May 14, June 9, October 26, 2018; January 19, 2019. July 14, 2019.

How NOT to Build an Infinite Lottery Machine

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The sustained failure of efforts to design an infinite lottery machine using ordinary probabilistic randomizers is traced back to a problem familiar to set theorists: we have no constructive prescriptions for probabilistically non-measurable sets. Yet construction of such sets is required if we are to be able to read the result of an infinite lottery machine that is built from ordinary probabilistic randomizers. All such designs face a dilemma: they can provide an accessible (readable) result with probability zero; or an inaccessible result with probability greater than zero.

1. Introduction

Norton (2018) and Norton and Pruss (2018a) explored the physical possibility of designing an infinite lottery machine.[2] Such a machine would choose without favor among a countable infinity of outcomes 1, 2, 3, … A curious anomaly appeared in these explorations. It proved possible to find successful designs using plausible custom physics, such as the quantum

[1] My thanks to Alexander R. Pruss, Geoffrey Hellman and James R. Brown for helpful discussion.

[2] Why should philosophers of science be concerned with such machines? De Finetti (1972, §5.17) uses the chances they induce to justify the transition from countably additive probability measures to finitely additive measures in his subjective Bayesianism. Norton (manuscript) uses them to motivate a variant, non-additive logic of induction.

randomizer of Norton (2018, §10). However all efforts to devise a machine using familiar probabilistic randomizers failed. The proposals reported in Norton (2018), corrected by Norton and Pruss (2018), were just a representative sample of many tried. Success could not be secured even allowing for quite exotic processes. The best designs that employed familiar probabilistic randomizers operated successfully only with probability zero. The recalcitrant nature of the failure raised the possibility that the obstacle was not merely a lack of imagination in design. Rather its recalcitrance suggested that the failure results from some matter of principle.

The goal of this note is to demonstrate that this is so. In any design, the probabilistic randomizers provide us with a probability space large enough to host the countable infinity of outcomes of the infinite lottery machine that encode “1”, “2” and so on. These infinite lottery outcomes must, in a general sense, be equal chance outcomes. If that equality of chances is expressed as an equality of probabilities, then these probabilities must each be zero valued. By countable additivity, it follows that successful operation, that is, the realization of any one of them, is a zero probability event. The escape that allows a non-zero probability of success is to employ probabilistically nonmeasurable outcome sets for the infinite lottery machine.

The failure of the design is now assured by a well-known problem in set theory. If they are required, probabilistically nonmeasurable sets must be assumed to exist, without being displayed constructively. All known examples of nonmeasurable sets are non-constructible in the sense that an explicit definition cannot be provided for them. There is no complete demonstration that this non-constructibility holds universally. There are missing pieces. One is the need to assume the existence of certain exotic cardinal numbers, whose existence is increasingly accepted in the literature. This non-constructibility of known examples has remained unbreached for nearly a century and there is little expectation that this will change. That it reflects a principled impossibility will be assumed below.

This non-constructibility leads to a problem for infinite lottery machines, derived from probabilistic randomizers. To know that the end state resulting from the operation of the machine lies in an outcome set encoding “1” or “2” or so on, we need to be able to specify which are these sets. But if the outcome sets are nonmeasurable, we cannot do it. A fatal tension ensues is the form of a dilemma for probabilistic, infinite lottery machines: these machines cannot both provide a result we can read and also operate successfully. That is:

- If such an infinite lottery machine employs measurable outcome sets, then they are sets of probability zero and the machine operates successfully only with zero probability.
- If an infinite lottery machine employs nonmeasurable outcome sets, then these outcomes cannot be defined explicitly and the result drawn by the infinite lottery machine is not accessible.

It is a cruel twist, reminiscent of "Catch-22." A design can provide a result that we can read, only if the machine operates successfully with probability zero.

The following section will present a motivating example that illustrates the incompatibility of success and accessibility. Sections 3, 4 and 5 develop a more general characterization of infinite lottery machines based on probabilistic randomizers. It will be sufficient to establish the general incompatibility of success and accessibility. Section 6 presents another illustration of the failure in a different design. Section 7 states conclusions. The analysis only impugns infinite lottery machines derived from probabilistic randomizers. It leaves untouched the possibilities of other designs. The conclusion presents an illustration of a quantum mechanical infinite lottery machine. An appendix provides supporting material for the characterization of Section 4.

In other work (Norton, 2018, §5; manuscript), the chance properties of an infinite lottery are required to conform with a strong invariance condition, "label independence." The analysis here does not impose this strong invariance condition. It relies only on the weaker requirement that each individual lottery outcome "1," "2," "3,"… has the same chance.

2. An Illustration: A Spin of a Pointer on a Dial

2.1 A Design with Probability Zero of Successful Operation

The main ideas to be developed here appear in the following illustration of a candidate infinite lottery machine, described in Norton (2018, §2.3). It consists of a pointer spun on a dial, such that the pointer will come to rest with a uniform probability distribution over all angles from 0 to 360^{o}. If the pointer halts on a rational angle, then it can be used as an infinite lottery machine. For there are only countably many rational angles and they can be mapped one-one to the natural numbers. The difficulty, however, is that the probability of selecting any particular rational angle r out of the infinity of possible angles, rational or real, is zero:

$$P(r) = 0 \quad (1)$$

Since there is only a countable infinity of rational angles *r*, the probability that any rational angle at all is selected is the sum of a countable infinity of zeros, which is zero.

$$P(success) = \sum_{0 \le r < 1} P(r) = 0 \quad (2)$$

The pointer will almost always select an irrational angle. That is, the randomizer will operate successfully only with probability zero. For all practical purposes, it does not function.

2.2 The Extended Design: Improving the Probability of Success

The difficulty would seem to be easily solved. We take the infinite lottery outcome sets, that is, those sets of randomizer outcomes to which the lottery outcomes 1, 2, 3, … are associated. We enlarge or extend them to include irrational angles. To each rational angle *r*, we attach some suitable set of irrational angles *extend*(*r*), such that the sets *extend*(*r*) partition the set of all angles. Then each spin of the pointer must halt in one of these sets *extend*(*r*). The associated rational *r* is then read off as the result of the infinite lottery machine. This extended design will always succeed, for every angle must belong to just one of the infinite lottery outcome sets, *extend*(*r*).

A concern is that we cannot assign any definite probability to each outcome set, *extend*(*r*). For if we assign zero probability to each set, then the probability of successful operation is zero, as (1) and (2) show. We would now also contradict the normalization of the probability distribution to unity. If we assign a probability greater than zero, say $\varepsilon > 0$, no matter how small:

$$P(extend(r)) = \varepsilon > 0 \quad (3)$$

then the sum of only finitely many of these probabilities will exceed one, in contradiction with the normalization of the probability distribution to unity. Choose any natural number $N > 1/\varepsilon$. Summing over *N* infinite lottery outcomes yields

$$\sum_{N} P(extend(r)) > \varepsilon.(1/\varepsilon) = 1 \quad (4)$$

We are caught in a dilemma:

> *Probability assignment dilemma*: the probability of each infinite lottery outcome is either zero or non-zero. In both cases, if the outcome sets partition the outcome space, the normalization to unity of the probability distribution is contradicted.

What follows will provide an escape from dilemma.

2.3 Extension by Vitali Sets

A simple scheme for this extension is provided by the Vitali sets. To implement the scheme, we take the angles from 0 to 360^o and rescale them to 0 to 1.[3] We partition the set of angles into equivalence classes, such that two angles belong to the same equivalence class just if they differ by a rational angle. That is, angles x and y are in the same equivalence class, just if $y = x \oplus r$, where r is a rational angle. (Addition "$\oplus$" is modulus 1 addition, so all angles x, y remain constrained as $0 \leq x,y < 1$.) Writing "$[x]$" for the equivalence class that contains x, we can give the formal definition:

$$[x] = \{y \mid \text{there is a rational } r \text{ such that } y = x \oplus r\}$$

All the rational angles form one such equivalence class, [0]. Irrational angles form other equivalence classes such as $[1/e]$ and $[1/\sqrt{2}]$.

A Vitali set is formed as the set assembled from the following selections:

Choice: select just one angle from each of these equivalence classes $[x]$.

Just how this selection is made will prove to be the essential point, to which we will return shortly. Call *Vit*(0) a Vitali set formed under some selection that contains the rational angle 0. We can generate another Vitali set from this first Vitali set merely by adding a rational angle r, modulo 1, for some $0 \leq r < 1$, to each element of *Vit*(0). Call this new set *Vit*(r). Each real angle r in $0 \leq r < 1$ defines a distinct Vitali set.[4]

Since the collection of Vitali sets *Vit*(r) partitions the full set of angles, we can effect the extension envisaged by setting *extend*(r) = *Vit*(r), for all $0 \leq r < 1$. A key property of the Vitali sets is that, by construction, they are invariant under a rotation by any rational angle. That is, any Vitali set *Vit*(r) can be mapped to any other Vitali set *Vit*(s) by a rotation through the angle s-r. The probability distribution that governs the spins is also invariant under these rotations. If a subset of angles has probability p, then any subset produced by rotating that original subset by

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[3] To avoid duplication with 0, 360^o and 1 are excluded.

[4] For otherwise if there were two such rationals r and s such that *Vit*(r) = *Vit*(s), then the set would contain two distinct rationals r and s drawn from [0], contrary to the definition of a Vitali set.

any angle will also have probability p. It follows that an infinite lottery machine, operating with these extended outcome sets, is choosing its outcome without favor.

The discussion above rehearses a familiar starting point in treatments of measure theory.[5] The Vitali sets provide the standard, introductory example of outcome sets to which no probability measure can be assigned. For, by rotational invariance, we must assign the same probability to each Vitali set. Replicating the computations of (1)-(4), if we assign zero probability to each, then their countably infinite sum is zero, contradicting the requirement that the probability of the total space is unity. If we assign any probability greater than zero to each, then summing finitely many will be sufficient to yield a sum greater than one, once again contradicting the requirement that the probability of the full outcome space is unity.

That the Vitali sets are nonmeasurable is usually taken as a negative result, restricting the scope of measure theory. Here it is a positive result. It provides a path between the horns of the *Probability assignment dilemma* above. It supplies infinite lottery outcomes to which no probabilities need to be assigned. The dilemma no longer troubles the extended design. The device is still an infinite lottery machine. For its indeterministic dynamics selects without favor among the infinite lottery outcomes in virtue of the rotational symmetry of the dynamics.

2.4 Inaccessible Results

A problem remains for the extended design; and it is fatal. If an infinite lottery machine is to operate successfully, we must be able to read its result. Let us assume that we can read the exact angle on which the pointer halts. Then the result of the original infinite lottery machine of Section 2.1 can be read. However we have no way of reading the result of the infinite lottery machine with the extended design. For the prescription ("*Choice*") above does not give us an explicit definition of the Vitali sets used. Rather their existence only is inferred from the assumption that such a choice is possible. Thus, even if infinite precision measurements tell us that the spinning pointer halted on the irrational angle $1/e$, we cannot know which of the Vitali sets $Vit(r)$ the angle $1/e$ is a member. Thus we cannot read the outcome of the infinite lottery machine with the extended design. The machine does not operate successfully.

The problem is grave. It is worse than the technical problem of the infinite precision needed if a measurement is to read the specific angle on which the pointer halts. For this angle

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[5] See Rosenthal (2006, pp. 3-4), Wagon (1985, pp. 7-8).

has a definite value. It is there to be read, if only we can figure out how to do it. The Vitali sets *Vit*(*r*) of the above construction are not uniquely specified. Each Vitali set contains an uncountable infinity of elements, each chosen from a countable infinity of angles. There are very, very many sets that could serve as each *Vit*(*r*). The prescriptions employed above place no restriction on which is implemented. Even extravagant idealizations of our powers of measurement are ineffective if the target of our measurement is not uniquely specified.

One might imagine that this lack of unique specification is a minor obstacle. Might not more careful attention to the construction give us explicitly defined sets *extend*(*r*)? It turns out that no way has been found of providing explicit definitions of nonmeasurable sets like *extend*(*r*). Their existence is always assumed without the provision of an explicit recipe for constructing them. That the selections of *Choice* are possible without explicit prescription is widely accepted. It is one of the axioms of Zermelo-Fraenkel set theory, the "axiom of choice." It was formulated over a century ago in Zermelo (1904) and was controversial from the outset. Zermelo (1908) had to mount a vigorous defense of its use. As we shall see below, the ingenuity of generations of mathematicians since has failed to find explicit recipes that can specify how the choices should be made. Theorems in set theory to be discussed below suggest but do not definitely prove that this failure is a necessity of the mathematics.

In sum, infinite lottery machines derived from the probabilistic randomizer, the spin of a pointer on a dial, must fail. A successful design must employ infinite lottery outcome sets that are probabilistically nonmeasurable, since otherwise the machine operates successfully only with probability zero. However, if we employ nonmeasurable outcome sets in the design, then we cannot read the result. The machine fails to operate successfully once again.

3. The Physical Description

3.1 The Specification

The failure of this last design of an infinite lottery machine does not derive, I contend, from a poor choice of the design specifics. Rather, any design for an infinite lottery machine based on probabilistic randomizers will fail in the same way. To arrive at this conclusion, we will first see a general physical description of the common features of all such lottery machines.

The randomizer. The machine consists of a device that is initialized in one initial state and then evolves according to a known physical theory (classical mechanics, quantum theory, etc.) to one of many possible end states, whose totality forms the randomizer outcome space.

Probabilities. The stochastic properties of the time evolution of the device[6] induce a countably additive probability measure over the randomizer outcome space.

Infinite lottery outcomes. A countable infinity of disjoint sets of end states is designated as the set of outcomes comprising the possible selections of the infinite lottery machine.

Dynamical symmetries. These are functions that map invertibly the outcome space back onto itself, while preserving probabilities. That is, each symmetry maps any subset of the outcome space for which a probability is induced by the dynamics to another subset of the same probability.

Equal favoring of lottery outcomes. For any pair of infinite lottery outcomes, there is a dynamical symmetry that maps the first onto the second.

Accessibility of the result. There is a unique prescription for numbering the infinite lottery outcomes as 1, 2, 3, … so that a definite result of the operation of the infinite lottery machine can be read.

Successful operation. The operation of the machine returns a result with probability greater than zero.

This description is already rich enough to realize the *Probability assignment dilemma* above. The only probability assignment that conforms with the equality of chances and the normalization to unity of the probability distribution, is one that assigns zero probability to each of the infinite lottery outcomes:

$$P(1) = P(2) = P(3) = \ldots = 0$$

Since successful operation requires that one of the infinite lottery outcomes be realized, the infinite lottery machine can operate successfully only with probability zero.

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[6] While familiar randomizers governed by classical physics (coin tosses, die rolls, etc.) are only pseudo-randomizers, I accept the conclusion of Poincaré's method of arbitrary functions, such as elaborated in Myrvold (2016), that these randomizers provide probabilities objective enough for present purposes.

$$P(success) = P(1) + P(2) + P(3) + \ldots = 0 \quad (5)$$

Following the escape described in Section 2.3 above, we arrive at a general result that will prove of central importance:

> *Infinite lottery outcomes are nonmeasurable*. If an infinite lottery machine derived from probabilistic randomizers operates successfully with greater than zero probability, it employs probabilistically nonmeasurable infinite lottery outcomes.

3.2 A Restriction to Finite Additivity Does Not Help

The analysis here presumes countable additivity of the probability measures. It does that since this is the common assumption for probability measures and the one that applies to all familiar probabilistic randomizing machines. One might well ask whether dropping it in favor of mere finite additivity will allow a more appealing escape from the dilemma of Section 1 above. Finite additivity allows us to sum the probabilities of only finitely many disjoint outcomes to arrive at the probability of their disjunction. The summations (2) and (5), however, sum a countable infinity of such outcomes and are no longer permitted.

A restriction to finite additivity escapes the probability assignment dilemma of Section 2. We can assign zero probability to each of a countable infinity of disjoint outcomes of the lottery, "1", "2," "3," ... However we can still assign unit probability to their disjunction. That is, with the weaker finitely additive probability measure, can no longer infer that the device operates successfully only with zero probability. Merely finitely additive probability measures appear to open a place for infinite lottery machines in probability theory.[7] De Finetti (1972, §5.17) used this fact to motivate a restriction to finite additivity. A further appeal of finitely additive measures is that every outcome in them can be assigned a probability. There are no probabilistically nonmeasurable outcomes.

Promising as this sounds initially, a restriction to finite additivity provides no respite from the problems of the dilemma of Section 1. The problem is that the outcome sets associated with the infinite lottery outcomes "1," "2," "3," … are still not constructible. That a merely

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[7] It is argued in Norton (manuscript) that this is a misleading appearance. An infinite lottery machine respects label independence. It precludes even finite additivity for the infinite sets of outcomes.

finitely additive measure can assign a probability to them has not expanded our capacities for identifying sets. These non-constructible sets remain as non-constructible as before.

The reasoning that led to their non-constructibility can be restored, but now using countably additive probability measures merely as mathematical adjuncts. This is permissible, since the inferences that led to non-constructibility do not require that the countably additive probability measure be the true measure of the stochastic dynamics of the system.

Consider some candidate infinite lottery machine. Its operation employs a dynamics that leads it to settle into some end state within a larger outcome space. The outcome space contains disjoint infinite lottery outcome sets "1," "2," "3," … Since the lottery machine is fair, its dynamics makes it equally likely that the end state is any of these infinite lottery outcomes sets. This is expressed as a symmetry over the outcome space: we can switch around the number labels on these outcomes without affecting the chances of the outcomes associated with the labels. By supposition, the chances are expressed by a finitely additive probability measure that is induced by the dynamics. Since it is induced by the dynamics, the measure shares the same symmetries.

For example, the pointer on a dial randomizer comes to rest in way that favors no part of the dial. Infinite lottery outcome sets associated with "m" and "n" are so chosen that none are favored. The symmetry that expresses this is just a rotation of all outcomes by some fixed angle. It is implemented by adding a rational number r modulo 1 to all the angles. That means that whichever outcome set is associated with some "m" can be taken to an outcome set that is associated with some other outcome "n" merely by a rotation. The induced probability measure also respects this symmetry. It follows that the probability associated with infinite lottery outcome "m" must equal that associated with outcome "n."

Returning to the general case, the finitely additive measure must assign zero probability to each of the infinite lottery outcomes. We construct a new, countably additive measure from the finitely additive measure by eliminating just sufficient of its probability assignments that the remaining assignments can be consistently extended by countable addition. This elimination must remove the probabilities assigned to at least some of these infinite lottery outcomes. Otherwise the countably infinite sum of the zero probabilities of each the infinite lottery outcomes would be zero. That zero would contradict the non-zero probability of the outcome space or the subspace partitioned by the infinite lottery outcomes. But now we have concluded that at least some of the infinite lottery outcomes are non-measurable in an inextendible, countably additive probability measure. Such outcome sets are non-constructible.

We can extend this reasoning from the case of some infinite lottery outcomes to all by employing different adjunct, countably additive measures. To do this we employ the symmetry transformation described above. If we have a countably additive measure that leaves infinite lottery outcome "*m*" nonmeasurable, choose a symmetry transformation that takes "*m*" to outcome "*n*" and apply it to the countably additive measure. The result is a countably additive measure in which infinite lottery outcome "*n*" is non-measurable and thus also non-constructible.

For example, in the pointer on a dial randomizer, a finitely additive measure must assign zero probability to each of the *extend*(r) outcome sets that partition the outcome space. To recover a countably additive measure from this finitely additive measure, we must eliminate the probability assignments to at least some of the sets *extend*(r), rendering them nonmeasurable in an inextendible, countably additive measure and thus non-constructible. Assume some particular *extend*(r) is nonmeasureable with respect some countably additive probability measure. What of another set *extend*(r')? We apply a rotation by adding a rational number $r' - r$ (modulo 1) to the angles that will map the set *extend*(r) to the set *extend*(r'). This set *extend*(r') will be nonmeasurable with respect to the rotated countably additive measure[8] and so non-constructible.

4. An Abstract Description of the Outcome Spaces

The description of infinite lottery machines given in Section 3.1 is incomplete. It employs terms with vague referents. Just what is a "device"? What are its initial and end states? What is the scope of "known physical theory"? We need a more precise, abstract description of the probabilistic randomizer outcome spaces. To that end, we make the further supposition that:

> *Outcome space I.* The probabilistic randomizer outcome space can be represented mathematically as sets, sets of sets, and so on. The set elements include natural numbers, rational numbers, real numbers, finite or infinite sequences of them and their finite-dimensioned Cartesian products. The outcomes that comprise the results of an infinite

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[8] Briefly, if the measure assigns probability P to some set of angles S, then the measure rotated by t assigns the same probability P to the rotated set $S = \{a' | a' = a \oplus t \text{ and } a \in S\}$.

lottery machine form a countable set of disjoint subsets within the probabilistic randomizer outcome space.

This supposition strengthens the description of infinite lottery machines given in Section 3.1 above in the particular aspects needed for the analysis that follow. Since the full scope of just which sets are included is still incompletely specified, this is the first version, labeled "I". It will be made more precise shortly.

The adequacy of this abstract description must be supposed. The vagueness of the description in Section 3.1 precludes anything more. However the supposition can be motivated by a review given in the Appendix. It shows that each of the probabilistically based designs for infinite lottery machines in Norton (2018) conform with the description. All the designs considered that provide accessible results operate successfully only with probability zero.

5. Nonmeasurability Precludes Accessibility

5.1 Zermelo-Fraenkel Set Theory

To proceed, we need greater clarity concerning the above abstract description of the outcome spaces in terms of mathematical sets. There are two, distinct questions: First, which structures exist? Among these will be found the probabilistic randomizer outcomes spaces. Second, which structures are explicitly definable? Among these will be found infinite lottery outcomes that can figure in an infinite lottery machine whose results are accessible.

Axiomatic set theory provides a well-developed answer to both questions. The list of axioms of Zermelo-Fraenkel set theory with choice ("ZFC") are given by Hrbacek and Jech (1999, Ch. 15) as

Existence, Extensionality, Schema of Comprehension, Pair, Union, Power Set, Infinity, Replacement, Foundation, Choice

Other texts, such as (Enderton, 1977, pp. 271-72) give equivalent formulations, but with slight variations in terminology. The standard project is to show that the existence and properties of structures used in familiar mathematics can be derived within this axiomatic system. The project has deceptively simple beginnings. Both Hrbacek and Jech (1999, Ch.3) and Enderton (1977, Ch.4) begin by defining the number 0 as the empty set $\varnothing$, so that $0=\varnothing$, where the existence of the empty set is asserted by the first axiom, the Axiom of Existence. The remaining numbers are then defined as $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, and so on. The natural number n+1 is defined as a union of sets $n+1 = n \cup \{n\}$. The existence of the union of sets invoked at each stage is assured by the Axiom of Union. The project continues with the rational numbers, the real numbers and well beyond. We have the assurance of Hrbacek and Jech (p. 268) of a far-reaching success that extends to the fundamental objects of topology, algebra and functional spaces, as well as demonstration of the widely accepted theorems that govern them.

5.2 Limits to What is Explicitly Definable

There is a strong temptation to replace the above characterization of the randomizer outcome spaces with something that is much more than ample: the sets that comprise the outcome spaces of the probabilistic randomizers are derivable within ZFC. Tempting as it may seem, this characterization cannot stand. For within this outcome space we must be able to specify accessible infinite lottery outcomes.

The difficulty that this further requirement brings will not be apparent in the early stages of the construction of the natural numbers sketched above. The sets comprising the natural numbers 1, 2, 3, … are defined explicitly. The number 3 is defined explicitly as the set $\{0, 1, 2\}$. The axiom system is especially amenable to explicit definition through the Axiom Schema of Comprehension or Separation. It lets us take any property $P(x)$ for an entity x (which will always be a set in axiomatic set theory) and use it to define a new set as a subset of a larger set: the defined set contains just those elements of the larger set that satisfy P. The possibilities for property P are very great. It can be anything that can be written in first order predicate logic using the predicates of axiomatic set theory. Is it an axiom schema, not an axiom, since each such property defines a new axiom. This descriptive flexibility is encouraging for efforts to specify the infinite lottery outcomes, for they are introduced as subsets of the larger outcome space of the probabilistic randomizers. For example, in the infinite lottery machine of Section 2.1, the set of lottery outcomes was introduced as a subset of real numbers in [0,1) by the property $P(x)$ = "x is a rational number." The individual lottery outcomes were introduced by restricting P to specific rational numbers.

The trouble starts with the "C" in ZFC. It designates the Axiom of Choice. A choice function for some system of sets is a function that maps each set in the system to one of that same set's elements. The axiom just says (Hrbacek and Jech, 1999, p. 268):

Axiom of Choice: Every system of sets has a choice function.

In more informal terms, the axiom just tells us that, if we have a collection of sets, we can form a new set by picking one element from each of the sets. At first look, this axiom seems plausible and innocent.

The hidden peril is that the axiom does not define which is that choice function. It merely asserts that, whenever we have a system of sets, we can assume the existence of a choice function. That is what made the axiom so controversial. It means that, if we use the axiom to assert the existence of a lottery outcome set, we have not specified which that set is, but only that it exists. An assurance of existence is cold comfort if we are trying to read the result of an infinite lottery machine. We may be assured that the outcome is in some infinite lottery outcome set, but since we do not know which these sets are, we cannot know the lottery outcome. The danger is real. We saw in Section 2.3 above that a choice function was used to construct the Vitali sets. That yielded a design for an infinite lottery machine whose result was inaccessible.

There is no simple escape. The axioms of Zermelo-Frankel set theory *excluding* the Axiom of Choice ("ZF") are too weak. We cannot replace the results derived from the Axiom of Choice by some more inventive or ingenious use of the axioms in ZF alone. Important theorems due to Kurt Goedel and Paul Cohen show that the Axiom of Choice is logically independent from the other axioms collected in ZF.[9] We can add either the Axiom of Choice or its negation to ZF, without contradiction, as long as ZF itself is already consistent. Either choice will lead consistently to different sets of result. Thus, to secure all the results derived from the Axiom of Choice, we have to add it to the other axioms.

We require the infinite lottery outcomes to be accessible, so that the results of the infinite lottery machine can be read. Characterizing the outcome space as all structures arising in ZFC opens the possibility that the infinite lottery outcomes of interest to us are not explicitly defined and thus inaccessible. The risk is quite real. The Vitali sets of Section 2.3 can be introduced in ZFC, yet all efforts to provide explicit definitions for them have failed. On the expectation that

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[9] See Hrbacek and Jech, 1999, p. 269. The literature on the axiom of choice is enormous. For more, see Jech (1973). The axiom's status remains troubled in part for its essential role in counterintuitive constructions such as used in the Banach-Tarski "paradox."

failures like this will persist, we have to narrow the characterization of the randomizer outcomes spaces:

> *Outcome space II*. The sets that comprise the probabilistic randomizer outcome space and the infinite lottery outcomes are derivable within ZF.

The scope of sets so definable is expansive. The Axiom Schema of Comprehension allow us to separate out sets by means of any set theoretic predicate, definable in first order logic.

5.3 Limits to Measurable Sets

The restriction to ZF alone may not seem so harmful. The damage becomes apparent when we consider nonmeasurable sets. We saw above in Section 3.1 that a successfully operating infinite lottery machine must employ infinite lottery outcomes that are nonmeasurable. Yet nonmeasurable sets cannot be derived within ZF.

This impossibility is a hard won realization of set theory. When set theory goes beyond ZF with the axiom of choice, it brings with it the possibility of nonmeasurable sets. That in turn enabled unexpected geometrical constructions, such as the Banach-Tarksi "paradox" or, better, the Banach-Tarski theorem, since there is no real paradox, just an odd result. Using as parts nonmeasurable sets authorized by the Axiom of Choice, it is possible to take a sphere in three dimensional Euclidean space, decompose it into five parts and then reassemble them into two spheres, each of the same size as the first.[10]

The weakness of the theorem is that suitable nonmeasurable sets are assumed to exist, under the Axiom of Choice, but are not defined explicitly. This provided a stimulus for resisting Banach and Tarski's result. At the same time, it gave strong motivation to efforts to give explicit definitions for the nonmeasurable sets. No such efforts succeeded. That they must fail was all but shown by a theorem due to Solovay (1970). He showed (subject to the qualification below) that the proposition that all subsets of the reals are measurable could be added to ZF without

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[10] The literature on Banach-Tarski is enormous. See Wagon (1985) for a thorough treatment and Wapner (2005) for a delightful, more popular account. Lest the theorem appear an affront to reason, at root it is no more bizarre that this construction. Take a countable infinity of marbles, numbered, 1, 2, 3, 4, … Divide them into the even and odd numbered sets, 2, 4, 6, 8, … and 1, 3, 5, 7, … Renumber the even set as 1, 2, 3, 4, … and the odd set as 1, 2, 3, 4, … We have now duplicated the original set of marbles.

contradiction, as long as ZF itself is consistent. Thus it is not possible to derive a proposition in ZF that asserts: "This subset *S* of real numbers is nonmeasurable." For that would contradict Solovay's added proposition.

Solovay's theorem "all but" shows the impossibility since there are loopholes. First, one of its premises is that there exists an uncountable, inaccessible cardinal number. Its existence is generally expected, but no proof of it is possible.[11] Second, even if the Axiom of Choice is needed in the larger logical system in which nonmeasurable sets arise, Solovay's result does not rule out the possibility that the strengthened system allows explicit definition of some nonmeasurable sets.

Much more can be said on these issues. For a review, see Wagon (1985, Ch. 13). What we have seen so far, however, is sufficient for present purposes. I will proceed with the presumption that nonmeasurable sets cannot be constructed in ZF and that the expansion to ZFC will not provide for nonmeasurable sets that are explicitly definable. Both these presumptions might be false, but their falsity would be surprising.

5.4 Accessibility and Measurability

We can now assemble the results of the set theoretic analysis. If an infinite lottery machine, based on a probabilistic randomizer, is to yield an accessible result, we must restrict our outcome space, abstractly described, to sets that are derivable in ZF. If we impose that restriction, then the infinite lottery outcomes will be measurable. If the infinite lottery outcomes are measurable, then (from Section 3.1) the infinite lottery machine operates successfully with at best zero probability.

These last inferences give us the precise basis for the incompatibility of accessibility and successful operation. If the outcome of an infinite lottery machine is accessible, the machine cannot operate successfully with more than probability zero. If the machine operates successfully with more than probability zero, its result is inaccessible and cannot be read by us.

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[11] The impossibility follows from Goedel's second incompleteness theorem. (I thank an anonymous referee for this clarification.)

6. Pruss' Well-Ordered Reals Infinite Lottery Machine

A design for an infinite lottery machine by Alexander Pruss (2014) illustrates the incompatibility just described. Assume that we have a countable infinity of randomizers each of which picks a single real number in (0,1) with a uniform probability density over (0,1). We may use a spin of a pointer on a dial. Or each randomizer may consist of a countably infinite sequence of coin tosses. If we encode heads and tails as 1 and 0, the ensuing sequence, read as a binary fraction, identifies a real number in (0,1). For example <H, T, H, T, H, T,…> is read as 0.10101010… which equals 2/3 in decimal notation. If the real numbers picked by randomizers 1, 2, …., n, … are $r_1, r_2, \ldots, r_n, \ldots$, then they form a set of real numbers in (0,1).[12] We can now proceed with two versions of the infinite lottery machine:

6.1 A Design that Provides an Accessible Result with Probability Zero

Among this set of real numbers, $\{r_1, r_2, \ldots, r_n, \ldots\}$, we choose the number that is arithmetically the smallest, say r_N. The number N of the randomizer that picked r_N is the outcome of the infinite lottery.

The complication is that most of these infinite sets of real numbers have no smallest element. There will be a smallest real in the set with probability zero. The pertinent outcome space consists of all sequences of randomizer numbers, ordered arithmetically. If that order gives us $r_{101} < r_3 < r_{24} = r_7 < r_{47} < \ldots$, then the outcome is the sequence <101, 3, {24, 7}, 47, …>. The measure zero case of two randomizers, 24 and 7, picking the same real number is accommodated by including those outcomes as a set in the sequence. The infinite lottery outcome N corresponds to all those sequences whose first term is N. Since a symmetry of the selection of the real numbers in $\{r_1, r_2, \ldots, r_n, \ldots\}$ is an arbitrary permutation of the order of the randomizers, each of these lottery outcomes must have the same probability. Since their sum

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[12] With probability zero, the pointer on a dial may return 0 and the coin tosses 0 = 0.00000… or 1 = 0.11111… We excise these cases manually by spinning or tossing again whenever they occur.

cannot exceed unity, it follows that the probability of each infinite lottery outcome N is zero.[13] Summing, the probability that the lottery machine returns any of these as a result is also zero.

6.2 A Design that Provides an Inaccessible Result with Probability One

This last failing of the lottery machine design can be remedied if we replace the arithmetic ordering by a "well-ordering" on (0,1). It is a transitive, irreflexive relation on (0,1) such that every subset of (0,1), including (0,1) itself, has a (unique) least element. The ordinary arithmetic "less than" relation is not a well-ordering on (0,1), since there is no arithmetically smallest real number in (0,1) or in any of its open subintervals. Under this well-ordering, every infinite set $\{r_1, r_2, \ldots, r_n, \ldots\}$ of reals selected by the randomizers has a unique least member.

A complication is that this least real may be the outcome chosen by more than one randomizer, say M and N, for which $r_M = r_N$. In this case, the outcome of the lottery is not unique. This confounding will only happen with probability zero. For the probability that any two nominated randomizers choose the same real number is zero; and there are only a countable infinity of pairs of randomizers. Thus, with probability one, this design of randomizer will return a result.

However, the individual infinite lottery outcomes can be seen to be nonmeasurable using the same argument as used in Section 2.3 for the Vitali sets. They partition the outcome space, setting aside the measure zero sector in which a unique lottery outcome fails to arise. Hence their probabilities must sum to unity. However each infinite lottery outcome must also have the same probability. Since there is a countable infinity of infinite lottery outcomes, neither a zero nor a

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[13] More directly, partition the real number interval (0,1) into a countable infinity of subintervals, …, [1/8, 1/4), [1/4, 1/2), [1/2, 1). Also divide up the infinity of randomizers into a countably infinite set of countably infinite subsets. Match the subsets of randomizers one-one to the intervals. With probability one, the subset of randomizers matched to [1/2, 1) will return a real in that interval at least once. For they fail to do so with probability $(1\text{-}1/2)^\infty = 0$. Continuing with the other intervals, there is a probability one that each interval contains a real returned by some randomizer. Combining, with probability one, each interval contains a real selected by a randomizer. In this probability one case, there is no smallest real.

non-zero value for the probability can lead to this unit sum. No probability can be assigned consistently to them.

If the design has probability one of success, then we should expect its result to be inaccessible. That is, even given a specification of the infinite set of real numbers chosen by the randomizers, $\{r_1, r_2, \ldots, r_n, \ldots\}$, it must turn out that we are unable to ascertain just which is the infinite lottery outcome. This proves to be the case, since it turns out that there is no finite way to specify the well-ordering of (0,1) needed. Indeed, in results tracing back to Zermelo (1904), it turns out that the existence of a choice function of the Axiom of Choice is equivalent to the existence of a well-ordering of the set.[14] Since a choice function is presumed to exist for (0,1) but cannot be displayed explicitly, the same is true of the well ordering of (0,1). Thus the outcome of the well-ordered reals infinite lottery machine may exist, but its value is inaccessible to us.

7. Conclusion

This analysis shows that the project of designing an infinite lottery machine using ordinary probabilistic randomizers is fraught with difficulties. However it does not impugn the very idea of such a machine. If "possible" means that the machine can be implemented in some plausible physical theory, then the quantum mechanical infinite lottery machine of Norton (2018, §10) suffices. Perhaps the simplest implementation of this quantum type of infinite lottery machine is provided by an ordinary quantum particle in a momentum eigenstate. If we treat just one dimension of space with coordinate x and time t, then the wave function of the particle with energy E and momentum p is spread over all space as

$$\Psi(x, \mathrm{t}) = \exp(2\pi \mathrm{i}(px - Et)/\mathrm{h})$$

where h is Planck's constant. Divide the possible spatial positions x into a countable infinity of intervals $[n, n+1)$ of equal size, where $n = \ldots -2, -1, 0, 1, 2, \ldots$ Taking the Born rule as our guide,

[14] Proof sketch: If a set is well ordered, then the least element of each subset defines a choice function. If there is a choice function on a set, then the value it assigns to the whole set is the first element in the well-ordering. The next element in the well-ordering is the value assigned to the set with that first element removed. And so on until the set is exhausted.

the chance that the particle manifests in any of these intervals is proportional to the integral of the norm of $\Psi(x, t)$ over this interval:

$$Chance([n,n+1]) = \int_n^{n+1} \Psi^*(x,t)\Psi(x,t)dx = \int_n^{n+1} 1.dx = 1$$

It is the same for all the intervals. If we use some scheme to number the intervals 1, 2, 3, 4, …, then the lottery outcome is just the number assigned to the interval in which the particle position manifests. Each arises with equal chance.

The notion of chance employed is not probabilistic. For the wave function $\Psi(x, t)$ cannot be normalized and, thus, a full application of the Born rule is not possible. For an elaboration of the notion of chance that is applicable, see Norton (manuscript).

Appendix: Abstract Descriptions of the Infinite Lottery Outcome Spaces

Spin of a pointer on a dial. [15] The outcome space consists of the angular position at which the pointer halts. The physical angles from 0 to 360^0 can be represented by the half open interval of real numbers, [0,1). There is a uniform probability distribution over this interval. The rational valued outcomes employed as infinite lottery outcomes are all probability zero. One of them arises only with probability zero.

The jumping flea.[16] A flea jumps from cells 1 to 2 to 3 to …, choosing to make each jump or not according to a probabilistic formula. The schedule of jumps is accelerated so that an infinite number can be made in finite time. If the flea halts on cell *n*, then the outcome is represented by the sequence <1, 2, …, *n*>. The full outcome space consists of all finite sequences: <1>, <1, 2>, <1, 2, 3>, <1, 2, 3, 4>, … and the infinite sequence <1, 2, 3, …>. Probability zero is assigned to each of the finite sequences and probability one to the infinite

[15] The accounts given here are minimal. For further details, see Norton (2018).

[16] Both this jumping flea and random walk design fail to meet the requirement of equality of chances, even though their lottery outcomes are all probability zero, since the dynamical evolutions leading to the lottery outcomes are not related by symmetries of the dynamics.

sequence. This last infinite sequence represents the case of the flea never halting. With probability one, this design fails to return a result.

The infinitely accelerated random walk. At each stage, a walker chooses probabilistically to step left ("-1"), step right ("+1") or stay put ("0") on an infinite road, marked off into a countable infinity of cells. The schedule of stages is accelerated so that infinitely many are completed in finite time. The probability distributions over walker position approaches arbitrarily closely to a uniform distribution as the number of stages grows large. If we take the outcome space just to consist of the final positions of the walker, as noted in Norton (2018, §8), we do not arrive at a well-defined space with a uniform probability measure. For almost all the motions fail to converge to a well-defined final position for the walker. A better choice of outcome space consists of all possible infinite sequences of -1, +1, 0, such as <+1, +1, 0, -1, 0, +1, …>, tracking the successive motions of the walker. Almost all of these sequences will correspond to a failure of the walker position to converge to one cell. Convergence is required since the resulting cell is the infinite lottery outcome. Convergence will arise only in cases of sequences with an infinite tail of 0's. For example <+1, +1, +1, -1, 0, 0, 0, 0, 0, 0, ….> corresponds to a walker halting at two positions to the right of the start. Each of these convergent sequences is a probability zero outcome. Since there is only a countable infinity of them,[17] there is a probability zero that any arises and, *a fortiori*, a probability zero that the infinite lottery machine returns any of the requisite infinite lottery outcomes.

The infinite array of coin tosses. The randomizer consists of an infinity of coin toss outcomes, arranged in one quadrant of a two dimensional array. Representing heads by "1" and tails by "0", the outcome space consists of infinite two dimensional arrays of 1 and 0. The infinite lottery outcomes are encoded in rows with particular configurations. The outcome *n* is encoded in a row whose first *n* elements are 1 and all of whose remaining elements are 0. Thus the row <1, 1, 1, 1, 1, 0, 0, 0, 0, 0, …> encodes the lottery outcome 5. The outcome provided by the lottery machine is the number encoded in the first row of the array that encodes a number. Once the correction of Norton and Pruss (2018) is accommodated, there is a probability of zero

[17] Proof: each convergent sequence has a finite initial sequence of +1, -1 and 0. Taking them as the digits of a base 3 ternary arithmetic and reading the initial sequence in reverse, each convergent sequence can be encoded by a unique natural number.

that there is such a row in the array, so that the machine operates successfully only with probability zero. For the probability that any nominated row encodes some fixed number is zero. Since there is a countable infinity of possible numbers to be encoded, there is a probability zero that some nominated row encodes any number. Finally, since there is a countable infinity of rows, there is a probability zero that any of them encode a number.[18]

Hansen's reversed supertask. An urn initially contains a countable infinity of numbered balls. In a reversed supertask, actions are undertaken at times …, $1/n$, …, 1/4, 1/3, 1/2. At time $1/n$, an urn that contains exactly n+1 balls arrives. One ball is chosen with equal probability and removed, so that an urn with only n balls is passed on to time $1/(n-1)$. At time 1, there is only one ball left in the urn and it has been chosen by a process that favors all balls equally. In spite of its ingenuity, this design is not successful since, as described in Norton (2018, §9), it fails to specify a way for the process to start at time 0 so that the urn at any time $1/n$ can pass on an urn containing just n balls. However its outcome space still conforms with the above description. If we write S_n for the n membered set of numbers of the n balls passed on at time $1/n$, then the abstract description of the outcome space consists of all possible infinite sequences of sets of natural numbers $<\ldots, S_n, S_{n-1}, \ldots, S_3, S_2, S_1>$, such that $\ldots \supset S_n \supset S_{n-1} \supset \ldots \supset S_3 \supset S_2 \supset S_1$.

If this is the total space, there is no well-defined probability measure over it that conforms with the design specification. The specification only provides conditional probabilities connecting successive stages. For example, for any specific outcome "k", that is, that $S_1 = \{k\}$, for k any nominated natural number, we must have the conditional probability

$$P(S_1 = \{k\} \mid k \in S_n) \quad = \quad 1/n$$

since the dynamics of ball removal does not favor any ball. These conditional probabilities cannot be combined into an unconditional probability measure. For such an unconditional probability measure over the whole space, if it exists, would satisfy

$$P(S_1 = \{k\}) = \sum P(S_1 = \{k\} \mid k \in S_n) \text{ x } P(k \in S_n) = \ 1/n \sum \ P(k \in S_n)$$

[18] There is a probability one that first row encodes no number; and so on for the rest of the rows. Therefore the probability that none the rows encode a number is the product of infinitely many of these unit probabilities, which is one.

where the summation extends over all n-member sets S_n containing k. Since the summed probabilities in this formula must be less than or equal to one, it follows that $P(S_1 = \{k\}) \le 1/n$. Since there are infinitely many stages, n can be set as large as we like, so that $P(S_1 = \{k\}) = 0$. Since there are only countably many k and one of them must be realized, these probabilities $P(S_1 = \{k\})$ must sum to unity if the unconditional probability measure is to normalize. However since all $P(S_1 = \{k\})$ equal zero, they sum to zero.[19]

This failure is not the same as the nonmeasurable character of the Vitali sets. For in the latter case there is a probability measure over the entire space in which the Vitali sets are subsets. In the present case, what does not exist is the probability measure over the entire space.

References

de Finetti, Bruno. 1972. *Probability, Induction and Statistics*. London: Wiley.

Enderton, Herbert B. 1977. *Elements of Set Theory*. New York: Academic Press.

Hrbacek, Karel and Jech, Thomas J. 1999. *Introduction to Set Theory*. 3rd ed. Now York and Basel: Marcel Decker, Inc.

Jech, Thomas J. 1973. *The Axiom of Choice*. Amsterdam: North Holland.

Myrvold, Wayne. 2016. "Probabilities in Statistical Mechanics," Ch. 27 in *The Oxford Handbook of Probability and Philosophy*. Eds Alan Hajek and Christopher Hitchcock. Oxford: Oxford University Press.

Norton, John D. 2018. "How to Build an Infinite Lottery Machine." *European Journal for Philosophy of Science*. 8: 71-95.

Norton, John D. and Pruss, Alexander R. 2018a. Correction to John D. Norton "How to Build an Infinite Lottery Machine, " *European Journal for Philosophy of Science*. 8: 143-44.

Norton, John D. manuscript. "Infinite Lottery Machines," Ch. 13 in *The Material Theory of Induction*. http://www.pitt.edu/~jdnorton/homepage/cv.html#material_theory

[19] This analysis supercedes the conclusion of Norton (2018, §9) that the machine operates successfully only with probability zero. The present analysis shows that no well-defined probability can be assigned to successful operation.

Pruss, Alexander R. 2014. "How to Run an Infinite Fair Lottery," http://alexanderpruss.blogspot.com/2014/04/how-to-run-infinite-fair-lottery.html

Rosenthal, Jeffrey. 2006. *A First Look at Rigorous Probability Theory*. 2nd ed. Singapore: World Scientific.

Solovay, Robert M. 1970. "A Model of Set-Theory in Which Every Set of Reals is Lebesgue Measurable," *Annals of Mathematics*, Second Series, 92: 1-56

Wagon, Stan. 1985. *The Banach-Tarski Paradox*. Cambridge: Cambridge University Press.

Wapner, Leonard M. 2005. *The Pea and the Sun: A Mathematical Paradox*. Wellesley, MA: A. K. Peters, Ltd.

Zermelo, Ernst. 1904. "Proof that Every Set Can be Well Ordered." In, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931,* J. van Heijenoort, ed. 139-141. Cambridge MA: Harvard University Press, 1967.

Zermelo, Ernst. 1908. "A New Proof of the Possibility of a Well-Ordering," In, *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931,* J. van Heijenoort, ed. 183-198. Cambridge MA: Harvard University Press, 1967.