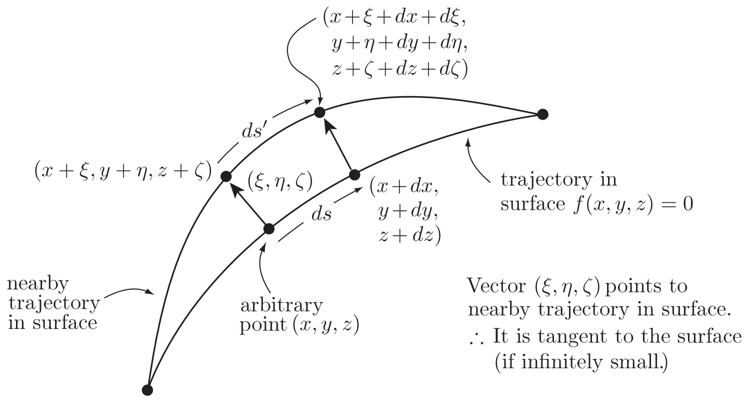
The details for those who want them:



Compare corresponding lengths on two curves and find

$$\boxed{ds' - ds = (\dot x\,\dot\xi + \dot y\,\dot\eta + \dot z\,\dot\zeta)\,ds \quad \text{where } \dot{} = d/ds}$$

$$\begin{aligned}
\text{since } (ds')^2 &= [(x+\xi+dx+d\xi)-(x+\xi)]^2 + \cdots = (dx+d\xi)^2 + \cdots \\
&= (dx^2 + 2\,dx\,d\xi + d\xi^2) + \cdots \\
&= \underbrace{(dx^2+dy^2+dz^2)}_{ds^2} + 2\,dx\,d\xi + 2\,dy\,d\eta + 2\,dz\,d\zeta \text{ in } 2^{\text{nd}} \text{ order quantities} \\
&= ds^2\left(1 + 2\,\frac{dx}{ds}\cdot\frac{d\xi}{ds} + \cdots\right) = ds^2(1 + 2\,\dot x\,\dot\xi + \cdots)
\end{aligned}$$

$$\therefore ds' = ds(1+\dot x\dot\xi + \cdots) \qquad \therefore ds' - ds = (\dot x\dot\xi + \cdots)ds$$

The condition for the motion to be force free, excepting the constraint to a surface $f(x,y,z)$,

$$m(a_x,a_y,a_z) = m\left(\frac{d^2x}{dt^2},\frac{d^2y}{dt^2},\frac{d^2z}{dt^2}\right) = \lambda\underbrace{\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)}_{\text{orthogonal to surface}}$$

is

$(x(s),y(s),z(s))$ has extremal length	$\Leftrightarrow$	(a_x,a_y,a_z) is orthogonal to the surface tangent vector (ξ,η,ζ)

The condition follows since the variation $\delta\int ds$ must vanish if the curve is extremal in length (a geodesic);

$$\delta\int ds = \underbrace{\int ds' - \int ds}_{\text{same start and finish}} = \int(\dot x\dot\xi + \dot y\dot\eta + \dot z\dot\zeta)\,ds$$

$$= \underbrace{\int\frac{d}{ds}(\dot x\xi + \cdots)\,ds}_{\substack{(\dot x\xi+\cdots)|_{\text{start}}^{\text{end}} = 0 \\ \text{since} \\ (\xi,\eta,\zeta)(\text{start}) = (\xi,\eta,\zeta)(\text{end}) = 0}} - \underbrace{\int(\xi\ddot x + \eta\ddot y + \zeta\ddot z)\,ds}_{\substack{\text{vanishes in general if} \\ (\ddot x,\ddot y,\ddot z) \text{ orthogonal} \\ \text{to } (\xi,\eta,\zeta)}}$$

The orthogonality of $(\ddot x,\ddot y,\ddot z)$ to (ξ,η,ζ) implies the orthogonality of (a_x,a_y,a_z) to (ξ,η,ζ) because constrained motion has constant kinetic energy, and hence $s \propto t$.

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