_{i+1/2,j} &= \tfrac{1}{2}\left[(K_{11})_{i+1/2,j-1/2} + (K_{11})_{i+1/2,j+1/2}\right]\tilde{u}^x_{i+1/2,j} \\ &\quad + \frac{1}{2(h^x_i + h^x_{i+1})}\Big\{\Big[(K_{12})_{i+1/2,j-1/2}\tilde{u}^y_{i+1,j-1/2} \\ &\qquad + (K_{12})_{i+1/2,j+1/2}\tilde{u}^y_{i+1,j+1/2}\Big] h^x_i \\ &\qquad + \Big[(K_{12})_{i+1/2,j-1/2}\tilde{u}^y_{i,j-1/2} + (K_{12})_{i+1/2,j+1/2}\tilde{u}^y_{i,j+1/2}\Big] h^x_{i+1}\Big\} \\ &\quad + O(h^2), \quad i = 1, \dots, N_x - 1, \end{aligned} \tag{5.1.8a}$$

$$\begin{aligned} u^x_{i+1/2,j} &= \tfrac{1}{2}\Big\{\left[(K_{11})_{i+1/2,j-1/2} + (K_{11})_{i+1/2,j+1/2}\right]\tilde{u}^x_{i+1/2,j} \\ &\qquad + \Big[(K_{12})_{i+1/2,j-1/2}\tilde{u}^y_{\hat{\imath}+1,j-1/2} + (K_{12})_{i+1/2,j+1/2}\tilde{u}^y_{\hat{\imath}+1,j+1/2}\Big]\Big\} \\ &\quad + O(h), \quad i = \hat{\imath} = 0 \text{ or } i = \hat{\imath} + 1 = N_x, \end{aligned} \tag{5.1.8b}$$

where the last $O(h)$ becomes $O(h^2)$ if K is diagonal. The coefficients in (5.1.7a) and (5.1.8a) differ only in the weights h^x_i and h^x_{i+1}.

If K is diagonal,

$$|U^x_{i+1/2,j} - u^x_{i+1/2,j}| \le C|\tilde{U}^x_{i+1/2,j} - \tilde{u}^x_{i+1/2,j}| + O(h^2),$$

and so (5.1.4) follows in this case.

If K is a full tensor, we add and subtract h^x_{i+1} and h^x_i to the weights of the second and third term on the right side of (5.1.8a), respectively. Subtracting from (5.1.7a), we have

$$
\begin{aligned}
&|U^x_{i+1/2,j} - u^x_{i+1/2,j}| \\
&\quad \le C\{h^2 + |\tilde U^x_{i+1/2,j} - \tilde u^x_{i+1/2,j}| \\
&\qquad + |\tilde U^y_{i+1,j-1/2} - \tilde u^y_{i+1,j-1/2}| + |\tilde U^y_{i,j-1/2} - \tilde U^y_{i,j-1/2}| \\
&\qquad + |\tilde U^y_{i+1,j+1/2} - \tilde u^y_{i+1,j+1/2}| + |\tilde U^y_{i,j+1/2} - \tilde U^y_{i,j+1/2}|\} \\
&\qquad + \frac{1}{4}\left\{\left|(K_{12})_{i+1/2,j-1/2}\frac{\partial \tilde u^y}{\partial x}(\bar x', y_{j-1/2})\right|\right. \\
&\qquad\quad \left. + \left|(K_{12})_{i+1/2,j+1/2}\frac{\partial \tilde u^y}{\partial x}(\bar x'', y_{j+1/2})\right|\right\} |h^x_{i+1} - h^x_i|, \quad i = 1, \dots, N_x - 1,
\end{aligned} \tag{5.1.9}
$$

where $\bar x'$ and $\bar x''$ are points between x_i and x_{i+1}. We now combine estimates (5.1.9), (5.2), and (5.1.3) to conclude (5.1.4) in the strict interior of Ω with $r = 2$ if the grids are generated by a C^2 map and $\bar e \cap \bar\Gamma^D = \emptyset$, and otherwise $r = 1$. On $\partial\Omega$, (5.1.7b) and (5.1.8b) imply the estimate (5.1.4) only for $r = 1$.

In a similar way we estimate $|U^y_{i,j+1/2} - u^y_{i,j+1/2}|$. ∎

COROLLARY 5.4. *For the* $\mathbf{U} \in V_h$, $\tilde{\mathbf{U}} \in V_h$, $P \in W_h$, *and* $\lambda^* \in \Lambda^F_h$ *in Lemma* 5.3, *there exists a constant* C, *independent of* h, *such that*

$$\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}\|_{\mathbf{TM}} \le Ch^{\tilde r} \quad \text{and} \quad \|\mathbf{U} - \mathbf{u}\|_{\mathbf{TM}} \le Ch^r,$$

where $\tilde r = 2$ *if* $\Gamma^D = \emptyset$ *and* $\tilde r = 3/2$ *otherwise and where* $r = 2$ *if* K *is diagonal and* $\Gamma^D = \emptyset$, $r = 3/2$ *if* K *is diagonal or the grids are generated by a* C^2 *map, and* $r = 1$ *otherwise.*

The proof is immediate, since $\partial\Omega$ is a set of dimension one less than that of Ω.

5.2. Estimates for the vectors u and ũ.

LEMMA 5.5. *If* $0 \le \alpha \in W^{1,\infty}(\Gamma^R)$, $\varphi \in H^1(\Gamma^R)$, *and* $\mu \in \Lambda^R_h$, *then*

$$(\alpha(\mathcal{Q}_h\varphi - \varphi), \mu)_{\Gamma^R} \le C\|\mathcal{Q}_h\varphi - \varphi\|_{0,\Gamma^R}\,\|\mu\|_{0,\Gamma^R}\,h.$$

Proof. It is well known that the difference of two weighted L^2-projections are superclose. In our case, for any $\chi_1, \chi_2 \in \Lambda^R_h$, $\chi_1\chi_2 \in \Lambda^R_h$; therefore,

$$
\begin{aligned}
(\alpha(\mathcal{Q}_h\varphi - \varphi), \mu)_{\Gamma^R} &= ((\alpha - \mathcal{Q}_h\alpha)(\mathcal{Q}_h\varphi - \varphi), \mu)_{\Gamma^R} \\
&\le C\|\alpha - \mathcal{Q}_h\alpha\|_{0,\infty,\Gamma^R}\,\|\mathcal{Q}_h\varphi - \varphi\|_{0,\Gamma^R}\,\|\mu\|_{0,\Gamma^R} \\
&\le C\|\alpha\|_{1,\infty,\Gamma^R}\,\|\mathcal{Q}_h\varphi - \varphi\|_{0,\Gamma^R}\,\|\mu\|_{0,\Gamma^R}\,h,
\end{aligned}
$$

since $\mathcal{Q}_h$ approximates optimally in L^∞ as well as in L^2. ∎

THEOREM 5.6. *Assume* (H1)–(H5) *and that* $g^R_1 \ge \gamma > 0$ *on* Γ^R. *There exists a constant* C, *independent of* h, *such that*

$$\|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 = \|\nabla\cdot(\mathbf{u} - \Pi\mathbf{u})\|_0 \le Ch,$$

and, if $g^R_1 \in W^{1,\infty}(\Gamma^F)$, $p \in C^{3,1}(\bar\Omega)$, $\mathbf{u} \in (C^1(\bar\Omega) \cap W^{2,\infty}(\Omega))^d$, *and* $K \in (C^1(\bar\Omega) \cap W^{2,\infty}(\Omega))^{d\times d}$, *then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM}} + \|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_{\mathbf{TM}} + \|\mathcal{Q}_h\lambda - \lambda_h\|_{0,\Gamma^R} + \|(\Pi\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \le Ch^r,$$

where $r = 2$ if K is diagonal and $\Gamma^D = \emptyset$, $r = 3/2$ if K is diagonal or the grids are generated by a C^2 map, and $r = 1$ otherwise.

Proof. Subtracting (4.1) from (1.3a), (5.1.1), and (1.3d) extended to Γ^F gives the error equations

$$(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w) = 0, \qquad w \in W_h, \tag{5.2.1a}$$

$$(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h, \mathbf{v})_{\mathbf{TM}} = (P - p_h, \nabla \cdot \mathbf{v}) - (\lambda^* - \lambda_h, \mathbf{v} \cdot \nu)_{\Gamma^F}, \quad \mathbf{v} \in V_h, \tag{5.2.1b}$$

$$(\mathbf{U} - \mathbf{u}_h, \tilde{\mathbf{v}})_{\mathbf{TM}} = (K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{v}})_{\mathbf{T}}, \qquad \tilde{\mathbf{v}} \in V_h, \tag{5.2.1c}$$

$$((\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu, \mu)_{\Gamma^F} = (g_1^F(\lambda - \lambda_h), \mu)_{\Gamma^F}, \qquad \mu \in \Lambda_h^F. \tag{5.2.1d}$$

Equation (5.2.1a) and (3.4) imply $\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h) = 0$, giving the first part of the theorem with (3.1).

Continuing, let $\mathbf{v} = \Pi\mathbf{u} - \mathbf{u}_h$, $\tilde{\mathbf{v}} = \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h$, and $\mu = \lambda^* - \lambda_h$ in (5.2.1b)–(5.2.1d), and combine to obtain

$$\begin{aligned}(K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)_{\mathbf{T}} &+ (g_1^F(\lambda^* - \lambda_h), \lambda^* - \lambda_h)_{\Gamma^F} \\ &= (\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h, \mathbf{U} - \Pi\mathbf{u})_{\mathbf{TM}} + (g_1^F(\lambda^* - \lambda), \lambda^* - \lambda_h)_{\Gamma^F}.\end{aligned}$$

The last term above is estimated as

$$\begin{aligned}&(g_1^F(\lambda^* - \lambda), \lambda^* - \lambda_h)_{\Gamma^F} \\ &\quad = (g_1^F(\mathcal{Q}_h\lambda - \lambda), \lambda^* - \lambda_h)_{\Gamma^F} + (g_1^F(\lambda^* - \mathcal{Q}_h\lambda), \lambda^* - \lambda_h)_{\Gamma^F} \\ &\quad \le C\{\|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^R}\, h + \|\lambda^* - \mathcal{Q}_h\lambda\|_{0,\Gamma^R}\}\|\lambda^* - \lambda_h\|_{0,\Gamma^R},\end{aligned}$$

using Lemma 5.5. Thus, applying the Schwarz inequality, using that $g_1^R \ge \gamma > 0$, and hiding some terms,

$$\begin{aligned}&\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathbf{TM}} + \|\mathcal{Q}_h\lambda - \lambda_h\|_{0,\Gamma^R} \\ &\quad \le C\{\|\mathbf{U} - \Pi\mathbf{u}\|_{\mathbf{TM}} + \|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^R}\, h + \|\lambda^* - \mathcal{Q}_h\lambda\|_{0,\Gamma^R}\}.\end{aligned}$$

Now, $\tilde{\mathbf{v}} = \mathbf{U} - \mathbf{u}_h$ in (5.2.1c) gives

$$\|\mathbf{U} - \mathbf{u}_h\|_{\mathbf{TM}} \le C\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathbf{TM}},$$

and $\mu = (\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu$ in (5.2.1d) gives

$$\|(\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu\|_{0,\Gamma^F}^2 = (g_1^F(\mathcal{Q}_h\lambda - \lambda_h), (\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu)_{\Gamma^F} + (g_1^F(\lambda - \mathcal{Q}_h\lambda), (\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu)_{\Gamma^F},$$

and thus with Lemma 5.5,

$$\|(\Pi\mathbf{u} - \mathbf{u}_h) \cdot \nu\|_{0,\Gamma^F} \le C\{\|\mathcal{Q}_h\lambda - \lambda_h\|_{0,\Gamma^R} + \|\lambda - \mathcal{Q}_h\lambda\|_{0,\Gamma^R}\, h\}.$$

An application of Corollary 5.4, (5.1.5), (3.3), and the known estimates

$$\|\Pi\mathbf{u} - \mathbf{u}\|_{\mathbf{TM}} + \|\Pi\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\|_{\mathbf{TM}} \le Ch^2 \tag{5.2.2}$$

(see [25], [16], [17], [19]) completes the proof. ☐

Remark. Theorem 5.6 states that, in the case of a full tensor, both $\mathbf{u}_h$ and $\tilde{\mathbf{u}}_h$ are superconvergent to the true solution in the discrete seminorm defined at the nodal points (for each velocity component, in its direction at the centers of the edges or faces) under our mild assumption on the grid. We also recovered the superconvergence result for diagonal K [32] (extended to Robin conditions).

5.3. Estimates for the scalars p and λ. We now consider bounding the error in pressure for the finite difference method. From (1.3) and (4.1) we get the error equations

$$(\nabla \cdot (\mathbf{u} - \mathbf{u}_h), w) = 0, \qquad w \in W_h, \tag{5.3.1a}$$

$$\begin{aligned}(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \mathbf{v}) = (\mathcal{P}_h p - p_h, \nabla \cdot \mathbf{v}) - (\mathcal{Q}_h \lambda - \lambda_h, \mathbf{v} \cdot \nu)_{\Gamma^F} \\ - E_{\mathbf{TM}}(\tilde{\mathbf{u}}_h, \mathbf{v}),\end{aligned} \qquad \mathbf{v} \in V_h, \tag{5.3.1b}$$

$$(\mathbf{u} - \mathbf{u}_h, \tilde{\mathbf{v}}) = (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \tilde{\mathbf{v}}) - E_{\mathbf{TM}}(\mathbf{u}_h, \tilde{\mathbf{v}}) + E_{\mathbf{T}}(K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}}), \quad \tilde{\mathbf{v}} \in V_h, \tag{5.3.1c}$$

$$((\mathbf{u} - \mathbf{u}_h) \cdot \nu, \mu)_{\Gamma^F} = (g_1^F (\lambda - \lambda_h), \mu)_{\Gamma^F}, \qquad \mu \in \Lambda_h^F, \tag{5.3.1d}$$

where

$$E_{\mathbf{Q}}(\mathbf{q}, \mathbf{v}) = (\mathbf{q}, \mathbf{v}) - (\mathbf{q}, \mathbf{v})_{\mathbf{Q}}, \quad \mathbf{Q} = \mathbf{TM} \text{ or } \mathbf{T}.$$

It is well known [14] that the error in approximating an integral by either of these rules is of order h^2:

$$|E_{\mathbf{Q}}(\mathbf{q}, \mathbf{v})| \le C \sum_E \sum_{|\alpha|=2} \left\| \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} (\mathbf{q} \cdot \mathbf{v}) \right\|_{0,1,E} h^2. \tag{5.3.2}$$

We will need the following lemma.

LEMMA 5.7. *If the lowest-order RTN spaces on rectangles are used, then for any* $\mathbf{q} = (q^x, q^y) \in H^1(\Omega)$ *and* $E \in \mathcal{T}_h$,

$$\left\| \frac{\partial}{\partial x} (\Pi \mathbf{q})^x \right\|_{0,E} \le \left\| \frac{\partial q^x}{\partial x} \right\|_{0,E} \quad \text{and} \quad \left\| \frac{\partial}{\partial y} (\Pi \mathbf{q})^y \right\|_{0,E} \le \left\| \frac{\partial q^y}{\partial y} \right\|_{0,E}.$$

Proof. Without loss of generality, assume that E is the unit square. By definition, Π satisfies on each edge e of E

$$\int_e (\mathbf{q} \cdot \nu - \Pi \mathbf{q} \cdot \nu) \, ds = 0.$$

Writing this for the two vertical edges, we have

$$\int_0^1 [q^x(1, y) - (\Pi \mathbf{q})^x(1, y)] \, dy = 0 \quad \text{and} \quad \int_0^1 [q^x(0, y) - (\Pi \mathbf{q})^x(0, y)] \, dy = 0.$$

Subtraction of the above equations and the fundamental theorem of calculus imply

$$\int_0^1 \int_0^1 \left[\frac{\partial}{\partial x} q^x(x, y) - \frac{\partial}{\partial x} (\Pi \mathbf{q})^x(x, y) \right] \frac{\partial v^x}{\partial x} \, dx \, dy = 0, \quad \mathbf{v} \in V_h.$$

Therefore $(\Pi \mathbf{q})^x$ is the H_0^1-projection of q^x in the x direction. Similarly, $(\Pi \mathbf{q})^y$ is the H_0^1-projection of q^y in the y direction, which proves the lemma. ∎

THEOREM 5.8. *Assume* (H1)–(H5) *and that* $g_1^R \ge \gamma > 0$ *on* Γ^R. *If the grids are quasiuniform,* $p \in C^{3,1}(\bar{\Omega})$, $\mathbf{u} \in (C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega))^d$, *and* $K \in (C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega))^{d \times d}$, *then there exists a constant* C, *independent of* h, *such that*

$$\|\mathcal{P}_h p - p_h\|_0 + \|\mathcal{Q}_h \lambda - \lambda_h\|_{-1/2,\Gamma^F} \le C h^2.$$

Proof. To estimate the pressure error, we use a duality argument. Again, let $\rho \in L^2(\Omega)$, $\psi \in H^{1/2}(\Gamma^F)$, and $\varphi \in H^2(\Omega)$ solve (3.13), set $\mathbf{v} = -\Pi K \nabla\varphi$, and note that $\nabla\cdot\mathbf{v} = \mathcal{P}_h\rho$, $\mathbf{v}\cdot\nu = \mathcal{Q}_h(\psi + g_1^F\varphi)$, and (3.14)–(3.15) hold. We have with (5.3.1b),

$$\begin{aligned}&(\mathcal{P}_h p - p_h, \rho) - (\mathcal{Q}_h\lambda - \lambda_h, \psi)_{\Gamma^F}\\ &\quad = -(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h, \Pi K\nabla\varphi - K\nabla\varphi) - (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \nabla\varphi - \Pi\nabla\varphi)\\ &\qquad - (K(\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h), \Pi\nabla\varphi) + (\mathcal{Q}_h\lambda - \lambda_h, g_1^F\varphi)_{\Gamma^F} + E_{\mathbf{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi),\end{aligned}$$

which is analogous to (3.27). Thus, as in (3.28) for $s = 0$,

$$\begin{aligned}&(\mathcal{P}_h p - p_h, \rho) - (\mathcal{Q}_h\lambda - \lambda_h, \psi)_{\Gamma^F}\\ &\quad \le C\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 + \|(\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F}\\ &\qquad + \|\lambda - \lambda_h\|_{0,\Gamma^R} + \|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^F}\}\|\varphi\|_2\, h\\ &\quad + |E_{\mathbf{TM}}(\mathbf{u}_h, \Pi\nabla\varphi)| + |E_{\mathbf{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi)| + |E_{\mathbf{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi)|.\end{aligned} \tag{5.3.3}$$

Using (5.3.2) and the fact that the functions are in the discrete space, we have

$$\begin{aligned}|E_{\mathbf{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi)| \le C\sum_E \Bigg\{&\left\|\frac{\partial \tilde{u}_h^x}{\partial x}\right\|_{0,E}\left\|\frac{\partial}{\partial x}(\Pi K\nabla\varphi)^x\right\|_{0,E}\\ &+ \left\|\frac{\partial \tilde{u}_h^y}{\partial y}\right\|_{0,E}\left\|\frac{\partial}{\partial y}(\Pi K\nabla\varphi)^y\right\|_{0,E}\Bigg\} h^2.\end{aligned} \tag{5.3.4}$$

We observe by the inverse inequality (valid for quasi-uniform grids) and Lemma 5.7 that

$$\begin{aligned}\left\|\frac{\partial \tilde{u}_h^x}{\partial x}\right\|_{0,E} &= \left\|\frac{\partial}{\partial x}(\tilde{u}_h^x - (\Pi\tilde{\mathbf{u}})^x)\right\|_{0,E} + \left\|\frac{\partial}{\partial x}(\Pi\tilde{\mathbf{u}})^x\right\|_{0,E}\\ &\le C\|\tilde{u}_h^x - (\Pi\tilde{\mathbf{u}})^x\|_{0,E}h^{-1} + \left\|\frac{\partial \tilde{u}^x}{\partial x}\right\|_{0,E}\\ &\le C\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_{0,E}h^{-1} + \|\tilde{\mathbf{u}}\|_{1,E}.\end{aligned}$$

A similar expression holds for the y direction. Now, from (5.3.4) and again using Lemma 5.7,

$$\begin{aligned}&|E_{\mathbf{TM}}(\tilde{\mathbf{u}}_h, \Pi K\nabla\varphi)|\\ &\quad \le C\sum_E \{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_{0,E}h^{-1} + \|\tilde{\mathbf{u}}\|_{1,E}\}\|K\nabla\varphi\|_{1,E}h^2\\ &\quad \le C\sum_E \{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_{0,E} + \|\tilde{\mathbf{u}}\|_{1,E}\, h\}\|\varphi\|_{2,E}\, h\\ &\quad \le C\{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_0 + \|\tilde{\mathbf{u}}\|_1\, h\}\|\varphi\|_2\, h.\end{aligned} \tag{5.3.5}$$

We bound the other two quadrature error terms similarly:

$$\begin{aligned}|E_{\mathbf{TM}}(\mathbf{u}_h, \Pi\nabla\varphi)| &\le C\{\|\mathbf{u}_h - \Pi\mathbf{u}\|_0 + \|\mathbf{u}\|_1\, h\}\|\varphi\|_2\, h,\\ |E_{\mathbf{T}}(K\tilde{\mathbf{u}}_h, \Pi\nabla\varphi)| &\le C\{\|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_0 + \|\tilde{\mathbf{u}}\|_1\, h\}\|\varphi\|_2\, h,\end{aligned}$$

noting that the constants depend on $\|K\|_{2,\infty}$. From (5.3.3), then,

$$\begin{aligned}(\mathcal{P}_h p - p_h, \rho) &- (\mathcal{Q}_h\lambda - \lambda_h, \psi)_{\Gamma^F} \\ &\le C\{\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}_h\|_0 + \|\mathbf{u} - \mathbf{u}_h\|_0 + \|\tilde{\mathbf{u}}_h - \Pi\tilde{\mathbf{u}}\|_0 + \|\mathbf{u}_h - \Pi\mathbf{u}\|_0 \\ &\quad + \|\nabla\cdot(\mathbf{u} - \mathbf{u}_h)\|_0 + \|(\mathbf{u} - \mathbf{u}_h)\cdot\nu\|_{0,\Gamma^F} \\ &\quad + \|\lambda - \lambda_h\|_{0,\Gamma^R} + \|\mathcal{Q}_h\lambda - \lambda\|_{0,\Gamma^F} + h\}\|\varphi\|_2\, h,\end{aligned} \tag{5.3.6}$$

which, combined with Theorem 5.6, the projection error estimates, (5.1), (5.2.2), and (3.14), proves the result. □

5.4. Interior estimates for the vectors u and ũ. We finally establish interior estimates of $\mathbf{u}-\mathbf{u}_h$ and $\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h$ that show in all cases essentially second-order superconvergent rates of convergence. We will need $\tilde{\Pi}_{\mathbf{T}}$, the element-by-element trapezoidal (i.e., corner) interpolation operator into V_h. We will need the following lemma.

LEMMA 5.9. *If $\Phi \in C^\infty(\Omega)$ and $\mathbf{v} \in V_h$, then*

$$\|(1-\Pi)(\Phi\mathbf{v})\|_{\mathbf{T}} \le C\|\mathbf{v}\|_{\mathbf{TM}}\, h,$$

where C depends on Φ.

Proof. By (3.6), for any edge or face e with unit normal ν,

$$\Pi(\Phi\mathbf{v})\cdot\nu|_e = \left(\frac{1}{|e|}\int_e \Phi\, ds(\mathbf{x})\right)\mathbf{v}\cdot\nu|_e,$$

where $|e|$ is the length or area of e; thus, at an element corner point $\xi \in \partial e$,

$$(1-\Pi)(\Phi\mathbf{v}\cdot\nu)(\xi) = \left(\Phi(\xi) - \frac{1}{|e|}\int_e \Phi\, ds(\mathbf{x})\right)\mathbf{v}\cdot\nu(\xi)$$

is first-order accurate. □

THEOREM 5.10. *Assume* (H1)–(H5) *and that $g_1^R \ge \gamma > 0$ on Γ^R. Let Ω' be compactly contained in Ω. For any $\epsilon > 0$, there exists a constant C_ϵ, independent of h, such that if $p \in C^{3,1}(\bar{\Omega})$, $\mathbf{u} \in (C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega))^d$, $K \in (C^1(\bar{\Omega}) \cap W^{2,\infty}(\Omega))^{d\times d}$, and either K is diagonal or the grids are generated by a C^2 map, then*

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{TM},\Omega'} + \|\tilde{\mathbf{u}}-\tilde{\mathbf{u}}_h\|_{\mathbf{TM},\Omega'} \le C_\epsilon h^{2-\epsilon}.$$

LEMMA 5.11. *If Ω'' is compactly contained in Ω and either K is diagonal or the grids are generated by a C^2 map, then*

$$\|\mathbf{U}-\mathbf{u}\|_{\mathbf{TM},\Omega''} + \|\tilde{\mathbf{U}}-\tilde{\mathbf{u}}\|_{\mathbf{TM},\Omega''} \le Ch^2,$$

where C is independent of Ω''.

This is a corollary of Lemma 5.3.

Proof of Theorem 5.10. For $i = 1, 2, \ldots$, fix domains Ω_i such that

$$\Omega' \subset\subset \Omega_{i+1} \subset\subset \Omega_i \subset\subset \Omega_0 = \Omega,$$

and let $0 \le \Phi_{i+1} \in C_0^\infty(\Omega_i)$ with $\Phi_{i+1} \equiv 1$ on Ω_{i+1}.

We will analyze the error equations (5.2.1). First note that (5.2.1) holds with the **TM** quadrature rule replaced by the **T** rule. We have, using (5.2.1c) for some $c > 0$,

$$\begin{aligned}c\|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h)\|^2_{\mathbf{T},\Omega_i} &\le (K(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h), \Phi_{i+1}(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h))_{\mathbf{T},\Omega_i} \\ &= (K(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h), \tilde{\Pi}_{\mathbf{T}}(\Phi_{i+1}(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h)))_{\mathbf{T},\Omega_i} \\ &= (\mathbf{U}-\mathbf{u}_h, \tilde{\Pi}_{\mathbf{T}}(\Phi_{i+1}(\tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h)))_{\mathbf{T},\Omega_i} \\ &= (\Phi_{i+1}(\mathbf{U}-\mathbf{u}_h), \tilde{\mathbf{U}}-\tilde{\mathbf{u}}_h)_{\mathbf{T},\Omega_i}.\end{aligned} \tag{5.4.1}$$

Now (5.2.1b) with $\mathbf{v} = \Pi(\Phi_{i+1}(\Pi\mathbf{u} - \mathbf{u}_h))$, the fact that $\nabla \cdot (\Pi\mathbf{u} - \mathbf{u}_h) = 0$, and Lemma 5.9,

$$
\begin{aligned}
&(\Phi_{i+1}(\mathbf{U} - \mathbf{u}_h), \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)_{\mathbf{T},\Omega_i} \\
&\quad = (\Phi_{i+1}(\mathbf{U} - \Pi\mathbf{u}), \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)_{\mathbf{T},\Omega_i} \\
&\qquad + ((1-\Pi)(\Phi_{i+1}(\Pi\mathbf{u} - \mathbf{u}_h)), \tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)_{\mathbf{T},\Omega_i} \\
&\qquad + (P - p_h, \nabla\Phi_{i+1} \cdot (\Pi\mathbf{u} - \mathbf{u}_h))_{\Omega_i} \\
&\quad \le \|\Phi_{i+1}^{1/2}(\mathbf{U} - \Pi\mathbf{u})\|_{\mathbf{T},\Omega_i} \|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)\|_{\mathbf{T},\Omega_i} \\
&\qquad + C_{i+1}\|\Pi\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM},\Omega_i}\{\|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathbf{TM},\Omega_i} h + \|P - p_h\|_0\} \\
&\quad \le \tfrac{1}{2}c\|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)\|^2_{\mathbf{T},\Omega_i} \\
&\qquad + C_{i+1}\{\|\mathbf{U} - \Pi\mathbf{u}\|^2_{\mathbf{T},\Omega_i} + \|\Pi\mathbf{u} - \mathbf{u}_h\|^2_{\mathbf{TM},\Omega_i} h \\
&\qquad\quad + \|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|^2_{\mathbf{TM},\Omega_i} h + \|\Pi\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM},\Omega_i}\|P - p_h\|_0\}.
\end{aligned}
\tag{5.4.2}
$$

Thus, using (5.1), (5.1.2), Corollary 5.4, Theorem 5.6, (5.2.2), and Lemma 5.11,

$$
\begin{aligned}
\|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)\|_{\mathbf{T},\Omega_i} &\le C_{i+1}\, h\{h + \|\mathbf{U} - \mathbf{u}_h\|^{1/2}_{\mathbf{TM},\Omega_i}\} \\
&\le C_{i+1}\, h\{h + \|\Phi_i^{1/2}(\mathbf{U} - \mathbf{u}_h)\|^{1/2}_{\mathbf{T},\Omega_{i-1}}\}.
\end{aligned}
\tag{5.4.3}
$$

Likewise, (5.2.1c) gives for any $\eta > 0$,

$$
\begin{aligned}
&\|\Phi_{i+1}^{1/2}(\mathbf{U} - \mathbf{u}_h)\|^2_{\mathbf{T},\Omega_i} \\
&\quad = (\mathbf{U} - \mathbf{u}_h, \tilde{\Pi}_{\mathbf{T}}(\Phi_{i+1}(\mathbf{U} - \mathbf{u}_h)))_{\mathbf{T},\Omega_i} \\
&\quad = (K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \tilde{\Pi}_{\mathbf{T}}(\Phi_{i+1}(\mathbf{U} - \mathbf{u}_h)))_{\mathbf{T},\Omega_i} \\
&\quad = (K(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h), \Phi_{i+1}(\mathbf{U} - \mathbf{u}_h))_{\mathbf{T},\Omega_i} \\
&\quad \le C\|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)\|^2_{\mathbf{T},\Omega_i} + \tfrac{1}{2}\|\Phi_{i+1}^{1/2}(\mathbf{U} - \mathbf{u}_h)\|^2_{\mathbf{T},\Omega_i};
\end{aligned}
\tag{5.4.4}
$$

therefore, with (5.4.3),

$$
\begin{aligned}
\|\Phi_{i+1}^{1/2}(\mathbf{U} - \mathbf{u}_h)\|_{\mathbf{T},\Omega_i} &\le C\|\Phi_{i+1}^{1/2}(\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h)\|_{\mathbf{T},\Omega_i} \\
&\le C_{i+1}\, h\{h + \|\Phi_i^{1/2}(\mathbf{U} - \mathbf{u}_h)\|^{1/2}_{\mathbf{T},\Omega_{i-1}}\}.
\end{aligned}
\tag{5.4.5}
$$

Since Theorem 5.6 and Lemma 5.11 give

$$
\|\Phi_i^{1/2}(\mathbf{U} - \mathbf{u}_h)\|_{\mathbf{T},\Omega_{i-1}} \le C_i\|\mathbf{U} - \mathbf{u}_h\|_{\mathbf{TM},\Omega_{i-1}} \le C_i h^{3/2},
$$

applying (5.4.5) recurrently, we see that

$$
\|\mathbf{U} - \mathbf{u}_h\|_{\mathbf{TM},\Omega'} + \|\tilde{\mathbf{U}} - \tilde{\mathbf{u}}_h\|_{\mathbf{TM},\Omega'} \le C_\epsilon h^{2-\epsilon},
$$

and the theorem follows from Lemma 5.11. ◻

6. Some numerical experiments. In this section we present the results of some numerical tests on the cell-centered finite difference method defined in section 4 which confirm some of the theoretical results from the previous section. We solve a

Table 6.1

Convergence rates for the relatively simple Dirichlet problem $\|p - p_h\|_{\mathbf{M}} \le C_p\, h^{\alpha_p}$ *and* $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM}} \le C_u\, h^{\alpha_u}$.

Tensor	Grid	C_p	α_p	C_u	α_u
D	uniform	0.367	1.948	0.331	1.788
D	C^2-grid	0.050	2.127	0.104	2.069
D	nonuniform	0.275	1.997	0.186	1.625
N	uniform	0.394	1.950	0.287	1.762
N	C^2-grid	0.040	2.104	0.091	2.065
N	nonuniform	0.291	1.993	0.161	1.542

problem on the unit square with either a Dirichlet or a Neumann boundary condition. To stabilize the Neumann condition, we add p to the left-hand side of (1.1a).

The numerical tests are divided into three subsections. First, we consider a relatively simple example, for which the loss of superconvergence due to the boundary conditions is only very slight. Second, a relatively hard example, with a loss of $h^{1/2}$ superconvergence near the boundary, is shown. Third, the harder example is solved with a nonsymmetric variant of our cell-centered finite difference method that recovers most of the superconvergence near the boundary.

6.1. A relatively simple example. In these numerical experiments, the conductivity tensor is either diagonal or full (nondiagonal) with variable components:

$$K = D = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad K = N = \begin{pmatrix} (x+2)^2 + y^2 & \sin(xy) \\ \sin(xy) & 1 \end{pmatrix}.$$

The true solution is

$$p = x^3 y + y^4 + \sin(x)\cos(y),$$

with f and g^D or g^N defined accordingly by (1.1).

Convergence rates are established by running cases for six levels of grid refinement, starting with $h = 1/5$ on level one and refining by a factor of two for each successive level until $h = 1/160$ for level six. Assuming that the error takes the form Ch^{α}, C and α are determined to give the best least squares fit to the data. We consider three types of grids, a uniform grid, a C^2-grid generated by a C^2 mapping of the uniform grid, and a nonuniform grid that is a random perturbation of the uniform grid. The C^2 map is defined by

$$F^x(x) = \frac{e^{-2x} - 1}{e^{-2} - 1}, \quad F^y(y) = \frac{25 - (5 - 4y)^2}{24}.$$

The results for the Dirichlet and Neumann problems are presented in Tables 6.1 and 6.2, respectively. Quadrature rules are used for calculating the error; that is, the results presented are for the norm $\|\cdot\|_{\mathbf{M}}$ for pressure and $\|\cdot\|_{\mathbf{TM}}$ for velocity. The convergence rate for the pressure is $O(h^2)$ in all cases. The Neumann problem's velocity is also $O(h^2)$ convergent for both the diagonal and the nondiagonal tensors. A slight loss of convergence is observed for the Dirichlet problem for all but the C^2-grid, but it is still at least $O(h^{3/2})$.

The C^2-grid provides the best results since we refined more where the solution is large (near the corner at (1,1)). Thus, the map induces a somewhat better than halving of the grid size where the solution is difficult to approximate. The loss of superconvergence for the velocity in these examples due to the boundary conditions is slight, since the boundary flux is nearly in the normal direction.

TABLE 6.2
Convergence rates for the relatively simple Neumann problem $\|p - p_h\|_{\mathbf{M}} \le C_p\, h^{\alpha_p}$ *and* $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM}} \le C_u\, h^{\alpha_u}$.

Tensor	Grid	C_p	α_p	C_u	α_u
D	uniform	2.091	2.005	0.069	1.996
D	C^2-grid	0.564	2.107	0.012	2.089
D	nonuniform	1.370	2.081	0.047	2.063
N	uniform	7.565	2.005	0.182	1.946
N	C^2-grid	1.565	2.105	0.022	2.072
N	nonuniform	5.194	2.094	0.097	1.892

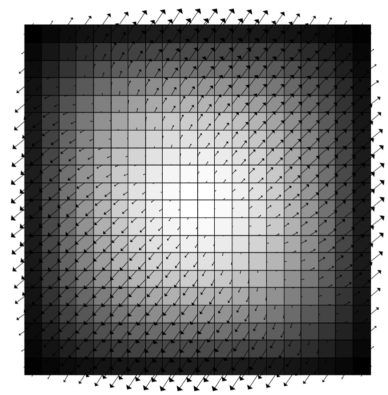


FIG. 6.1. *The computed pressure and velocity for the relatively hard problem with a large tangential flux* ($h = 1/20$).

TABLE 6.3
Convergence rates for the relatively hard problem $\|p - p_h\|_{\mathbf{M}} \le C_p\, h^{\alpha_p}$ *and* $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM}} \le C_u\, h^{\alpha_u}$.

BC	Levels	C_p	α_p	C_u	α_u
Dirichlet	1–7	2.339	2.000	0.759	1.445
Dirichlet	6–7	2.346	2.000	1.005	1.496
Neumann	1–7	2.416	1.936	0.386	1.333
Neumann	6–7	3.340	1.998	0.839	1.472

6.2. A relatively hard example. In the following experiments, we show that the loss of a half power of h is genuine if the tangential component of the flux through the boundary is large and that this loss occurs strictly on the boundary. The problem has the true solution

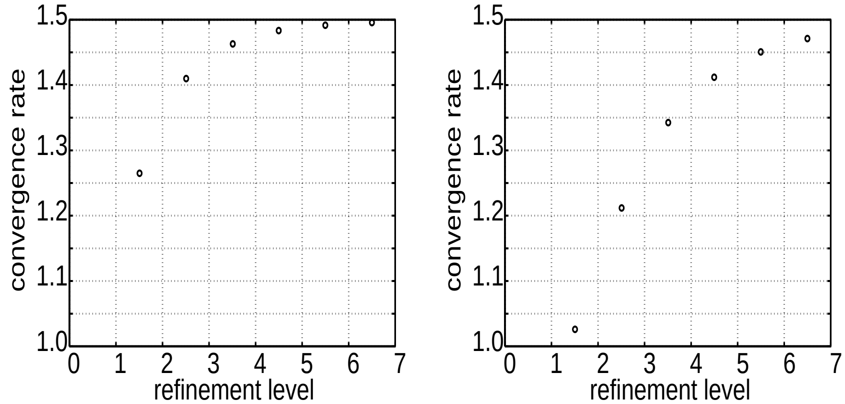
$$p = (x - x^2)(y - y^2)$$

and the conductivity tensor

$$K = \begin{pmatrix} 11 & 9 \\ 9 & 13 \end{pmatrix}.$$

Note that this conductivity causes a large diagonal component to the flow (see Fig. 6.1); that is, there are significant tangential fluxes on the element boundaries.

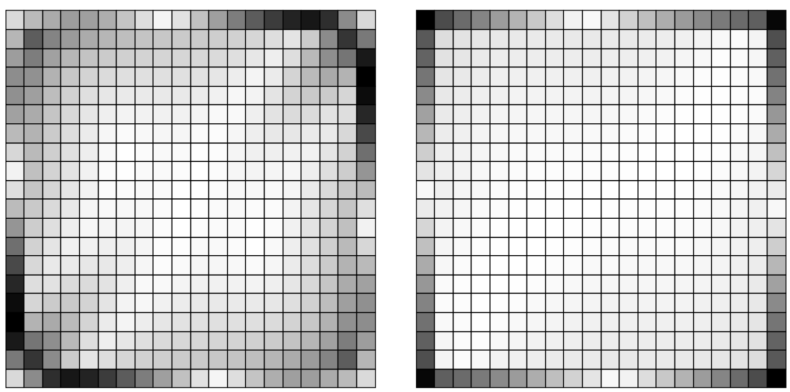
The rates of convergence on uniform grids are shown in Table 6.3. They are computed by a least squares fit to the data from the seven levels of refinement starting



A. Dirichlet problem

B. Neumann problem

Fig. 6.2. *Velocity convergence rates for the relatively hard problem.*



A. Neumann problem

B. Dirichlet

Fig. 6.3. *Magnitude of the velocity error in the relatively hard problem with a large tangential flux ($h = 1/20$).*

Table 6.4

Interior convergence rates for the relatively hard problem $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{TM},\Omega'_h} \leq C_u\, h^{\alpha_u}$.

BC	Levels	C_u	α_u
Dirichlet	1–7	0.464	1.818
Dirichlet	6–7	0.959	1.950
Neumann	1–7	0.471	1.587
Neumann	6–7	1.897	1.836

with $h = 1/10$ and also to the data from only the last two levels. The results indicate that a relatively fine mesh is needed for this problem to be in the asymptotic range of convergence for the velocity. This can also be seen in Fig. 6.2, where the velocity convergence rates are plotted, computed by comparing the error from any two successive levels. Fig. 6.3 shows that the velocity error is concentrated near the boundary.

We did not compute interior estimates for a fixed domain Ω'; rather, we used Ω'_h, the full domain less the elements touching $\partial\Omega$. In Table 6.4, these interior estimates for the velocity nearly show the full $O(h^2)$ superconvergence.

6.3. A nonsymmetric cell-centered finite difference method. In this subsection we present a nonsymmetric variant of our finite difference method that appears to be superconvergent even at the boundary of the domain. We define it only for Dirichlet or Neumann boundary conditions (i.e., assume $\Gamma^R = \emptyset$). The nonsymmetric method is based on an extension of the domain. The method is unmodified in the interior; thus, the nonsymmetry appears only near the boundary. We begin with some notation.

We assume explicitly that Ω is a rectangular parallelepiped. Let $\hat{\Omega}_h$ denote the extension of Ω by one element beyond $\partial\Omega$. This extension should respect any underlying C^2 map that generates the grid. Let $\hat{\Omega}_h$ be decomposed into the union $\hat{\Omega}_h = \Omega \cup \Omega_h^D \cup \Omega_h^N \cup \Omega_h^C$, disjoint except for edge or face overlap, where Ω_h^D and Ω_h^N consist of the elements outside Ω that are adjacent to Γ^D and Γ^N, respectively, and Ω_h^C denotes the elements outside Ω but not adjacent to $\partial\Omega$ (i.e., the "corner" elements).

Let $\hat{V}_h \times \hat{W}_h$ denote the lowest order Raviart–Thomas space defined on $\hat{\Omega}_h$ such that for any $\mathbf{v} \in \hat{V}_h$, $\mathbf{v} \cdot \nu_h = 0$, where ν_h is the outer unit normal to $\partial\hat{\Omega}_h$. Let $\hat{V}_h^C$ denote the subset of $\hat{V}_h$ for which $\mathbf{v} \cdot \nu|_e = 0$ on any edge or face e strictly outside $\Omega \cup \Omega_h^C$; that is, $\hat{V}_h^C$ has possibly nonzero nodal degrees of freedom only on the edges or faces of Ω and the corner elements of Ω_h^C. Further, let $\hat{W}_h^{N,C}$ denote the subset of $\hat{W}_h$ for which $w = 0$ on Ω_h^D.

On an element E with area or volume $|E|$, let

$$f_h(\mathbf{x}) = \begin{cases} f & \text{if } \mathbf{x} \in E \subset \Omega, \\ -\frac{1}{|E|} \int_{\partial E \cap \Gamma^N} g^N \, ds & \text{if } \mathbf{x} \in E \subset \Omega_h^N, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3.1}$$

The nonsymmetric method is to find $\mathbf{u}_h \in \hat{V}_h^C$, $\tilde{\mathbf{u}}_h \in \hat{V}_h$, and $p_h \in \hat{W}_h$ such that

$$(\nabla \cdot \mathbf{u}_h, w)_{\Omega_h} = (f_h, w)_{\Omega_h}, \qquad w \in \hat{W}_h^{N,C}, \tag{6.3.2a}$$

$$(\tilde{\mathbf{u}}_h, \mathbf{v})_{\mathbf{TM},\Omega_h} = (p_h, \nabla \cdot \mathbf{v})_{\Omega_h}, \quad \mathbf{v} \in \hat{V}_h, \tag{6.3.2b}$$

$$(\mathbf{u}_h, \tilde{\mathbf{v}})_{\mathbf{TM},\Omega_h} = (K\tilde{\mathbf{u}}_h, \tilde{\mathbf{v}})_{\mathbf{T},\Omega_h}, \quad \tilde{\mathbf{v}} \in \hat{V}_h^C, \tag{6.3.2c}$$

where we must further define p_h on Ω_h^D. For $E_1 \subset \Omega_h^D$ such that $E_2 \subset \Omega$ shares its edge or face e on $\partial\Omega$, let $h_i = |E_i|/|e|$, $i = 1, 2$, and then define by extrapolation

$$\frac{p_h|_{E_1} - p_h|_{E_2}}{\frac{1}{2}(h_1 + h_2)} = \frac{\frac{1}{|e|}\int_e g^D \, ds - p_h|_{E_2}}{\frac{1}{2}h_2};$$

that is,

$$p_h|_{E_1} = \frac{h_1 + h_2}{h_2|e|} \int_e g^D \, ds - \frac{h_1}{h_2} p_h|_{E_2}. \tag{6.3.2d}$$

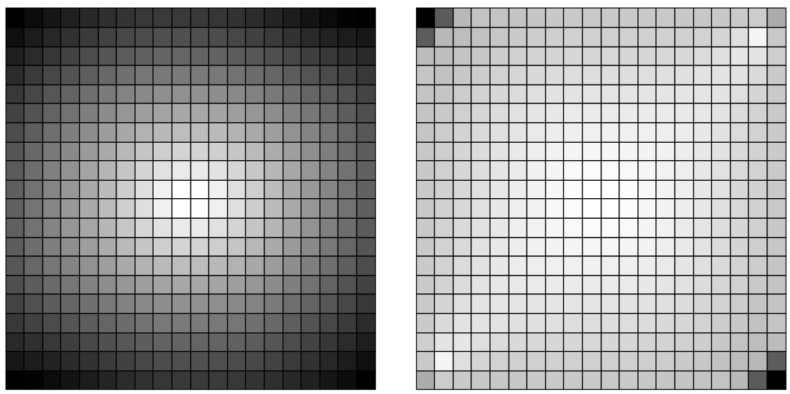
For an element $E \subset \Omega_h^N$, let $e = \partial E \cap \partial\Omega$, and set w equal to the characteristic function of E. Then (6.3.2a) implies that

$$\int_e g^N \, ds = -\int_{\partial E} \mathbf{u}_h \cdot \nu_{\partial E} \, ds = \mathbf{u}_h \cdot \nu_{\partial\Omega}|e| - \int_{\partial E \cap \partial\Omega_h^C} \mathbf{u}_h \cdot \nu_{\partial E} \, ds;$$

TABLE 6.5

Convergence rates for the relatively hard problem using the nonsymmetric method $\|p-p_h\|_{\mathbf{M}} \le C_p\, h^{\alpha_p}$ *and* $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{TM}} \le C_u\, h^{\alpha_u}$.

BC	Levels	C_p	α_p	C_u	α_u
Dirichlet	1–5	1.971	2.091	0.831	1.741
Dirichlet	4–5	1.386	1.998	1.887	1.957
Neumann	1–5	0.825	1.727	0.839	1.963
Neumann	4–5	2.153	1.979	0.938	1.992



A. Neumann problem B. Dirichlet

FIG. 6.4. *Magnitude of the velocity error in the relatively hard problem with a large tangential flux using the nonsymmetric method* ($h = 1/20$).

thus, $\mathbf{u}_h \cdot \nu$ is set properly on Neumann edges or faces that do not touch the corners. Indeed, for an edge or face $e \subset \Gamma^N$ that does touch a corner point or line, $\mathbf{u}_h \cdot \nu$ may not be the average of g^N; rather, only the net flux into the corner element is correct (by (6.3.2a)).

If we repeat the relatively hard experiments of the previous subsection, we recover the full $O(h^2)$ superconvergence of the velocity with this nonsymmetric method in the asymptotic regime. The rates of convergence are presented in Table 6.5. The error seems to be concentrated near the corners of the domain, as seen in Fig. 6.4.

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