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THE LOGICAL INCONSISTENCY OF THE OLD QUANTUM THEORY OF BLACK BODY RADIATION*

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The old quantum theory of black body radiation was manifestly logically inconsistent. It required the energies of electric resonators to be both quantized and continuous. To show that this manifest inconsistency was inessential to the theory's recovery of the Planck distribution law, I extract a subtheory free of this manifest inconsistency but from which Planck's law still follows.

1. Introduction. The old quantum theory of black body radiation emerged in the first decade of this century, when it was found that the conjunction of

1. Thermodynamics.
2. Statistical mechanics.
3. Classical electrodynamics.

led to the Rayleigh-Jeans law for the distribution of energy over the component frequencies of black body radiation, rather than the empirically verified Planck distribution law. To recover the Planck law, the old quantum theorists simply conjoined

4. Quantum postulate.

to the above theories to form the old quantum theory of black body radiation.

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To the critical eye of the classical logician, this strategy is self-defeating and the new theory's success suspect. For the augmented theory had been rendered manifestly logically inconsistent. The quantum postulate 4. contradicts the continuity of energy levels posited by classical electrodynamics 3. *Any* proposition, including the Planck law or for that matter any other distribution law, can be derived from a contradiction.

In this paper I will show that the manifest inconsistency produced by conjoining 4. to 1., 2. and 3. was inessential to the old quantum theory's recovery of the Planck distribution law and the results leading up to it. To do this I will extract a subtheory from 3. which no longer posits continuity of the relevant energies and show that the Planck distribution law can still be recovered from the conjunction of it with 1., 2. and 4. The resulting subtheory of the old quantum theory of black body radiation will be free of manifest inconsistency and I conjecture its consistency.

Of course in the event, the old quantum theory did not fall victim to the logical anarchy of inconsistency. It was avoided by isolating the results of classical electrodynamics from those derived from quantum discontinuity in two domains of calculation. I now outline the main results of each domain.

Classical Electrodynamic domain. This domain contained two results constraining the functional form of the distribution law. They were the Stefan-Boltzmann law of 1879/1884 and the Wien displacement law of 1894, a more general law entailing the former law. The Stefan-Boltzmann law asserted that the total energy density of black body radiation u is proportional to the fourth power of temperature T:

$$u = \sigma T^4 \tag{1}$$

where σ is a constant. The Wien displacement law amounted to the assertion that the energy density of black body radiation of temperature T at a given frequency f, u_f, must be a function g of a single variable

$$u_f(T) = f^3 g(f/T) \tag{2}$$

Upon integration over all values of f, equation (2) reduces to equation (1).

There were two further relations in this domain. Planck's celebrated analysis (1900) of black body radiation assumed that the radiation was in equilibrium with a large number of electric resonators within the cavity enclosing the radiation. The energy density u_f of the radiation at frequency f in equilibrium with a resonator of energy U had been determined from classical electrodynamics in Planck's earlier work, where he assumed that the energy U could vary continuously. He found

$$u_f = (8\pi f^2/c^3)\, U \tag{3}$$

In an alternate approach, used by Rayleigh (1900) for his derivation of the Rayleigh-Jeans distribution law, one took enclosed radiation at equilibrium to consist of a superposition of independent "normal modes of vibration" or, as I shall call them, radiation oscillators. The density of radiation oscillators in the cavity was calculated essentially by requiring that the wavelength of each oscillator present fit an integral number of times into the cavity. The volume density n_f of oscillators per unit frequency is

$$n_f = 8\pi f^2/c^3 \tag{4}$$

Quantum domain. In this domain, the average energy of a resonator U or radiation oscillator E in thermal equilibrium at temperature T is determined by assuming that each can only adopt, with equal *a priori* probability, energy levels which are an integral multiple of some energy element q. It follows that

$$U = q/(\exp(q/kT) - 1) \tag{5}$$

and similarly for E. (k is Boltzmann's constant.) Classical electrodynamics requires that the energy levels of both types of system can vary continuously, that is $q = 0$. With this condition, (5) reduces to the classical equipartition result for a system with two degrees of freedom, $U = kT$. The inconsistency of the quantum and classical electrodynamic domains follows directly from the nonzero value accorded q in the quantum domain.

The Planck distribution law was derived by importing the results of the classical electrodynamic domain into the quantum domain. According to whether the quantization was carried out over resonators or radiation oscillators, results (3) or (4) respectively were conjoined with (5) to give an expression for u_f. Comparison with the Wien displacement law (2) then required q to be set proportional to frequency f, the constant of proportionality h being given the value of Planck's constant by comparison with experimental results. Thus the Planck distribution law

$$u_f = (8\pi h f^3/c^3)\cdot(1/(\exp(hf/kT) - 1)) \tag{6}$$

was recovered.

Derivations of this type involving quantization over resonators were given by Lorentz (1910) and Larmor (1909), for example, and those involving direct quantization of the radiation oscillators were given by Ehrenfest (1906) and Debye (1910).

The response of the physicists concerned to the inconsistency inherent in their theory was somewhat varied. Planck's own attitude has become the focus of some attention following Kuhn's disputed claim (1978) that Planck believed his analysis of 1900 and 1901 to be thoroughly classical, whatever quantum discontinuity it contained being a result of the misapplication of Boltzmann's method. For discussion, see Klein, Shimony and Pinch (1979).

Perhaps Einstein's attitude was the most enlightened. In his 1905 light quantum paper (1905), he stated more clearly than anyone else the failure of a purely classical analysis and concluded that classical electrodynamics could only hold for the time average of quantities. Thus he noted elsewhere (1906), (1909) that strictly speaking Planck's relation (3) must be posited independently, although its applicability for equilibrium calculations was justified by the assumed correctness of classical electrodynamics for time average quantities. His later "*A* and *B* coefficient" derivation of the Planck distribution law (1916) tried to solve the consistency problem through replacing use of the results of classical electrodynamics by statistical analysis of quantized absorption and emission of radiation. But he still invoked the Wien displacement law without offering a non-classical derivation of it.

The most widespread attitude seems to have been the one familiar to us through the well-known textbook accounts of the theory, such as Becker (1982, pp. 277–292). One was free to pick and choose from the results of classical electrodynamics. The logical inconsistency of this procedure was more an inconvenience around which one negotiated and which was to be eliminated—eventually.

What made this attitude possible and enabled any application of the theory at all was a generally careful but inarticulated control on which results could be exported from one domain to the other. Whilst the Wien displacement law (2), the Planck resonator formula (3) and radiation oscillator density relation (4) could be imported freely by the quantum domain, other classical results involving an explicit assertion of the continuity of energy levels of resonators, for example, were prohibited entry.

However since the rules governing these exchanges were never made explicit, the soundness of the strategy and therefore of the theory as a whole was by no means obvious.

Rescher and Brandom (1980) have offered a nonstandard semantics that can tolerate logical inconsistency. In their system, the truth of a proposition *P* in one consistent collection of propositions and the truth of its negation in another consistent collection does not enable us to infer the truth of the conjunction (*P* & *not-P*) in the combined collections, even though the syntactic derivability relation still allows *P*, *not-P* $\vdash$ (*P* & *not-P*). Their nonstandard semantics distinguishes the distributed truth of two propositions P and Q of two different collections from the collective truth of the conjunction (P & Q). The distributed truth of P and Q does not guarantee the collective truth of (P & Q). This device enables logical inconsistency to be tolerated without anarchy and provides the kind of rule needed to govern exchanges of propositions between the classic and quantum domains.

Unfortunately the old quantum theorists did not adhere to Rescher and Brandom's nonstandard semantics. For example, they took (2) and (3) from the classical domain and (5) from the quantum domain and conjoined them to arrive at the Planck distribution law (6). From the distributed truth of (2), (3) and (5), they immediately inferred the collective truth of the conjunction in (6).

2. The Subtheory Outlined. The key to constructing the subtheory is the recognition that the physicists of the old quantum theory extravagantly overcommitted themselves in incorporating all of classical electrodynamics into their theory. One only needs a minimal characterization of electromagnetic radiation to enable the recovery of the Planck distribution law and the results leading up to it. Thus the subtheory is formed as the conjunction of

1. Thermodynamics
2. Statistical mechanics

with a short list of properties of a form of matter to be called "radiation". Planck's electric resonators are also introduced in the list as "generalized Planck resonators". This list is a subtheory of 3. Classical electrodynamics and 4. Quantum postulate, which it replaces. It is summarized in Table 1 along with the results that can be derived from it.

The manifest inconsistency of the original theory stemmed from the assumption in 3. Classical electrodynamics, of continuous energy levels in contradiction with 4. Quantum postulate. This inconsistency no longer troubles the subtheory since no assumption is made about the continuity of energy levels in the list, apart from the quantum postulate itself. The simplicity and parsimony of the list make plausible the conjecture that the subtheory is consistent.

It is convenient to think of the subtheory as the thermodynamic analysis of the general forms of matter described by the list. Statistical mechanical arguments are used only at one point in the analysis. In Section 8 for reasons given there, I shall indicate how they can be dispensed with.

We shall see that radiation comprises a more general class of zero rest mass matter than the electromagnetic radiation of classical theory. It is characterized by a two-parameter family of types, the two parameters being "f-type" and "h-type". The f parameter corresponds to the fre-

TABLE 1. PROPERTIES POSITED FOR RADIATION AND GENERALIZED PLANCK RESONATORS AND THE RESULTS RECOVERED FROM THEM

PROPERTY	RESULTS RECOVERED
Radiation	
(*i*) *Rest mass property:* Radiation has zero rest mass.	
(*ii*) *Frequency Property:* There is a family of types of radiation, parameterized by the positive real-valued index "frequency" f.	
(*iii*) *Spatial superposition property:* The mixing of radiation of different frequencies and of different directions occurs reversibly, without requiring or releasing energy.	*From* (*i*), (*ii*), *and* (*iii*): Kirchhoff's laws Stefan-Boltzmann law
(*iv*) *Frequency characterization:* The ratio of energy E to frequency f of a unidirectional, monochromatic element of radiation is a Lorentz invariant.	*From* (*i*), (*ii*), (*iii*) *and* (*iv*): Wien displacement law (*WDL*) $u_f = f^3 g(f/T)$ for some function g
(*v*) *Quantum postulate:* *a.* Radiation oscillators can only take energies that are an integral multiple of an energy element q, that is $0, q, 2q, 3q, \ldots$	*From WDL and* (*v*): Average radiation oscillator energy (*AROE*) $E_f = hf/(\exp(hf/kT) - 1)$ and $q = hf$
b. In determining the statistical equilibrium of a system of radiation oscillators, each of the accessible energy levels has equal *a priori* probability.	
(*vi*) *Classical limit property:* Radiation for which $q = h = 0$ is classical electromagnetic radiation. (See Section 8 for alternate macroscopic quantum postulate)	*From* (*i*), (*ii*), (*iii*), (*iv*) *and* (*vi*): Radiation oscillator density formula (*RODF*) $n_f = 8\pi f^2/c^3$ *From AROE and RODF:* Planck distribution law (*PDL*) $u_f = (8\pi f^2/c^3)(hf/(\exp(hf/kT) - 1)$

quency of radiation of the classical theory, although very few of the characteristically wavelike properties associated with frequency are posited for radiation. The value of the h parameter determines whether the radiation energies are continuous or quantized and to which degree. $h = 0$ radiation is not quantized. Radiation for which h takes the value of Planck's constant, that is $h = 6.63 \times 10^{-27}$ erg sec, is quantized in accord with our experience of the real world.

Similarly, generalized Planck resonators represent a general class of nonzero rest mass systems which can exchange energy with radiation. They are characterized analogously by "f_r-type" and "h_r-type", where f_r corresponds to the resonant frequency of classical electric resonators and

TABLE 1. (Continued)

PROPERTY	RESULTS RECOVERED
Generalized Planck Resonators	
(i') *Rest mass property:* Generalized Planck resonators are systems containing energy of nonzero rest mass.	
(ii') *Resonant frequency property:* There is a family of types of generalized Planck resonators, parameterized by the positive real valued index "resonant frequency", f_r.	
(iv') *Resonant frequency characterization:* The ratio of energy U to resonant frequency f_r of a generalized Planck resonator is a Lorentz invariant.	*From* (i'), (ii') *and* (iv'): Wien displacement law for resonators (*WDLR*) $U = g(f_r/T)$ for some function g
(v') *Quantum postulate for generalized Planck resonators:* *a.* Generalized Planck resonators can only take energies that are an integral multiple of an energy element q_r, that is $0, q_r, 2q_r, 3q_r, \ldots$ *b.* In determining the statistical equilibrium of a system of generalized Planck resonators, each of the accessible energy levels has equal *a priori* probability.	*From WDLR and* (v'): Average generalized Planck resonator energy (*AGPRE*) $U = h_r f_r/(\exp(h_r f_r/kT) - 1)$ and $q_r = h_r f_r$
(vii) *Compatibility conditions:* *a.* A generalized Planck resonator of type h_r can only interact with a radiation system of type h, if $h_r = h$. *b.* A generalized Planck resonator of resonant frequency f_r can only come to equilibrium with radiation of frequency f, provided $f_r = f$ in the resonator's rest frame.	*FROM PDL, AGPRE and* (vii): Planck resonator formula (*PRF*) $u_f = (8\pi f^2/c^3)U$

h_r determines the degree of quantization. The h_r and h values of resonators and radiation which interact are equal, so that the degree of quantization of each can be represented by the value of h alone.

The subtheory relies on invoking the requirement of Lorentz invariance. This requirement did not always figure explicitly in work in electrodynamics and the old quantum theory in the relevant historical period. However such work was typically implicitly Lorentz invariant. Here I am concerned only with those versions of the old quantum theory that were Lorentz invariant, whether implicitly or explicitly so.

There has been some confusion over the correct Lorentz transformation formulae for temperature and heat, with Ott (1963) challenging the traditional formulae of Planck and Einstein. I shall avoid adopting any particular transformation for temperature and heat. The required results will be derived from the assumed Lorentz invariance of entropy. The latter assumption agrees with both Ott and the practice of the old quantum theory. It can be derived by demonstrating that one can construct the Lorentz boost of a system by a reversible adiabatic acceleration. For further discussion of this general area, see Earman (1980).

How does the reconstruction vindicate the original theory?

First, it shows that the original theory's manifest inconsistency was inessential to its recovery of the Planck distribution law and the results leading up to it. For these results can be derived from a subtheory, free of manifest contradiction.

Second, it explains retrospectively how the old quantum theorists came to introduce inconsistency into their theory and why this move was not fatal to the theory. Their task was the discovery of the behavior of quantized systems with h equal to Planck's constant. However their existing theoretical knowledge was almost exclusively limited to classical systems of $h = 0$ in the form of classical electrodynamics. Fortunately certain crucial results already recovered for $h = 0$ systems—such as the Wien displacement law (2) and the Planck resonator formula (3)—turned out to hold for systems of arbitrary h as well. In the absence of a general characterization of arbitrary-h systems and troubled by the departure from classical ideas in quantization, they took an expedient course: they ignored that these results then had only classical derivations and applied them to quantum systems as well. Naturally only very few classically derived results could be treated this way. Therefore, they required the tacit introduction of domains of calculation and a careful if inarticulated control over just which results could be exported from the classical to the quantum domain.

Third, the reconstructed theory provides a simple and now obvious rule for governing the exchange of results between these domains:

> *Only results that hold for systems of all h can be exchanged freely between classical and quantum domains.*

This rule does not eliminate the fallacy of using classical derivations for quantum results. Rather it enables us to review the manipulations of the old quantum theorists and to decide whether a result transferred fallaciously from a classical to a quantum domain is true in the latter domain.

For the remainder of the paper, I turn to the task of showing that the properties listed in Table 1 are sufficient to yield the Planck distribution law and the results leading up to it.

3. Kirchhoff's Laws and the Stefan-Boltzmann Law. Radiation is assumed to have the following properties:

(*i*) *Rest mass property: Radiation has zero rest mass.*

It follows immediately that radiation propagates at the invariant speed c in empty space. To see this well known result recall that the zero rest mass condition asserts merely that the energy-momentum vector of a small element of unidirectional radiation is a null vector. Therefore its four-velocity vector is also a null vector. No assumptions are made as yet about the interaction between radiation and the matter upon which it is incident. In general it may be absorbed—partially, completely or not at all.

(*ii*) *Frequency Property: There is a family of types of radiation, parameterized by the positive real valued index "frequency" f.*

(*iii*) *Spatial superposition property: The mixing of radiation of different frequencies and of different directions occurs reversibly, without requiring or releasing energy.*

A familiar instance of the reversibility of this mixing is the separation of visible light into its spectral components by a prism and their subsequent recombination by a lens and second prism. A surface is a reflector *by definition,* if radiation incident upon it when it is at rest is not absorbed but reradiated with its frequency unchanged. The index f is virtually undetermined by these properties, which permit reparameterization by arbitrarily many new indices generated through appropriate functions of f.

It follows from Property (*iii*) that, when a system of radiation is generated by mixing components of various frequencies and directions, the energy of each component retains its identity in the sense that it can be recovered by a reversible process requiring or releasing no work and in the form of radiation of the same frequency and direction. Thus we can speak of the energy and entropy of each component of the mixed system and note that each component behaves like an independent thermodynamic system. Energy can only be exchanged between different components, for example, through some external medium such as an enclosing cavity wall. In the context of the quantum postulate, these components will also be called "radiation oscillator".

Properties (*i*), (*ii*) and (*iii*) are sufficient for the derivation of Kirchhoff's laws by the well-known thermodynamic arguments. See, for example, Sommerfeld (1955, pp. 136–138). These arguments enable us to infer that radiation can be emitted by matter, provided that this matter is capable of absorbing radiation. More specifically we conclude that: for isotropic radiation enclosed in an isothermal cavity with at least a small quantity of absorbing matter ("cavity radiation"), the volume energy density per unit frequency u_f is a function of f and T, the temperature of the cavity wall, but not of its composition or shape; that the ratio of absorptivity to emissivity of a given body is independent of its composition; and that cavity radiation has the same constitution as radiation emitted by a perfectly absorbing body of the same temperature, that is, as "black body radiation".

For a given total energy, the spectral energy distribution of black body radiation has the maximum entropy. This follows since radiation of any other spectral energy distribution, enclosed in a cavity with reflecting walls, can be made to revert irreversibly to the black body distribution by the introduction of a minute speck of radiation absorbing matter. Thus the entropy S of a fixed volume of black body radiation satisfies $\delta S = 0$ where δ is any variation of the spectral energy distribution under the constraint of constant total energy. It follows from the usual methods that black body radiation satisfies[1]

$$\delta s_f / \delta u_f = 1/T \tag{7}$$

where s_f is the entropy density per unit frequency.

To recover the Stefan-Boltzmann law (1), we can use the well known derivation of Sommerfeld (1959, pp. 139–140). We consider the reversible adiabatic expansion of enclosed black body radiation, driven by radiation pressure, and arrive at (1) from the requirement of the exactness of the entropy differential, expressed as a function of pressure and temperature changes. In this standard derivation, classical electrodynamics is needed only to provide the crucial expression for radiation pressure p as a function of total radiation energy density u,

$$p = u/3 \tag{8}$$

Here however we recover (8) directly from property (i) and the assumed isotropy of the instances of radiation in question. We write the components of the stress-energy tensor $T_k^{\ i}$ for radiation in space-time coordi-

[1]After Planck (1959, pp. 92–93). Alternately consider a cavity with reflecting walls containing radiation of frequency f only. It communicates with a system of black body radiation at T through a filter that passes radiation of frequency f only. T increases infinitely slowly. Result (7) follows from writing the rate of increase of entropy density s with energy density u_f in the cavity, using the relation $ds/du_f = 1/T$.

nates appropriately adapted to the world line of the observer[2] and can read off the locally observed radiation energy density u and pressure p as

$$u = T_0{}^0$$

$$-p = T_1{}^1 = T_2{}^2 = T_3{}^3$$

The equality of these last three components follows immediately from isotropy. But property (i) asserts that the trace of $T_k{}^i$ vanishes, that is

$$T_0{}^0 + T_1{}^1 + T_2{}^2 + T_3{}^3 = 0$$

from which (8) follows.

4. The Wien Displacement Law. Properties (i), (ii) and (iii) are not sufficient for recovery of the Wien displacement law (2). These properties admit many reparameterizations of frequency f. But the majority of them will be inconsistent with (2), if (2) held for the original parameterization. In traditional treatments such as Born (1960, pp. 453–455), which follow Wien's original method (1894), the further characterization of f needed for recovery of (2) is provided by assuming that the frequency of radiation is Doppler shifted upon reflection from a moving mirror. (2) then follows from the consideration of the reversible, adiabatic expansion of black body radiation in a mirrored cavity.[3]

However characterization of f in terms of the Doppler effect, as it is usually understood, commits us to assumptions about the wavelike nature of radiation which are unnecessary for the derivation of (2). Rather, we require only the following property.

(iv) *Frequency characterization: The ratio of energy E to frequency f of a unidirectional, monochromatic element of radiation is a Lorentz invariant.*

The sufficiency of this property to yield the Doppler shift behavior for radiation and thus also the Wien displacement law follows from two results already noted by Einstein in his original 1905 paper on special relativity: light energy and frequency share the same Lorentz transformation law and a Doppler shifted light ray is just a Lorentz boosted light ray. An alternate motivation for property (iv) is the fact that E/hf, where h is

[2]These coordinates are normal coordinates, where the timelike vector of the orthonormal tetrad defining them as tangent to the observer's world line.

[3]An alternate procedure is based on a dimensional analysis of the quantities concerned and rests heavily on assumption about the number and nature of the physical constants involved. See Sommerfeld (1955, pp. 140–145).

Planck's constant, is just the number of quanta contained in the energy element.[4]

Derivation of the Wien Displacement Law. The derivation given is based on a simple device. I will describe it in some detail here since similar devices will be employed twice more in the sections following.

The entropy S of a unidirectional monochromatic element of radiation must be some function of its energy E and frequency f.[5] That is

$$S = G(E,f) \tag{9}$$

for some function G of two variables. But since this law must obey Lorentz invariance, the Lorentz invariant S in (9) must be set equal to another Lorentz invariant. Thus we can reduce the function G to a function G^* of one variable

$$S = G^*(E/f) \tag{10}$$

since E/f is the only Lorentz invariant that can be formed out of E and f (up to an arbitrary function of E/f, which can be trivially absorbed into G^*).

This reduction in the number of variables is essentially all the content of the Wien displacement law. This derivation can be modified to yield the more familiar but less transparent form of the law (2) for the case of black body radiation. The modification requires the three steps indicated and the supplying of the missing mathematical detail.

(*a*) *The replacement of entropy S and energy E with the appropriate entropy and energy densities,* s_{af} and u_{af}, the volume densities per unit frequency and solid angle. Consider an element of unidirectional, monochromatic radiation. s_{af} will be some function of u_{af} and f. This functional relation is required to be Lorentz invariant. Therefore the same relation will hold for the corresponding quantities s'_{af}, u'_{af} and f' of the image under Lorentz boost of the original element of radiation to velocity w in its direction of propagation. It follows that

$$Ds_{af} = (\partial s_{af}/\partial u_{af})Du_{af} + (\partial s_{af}/\partial f)Df$$

where D is the operator $(d/dw)|_{w=0}$. Substituting for Ds_{af}, Du_{af} and Df

[4]Ehrenfest (1911), (1923) also derived the Wien displacement law on the basis of the *adiabatic* invariance of this same ratio E/f. Property (iv) entails the adiabatic invariance of E/f for radiation enclosed in a mirrored chamber.

[5]I ignore the possibility of volume dependence, which will be ruled later by the quantum postulate (v). In any case it is irrelevant to the density form of the law (12) and (13).

using the values in the appendix, we recover

$$(\partial s_{af}/f^2)/(\partial \ln u_{af}) + (\partial s_{af}/f^2)/(\partial \ln f^3) = 0 \tag{11}$$

which is equivalent to requiring that s_{af} satisfy[6]

$$s_{af}/f^2 = H(u_{af}/f^3) \tag{12}$$

where H is an undetermined function of one variable. Equation (12) is the analog of equation (10) with the Lorentz invariants s_{af}/f^2 and u_{af}/f^3 replacing the Lorentz invariants S and E/f.

(*b*) *Conversion to the case of isotropic radiation,* through the substitution of the relations $s_f = 4\pi s_{af}$ and $u_f = 4\pi s_{af}$ into (12). We recover

$$s_f/f^2 = K(u_f/f^3) \tag{13}$$

for some undetermined function K.

(*c*) *Conversion to the case of black body radiation,* which, unlike isotropic radiation *simpliciter,* has a definite temperature T. Using (7) to replace the entropy density s_f in (13) with T, we recover

$$1/T = \partial s_f/\partial u_f = (1/f)K'(u_f/f^3)$$

where K' is the derivative of K with respect to its argument. Inverting the function K' in the second equality yields the Wien displacement law in the form (2).

Unfortunately expressing the law in a form that holds only for isotropic radiation, masks the fundamental connection between it and Lorentz invariance. For example, (13), unlike (10) and (12), no longer explicitly relates a Lorentz invariant to a Lorentz invariant. Neither S_f/f^2 nor u_f/f^3 is a Lorentz invariant. Also the form of the law now varies with the dimensionality of space, unlike the simple form (10). The three-dimensionality of space enters this derivation through the use of the solid angle densities s_{af} and u_{af}. In a two-dimensional space (planar angle densities) or one-dimensional space (no angle densities), the forms of the law corresponding to (13) and (2) would be altered by factors of f and f^2.

Laue (1943) was the first to derive the Wien displacement law from Lorentz invariance arguments. Here I have tightened his procedure somewhat by replacing his juggling of Lorentz invariants with a mechanical procedure which yields the differential equation (12). Unfortunately Laue's

[6]To see the equivalence, note that (12) entails (11) through direct evaluation of the partial derivatives. The converse entailment follows from re-expressing the partial derivatives in (11) as partial derivatives of the two new variables $\ln (u_{af}/f^3)$ and x, where x is any suitably different function of u_{af} and f. Equation (11) then reduces to $(\partial/\partial x)(s_{af}/f^2) = 0$. This entails that s_{af}/f^2 is a function of the variable $\ln (u_{af}/f^3)$ alone, from which (12) follows.

derivation seems to have drawn little lasting interest, although it was probably well known informally in earlier years. In his 1921 *Encyklopaedie* article on relativity for example, Pauli (1958, p. 95) refers to the invariance of E/f with a remarkably cryptic single sentence, "Wien's law is connected with it."

5. Radiation Oscillators. We can recover the expression (4) for the density of radiation oscillators at frequency f up to a multiplicative constant from the properties of radiation posited so far. We consider n_{af}, the volume density of these oscillators per unit frequency and solid angle, and use the following two-step argument.

(*a*) *n_{af} is a function of frequency f only,* even though in principle it could also depend on u_{af} and direction and indirectly on the shape, velocity or other properties of an enclosing cavity.

To see this result, note that in the case of black body radiation, components of the maximum possible number of different frequencies and directions will be present. Therefore n_{af} will take a maximum value. Consider a system of black body radiation whose radiation oscillators cannot exchange energy through any external medium. Now increase or decrease the energy of any of the radiation oscillators present. It follows from property (*iii*) that this change cannot affect the value of n_{af}, since no new oscillators can be formed. Of course we can reduce the energy of any given oscillator to zero. But provided we allow that n_{af} counts such zero energy oscillators, the value of n_{af} still remains unaffected. Since such a manipulation can convert the black body energy distribution into any nominated equilibrium distribution u_{af}, it follows that n_{af} is independent of u_{af}. Finally we have that black body radiation is isotropic and its n_{af} is independent of the properties of an enclosing cavity. Hence the same holds for n_{af} in the general case. Thus n_{af} can depend only on f.

(*b*) *The requirement of Lorentz invariance,* yields the functional dependence of n_{af} on f. In brief the only Lorentz invariant that can be formed out of n_{af} and f is n_{af}/f^2, which must equal a constant. That is, n_{af} must be proportional to f^2. This result is now derived more mechanically.

A system of radiation at equilibrium is Lorentz boosted to velocity w. The density n_{af} of unidirectional radiation oscillators whose directions lie in the direction of the boost must satisfy

$$Dn_{af} = (dn_{af}/df)Df$$

where $D = (d/dw)|_{w=0}$. Substituting for the D terms from the appendix, we have

$$dn_{af}/df = 2n_{af}/f$$

which can be solved directly to yield $n_{af} = A'f^2$ where A' is a constant. Finally from the isotropy of n_{af} we have $n_f = 4\pi n_{af}$ and therefore

$$n_f = Af^2 \tag{14}$$

where $A = 4\pi A'$.

The essential result leading up to (14)—that n_{af} and n_f depend only on f—is already familiar from classical electromagnetism. There n_f is determined solely from the requirement that the wavelength ($=c/f$) of any radiation oscillator present fit an integral number of times into the enclosing cavity. The independence of the result from the shape of the cavity follows from Kirchoff's laws as above. See Bohm (1951, Ch.1), for example.

6. The Quantum Postulate. In order to introduce the quantum postulate in the traditional manner and complete the derivation of the Planck distribution law, we depart from a purely macroscopic thermodynamics and posit that thermodynamic quantities, such as energy, entropy and temperature, are derived from the most probable behavior of a large number of systems. The quantum postulate for radiation is:

(*v*) *Quantum postulate:*
- a. *Radiation oscillators can only take energies that are an integral multiple of an energy element q, that is 0, q, 2q, 3q, . . .*
- b. *In determining the statistical equilibrium of a system of radiation oscillators, each of the accessible energy levels has equal* a priori *probability.*

Consider a system of radiation oscillators that have come to a thermodynamic equilibrium at temperature T. From the usual methods, it follows that the probability $P_f(i)$ that a radiation oscillator of frequency f has energy iq, for i a nonnegative integer, is proportional to $\exp(-iq/kT)$. The average energy E_f of each radiation oscillator of frequency f is $\Sigma\, iqP_f(i)$ which evaluates to expression (5). Combining with the expression (4) for the density of radiation oscillators, we find that the spectral energy distribution of black body radiation at temperature T is

$$u_f = n_f E_f = Af^2(q/(\exp(q/kT) - 1))$$

From a comparison with the form of the distribution law specified by the Wien displacement law (2) we can now conclude that q must depend on f according to

$$q = hf \tag{15}$$

for h a nonnegative constant.

Setting h to some allowed value determines the value of q at any given frequency. Thus we can use h to classify radiation into a one-parameter family of "h-types". $h = 0$ gives us the nonquantized limiting case. Setting $h = 6.63 \times 10^{-27}$ erg sec, the value of Planck's constant, gives us the radiation of our actual world.

The black body radiation law now becomes

$$u_f = Ahf^3\,(1/(\exp(hf/kT) - 1)) \tag{16}$$

The value of the constant A, first introduced in the radiation oscillator density equation (14) above, can be determined by a limiting property of radiation:

(vi) *Classical limit property: Radiation for which* $q = h = 0$ *is classical electromagnetic radiation.*

By comparing equation (14) with its classical limiting case, equation (4), we conclude that $A = 8\pi/c^3$ through which equation (16) reverts to the Planck distribution law (6).

7. Generalized Planck Resonators. The electric resonators of the old quantum theory shall be represented here as "generalized Planck resonators", with the following properties:

(i') *Rest mass property: Generalized Planck resonators are systems containing energy of nonzero rest mass.*

(ii') *Resonant frequency property: There is a family of types of generalized Planck resonators, parameterized by the positive real valued index "resonant frequency,"* f_r.

(iv') *Resonant frequency characterization: The ratio of energy* U *to resonant frequency* f_r *of a generalized Planck resonator is a Lorentz invariant.*

As the numbering indicates, the properties of generalized Planck resonators are closely analogous to those of radiation. The principal difference lies in their differing rest mass properties. In addition I have not specified any properties pertaining to the spatial superposition of the resonators, since we shall only deal with resonators at different spatial locations. Thus no assumptions analogous to property (iii) for radiation need be made.

Proceeding as in the case of radiation, we can now derive a result corresponding to the Wien displacement law. In general, the entropy S of a generalized Planck resonator can be a function of its energy U, its resonant frequency f_r and its velocity v.

$$S = G(U, f_r, v)$$

for some function G. As before, the requirement of Lorentz invariance will lead us to replace this general function G by a function of Lorentz invariants only, so that the Lorentz invariant S is itself set equal to a Lorentz invariant.

The need to consider a velocity as an argument of the entropy function did not arise in the radiation case, which makes the present calculation more complex. But the result is much the same. For we shall see that the requirement of Lorentz invariance eliminates v from the arguments of G and reduces the function to one of a single variable, U/f_r. This last quantity is the only Lorentz invariant that can be constructed from the three quantities U, f_r and v—a result which is by no means obvious, but returned to us quite mechanically from the calculation which follows.

Consider a generalized Planck resonator Lorentz boosted by a velocity w in the direction of its own velocity v. From the requirement of Lorentz invariance, it follows that

$$DS = (\partial S/\partial U)DU + (\partial S/\partial f_r)Df_r + (\partial S/\partial v)Dv$$

where $D = (d/dw)|_{w=0}$ as before. We have from the appendix that

$$DS = DU = Df_r = 0, \quad Dv = 1 - v^2/c^2$$

After substituting these values we recover

$$\partial S/\partial v = 0$$

from which it follows that S is not a function of v.

To complete the calculation, we must consider D^2S, where $D^2 = (d^2/dw^2)|_{w=0}$. Allowing that $DU = Df_r = 0$, we have

$$D^2S = (\partial S/\partial U)D^2U + (\partial S/\partial f_r)D^2f_r$$

Substituting for the D^2 terms from the appendix, we have

$$\partial S/\partial \ln U + \partial S/\partial \ln f_r = 0$$

This is equivalent to requiring that S be some function H of the single variable U/f_r

$$S = H(U/f_r) \tag{17}$$

which is the simplest form of the Wien displacement law for resonators, analogous to equation (10) for radiation. We can now replace S by the temperature T through the relation $dS/dU = 1/T$. Differentiating (17) with respect to U we have

$$1/T = f_r H'(U/f_r)$$

where H' is the derivative of H with respect to its argument. Inverting the function H' we recover the Wien displacement law for generalized Planck resonators in its final form

$$U/f_r = g(f_r/T) \tag{18}$$

where $g = (H')^{-1}$ is some function of a single variable.

Generalized Planck resonators are also governed by a quantum postulate:

(v') *Quantum postulate for generalized Planck resonators:*
- *a. Generalized Planck resonators can only take energies that are an integral multiple of an energy element q_r, that is 0, q_r, $2q_r$, $3q_r$, . . .*
- *b. In determining the statistical equilibrium of a system of generalized Planck resonators, each of the accessible energy levels has equal* a priori *probability.*

Following a procedure analogous to that of the previous section we now find that the average energy U of a set of resonators at temperature T is given by

$$U = q_r/(\exp(q_r/kT) - 1)$$

It follows immediately from the Wien displacement law for resonators (18) that q_r must depend on f_r according to

$$q_r = h_r f_r \tag{19}$$

for h_r a nonnegative constant. Thus U is equal to

$$U = h_r f_r(\exp(h_r f_r/kT) - 1) \tag{20}$$

We can now classify generalized Planck resonators according to the value of h_r into h_r-types. Planck's original classical resonators are of type $h_r = 0$, since they can adopt a continuous range of energy levels.

So far, generalized Planck resonators have been treated independently of their interaction with radiation. To complete their treatment we now need to recover the Planck resonator formula (3) and show that it holds for both quantized and classical systems.

This formula applies to resonators in equilibrium with radiation. To recover it, it must be posited that resonators in equilibrium with radiation satisfy two compatibility conditions. The first ensures that the quantization of any resonator and radiation system which interact are well adapted. It will forbid, for example, any interaction between a nonquantized resonator ($h_r = 0$) and quantized radiation ($h > 0$). The second determines which f_r-types of resonators can come to equilibrium with which f-types of radiation.

(*vii*) *Compatibility conditions:*

a. A generalized Planck resonator of type h_r can only interact with a radiation system of type h, if $h_r = h$.

b. A generalized Planck resonator of resonant frequency f_r can only come to equilibrium with radiation of frequency f, provided $f_r = f$ in the resonator's rest frame.

Since the *a.* condition ensures that the h_r and h values of any interacting system of resonator and radiation will agree, we can speak of the h value and h-type of any such system as a whole and for most purposes drop the distinction between h_r and h.

The *b.* condition must be limited to the rest frame of the resonator since the equation $f_r = f$ can only hold in one frame, due to the different transformation properties of f_r and f. The two conditions combined entail that $q_r = q$ in the resonator rest frame. It also follows from *b.* that a resonator can only reach equilibrium with radiation oscillators of a single frequency (measured in the resonator rest frame) but with arbitrary directions of propagation.

Consider a volume V of radiation of frequency f in equilibrium with N generalized Planck resonators at rest with respect to one another. Let the system increase in temperature quasi-statically so that the radiation frequency remains constant and the radiation remains in equilibrium with the resonators. The rate of increase of total radiation energy with temperature T is Vdu_f/dT and the rate of increase of total resonator energy is NdU/dT. These quantities can be evaluated through equations (6) and (20) respectively. With the application of the compatibility conditions, it then follows from these evaluations that

$$(c^3/8\pi f^2)(du_f/dT) = d/dT(hf/(\exp(hf/kT) - 1)) = dU/dT$$

in the resonators' rest frame. It then follows that

$$du_f/dU = 8\pi f^2/c^3$$

Upon integration, recalling that the frequency f is held constant during the change of temperature, we recover

$$u_f = (8\pi f^2/c^3)\, U + \text{constant}$$

The constant of integration can be set to zero to recover (3) by noting that as T approaches zero, it follows from equations (6) and (20) that u_f and U also approach zero.

8. A Macroscopic Quantum Postulate. Both the original and reconstructed theories of radiation are essentially macroscopic thermodynamical theories. However with the introduction of the quantum postulate $(v)/(v')$ to enable derivation of the energy formula (5), an excursion is made into the statistical mechanical reduction of thermodynamics. What is not entirely satisfactory about this excursion is the paucity of the microscopic picture involved. For example there is no clear picture of the time evolution of each microscopic component system. This picture is sometimes provided by adopting the phase space of the corresponding classical system and dividing it into quantum cells. But the use of this hybrid phase space simply restates the original inconsistency in more sophisticated form. Thus I shall now ask how the statistical quantum postulate $(v)/(v')$ can be replaced by a purely macroscopic postulate.

An elegant macroscopic postulate can be found for the classical limiting case of $q = 0$ of the quantum postulate:

> *Equipartition postulate: The energy of a generalized Planck resonator and radiation oscillator depends only on its temperature.*

This assumption is made routinely but rarely explicitly in the standard statistical derivation of the equipartition theorem. Given the Wien displacement law, the above equipartition postulate already forces the equipartition energy up to a multiplicative constant. In the case of a generalized Planck resonator, the independence of energy U from resonant frequency f_r enables the identification of the undetermined function g in the Wien displacement law (18) as the inverse function (up to a multiplicative constant). Thus $U = KT$, where K is an undetermined constant. A similar argument holds in the case of radiation oscillators.

I have been unable to find a macroscopic quantum postulate that is equally simple. Perhaps the quantum postulate finds its most informative expression in the statistical form of postulate $(v)/(v')$. Indeed it has been known from the early days of quantum theory that the discreteness of energies in postulate $(v)/(v')$ is not only sufficient but *necessary* for the Planck distribution law, under a quite general statistical model. See Poincaré (1911), (1912) and Ehrenfest (1911).

To see the result, let $g(E)$ be the relative *a priori* probability or weight of energy E for a radiation oscillator.[7] Consider a system of oscillators

[7]The considerations of the remainder of this section apply equally to generalized Planck resonators. Their differing transformation properties are irrelevant here.

of the same frequency at thermal equilibrium. Defining the partition function

$$Z(t) = \int_0^\infty g(E) \exp(-Et)\, dE \tag{21}$$

where $t = 1/kT$, it follows that the average energy of each oscillator is $E = -d/dt(\ln Z)$. If E has the quantum value (5), it follows that

$$\begin{aligned} Z(t) &= 1/(1 - \exp(-qt)) \\ &= 1 + \exp(-qt) + \exp(-2qt) + \exp(-3qt) + \ldots \end{aligned} \tag{22}$$

up to an arbitrary multiplicative constant. We recover the weight function associated with this partition function by noting that $Z(t)$ as defined in (21) is just the Laplace transform of $g(E)$. Inverting (22) term by term we recover

$$g(E) = \delta(E) + \delta(E - q) + \delta(E - 2q) + \delta(E - 3q) + \ldots \tag{23}$$

where δ is the Dirac delta function. (23) amounts to the quantum postulate (v).

The fact that $Z(t)$ is just the Laplace transform of $g(E)$ affords a natural means of finding a macroscopic quantum postulate.[8] We can take the micro-quantum postulate expressed in terms of $g(E)$ in (23) and use the Laplace transformation to translate it into a macro-postulate in terms of $Z(t)$.[9]

First note that (23) is ec ent to the two requirements

I. $g(E)$ has period q: $g(E + q) = g(E)$ for $E \geq 0$
II. The period integral $J = \int_0^q \exp(-Et)\, g(E)\, dE$ is independent of temperature t.

The equivalence with (23) depends on the fact that the Laplace transform of the periodic condition I. is

$$Z(t) = J/(1 - \exp(-\mathrm{qt})). \tag{24}$$

Second, given a system with *any* $E(T)$ and corresponding $Z(t)$, we can

[8] An alternate route to a macro quantum postulate rests on positing that the inverse temperature heat capacity dU/dt is independent of frequency in the equipartition case and varies linearly with it in the general quantum case. Its connection with the traditional micro-postulate is less clear however.

[9] In spite of its micro origins, the partition function $Z(t) = -\exp \int E(t)dt$ is just as much a macro-quantity as energy and entropy.

describe a "q-shifted" system satisfying $E^q(T) = E(T) + q$ for all T by setting

$$Z^q(t) = \exp(-qt)Z(t) \tag{25}$$

Finally, combining (24), the Laplace transform of the periodic condition I., with (25) and condition II., we recover a macroscopic translation of (23):

Macroscopic quantum postulate: Each radiation oscillator has a characteristic energy element q such that $(Z^q - Z)/q$ is independent of temperature.

In the limit of $q = 0$, this postulate yields an alternate equipartition postulate in which the macro-postulate's finite difference constraint is replaced by a differential constraint:

Equipartition postulate: $(dZ^q/dq)|_{q=0}$ is independent of temperature.

This macro version of the equipartition postulate is a translation of the equivalent micro-postulate, which requires that $g(E)$ is a constant.

9. Conclusion. This case study shows that the logical inconsistency of a theory need not automatically render it physically uninteresting. The analysis provides a strategy for dealing with such inconsistency: the elimination of the inconsistency discovered by the extraction of a subtheory free of it but which still contains the results of interest.

This stragey must be used with some care. Classically any inconsistent theory contains all propositions. Therefore trivially every inconsistent theory contains a subtheory with any nominated result of interest and which is potentially free of inconsistency. For example, every inconsistent theory contains a subtheory whose sole proposition is the Planck distribution law.

This trivial strategy was not used here. The old quantum theory of black body radiation was logically inconsistent in so far as it contained both a proposition P and its negation *not-P*. But one could not derive any proposition within the theory because of the tacit introduction of a nonclassical device, the two domains of calculation with inarticulated restrictions on the exchange of results between them.

In any case, the subtheory was constructed prior to the appearance of the manifest logical inconsistency. This inconsistency arose only with the conjoining of the original theory's four components described in the introduction. The subtheory was produced by first taking a smaller subtheory of the third component, classical electrodynamics, and only then conjoining the four components, their conjunction now being free of the original inconsistency.

APPENDIX: LORENTZ BOOST FORMULAE

An element of monochromatic radiation of zero rest mass energy E, entropy S, volume V and frequency f has its directions of propagation enclosed within a very small conically shaped solid angle a and contains n radiation oscillators. Under a Lorentz boost to a velocity w in the average direction of propagation, these quantities take the primed values

$$E' = E\sqrt{(1 + w/c)/(1 - w/c)} \quad f' = f\sqrt{(1 + w/c)/(1 - w/c)} \quad S' = S$$

$$V' = \sqrt{V(1 - w/c)/(1 + w/c)} \quad a' = a(1 - w/c)/(1 + w/c) \quad n' = n$$

Note that the expression for V' is peculiar to volumes that propagate at c. Thus the volume densities per unit frequency and solid angle of E, S, and n transform respectively as

$$u'_{af} = u_{af}((1 + w/c)/(1 - w/c))^{3/2}$$

$$s'_{af} = s_{af}(1 + w/c)/(1 - w/c)$$

$$n'_{af} = n_{af}(1 + w/c)/(1 - w/c)$$

It follows that u_{af}/f^3, s_{af}/f^2 and n_{af}/f^2 are Lorentz invariants.

If U is the energy of a generalized Planck resonator of resonant frequency f_r and velocity v, then under Lorentz boost by velocity w in the direction of v, we have:

$$U' = U/\sqrt{1 - w^2/c^2} \quad f_r' = f_r/\sqrt{1 - w^2/c^2} \quad v' = (v + w)/(1 + vw/c^2)$$

Defining the operators $D = (d/dw)|_{w=0}$ and $D^2 = (d^2/dw^2)|_{w=0}$, we find

$$Du_{af} = 3u_{af}/c \quad Ds_{af} = 2s_{af}/c \quad Dn_{af} = 2n_{af}/c$$

$$DU = 0 \quad D^2U = U/c^2 \quad Df = f/c$$

$$Df_r = 0 \quad D^2f_r = f_r/c^2 \quad Dv = 1 - v^2/c^2$$

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