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THE DETERMINATION OF THEORY BY EVIDENCE: THE CASE FOR QUANTUM DISCONTINUITY, 1900–1915

ABSTRACT. The thesis that observation necessarily fails to determine theory is false in the sense that observation can provide overwhelming evidence for a particular theory or even a hypothesis within the theory. The saga of quantum discontinuity illustrates the power of evidence to determine theory and shows how that power can be underestimated by inadequate caricatures of the evidential case. That quantum discontinuity can save the phenomena of black body radiation is the widely known result, but it leaves open the possibilities of other accounts. That these phenomena, with the aid of minimal assumptions, entail quantum discontinuity is the crucial but now largely forgotten result. It was first demonstrated by Ehrenfest and Poincaré in 1911 and 1912.

1. INTRODUCTION AND SUMMARY

1.1. *Practical Science and the Underdetermination Thesis*

There is a serious contradiction between a thesis increasingly popular amongst philosophers of science and the proclamations of scientists themselves. The underdetermination thesis asserts that a scientific theory cannot be fully determined by all possible observational data. Scientists, however, are not so pessimistic about the power of observational data to guide theory selection. The history of science is full of cases in which they urge that the weight of observational evidence forces acceptance of a definite theory and no other. Thus our science text books teach us to accept the approximate sphericity of the earth, the heliocentric layout of planetary orbits, the oxygen theory of combustion, and a host of other theoretical claims simply because the evidence admits no alternatives.

The case for the underdetermination thesis depends in large measure on an impoverished picture of the ways in which evidence can bear on theory. The thesis is commonly advanced with tacit use of some kind of a hypothetico-deductive view of confirmation. Consider, for example, the version of the underdetermination thesis laid out for dissection in Quine (1975, p. 313):

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> Scientists invent hypotheses that talk of things beyond the reach of observation. The hypotheses are related to observation only by a kind of one-way implication; namely, the events we observe are what a belief in the hypotheses would have led us to expect. These observable consequences of the hypotheses do not, conversely, imply the hypotheses. Surely there are alternative hypothetical substructures that would surface in the same observable ways.

The underdetermination discussed depends on this shielding of the hypotheses from observation by the barrier of one-way implication. The supposition of this barrier gives the theorist sufficient latitude to entertain other hypotheses logically incompatible with the original but equally able to save the phenomena.

My thesis in this paper is that this barrier of one-way implication can be broken and that one can effect the converse implication *from* observation *to* hypotheses or *from* evidence *to* theory. The result is that a body of evidence can point to a definite theory or even individual theoretical hypothesis. In the case study that follows I shall show that this converse implication was used within recent science explicitly for the purpose of defeating the underdetermination thesis. To carry out this converse implication one needs to supplement the observations or evidence with further hypotheses. In the case study that follows we shall see that these further hypotheses can be of such a general and uncontroversial nature that the acceptance of the theory picked out is placed beyond reasonable doubt.

1.2. *The Case of Quantum Discontinuity*

I shall illustrate the power of evidence to determine theory with a case study of the advent of quantum discontinuity in the early part of this century. The theoretical result at issue is usually associated with the name of Planck and loosely formulated as the result that systems on an atomic scale can only adopt a discrete spectrum of energy levels. This result stands in contradiction to the basic suppositions of classical physics. Its acceptance marked the demise of classical physics and ushered in the quantum era.

The case study is especially interesting for a number of reasons. One might expect a case for the necessity of a theoretically deep result such as quantum discontinuity on the observational evidence to be exceedingly complicated. It turns out that the case can be based on a small group of arguments of great simplicity and generality yet startling in their power to force a definite result. That result, quantum discontinuity, was of revolutionary character and strongly resisted. As a result the arguments for it had to – and did – survive critical scrutiny by some of the most eminent and capable physicists of the day, who deemed all manner of conservative alternative to be preferable to quantum discontinuity.

The evidence will also be seen to pick out not just a theory but a particular result that must obtain in any theory that may be applied in the domain. This gives us a concrete and non-trivial instance of how evidence can bear not only on a theory as a whole but also on a particular result within that theory, in contradiction with holistic views of evidence such as the Duhem–Quine thesis.

Finally this case will show us how the collective amnesia of science readily leads later researchers to forget just how powerful and uncompromising the original case was for the result in question so that the later reconstructions of the case can be oversimplified and weakened. Broadly speaking, three classes of results are of importance in the case for quantum discontinuity. The first two are:

> I. Classical physics fails to account for certain phenomena, including the distribution of energy in black body radiation and the specific heats of substances at low temperatures.
> II. The supposition of quantum discontinuity enables a simple and elegant treatment of these same phenomena.

These first two groups of results figure in the cases for quantum discontinuity now to be found in text book expositions. Typical of these treatments is that found in Bohm's (1951) well-regarded text of the 1950s. His first chapter, "The Origin of the Quantum Theory", gives a lucid exposition of the results of I and II and announces in summary (p. 22): "We may conclude that all systems which oscillate harmonically are quantized with $E = hn\nu$ whether these systems be material oscillators, sound waves or electromagnetic waves".[1] A critical reader, such as a philosopher of science concerned with the underdetermination thesis, may well be unconvinced. The failure of the classical account can be accepted. The sufficiency of quantum discontinuity to account elegantly for the phenomena can be accepted. But what guarantee is there that there is no other account that does as well or even better? Does the evidence uniquely pick out the conclusion claimed?

In 1910–1912, this very question became the focus of attention of a number of researchers, notably Paul Ehrenfest, James Jeans, and Henri Poincaré. They were able to rule out conclusively such alternatives to quantum discontinuity with a powerful group of results:

> III. On a very general statistical model of matter one can infer directly *from* the observed distribution of energy in black body radiation – or even just its finiteness – *to* quantum discontinuity.

What made the results especially powerful was that they were essentially immune to experimental errors that might be made in measuring the observed distribution. The strongest result of the group took the observed distribution to be exactly that of the celebrated Planck formula. Other results, however, showed that quantum discontinuity still followed in a weaker form even if the correct distribution deviated significantly from the Planck formula, as long as the total energy density of the radiation was finite. As we shall see below, this group of results became central to the case made for quantum discontinuity in the early 1910s, the time in which this discontinuity came to the forefront of discussion in physics. Yet the results, for all their significance, are now largely forgotten, except by a few historians of science.

1.3. *The Relocation of Inductive Risk from Rules to Premises*

The results in III exemplify a powerful strategy for assessing the bearing of evidence on theory. As I have indicated, the results in I and II leave us uncertain of the precise bearing of the evidence of the observed black body spectrum on quantum discontinuity and especially of the degree to which a unique result is determined. To see this more clearly, notice that in II we see that quantum discontinuity in conjunction with suitable auxiliary hypotheses and boundary conditions entails a description of the observed phenomena in a deductive argument of the form:

ARGUMENT 1:

Quantum discontinuity
Auxiliary hypotheses and boundary conditions
——————————————————— (deduction)
Observed black body spectrum.

Thus the observed black body spectrum stands as evidence for quantum discontinuity according to the hypothetico-deductive (HD) model:

ARGUMENT 2:

Observed black body spectrum
——————————————— (inductive support, HD model)
Quantum discontinuity.

Because of a notorious weakness of the HD scheme, this second inductive argument leaves completely open the question of whether that same body of evidence might support a competing theoretical result equally well or better. For example, the premises of arguments like Argument 1 can often be modified in not too contrived a manner without compromising the entailment of the observational data. In such cases we can generate several alternative hypotheses, each equally able to save the relevant phenomena. Thus, according to the HD scheme, each of these alternatives is supported by the observational evidence and the scheme provides no way to pick between them. Because of this deficiency, the HD scheme readily invites precisely the underdetermination of theory by evidence at issue here.

The results of III enable a much clearer evaluation of the bearing of the evidence on the theoretical result. We now have the following deductive argument:

ARGUMENT 3: Observed black body spectrum
General statistical model of matter
Auxiliary hypotheses
——————————————— (deduction)
Quantum discontinuity.

It shows that acceptance of the observed black body spectrum *necessitates* acceptance of quantum discontinuity, provided one is prepared to accept the general statistical model of matter and the auxiliary hypotheses. The question of whether the observed black body spectrum does determine a unique theoretical result is now reduced to an assessment of the acceptability of the general statistical model and the auxiliary hypotheses. It will turn out, as we shall see in Section 8, these latter results are of sufficiently weak and general form that no credible challenge to their acceptance could be mounted. As a result, the necessity of quantum discontinuity on the evidence of the observed black body spectrum was deemed unavoidable.

The crucial move in this reassessment of the burden of evidence was a relocation of the inductive risk taken in inferring from the observed black body spectrum to quantum discontinuity. In Argument 2 that risk is located in the HD scheme itself, in so far as an assessment of the strength of the inference depends on an assessment of the strength of the HD scheme as applied to this example. Such an assessment is extremely problematic, throwing us into the murky depths of general confirmation theory. In Argument 3, however, the inductive risk is relocated in the premises added to convert Argument 2 into a deductive argument. The assessment of the inductive risk taken in accepting these new premises is by no means easy, but in this case it proved to be a great deal simpler than assessing the viability of a general inductive argument scheme.

This strategy of relocating inductive risk from rules to premises has a place in the traditional literature on inductive inference. Argument 3 is deductive and thus demonstrative. Nonetheless, arguments of this form have been classified with inductive argumentation where they are known as "demonstrative induction" (Johnson, 1964, p. 210). In such arguments one infers from premises of lesser generality and premises of greater generality to conclusions of intermediate generality. In the example, the premises of lesser generality are those specifying the observed black body spectrum. Those of greater generality specify the general statistical model of matter and the auxiliary hypotheses. The conclusion of intermediate generality is quantum discontinuity.

Demonstrative induction is closely related to another form of inference, eliminative induction. Viewed extensionally, the premises of greater generality of a demonstrative induction can be seen to specify a universe of candidate theories. The premises of lesser generality eliminate all but a select few of these theories that are specified in the conclusion. Using the eliminative induction form, we can express the essential content of Argument 3 in a compelling and telegraphic manner: of all possible statistical theories of black body radiation, only those that posit quantum discontinuity can do justice to the observed spectrum or even just its finiteness.[2]

2. BACKGROUND TO THE PROBLEM OF QUANTUM DISCONTINUITY CIRCA 1910[3]

The problem that gave rise to the introduction of quantum discontinuity concerned the determination of the density of energy u_ν at frequency ν in black body radiation of temperature T. By 1900, it was accepted that the function f, in the general form of the distribution law

$$(1) \qquad u_\nu = f(\nu, T)\,,$$

was constrained in the following ways by the electromagnetic character of the heat radiation. The Stefan–Boltzmann law of 1879, 1884 required the total energy density to increase with the fourth power of temperature

$$(2) \qquad u = \int_0^\infty u_\nu \, \mathrm{d}\nu = \sigma T^4$$

for some constant σ, and the Wien displacement law of 1894, which entailed the Stefan–Boltzmann law, in effect required that the function $f(\nu, T)$ could be replaced by a function $f(\nu/T)$ of a single variable according to

$$(3) \qquad u_\nu = \nu^3 f\left(\frac{\nu}{T}\right).$$

Further, a direct application of the principles of classical electrodynamics and statistical mechanics could then lead to a definite function for f.

In the most direct approach, Rayleigh (1900, 1905) and Jeans (1905a) pictured a system of black body radiation as a superposition of electromagnetic waves of all frequencies, freely interchanging energy. Each "mode of vibration" or, as I shall call it, 'radiation oscillator' behaves as the individual molecules of a kinetic gas so that the standard methods of statistical mechanics can be used to determine their average energy. The function f in (3) is determined by this average energy in conjunction with the result that the volume density n_ν of radiation oscillators is

$$(4) \qquad n_\nu = \frac{8\pi\nu^2}{c^3}\,,$$

where c is the speed of light. In the research leading up to his seminal contribution of 1900, Planck (1900a, 1990b) had modeled black body radiation enclosed within a cavity as a system of electromagnetic radiation in equilibrium with Hertzian electric resonators within the cavity. Using principles of classical electrodynamics he derived a relationship between the energy density u_ν of black body radiation and the energy U of a resonator with the resonant frequency ν:

(5) $$u_\nu = \frac{8\pi\nu^2}{c^3} U$$

so that once the average energy U of the resonators at each frequency was known, the function f of (3) could be inferred.

Both approaches yield the same result.[4] If the phase space of an individual radiation oscillator or Hertzian resonator has canonical variables $x_1, \ldots, x_n$, then its energy E at temperature T will be distributed according to the Boltzmann distribution. The probability dW that the system is in the volume element $dV = dx_1 \ldots dx_n$ of phase space is

(6) $$dW = C \exp\left(-\frac{E}{kT}\right) dV,$$

where k is Boltzmann's constant and C is a normalizing constant set to ensure that the total probability sums to unity. Alternatively, this can be expressed as the probability dW that the system has energy between E and $E + dE$

(7) $$dW = C \exp\left(-\frac{E}{kT}\right) \omega(E)\, dE,$$

where $\omega(E) = dV/dE$ and V is the volume of phase space enclosed by the surface of constant energy E.

The Boltzmann distribution entails the result that dominates all classical treatments of black body radiation, the equipartition theorem. For systems whose energy E is a quadratic function of the phase space variables

(8) $$E = \alpha_1 x_1^2 + \cdots + \alpha_n x_n^2$$

for constants $\alpha_1, \ldots, \alpha_n$, it asserts that the mean energy of each system is

(9) $$\bar{E} = n\frac{kT}{2},$$

or, in slogan form, there is an energy of $kT/2$ for each of the n degrees of freedom of the system. Special cases of the theorem are classical radiation oscillators or Hertzian resonators, which have two degrees of freedom. For either case we have

$$(10) \qquad \omega(E) = \text{constant},$$

and the mean energy is given by

$$(11) \qquad \bar{E} = \int_0^\infty EC \exp\left(-\frac{E}{kT}\right) \mathrm{d}E = kT.$$

This mean energy, in conjunction with either (4) or (5), yields the so-called Rayleigh–Jeans law for (1)

$$(12) \qquad u_\nu = \frac{8\pi\nu^2}{c^3} kT.$$

The law is immediately unsatisfactory, as both Rayleigh and Jeans recognized. It entails that the total energy density in black body radiation and its heat capacity is infinite. This infinitude actually follows directly from the equipartition theorem (9) and the fact that radiation has infinitely many degrees of freedom:

Total energy = ($kT/2$ per degree of freedom) ×
(infinitely many degrees of freedom),

so that an escape from this infinity must involve an escape from the equipartition theorem itself. Nonetheless, the Rayleigh–Jeans law fitted with the observed energy densities for small ν/T.

The best-known formula for the distribution law that did agree with the observed values and led to a finite total energy was introduced by Planck (1900a, 1900b). He had previously favored a distribution law due to Wien,

$$(13) \qquad u_\nu = \frac{8\pi h\nu^3}{c^3} \exp\left(-\frac{h\nu}{kT}\right),$$

where h is Planck's constant. Planck modified this law to obtain agreement with experimental values in the domain of small ν/T to yield his celebrated formula

$$(14) \qquad u_\nu = \frac{8\pi\nu^2}{c^3} \frac{h\nu}{\exp\frac{h\nu}{kT} - 1}.$$

Planck was able to derive this formula by using Boltzmann's statistical methods to conclude that the mean energy of a Hertzian resonator is

$$\bar{E} = \frac{h\nu}{\exp\frac{h\nu}{kT} - 1}. \tag{15}$$

What Planck's (1900b, 1901) early expositions of this derivation did not make clear was that the disastrous classical equipartition result (9) was avoided by introduction of the supposition of quantum discontinuity: the energies accessible to the system are limited to the discrete set 0, ϵ, 2ϵ, 3ϵ . . . for some non-zero energy element ϵ, with equal a priori probability.[5] In a later notation we can represent this assumption of quantum discontinuity as

$$\omega(E) = \delta(E) + \delta(E-\epsilon) + \delta(E-2\epsilon) + \delta(E-3\epsilon) + \cdots, \tag{16}$$

where δ is the Dirac delta.[6] The sole effect of quantum discontinuity on the classical calculation is to replace the integration in (11) by a summation over discrete values

$$\bar{E} = \sum_{E=0,\epsilon,2\epsilon\ldots} EC\exp\left(-\frac{E}{kT}\right)\mathrm{d}E = \frac{\epsilon}{\exp\frac{\epsilon}{kt} - 1}. \tag{17}$$

Compatibility with the Wien displacement law (3) forces $\epsilon = h\nu$, for some constant h, whose value can be set so that (17) yields (15).

By 1910 the sufficiency of quantum discontinuity for enabling derivation of the Planck distribution law (14) was widely recognized. It could be applied directly either to the radiation oscillators themselves or to the Hertzian resonators, yielding the mean energy (17) and the Planck law then recovered through (4) or (5). Derivations of the former type were given, for example, by Ehrenfest (1906) and Debye (1910), and of the latter type by Einstein (1907) and Lorentz (1910).

3. THE ALTERNATIVE STRATEGY: BEYOND SAVING THE PHENOMENA

The results outlined in Section 2 above were widely known by 1910.[7] Of course what was in dispute was their significance. Was the ability of quantum discontinuity to save the phenomena sufficient grounds for its acceptance? A conservative mainstream answered no. Jeans (1905b, p. 294) conceded that "Planck's law is in good agreement with experiment if h is given a value different from zero". However, he continued, "this does not alter my belief that the [classical] value $h = 0$ is the only value which it is possible to take". Again, Lorentz (1909, §59) in his influential *Theory of Electrons* could present the basic result of quantum discontinuity and then pass immediately without excuse to a fully classical approach to the problem of heat radiation.

Ehrenfest (1906) displayed precisely the difficulty.[8] One could not rule out the possibility of many ways of saving these same phenomena of black body radiation and he proceeded to show precisely how this possibility might be realized. He recapitulated the standard statistical calculation used to derive the Rayleigh–Jeans distribution law (12) via the Boltzmann distribution (6) and (7). It involved seeking a distribution that would maximize the entropy of the total system of radiation oscillators subject to two constraints: constancy of the number of radiation oscillators ("I") and constancy of total energy ("II"). One could not rule out the addition of further constraints, he continued, that might express some special physical property of the system. As an illustration, Ehrenfest augmented the constraints I and II with a third arbitrarily chosen constraint, "III", containing an arbitrary function, Φ. The standard calculation now yielded an alternative to the distribution (6) and (7) containing the arbitrary function Φ. He concluded in emphasized text (p. 5):

> Any arbitrarily desired spectral distribution can be computed in infinitely many different ways by the adjoining of an appropriate relation III.

He continued to note that any new constraint III added must have a physical foundation. Ehrenfest then proceeded directly to the case of the Planck law (14). Of the arbitrarily many additional constraints III that would yield the law, he reported that Planck had chosen the supposition of "energy atoms".[9]

Nonetheless there still remained the empirical success of Planck's formula (14) and the curious result that it could be derived so simply from the strange supposition of quantum discontinuity. Even if this last result could not force acceptance of a supposition that contradicted the deepest foundations of the edifice of classical physics, it was one that could not be ignored. An alternative strategy was needed to interpret the significance of Planck's formula as evidence. That strategy was simple in concept. *Do not ask what suppositions entail the observed results. Rather, ask what can be inferred directly* from *the observed results.*

While he did not use it exclusively, the master of this strategy was Einstein. He had been using it on the observations concerning black body radiation since 1905 and had used it to introduce notions even more revolutionary than quantum discontinuity. Einstein (1905) assumed the Planck formula correct and from it inferred the volume dependence of the entropy of radiation in the high frequency domain. That volume dependence yielded directly his light quantum hypothesis: high frequency radiation behaves thermodynamically as a collection of mutually independent quanta of energy of size $h\nu$.[10] Later, Einstein (1909a), from the Planck law, inferred expressions for fluctuations in the energy and the radiation pressure of black body radiation and showed them to be a sum of two terms, one of wave character, one of particle character, displaying the necessity of both wave and particle pictures in any complete treatment of radiation. Einstein (1909b, pp. 495–96) summarized the strategy:[11]

> Is it not conceivable that the radiation formula given by Planck is correct but nevertheless that a derivation of it could be given that did not depend on so horrendous looking an assumption as Planck's theory? Would it not be possible to replace the light quantum hypothesis by another hypothesis with which one can still account for the known phenomena? . . . To clarify these matters, we would like to try to proceed in the direction opposite to that of Planck in his theory of radiation. We regard Planck's radiation formula as correct and ask ourselves whether something can be inferred from it concerning the constitution of radiation.

4. THE ARGUMENT OF JAMES JEANS

Einstein's strategy could be applied in the case of quantum discontinuity as embodied in (16) and with striking success. The earliest results of such an analysis were published in 1910 by James Jeans, who was soon to become one of the most outspoken proponents of quantum discontinuity, after several years of attempting to preserve a classical account.[12] The main purpose of Jeans (1910) was to demonstrate the impossibility of a classical equilibrium account of black body radiation through a derivation of a generalized form of the equipartition theorem from very weak and general classical assumptions. Unlike the standard form of (8) above, for example, he assumed that the energy of each system was just a homogeneous function of degree s in the coordinates recovering, for a system of m degrees of freedom,

$$\bar{E} = m\frac{kT}{s}$$

in place of the standard (9). Since this mean energy per degree of freedom was still a constant and heat radiation possessed infinitely many degrees of freedom, the infinitude of its energy was still derivable.

The paper concluded with an attempt to derive quantum discontinuity from the Planck law in accord with the strategy Einstein described above. That attempt lacked the rigor and the mathematical brilliance that would shortly be applied to the task. It amounted to a simple inversion of Planck's (1901) original derivation of the Planck law. Jeans began with the quantum expression (15) for the energy of a radiation oscillator. He considered the total energy E of N oscillators and, using the relation $\partial S/\partial E = 1/T$, recovered an expression for the entropy S of the whole system. An application of Boltzmann's $S = k \log W$ yielded an expression for the volume W of phase space associated with the state of the system:

$$\log W = \left(N + \frac{E}{h\nu}\right)\log\left(N + \frac{E}{h\nu}\right) - \frac{E}{h\nu}\log\frac{E}{h\nu} + \text{cons[tant]}. \tag{18}$$

In Planck's original derivation, this value of $\log W$ was generated by taking W to be the number of (presumed equiprobable) ways that an amount of energy E divided into $P = E/h\nu$ energy elements of size $h\nu$ could be distributed over N systems. (In Planck's case, they were Hertzian resonators.) Since Planck never takes the limit of these energy elements becoming vanishingly small, this marks the introduction of quantum discontinuity into his derivation. The correct expression for W is

$$W = \frac{(N + P - 1)!}{(N - 1)!P!} \tag{19}$$

and it yields the expression (18) for $\log W$ when the factorials are approximated by exponentials through Stirling's formula. Jeans (1910, p. 953) reversed this last step to recover an expression for W, urging that it "follows inevitably" from (15) and finally "the necessity for an indivisible unit of energy follows inevitably from [(19a), below], for Planck's assumption of this indivisible unit is known to lead to formula [(19a)], and there can be only one way of distributing the fluid [of system state points] in the generalized space so that W is a given function of E for all values of E".[13]

While the derivation Jeans intended is obvious, his exposition is marred by a lack of clarity. Instead of the formula (19), Jeans writes

$$W = \frac{C(N+P)!}{P!}, \tag{19a}$$

where C is a constant. Charity might allow us to assume that this formula approximates the correct (19) with N legitimately approximated by $N-1$ and the constant C absorbing the missing term $N!$. Unfortunately, this charity is disallowed us. Jeans repeats the derivation in both editions of his *Report on Radiation and the Quantum Theory*, retaining formula (19a) and announcing erroneously that "$(M+P)!/P!$ is the number of ways in which P particles can be put into M pigeon-holes" (Jeans, 1914, pp. 38–39; 1924, pp. 27–28). It is hard to imagine how such an apparent blunder could survive and appear in the second edition of a widely read text, unless there is some alternative, charitable reading.

5. THE ARGUMENTS OF HENRI POINCARÉ

We could understand that Jeans might not have scrutinized too closely his argument for the necessity of quantum discontinuity by the time of its restatement in the 1914 first edition of his *Report*. He acknowledged there (p. 33) that the problem had been solved very completely in recent work of Poincaré, whose pronouncements on the necessity of quantum discontinuity were quoted approvingly and with prominence. Poincaré's results formed a central part of the case Jeans advanced for the quantum theory in a work whose intended audience viewed the theory as "an object of suspicion", so that the report "had to be an *apologia* as well as an exposition" (Jeans, 1924, Preface to 2nd ed.). Jeans, however, offered an exposition of his own argument of 1910 in place of Poincaré's, because "unfortunately Poincaré's Paper is of such an abstruse mathematical nature that is impossible to do any sort of justice to it in an abstract . . ." (1910, p. 33).

Poincaré had turned to active work on the problems of quantum theory in the year preceding his death, in July 1912.[14] He was enthused about work on the theory by his participation in the celebrated Solvay conference in Brussels (30 October–3 November 1911), where many of the leading physicists of the era assembled to brood over the quantum theory. His major contribution to the theory appeared the following year in the January issue of *Journal de Physique* (Poincaré, 1912), though he had already given a synopsis of its essential content on 4 December 1911, at the Academy of Sciences (Poincaré, 1911).

In the introduction (Poincaré, 1912, p. 5), he explained that he undertook the research for the paper precisely because of the problem of evidence described in Section 3 above. Nernst, he reported, had proposed that Planck's law might be accounted for by a mechanics free of quantum discontinuity but in which the mass of a body would vary not just with velocity, as in relativity theory, but with acceleration as well. Poincaré asked if any mechanics could give a viable treatment of Planck's law without quantum discontinuity and, through the results of his paper, he answered that it could not. Poincaré's argument was set in a context that masked the generality of his results. He considered resonators of long period that behaved classically and those of short period that would eventually become quantized as one of Planck's Hertzian resonators. He first took the case of energy interchange between one of each type and then extended it to the case of many resonators of both types. In his analysis he gave yet another derivation of the inevitability of the equipartition theorem in the most general classical theory as well as the sufficiency of quantum discontinuity for the derivation of Planck's law. There were two results, however, for which the paper is especially remembered.

The first result is the one extracted by Planck from the paper in his appreciation of Poincaré's work (Planck, 1921) and described with some exaggeration by Fowler (1936, p. 200) as the "whole substance" of Poincaré's paper. The result is that the *only* weight function $\omega(E)$ compatible with the quantum mean energy formula (15) is the discontinuous (16). To arrive at this result, Poincaré considered the integral transform of the weight function $\omega(E)$

$$\Phi(\tau) = \int_0^\infty \omega(E)\exp(-E\tau)\,\mathrm{d}E, \qquad (20)$$

where $\tau = 1/kT$. Typically, in the transformation from the weight function $\omega(E)$ to its Laplace transform $\Phi(\tau)$, no information is lost so that if we know the form of $\Phi(\tau)$ we can recover the original $\omega(E)$ by an inversion of the transformation. The function $\Phi(\tau)$ is determined, in turn, up to a multiplicative constant by the mean energy E of the system according to

$$\bar{E} = \frac{1}{\Phi(\tau)} \int_0^\infty E\omega(E) \exp(-E\tau)\, \mathrm{d}E = -\frac{\mathrm{d}}{\mathrm{d}\tau} \log(\Phi(\tau)), \tag{21}$$

so that an integration of the quantum mean energy formula

$$\bar{E} = \frac{h\nu}{\exp\frac{h\nu}{kt} - 1} \tag{15}$$

yields

$$\Phi(\tau) = \exp\left[-\int \bar{E}\, \mathrm{d}\tau \right] = \frac{1}{1 - \exp(-\tau h\nu)} \tag{22}$$

up to a multiplicative constant. The crucial move in the argument is the inversion of this expression for $\Phi(\tau)$ to recover the weight function $\omega(E)$.

Using a procedure that was not Poincaré's, we can see informally how this inversion proceeds. We expand this expression as an infinite power series

$$\Phi(\tau) = 1 + \exp(-\tau h\nu) + \exp(-2\tau h\nu) + \exp(-3\tau h\nu) + \cdots$$

and notice that each term of the series is the Laplace transform of a Dirac delta function. That is, $\delta(E)$ transforms to the constant 1; $\delta(E - h\nu)$ transforms to $\exp(-\tau h\nu)$; $\delta(E - 2h\nu)$ transforms to $\exp(-2\tau h\nu)$; etc.; so the corresponding weight function is just

$$\omega(E) = \delta(E) + \delta(E - h\nu) + \delta(E - 2h\nu) + \delta(E - 3h\nu) + \cdots, \tag{16a}$$

which is just (16) with ϵ set equal to $h\nu$. It provides for discontinuous weights on the energies $0, h\nu, 2h\nu, 3h\nu \ldots$[15]

It is important for our purposes to display the assumptions needed to allow the inference from the Planck law (14) through to quantum discontinuity (16a). Poincaré's argument is applied to Planck's Hertzian resonators so that the Planck resonator formula (5) is needed to proceed from the Planck law (14) to the mean energy formula (15). Clearly an analogous argument can be mounted for radiation oscillators, this time, however, using the radiation oscillator density formula (4) to proceed from the Planck law (14) to the mean energy formula (15). The most important assumption used applies both to Hertzian resonator and to radiation oscillator, and I shall call it:

> *General Statistical Model*: The observed energy of the system is the mean of a system whose state is distributed probabilistically according to
>
> $$\mathrm{d}W = C \exp\left(-\frac{E}{kT}\right) \omega(E)\, \mathrm{d}E, \qquad (7)$$
>
> where the weight function $\omega(E)$ is undetermined.

In sum, we have two deductive arguments belonging to the group of results III of Section 1:

III-1a Planck law (14)
General statistical model for Hertzian resonators
Planck resonator formula (5)

Quantum discontinuity (16a) for Hertzian resonators.

III-1b Planck law (14)
General statistical model for radiation oscillators
Radiation oscillator density formula (4)

Quantum discontinuity (16a) for radiation oscillators.

Poincaré's second result responds to the most obvious weakness of results III-1a and III-1b: both depend on assuming the exact correctness of Planck's formula (14); yet associated with any such experimentally determined formula is an amount of experimental error. Is there another formula that also agrees with the observed data within the limits of experimental error but that does not lead to quantum discontinuity? What Poincaré's second result showed was that there is no such formula. Indeed, any distribution law – even one that lies well beyond the limits allowed by experimental error – will yield a discontinuity at the $E = 0$ energy level as long as that distribution entails that the total energy density of black body radiation is finite.

Poincaré's (1912, §8) analysis dealt with the case of Hertzian resonators. Assuming only that the distribution law (1) is restricted by the Wien displacement law (3) and that the energy density of black body radiation is related to the energy of Hertzian resonators by the Planck's resonator formula (5), he showed that the finiteness of the total energy density of black body radiation forced $\Phi(\infty) > 0$ for a Hertzian resonator in equilibrium with the radiation. However, he also showed that the continuity of $\omega(E)$ entailed $\Phi(\infty) = 0$, for, as he showed,

$$(23) \qquad \Phi(\infty) < \int_0^{E_0} \omega(E)\, \mathrm{d}E,$$

for any $E_0 > 0$. Thus the finiteness of the total energy density of black body radiation entailed the discontinuity of $\omega(E)$. The form of that discontinuity could be read directly from the inequality (23). Since E_0 could have any value greater than zero, the integral could only be guaranteed to exceed a $\Phi(\infty) > 0$ if $\omega(E)$ had a discontinuous weight, concentrated at $E = 0$, whose value was at least as great at $\Phi(\infty)$. In terms of Dirac delta functions, this means that the weight function must have the form (up to multiplicative constant):

$$(16b) \qquad \omega(E) = \delta(E), \text{ at } E = 0.$$

Poincaré's second result applied only to Hertzian resonators. As with the first result, it could be modified to apply to radiation oscillators by substituting the radiation oscillator density formula (4) for Planck's resonator formula (5).

As before, we can collect these results in the form of two arguments:

III-2a Finiteness of total radiation energy density
General statistical model for Hertzian resonators
Planck resonator formula (5)
Wien displacement law (3)
___
Quantum discontinuity (16b) for Hertzian resonators.

III-2b Finiteness of total radiation energy density
General statistical model for radiation oscillators
Radiation oscillator density formula (4)
Wien displacement law (3)
___
Quantum discontinuity (16b) for radiation oscillators.

6. THE ARGUMENTS OF PAUL EHRENFEST

Poincaré's paper was mathematically sophisticated and attracted wide attention. Yet from the physical point of view the paper was not so sophisticated. It was at least initially dependent on a very specific physical model of resonators of short and long wave lengths. The essential physical assumptions on which the final results were based were buried in a gradual slide from this particular model and a derivation of the equipartition theorem (earlier in the paper) to the general results on the necessity of quantum discontinuity (given later). Finally, the paper considered only quantization of Planck's Hertzian resonators. The more secure results III-1b and III-2b were introduced in the last section only on analogy with Poincaré's III-1a and III-2a.

In all these aspects, Poincaré's paper was the antithesis of another submitted by the Viennese physicist Paul Ehrenfest (1911) to the *Annalen* in July 1911.[16] The paper, 'Which Features of the Light Quantum Hypothesis Play an Essential Role in the Theory of Heat Radiation?', started with a careful compendium of various physical assumptions and physical properties on which the final results would be based. Ehrenfest listed constraints on the form of the distribution law (1), including the Wien displacement law and various requirements of differing strength on the limiting behavior of the law. He listed electromagnetic aids, including the radiation oscillator density formula (4) as well as a careful statement of the statistical mechanical argument needed to form the crucial probability distribution (7). Unlike Poincaré, Ehrenfest applied his analysis to radiation oscillators, a more secure physical arena in which to locate quantum discontinuity, for one could replace the problematic Planck resonator formula (5) with the more secure radiation oscillator density formula (4) (see Section 8 below). From these foundations Ehrenfest proceeded to derive a series of results embracing and extending III-1b and III-2b. He seemed to recognize the interchangeability of these results with the corresponding results III-1a and III-2a, speaking in his conclusions (p. 110) of a transformation "from the Rayleigh–Jeans '*proper vibration*' terminology to the '*resonator*' terminology preferred by Planck".

Ehrenfest clearly had found all of Poincaré's essential results – and more – before Poincaré and even before the Solvay conference, which Ehrenfest unfortunately did not attend. Yet his paper was all but ignored at the conference. Ehrenfest's priority and superiority was rarely mentioned later and then only scantily even by a sympathetic sponsor, Lorentz (1921, p. 308).

Ehrenfest's results focused on a particular way of expressing the relationship between the function $f(\nu/T)$ of the distribution law (3) and the weight function $\omega(E)$, which, Ehrenfest noted, is really a function of both E and ν so that it is better written as $\gamma(\nu, E)$. Using results about the reversible adiabatic compression of radiation that are essentially equivalent to the Wien displacement law, Ehrenfest showed that this weight function could be expressed as a product of two functions:

$$\gamma(\nu, E) = Q(\nu) \cdot G(E/\nu). \tag{24}$$

Using this result, the Wien displacement law (3), the distribution law (7), the radiation oscillator density formula (4), and the following substitutions of variables

$$\sigma = \frac{\nu}{T} \quad q = \frac{E}{\nu},$$

Ehrenfest arrived at the following expression for $f(\nu/T) = f(\sigma)$:

$$Cf(\sigma) = \frac{\int_0^\infty q \exp(-\sigma q) G(q)\, \mathrm{d}q}{\int_0^\infty \exp(-\sigma q) G(q)\, \mathrm{d}q} = \frac{P(\sigma)}{Q(\sigma)} = -\frac{\mathrm{d}}{\mathrm{d}\sigma} \log Q(\sigma), \tag{25}$$

where C is a constant.[17]

Taking equation (25) as his focus, Ehrenfest began asking how constraints on the form of $f(\sigma)$ were to be reflected in the weight function $Q(\sigma)$. Unlike Poincaré, he ascended no mathematical heights to recover his basic results, preferring the practical mathematics of the working physicist of 1911, pausing from time to time to give examples of specific functions to illustrate the results obtained. His first main result, for example, was that no continuous $G(q)$ could yield a distribution with finite total energy density, expressed in this case as the requirement that $f(\sigma)$ diminish faster than $1/\sigma^4$ for large σ. In a table he showed, with a series of families of functions, that moving the weight of $G(q)$ toward the origin $q = 0$ caused $f(\sigma)$ to diminish faster. However, if $Q(\sigma)$ was to remain continuous and integrable, it was impossible to bring sufficient mass into the vicinity of $q = 0$ to enable $f(\sigma)$ to diminish faster than $1/\sigma^{2+\delta}$. He then turned to consider discontinuous weight distributions.

We can represent Ehrenfest's results as spanning between III-2b and

III-1b. If we require that the distribution law (1) yield not just a finite total energy density but also come closer and closer in form to the Planck law (14), then we find that the weight function advances correspondingly from the form (16b) to (16a). These results, which have obvious analogs for Hertzian resonators, can be summarized in the argument scheme

III-3 Wien displacement law (3)
Constraints A, B, and C, respectively, on the distribution law (3)
General statistical model for radiation oscillators
Radiation oscillator density formula (4)

Quantum discontinuity (16A), (16B), and (16C), respectively, for radiation oscillators

To complete the scheme III-3, we can now state the constraints A, B, and C and the expressions for $G(q)$ that follow, noticing how the expressions for $G(q)$ move from (16b) closer to (16a) as the constraints applied bring us closer to the Planck law (14).

A. "*Avoidance of the Rayleigh–Jeans Catastrophe in the ultraviolet*", "*violet requirement*": $f(\sigma)$ must diminish faster that $1/\sigma^4$, so that $\lim_{\sigma = \infty} \{\sigma^4 f(\sigma)\} = 0$. It yields

(16A) $G(q)$ approaches $q = 0$ faster than:

$$G(q) = \delta(q) + Aq^2$$

for A some positive constant.

B. "*Strenghthened violet requirement*": Since distribution laws such as that proposed by Wien (13) and Planck (14) require $f(\sigma)$ to diminish exponentially, that is faster than $1/\sigma^n$, for any integer $n > 0$, we can strengthen constraint A to require $\lim_{\sigma = \infty}\{\sigma^n f(\sigma)\} = 0$. It yields

(16B) $G(q)$ approaches $q = 0$ faster than:

$$G(q) = \delta(q) + Aq^n$$

for A some positive constant.

C. "*Wien–Planck violet requirement*": We require that $f(\sigma)$ diminish exponentially for large σ, so that $\lim_{\sigma=\infty}\{f(\sigma)/\exp(-L\sigma)\} = M$, for some non-zero constants L and M. This yields[18]

(16C) $G(q) = \delta(q)$, for $0 \leq q \leq L$,

and if $f(\sigma)$ *equals* $M \cdot \exp(-L\sigma)$ for large σ, then we have a second discontinuity at $q = L$:

(16C′) $G(q) = A\delta(q) + B\delta(q)$, for $0 \leq q \leq L$

for constants A and B.

Ehrenfest clearly thought the results in III-2 and III-3 the most important. However, we do also know that he was in possession of the essential parts of III-1. He mentions toward the close of his paper that if one knows the exact functional form of $f(\sigma)$, then the weight function G can be determined. As what he calls an "illustration", he considers the cases of Wien's distribution law (13) and Planck's law (14), sketching very briefly how the familiar discontinuous distributions corresponding to each law can be recovered. For the Planck law, of course, he recovered (16a). Elsewhere in his paper he gave the weight function for the Wien law,

(26)
$$G(q) = \delta(q) + \delta(q-1) + (1/2!)\delta(q-2) + \cdots + (l/r!)\delta(q-r) + \cdots$$

Ehrenfest's case by case treatment of each of his results does not give us any unified overview of their origin. We can approach such an overview in a suggestive but informal way as follows. Inverting Ehrenfest's equation (25) we recover

(27)
$$Q(\sigma) = \exp\left(-\int Cf(\sigma)\,\mathrm{d}\sigma\right) = 1 - \int Cf(\sigma)\,\mathrm{d}\sigma + \frac{1}{2!}\left(\int cf(\sigma)\,\mathrm{d}\sigma\right)^2 + \cdots$$

The elements of Ehrenfest's results pertain to the members of the power series expansion. Recalling that $Q(\sigma)$ is the Laplace transform of the weight function $G(q)$, we can invert terms in this power series to give us terms in an expression for $G(q)$.

The zeroth-order term is the constant 1. It inverts to give us $\delta(q)$, the discontinuity at $q = 0$ arising earlier from the requirement of finiteness of energy density and common to all results in III-2 and III-3. The first-order term $-\int Cf(\sigma)\,d\sigma$ yields the additional terms in $G(q)$ that are discussed in Ehrenfest's results III-3. If $f(\sigma)$ diminishes as $1/\sigma^n$, for large σ, then its integral is proportional to $1/\sigma^{n-1}$ and it provides a polynomial term q^{n-2} in the expression for $G(q)$ in addition to $\delta(q)$.[19] This immediately gives us (16B). If we consider the cases of $n = 4$ and $n = 4 + \epsilon$, for ϵ, a small increment, we can arrive at (16A) and its q^2 term. Finally, if we take $f(\sigma) = M \cdot \exp(-L\sigma)$, we find $\int Cf(\sigma)\,d\sigma$ is proportional to $\exp(-L\sigma)$. Its inversion yields the second discontinuous weight $\delta(q - L)$ at $q = L$ of (16C′).

7. FOWLER'S CODA

Fowler (1936, p. 200) sketched briefly an interesting coda to the results on the necessity of quantum discontinuity. He applied the methods used above to a very simple system, that of a volume V of black body radiation. Assuming only that the system obeys the Stefan–Boltzmann law and the general statistical model, Fowler was able to infer that there still must be a discontinuity in the system's weight function ω at $E = 0$. Taking the system's mean energy to be

$$\text{(2)} \qquad \bar{E} = \sigma V T^4 = \frac{\sigma V}{k^4 \tau^4},$$

we can recover $\Phi(\tau)$ by applying the formula in the first part of (22), arriving at:

$$\Phi(\tau) = \exp\left(\frac{\sigma V}{3k^4}\frac{1}{\tau^3}\right) = 1 + \frac{\sigma V}{3k^4}\frac{1}{\tau^3} + \frac{1}{2!}\left(\frac{\sigma V}{3k^4}\right)^2 \frac{1}{\tau^6} + \cdots$$

Inverting term by term we recover the weight function:

$$\text{(28)} \qquad \omega(E) = \delta(E) + \frac{\sigma V}{3k^4}\frac{E^2}{2!} + \frac{1}{2!}\left(\frac{\sigma V}{3k^4}\right)^2 \frac{E^5}{5!} + \cdots$$

The first term shows the discontinuity at $E = 0$. The remaining terms, as Fowler remarks, are "too complicated to give us much information". We have in summary form:

III-4 Stefan–Boltzmann law (2)
General statistical model for a volume of black body radiation.

Quantum discontinuity (28) for black body radiation.

8. ASSESSMENT OF INDUCTIVE RISK

In Section 1.3, I described how the inductive risk that one takes in inferring quantum discontinuity by means of the argument schemes III is located in the premises. I now review briefly the risk taken in adopting each of the premises.

Planck law (*14*): The inevitability of experimental error makes it impossible for a finite data set to guarantee a particular formula. As late as 1913, there was still discussion of the possibility of alternative formulae performing as well as Planck's (14).[20] Results in III-2, III-3, and III-4 show that even a minimally acceptable distribution law – one that only yields finite total energy density – forces a discontinuity at $E = 0$ and that the discontinuity bccomes more like that of (16) as the law approaches the Planck law (14) in form. Notice that a law such as Wien's (13), which differs markedly from Planck's (14) in the domain of small (ν/T), requires discontinuities at $E = 0, h\nu, 2h\nu \ldots$, as in (26).

Radiation oscillator density formula (*4*): This result is generated by an essentially geometric argument in which the wave length of a radiation oscillator is required to fit an integral number of times into the enclosing cavity (see, for example, Bohm, 1951, Chap. 1). The derivation would be difficult to assail, for it assumes essentially no properties for radiation other than the notion of wave length. Even then, other derivations are possible that rely essentially only on the Lorentz invariance of the overall theory (Norton, 1987, §5).

Planck resonator formula (*5*): This formula was derived by Planck using the full resources of classical electrodynamics. If the formula is essentially dependent on classical electrodynamics, then it cannot be used consistently in a derivation of a result that entails the falsity of the classical theory. This reservation was stated by Poincaré (1912, pp. 29–30) in concluding his derivation of III-2a.[21] The safest course would be to dispense with the formula and limit the analysis to radiation oscillators alone, in which case quantum discontinuity will still emerge. Einstein (1909a, p. 188), however, felt that one could retain the formula in the quantum domain in spite of its classical origins, for the equation pertains only to the time average of quantities. The classical theory, he noted, yielded correct results for time averaged quantities, as its success in geometric optics showed.[22]

Wien displacement law (3) and Stefan–Boltzmann law (2): There was little scope for doubting these laws. Apart from thermodynamic considerations, derivation of the Wien displacement law required only the assumption that the frequency of radiation was Doppler shifted on reflection from a moving mirror. If this was doubted, the result could also be derived from the zero rest mass of radiation. Similarly, aside from thermodynamic considerations, the Stefan–Boltzmann law followed from the assumption that isotropic radiation exerted a pressure equal to one-third its energy density, a result that followed immediately from its zero rest mass. Alternatively, the law could be arrived at directly by an integration of the Wien displacement law (see Norton, 1987).

General statistical model: That the gross behavior of a system of black body radiation represented a statistical average and, therefore, that the analysis of the system required the methods of a statistical mechanics followed from a beautiful thought experiment of Einstein (1909a, pp. 189–90). He considered a system of black body radiation able to exchange energy with a kinetic gas by means of a movable mirror. If the radiation pressure failed to exhibit statistical fluctuations, there would be an uncontrolled transfer of energy from the radiation to the gas, resulting in a violation of the second law of thermodynamics.

Once the need for a statistical treatment is clear, the crucial probability distribution law (7) follows essentially only from the assumption that the observed gross behavior of the system coincides with its most probable behavior. To recover the law (7), the standard methods introduced by Boltzmann suffice except that one is freed from the need to make the most troublesome assumption of Boltzmann's approach, the equal probability of each of his complexions. The prior probability of the various micro-states, as expressed in the weight function $\omega(E) = \gamma(\nu, E)$, is precisely what the investigation is intended to determine!

Thus the weight function is left undetermined and the prior probability of a system with frequency ν having energy in the range E to $E + \mathrm{d}E$ is set by definition as $\omega(E)\,\mathrm{d}E = \gamma(\nu, E)\,\mathrm{d}E$. A slight modification of the standard calculation then yields the probability distribution (7) as the most probable. For this derivation, see Ehrenfest (1911, §3).

There remained one other option. One could accept that black body radiation is a statistical phenomenon but escape (7) and the equipartition theorem in a classical analysis by insisting that the black body radiation we observe has not come to equilibrium. This view was explored thoroughly by Jeans before he abandoned it to champion the quantum theory. Jeans, following similar accounts by Lorentz, sought to recover the properties of black body radiation from an analysis of the mechanism of emission of radiation by accelerated charges. After he finally abandoned this work, Jeans (1914, pp. 28–29) listed four objections to accounts of this type. Three of them noted how the electron collision time parameter of the specific model discussed would have to behave in ways incompatible with known results about electrons and their motion if justice were to be done to observation. The other revealed a serious difficulty for *any* non-equilibrium account. To recover the independence of the properties of black body radiation from the nature of the material emitting it, one would have to assume implausibly that the collision time parameter is the same for all substances. Any non-equilibrium account would be expected to find difficulty in freeing the final results from the specific properties of the emitter. This remarkable independence follows, however, immediately from the simplest thermodynamic analysis once equilibrium is granted. Indeed a non-equilibrium account must forgo the entire tradition of thermodynamic analysis of black body radiation dating back to Kirchhoff. Jeans (1905c, pp. 309–11) even had to resist the applicability of the second law of thermodynamics to black body radiation. Yet the non-equilibrium account must still explain all the successes of the equilibrium approach.

Finally, it must be stressed that the body of results described in this paper do not exhaust the evidence for quantum discontinuity or the pathways taken from the evidence to the theory. Evidence for the theory was derived from numerous phenomena, and inductive argument schemes of many types have been used to display the import of that evidence. At the time Ehrenfest's and Poincaré's papers were published, the success of the quantum theory in accounting for the behavior of specific heats at low temperatures was especially important. By the early 1920s, Jeans (1924), in the more confident second edition of his *Report on Radiation and the Quantum Theory*, described four groups of phenomena as providing evidence for the new theory: black body radiation, the spectra of elements, the photoelectric effect, and the specific heats of solids. He summarized the bearing of the evidence from these four areas using an argument form that Salmon (1984, pp. 213–27) identified in Perrin's work on the reality of atoms, an argument to a common cause. If one looks at the values of Planck's constant *h* that are computed via the quantum theory from data in each of the four areas, one arrives at essentially the same value, making clear, Jeans (1924, p. 61) concluded, "that they agree in pointing to the same new system of quantum-dynamics".

9. CONCLUSION

The underdetermination thesis tells us that theory remains underdetermined by any body of evidence, no matter how large, rich, and diverse. If it were true, the theoretician seeking to build a theory on a body of evidence might reasonably expect to be faced with a plethora of theories, all of which do justice to the evidence. Yet this is not the common experience. When the available evidence is substantial, theoreticians consider themselves lucky to find *any* theory that does justice to the evidence and, if they do find one, the construction of competitors with any long-term viability proves well-nigh impossible. Perhaps this phenomenon is due to prejudice, social conditioning, stupidity, deference to dictatorial authority, or a host of other distractions that are traditionally deplored as non-scientific. What this study shows, however, is that, at least in the instance of quantum discontinuity, whichever of these forces were in operation, the weight of evidence was sufficient to force the unique determination of a particular result, no matter how unpalatable that result might be to the community of physicists. Perhaps what is exceptional about this instance is that the arguments forcing the unique determination can be presented in such a compact manner; and that we have been able to see beyond an inadequate caricature of the evidential case to the fuller case that lies hidden in forgotten texts and journals. Otherwise, I do not believe that the case is exceptional. Rather, it merely illustrates a commonplace of the lore of science, the power of a sufficient body of evidence to determine a unique theory.[23]

NOTES

[1] As usual, E is energy, h Planck's constant, n a non-negative integer, and ν frequency.
[2] For further discussion of these two forms of inference and their relation to the underdetermination thesis, see Norton (1994). For an analysis of their importance in Einstein's discovery of general relativity, see Norton (1989). See also Dorling (1973; 1987).
[3] The emergence of quantum discontinuity has been treated extensively in the history of science literature. See, for example, ter Haar (1967), Jammer (1966), Kangro (1976), and Kuhn (1978).
[4] The analysis follows that of Einstein (1907), which was applied to the latter approach only.
[5] The point was soon noted by, for example, Jeans (1905b). Kuhn (1978) has argued that Planck did not recognize the decisive role of quantum discontinuity in his early works.
[6] The notational convenience of the Dirac (1991, §15) delta function, $\delta(x)$, which represents the discontinuous concentration of unit weight on $x = 0$, was not available in 1910 and was not used in the work of Jeans, Poincaré, Ehrenfest, Fowler, and others discussed below. Its use in this paper greatly simplifies the statement of results concerning quantum discontinuity.
[7] Of course these results are just a single thread of the complex web of the reception of quantum discontinuity. For broader discussion see Garber (1976) and Hendry (1980).
[8] It is not clear that he intended to make a skeptical point.
[9] Other systems appeared to display similar flexibility. Thomson (1907), for example, worked with a classical account of heat radiation as produced by the acceleration of charged particles. He computed (p. 230) how laws governing the motion of these particles would have to be modified in order for the theory to yield the Planck law. Larmor (1909, 1910) also contains suggestions for constructing a derivation of the Planck law free of quantum discontinuity.
[10] See Dorling (1987) for a reconstruction of Einstein's argument as a demonstrative induction.
[11] The translation is based on Einstein (1989, pp. 390–91).
[12] For a recent account of Jeans's views on radiation, see Hudson (1989).
[13] I need hardly point out that this last step is where Jeans's argument lacks rigor. The claim might well be correct. Because it is central to Jeans's argument, however, it ought to be established with more than this token wave of an arm.
[14] For an account of Poincaré's involvement with the quantum theory, a synopsis of the results he established and of their influence, see McCormmach (1967). The proceedings of the Solvay conference are in Eucken (1914).
[15] The argument just sketched is imprecise, since it assumes that term by term inversion commutes with convergence of the infinite series. For a more precise but more opaque treatment, see Poincaré (1912) and a more rigorous version in Fowler (1921) that employs a Stieltjes's integral in place of (20) to allow for the discontinuous behavior of ω.
[16] For a discussion of this paper, its background and reception, see Klein (1970, pp. 245–53).
[17] Ehrenfest's version of (25) was actually a little more complicated. He reserved the function $G(q)$ for the continuous part of the weight function and added a sum term to

allow for the discontinuous weights $G_0, G_1, G_2 \ldots$ at $q = q_0, q_1, q_2 \ldots$ so that he set $Q(\sigma) = \Sigma_{r=0}^{\infty} \exp(-\sigma q_r) G_r + \int_0^{\infty} \exp(-\sigma q) G(q)\, dq$ and a similar expression for $P(\sigma)$.

[18] Notice that the novelty of (16C) is that $G(q)$ must be zero in the interval $0 < q \leqslant L$.

[19] Conveniently we can set the lower limit of integration of $\int C f(\sigma)\, d\sigma$ at a sufficiently high value to obliterate that part of $f(\sigma)$ that differs significantly from $1/\sigma^n$.

[20] See the remarks of A. E. H. Love as cited in Jeans (1914, pp. 25–26). See also Warburg's, and Ruben's contributions in Eucken (1914, pp. 65–70, 72–75) and Section 2 of Eucken's appendix.

[21] In Norton (1987) I argue that the old quantum theory was not fatally compromised by the inclusion of formulas like (5), since such results could also be generated plausibly in a more general theory compatible with both classical and quantum behavior of radiation.

[22] Einstein (1906, p. 203) had been less sure of this argument for retaining the formula.

[23] I am grateful to an anonymous referee for helpful criticism.

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