

A Quantum Mechanical Supertask

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That quantum mechanical measurement processes are indeterministic is widely known. The time evolution governed by the differential Schrödinger equation can also be indeterministic under the extreme conditions of a quantum supertask, the quantum analogue of a classical supertask. Determinism can be restored by requiring normalizability of the supertask state vector, but it must be imposed as an additional constraint on the differential Schrödinger equation.

1. INTRODUCTION

It is known that systems of infinitely many particles in classical mechanics can display pathologies such as indeterminism and failure of energy and momentum conservation. The recent supertask literature has presented some simple and vivid illustrations (e.g., see Ref. 1). Following a query by John Earman, the present project is to determine whether analogous systems in quantum mechanics exhibit pathologies. We will see that translating the classical supertasks into the quantum context is not so straightforward but certainly achievable as long as one recalls that the simplest cases in classical mechanics might not correspond to the simplest in quantum theory. The resulting systems are interesting in their own right, displaying pathologies according to how we choose to define quantum systems. The principal conclusions are as follows.

- In the context of the supertask system we investigate, the differential form of the Schrödinger equation² $i(d/dt) \Psi = H\Psi$ allows indeterminism

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² The notation is standard: t is a real time coordinate, Ψ the state vector, and H the Hamiltonian. Units are chosen so that $\hbar/2\pi = 1$.

and is not equivalent to the integral form. The latter specifies a quantum state time evolution operator $U(t) = \exp(-iHt)$ and does not admit indeterminism, even with unnormalizable initial states. The differential form must be supplemented by a weak condition, such as the requirement that state vectors be normalizable at all times,³ in order to recover the integral form and expel the pathologies.

The indeterminism arising here is *not* the indeterminism of the quantum measurement process. In standard applications of quantum mechanics, the time evolution governed by the Schrödinger equation is deterministic. It is the latter time evolution that now also becomes indeterministic.

- The indeterminism arises through a mechanism directly connected with the infinite degrees of freedom of the supertask system. For example, if all its degrees of freedom are in their lowest-energy ground states, they can be spontaneously excited by a simple mechanism that begins at time $t = 0$: the first degree of freedom is excited by a faster excitation of the second, the second by a faster excitation of the third, and so on indefinitely, without any particular excitation being responsible for initiating the process. This mechanism would fail if there were only finitely many degrees of freedom, for then there would be a last degree of freedom which must initiate the process, but there would be no means to excite this last degree of freedom spontaneously. This hindrance does not arise in the infinite system since it has no last degree of freedom.

This mechanism arises from the structure of the infinite systems of equations that govern the supertask system. The differential Schrödinger equation can be written in an iterative form [see (17) below] in which the time dependence of each degree of freedom is given fully as a function of that of the lower-order degrees of freedom. Thus one can specify any, arbitrarily chosen, spontaneous excitation of the first degree of freedom. The behavior of the remaining degrees of freedom needed to enforce it is then just read off by iterating through the equation set.

Conservation of state vector norm is usually deduced from the Schrödinger equation, so it is surprising that we are free to derive it from an additional, independent assumption. It will be helpful to indicate briefly how the usual proof of conservation of state vector norm fails for quantum

³ This condition is *not* the stronger condition that the state vectors have unit norm but merely that the norms be finite. That they are constant over time will be derived from the properties of the Hamiltonian.

supertasks. In a familiar two-line proof, one usually shows that the differential Schrödinger equation entails that the time derivative of the square of the norm $d/dt(\langle \Psi | \Psi \rangle)$ vanishes:⁴

$$\begin{aligned} i \frac{d}{dt} \langle \Psi | \Psi \rangle &= i \langle \Psi | \frac{d}{dt} (|\Psi\rangle) + i \frac{d}{dt} (\langle \Psi |) |\Psi\rangle \\ &= i \langle \Psi | H | \Psi \rangle - i \langle \Psi | H | \Psi \rangle = 0 \end{aligned} \quad (1)$$

But these last two terms sum to zero only as long as the quantity $\langle \Psi | H | \Psi \rangle$ is well defined. We shall see in Sec. 5.5 [see Eq. (34)] that the supertask Hamiltonian H proves to be unbounded and the quantity $\langle \Psi | H | \Psi \rangle$ can be divergent so that the sum $i \langle \Psi | H | \Psi \rangle - i \langle \Psi | H | \Psi \rangle$ is undefined. Thus it will turn out that the time evolution governed by the differential Schrödinger equation need not conserve the norm. We will see cases in which a normed state develops to an unnormalizable state.

In Sec. 1 below, I review some classical supertasks, which will be the models for the quantum version. I introduce a new classical supertask—“masses and springs”—whose formal structure is closer to those investigated in quantum theory. In Secs. 2 and 3, I set up the vector state space for the quantum supertask that accommodates infinite degrees of freedom and define a simple Hamiltonian capable of sustaining interaction between these degrees of freedom. Section 4 briefly reviews the behavior of the system in the case in which the degrees of freedom are kept finite. This is to assure us that the pathologies to come derive directly from the transition to infinite degrees of freedom. In Sec. 5, I turn to the quantum supertask associated with infinitely many particles. In Sec. 5.1, I state and prove six propositions that govern the behavior of the system. In Secs. 5.2–5.4, I lay out a series of particular solutions that instantiate the various possibilities for types of solutions. The principal results that emerge in Secs. 5.1–5.4 are collected and described in Sec. 5.5. In Sec. 6, I consider an accelerated version of the quantum supertask which one might suppose would exhibit more pathological behavior. I give my reasons for believing that this accelerated quantum supertask introduces no qualitatively new effects. Section 7 contains some concluding remarks.

1.1. Classical Supertasks

In the simplest classical supertasks, infinitely many bodies complete an interaction in finite time. One would already suspect that conservation of

⁴ The adjoint form of the Schrödinger equation $id/dt |\Psi\rangle = H |\Psi\rangle$ is $-id/dt \langle \Psi | = \langle \Psi | H^\dagger = \langle \Psi | H$.

energy and momentum cannot be realized for such a system in exactly the same way as they can for systems of finitely many bodies. If all the bodies are alike and share the same motion, for example, then the system's energy and momentum will be infinite. In such cases, the requirement of conservation of energy and momentum, if it makes any sense at all, imposes far fewer restrictions: a system of infinite bodies of unit mass each moving with unit velocity has the same energy as the same system with each mass moving with two units of velocity. In each case, the total energy is infinite. With only a little further thought, one can see that such supertask systems can spontaneously acquire energy and momentum and sustain indeterministic time developments.

Laraudogoitia⁽²⁾ describes one of the simplest classical supertasks that illustrates this. In it, one arrays a countable infinity of elastic bodies of equal mass along the interval $(0, 1)$. The bodies need to decrease in size in order that they may all be accommodated. Therefore their density increases without bound. All the bodies are initially at rest. A new elastic body of equal mass approaches the first body at unit velocity as shown in Fig. 1. The familiar series of collisions ensues. The new body strikes the first body and comes to rest, while the first body is set into motion at unit velocity toward the second. The first strikes the second, the first comes to rest, and the second moves off at unit velocity toward the third. The chain of collision proceeds through all the bodies and the infinitely many collisions are completed in unit time since the effect propagates through the system of bodies at unit velocity. What is the final state after unit time? One's first reaction is to say that the last body is projected out at unit velocity. That conclusion would be correct if there were only finitely many bodies. In this case, there is no last body to be expelled. The final state is that all bodies are once again at rest: the first body came to rest after the second collision, the second after the third, the n th after the $(n + 1)$ th, and so on for all n .

Classical mechanics is time reversible. So we now consider the time reverse of this supertask. In it, infinitely many elastic bodies of equal mass are arrayed in the interval $(0, 1)$ and all at rest. Spontaneously, without any apparent cause, a disturbance propagates from the end where infinitely many bodies are accumulated and ends up with the ejection of the first

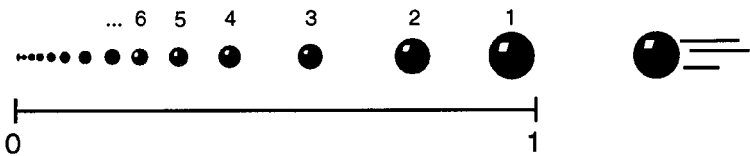


Fig. 1. A classical supertask.



Fig. 2. Masses and springs.

body. This is a violation of determinism and, under natural definitions, a violation of energy and momentum conservation.

This simple supertask is not so simple to translate into quantum theory since the initial state requires us to locate masses at rest (zero momentum) in arbitrarily small regions of space, in violation of the uncertainty relations. A related supertask—described here for the first time in the literature—is less problematic in this regard and closer in structure to the quantum case we will consider. In it, one sets up an infinite, linear array of unit masses with neighboring masses connected by a spring with spring constant k . See Fig. 2. A straightforward analysis shows that this system of masses and springs is indeterministic: it can be in an equilibrium state with all masses at rest and no springs stretched, but it can then spontaneously set its masses into motion. Let the masses be all of unit size and the spring constant k for each spring. Also, numbering the masses $1, 2, \dots, n, \dots$, we define x_n as the deviation in position of the n th mass from its position in the equilibrium state, that is, the state in which all masses are at rest and the springs neither extended nor compressed. We have immediately from Newton's second law of motion and the Hooke spring law that

$$\begin{aligned} \frac{d^2 x_2}{dt^2} &= k(x_2 - x_1), & \text{for } n = 1 \\ \frac{d^2 x_n}{dt^2} &= k(x_{n+1} - x_n) - k(x_n - x_{n-1}), & \text{for } n > 1 \end{aligned} \quad (1')$$

where t is the time coordinate. We can reorganize the terms to allow an iterative method of solution,

$$\begin{aligned} x_2 &= \frac{1}{k} \frac{d^2 x_1}{dt^2} + x_1, & \text{for } n = 1 \\ x_{n+1} &= \frac{1}{k} \frac{d^2 x_n}{dt^2} + 2x_n - x_{n-1}, & \text{for } n > 1 \end{aligned} \quad (2)$$

We solve this system of equations for the case of the system initially at $t = 0$ in its equilibrium state, that is,

$$x_n(0) = 0, \quad \frac{dx_n(0)}{dt} = 0, \quad \text{for } n \geq 1 \quad (3)$$

Systems (1') and (3) will be satisfied by *any* function $x_1(t)$ for which⁵

$$x_1(0) = 0, \quad \frac{d^n x_1(0)}{dt^n} = 0, \quad \text{for } n \geq 1 \quad (4)$$

(This function cannot be analytic, of course, unless it is everywhere zero.) With a suitable function $x_1(t)$ selected, the remaining functions $x_2(t)$, $x_3(t)$,... are computed iteratively from the system of Eqs. (2).

The indeterminism arises in this freedom to choose $x_1(t)$ and the resulting failure of conditions (3) to force a unique set of functions $x_1(t)$, $x_2(t)$,.... This failure depends essentially on the infinity of the system. If there were only N masses, then equation system (2) would not be infinite but would terminate in the final condition,

$$0 = \frac{1}{k} \frac{d^2 x_N}{dt^2} + x_N - x_{N-1} \quad (2')$$

This extra condition would prevent the iterative recovery of the functions $x_2(t)$, $x_3(t)$,... from an essentially arbitrary $x_1(t)$. The function $x_1(t)$ is further constrained in this case of finitely many masses by the additional condition that the functions generated iteratively via (2) from it must also yield function $x_{N-1}(t)$ and $x_N(t)$ that satisfy (2').

We can develop a sense of the physical mechanism that allows indeterminism in the infinite case if we choose a particular function for $x_1(t)$. A suitable choice is the nonanalytic function

$$x_1(t) = \frac{1}{t} \exp\left(-\frac{1}{t}\right) \quad (5)$$

which satisfies conditions (4). We may compute the remaining functions $x_2(t)$, $x_3(t)$,... by proceeding iteratively through the system of functions (2). The first three of these functions, for the case of $k = 100$, can be represented graphically as shown in Fig. 3.

The masses are all unenergized for $t \leq 0$ and become spontaneously energized at $t > 0$. The mechanism is as follows.

⁵ Proof sketch: First note that the following are easily proved by induction from (1'): (a) x_n is a linear sum of terms in x_1 , $d^2 x_1/dt^2$, ..., $d^{2n-2} x_1/dt^{2n-2}$, and (b) dx_n/dt is a linear sum of terms in dx_1/dt , $d^3 x_1/dt^3$, ..., $d^{2n-1} x_1/dt^{2n-1}$, for all $n \geq 2$. A second proof by induction readily shows that the boundary condition (3) leads via these last results to (4) as the sole constraint placed on $x_1(t)$ by the boundary condition.

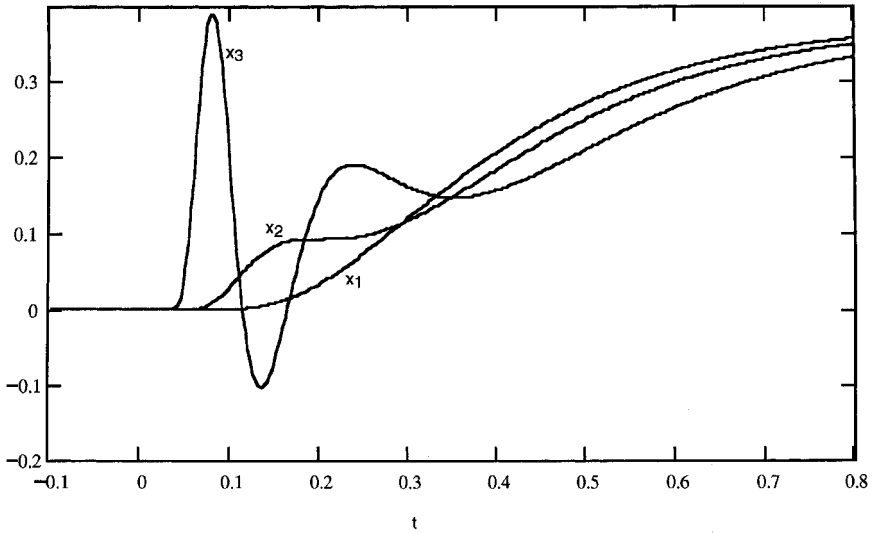


Fig. 3. Spontaneous energization of masses and springs.

The first mass is energized by a faster energization of the second mass. The second mass is energized by a faster energization of the third mass. The third mass is energized by a faster energization of the fourth mass, etc. (6)

This mechanism could not induce spontaneous energization of the entire system if there were only finitely many masses, for then there would be a last mass that would need to be energized by interacting with something outside the system in order to initiate the process. With infinitely many masses, however, there simply is no such last mass upon which to place this demand. Each mass is energized by the next and the result is a spontaneous energization of the entire system.

Note that this mechanism depends on the assumption that the springs communicate forces instantaneously. This assumption is built into the equations of motion via Hooke's law, for it is assumed that any change of position of the N th body is manifested immediately as forces acting on the $(N-1)$ th and $(N+1)$ th bodies.

Mechanism (6) is essentially similar to the one that allows the system of masses in Fig. 1 spontaneously to expel a body. The first body is set in motion by impact from the second, the second is set in motion by impact from the third, and so on for all masses. Since there is no last mass in this sequence, the process can arise spontaneously.

2. ON DEVISING A QUANTUM SUPERTASK

Is it possible to reproduce these sorts of supertasks in the context of quantum theory? Will they exhibit the same sorts of pathologies as they do in the classical context? In principle, it is always possible to take a classical system and construct another closely analogous to it in quantum theory. Thus we can take a system of infinitely many classical particles that interact by elastic collisions and model it as a system of infinitely many quantum particles interacting by means of very strong, short-range repulsive forces. Or we can model the masses and springs as a system of coupled harmonic oscillators. The analysis threatens to be complicated, however. Modeling action by contact, for example, requires spatial localization of the interacting particles; the interaction is “turned on” when the particles are sufficiently close spatially. But exactly this spatial localization forces the momentum of the particles to be poorly defined; that is, the particle states must be superpositions of momentum eigenstates, with the associated momenta spread over a very wide range. In pursuing the exchanges of momentum and energy in the interactions, we are forced from the start to deal with a complicated superposition of the simple case, the momentum eigenstates. Similar problems remain with the masses and springs, since the interaction potential will depend on position. So we may well wonder if it is really necessary to treat such close analogs in order to determine whether quantum theory admits pathological supertasks.

What is essential in the classical supertasks are three features.

- (a) The systems have infinitely many particles.
- (b) The interaction of the particles, when considered pairwise, is not pathological.
- (c) The interactions are sufficiently accelerated so that infinitely many particles are embroiled in the interaction in finite time.

The interest in supertasks lies in the emergence of pathological behavior, such as indeterminism and lack of energy and momentum conservation, in the transition from the interaction of finitely many particles to infinitely many.

In the quantum context, we can retain these essential features of the supertasks and avoid the inessential complications of position dependent interactions if we restrict our attention directly to the energy eigenstates or momentum eigenstates of some Hamiltonian. If we add terms to the Hamiltonian that represent generic interactions without consideration of the effect of spatial separation, we need not even consider the spatial disposition of the systems. To this end, we will represent the infinitely many particles of (a) within an infinite-dimensional vector space that is the

tensor product of the Hilbert spaces of the infinitely many component particles. We represent (b), the pairwise well-behaved interaction, by writing the interaction Hamiltonian for the entire system of particles as a sum of well-behaved pieces. It will turn out that there is no special need to accelerate the interactions as in (c) in order to allow the infinitely many particles to interact in finite time. Since we no longer allow spatial separation to switch off the interaction between particles, supertask-like behavior and pathology will emerge automatically. The system constructed will be conceptually and formally very similar to the masses and springs supertask, excepting that considerations of spatial position are eradicated.

3. THE SETUP OF A QUANTUM SUPERTASK

3.1. Particle States

We consider infinitely many particles, each with its own Hilbert space $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \dots$. The state vectors of particles in space \mathcal{H}_n for $n = 1, 2, \dots$, will be written as $|\Psi\rangle_n$. To keep the analysis as simple as possible, we will assume that each particle admits just two energy eigenstates $|0\rangle_n$ and $|1\rangle_n$ of the “particle Hamiltonian” H_n^{part} that act in space \mathcal{H}_n . They have unit norm ${}_n\langle 0|0\rangle_n = {}_n\langle 1|1\rangle_n = 1$. These eigenstates have eigenvalues 0 and 1, respectively, so that we have

$$H_n^{\text{part}}|0\rangle_n = 0|0\rangle_n = 0, \quad H_n^{\text{part}}|1\rangle_n = 1|1\rangle_n = |1\rangle_n \quad (7)$$

for $n = 1, 2, 3, \dots$. We consider states of infinitely many particles as given by vectors in the infinite-dimensional product space

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \dots \quad (8)$$

This space is not a Hilbert space since it admits vectors that cannot be normalized. In order to discern the effect of the transition to infinitely many particles, we also consider a system of N particles whose states are given by vectors in the finite-dimensional product space

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \dots \otimes \mathcal{H}_N \quad (9)$$

The natural basis vectors of the infinite-dimensional space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \dots$ are

$$|a_1\rangle_1 \otimes |a_2\rangle_2 \otimes |a_3\rangle_3 \dots$$

where $a_1 = 0$ or 1 , $a_2 = 0$ or 1 , $a_3 = 0$ or 1 , ... (10)

In the infinite-dimensional space (8) there are 2^{\aleph_0} binary decisions to be taken to fix each basis vector. Thus the set is uncountable.

3.2. The Interaction Hamiltonian

To mimic the interactions of the classical supertasks, we want an interaction Hamiltonian that allows particle 1 to interact with particle 2, particle 2 with particle 3, and so on. The simplest such interaction is one that creates excited state $|1\rangle_2$ of particle 2 while, at the same time, destroying excited state $|1\rangle_1$ of particle 1, and similarly for the rest. The disadvantage of this simplicity is that, over time, the interaction can raise states that are a superposition of any of the basis vectors of (10). That is, in the case of the infinite space (8), the state vector will visit within a space spanned by an uncountable basis. A simple modification of the interaction can restrict the space to a subspace with an infinite but countable basis. To describe it, we define vectors in the infinite vector space

$$|0\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \otimes \dots$$

$$|1\rangle = |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \otimes \dots$$

$$|2\rangle = |0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \otimes \dots$$

$$|3\rangle = |0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 \otimes |0\rangle_4 \otimes \dots$$

etc.

The interaction will⁶

destroy $|1\rangle$ and create $|2\rangle$; destroy $|2\rangle$ and create $|1\rangle$

destroy $|2\rangle$ and create $|3\rangle$; destroy $|3\rangle$ and create $|2\rangle$

⋮

destroy $|n\rangle$ and create $|n+1\rangle$; destroy $|n+1\rangle$ and create $|n\rangle$

⋮

and leave the ground state $|0\rangle$ unaffected.

These interactions have a nonlocal character in the sense that each involves all particle states. For example, the destruction of $|1\rangle$ and creation of $|2\rangle$ does not just affect the states of particles 1 and 2, but also involves

⁶ If the interaction destroys $|1\rangle$ and creates $|2\rangle$, then it follows from the Hermiticity of the Hamiltonian that it must also destroy $|2\rangle$ and create $|1\rangle$.

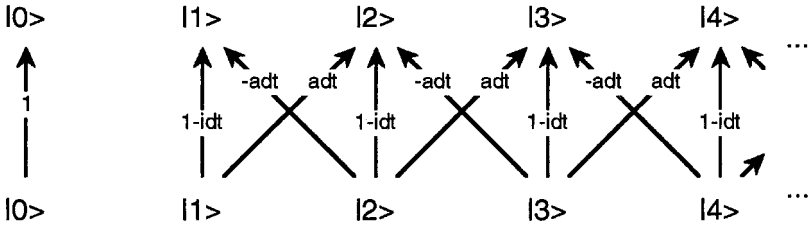


Fig. 4. Transition amplitudes for supertask Hamiltonian.

the creation of states $|0\rangle_3, |0\rangle_4, \dots$ entangled with the state $|1\rangle_2$ created for particle 2. While this sort of entanglement is common in quantum theory, if it troubles the reader, the reader is encouraged to think of the states $|0\rangle, |1\rangle, |2\rangle, \dots$ as the fundamental states with the individual particle states $|0\rangle_1, |1\rangle_1, \dots$ merely an intermediate used to define them.

The total Hamiltonian is then given as

$$H = H^{\text{part}} + H^{\text{int}} \tag{11}$$

where

$$H^{\text{part}} = \sum_{n=1}^{\infty \text{ or } N} |n\rangle \langle n| \tag{12}$$

$$H^{\text{int}} = \sum_{n=1}^{\infty \text{ or } N-1} ia |n+1\rangle \langle n| - ia |n\rangle \langle n+1| \tag{13}$$

The two ranges of summations in (12) and (13) are for the case of infinitely many particles and for N particles.⁷ The real, positive constant a is set arbitrarily and measures the strength of the interaction. The states $|0\rangle, |1\rangle, |2\rangle, \dots$ form a countable basis that spans an infinite dimensional subspace of the original infinite vector space (8). The Hamiltonian is so constructed as to map vectors in that subspace to vectors in that subspace, so that the time development it generates will remain in the subspace if it starts there. The Hamiltonian (11) is Hermitian.⁸

The transition amplitudes of this Hamiltonian are represented figuratively in Fig. 4. The transition amplitude $|n\rangle \rightarrow |m\rangle$ is defined as $\langle m| U(dt) |n\rangle$ where $U(dt) = \exp(-iHdt) \approx 1 - iHdt$ is the time evolution operator for small dt .

⁷ Caution! The summation in (12) should begin at $n = 1$. H^{part} cannot have a term $|0\rangle \langle 0|$ since it must annihilate the zero-energy ground state $|0\rangle$.

⁸ To see this, compute its matrix elements in the natural basis. $H_{mn} = \langle m| H |n\rangle = \delta_{mn} + ia\delta_{m-1,n} - ia\delta_{m,n-1} = \delta_{nm} + ia\delta_{n,m-1} - ia\delta_{n-1,m} = (\delta_{mn} + ia\delta_{n-1,m} - ia\delta_{n,m-1})^* = (H_{nm})^*$ so that $H = H^\dagger$, which is the condition of hermiticity.

3.3. The Schrödinger Equation

The Schrödinger equation requires

$$i \frac{d}{dt} |\Psi(t)\rangle = H |\Psi(t)\rangle \quad (14)$$

for a state vector $|\Psi(t)\rangle = \sum_{n=0}^{\infty \text{ or } N} C_n(t) |n\rangle$. Substituting this expression for $|\Psi(t)\rangle$ into (14) and invoking expressions (12) and (13) for the Hamiltonian, we recover a set of equations for $C_n(t)$, where the prime denotes d/dt .

$$\begin{aligned} iC'_0 &= 0 \\ iC'_1 &= C_1 - iaC_2 \\ iC'_2 &= C_2 + ia(C_1 - C_3) \\ &\vdots \\ iC'_n &= C_n + ia(C_{n-1} - C_{n+1}) \\ &\vdots \end{aligned} \quad (15)$$

For the infinite case, this set of equations continues indefinitely. For the case of N particles, it terminates with

$$iC'_N = C_N + iaC_{N-1} \quad (15')$$

Since the $|0\rangle$ mode interacts with no others, (15) requires

$$C_0(t) = C_0(0) \quad (16)$$

in all solutions. I typically suppress this equation in the systems and discussion to follow.

This system of Eqs. (15), (15') can be put into a simpler and more useful form by defining $f_n(t) = e^{it} C_n(t)$ for $n > 0$ and rearranging terms:

$$\begin{aligned} f_2 &= -(1/a) f'_1 \\ f_3 &= f_1 - (1/a) f'_2 \\ &\vdots \\ f_n &= f_{n-2} - (1/a) f'_{n-1} \\ &\vdots \end{aligned} \quad (17)$$

For the case of N particles only, the set terminates with

$$0 = f_{N-1} - (1/a) f'_N \quad (17')$$

In the form (17), (17'), one can already see immediately how indeterminism will arise in the infinite case. One can use the set as an iterative prescription for generating solutions to the Schrödinger equation. One starts with some function $f_1(t)$ and then uses each equation in turn to determine $f_2(t), f_3(t), \dots$. The result will be a full set of coefficients $C_n(t) = e^{-it} f_n(t)$ for $n = 1, 2, \dots$. These coefficients in turn fully specify the state $|\Psi(t)\rangle$ and its time development. In the finite case, the final equation (17') places a restriction on the choice of $f_1(t)$; we cannot insert any $f_1(t)$ into the scheme for it will most likely fail to generate functions f_{N-1} and f_N that satisfy (17'). But in the infinite case, there is no such restriction. The broad freedom to choose the function $f_1(t)$, tempered only by the need to satisfy initial conditions, is translated below into an indeterminism.

4. THE CASE OF N PARTICLES IS WELL BEHAVED

Before proceeding with the pathologies of the case of infinitely many particles, it will be helpful to ascertain that the system described is well behaved as long as we restrict ourselves to finitely many particles. This is already suggested by a crude count of the component equations (15), (15') or (17), (17') that comprise the Schrödinger equation. We are solving for N functions of times, $C_1(t), \dots, C_N(t)$, and we have a system of N first-order differential equations to solve. This suggests that all we need are N initial conditions to fix these N functions completely. These N initial conditions are supplied by setting the N values of $C_1(0), \dots, C_N(0)$. That is, the state of the system at time $t = 0$, $|\Psi(0)\rangle$, determines its future time development. Impatient readers may be satisfied with this observation. They are urged to jump ahead to the discussion of the case of infinitely many particles. Others are invited to review the behavior of the system for $N = 2$ and $N = 3$.⁹

⁹ For completeness I note that the energy eigenvalue problem admits a well-behaved solution. For the case of N particles there are N linearly independent solutions of the eigenvalue equation $H|E\rangle = |E\rangle$, for energy eigenvalues E , so that the energy spectrum is discrete. For the case of $N = 2$, the energy eigenvectors are $|E = 1 + a\rangle = (1/\sqrt{2})(|1\rangle + i|2\rangle)$ and $|E = 1 - a\rangle = (1/\sqrt{2})(|1\rangle - i|2\rangle)$. For $N = 3$, they are $|E = 1\rangle = (1/\sqrt{2})(|1\rangle + |3\rangle)$, $|E = 1 + a\sqrt{2}\rangle = \frac{1}{2}(|1\rangle + i\sqrt{2}|2\rangle - |3\rangle)$, and $|E = 1 - a\sqrt{2}\rangle = \frac{1}{2}(|1\rangle - i\sqrt{2}|2\rangle - |3\rangle)$.

4.1. $N=2$: The Case of Two Particles

For $N=2$, the Schrödinger equation (17), (17') reduces to

$$f_2 = -(1/a) f'_1, \quad 0 = f_1 - (1/a) f'_2$$

This is solved easily to give

$$f''_2 + a^2 f_2 = 0, \quad f_1 = (1/a) f'_2$$

These two differential equations admit the general solution

$$f_1(t) = A \cos at + B \sin at, \quad f_2(t) = -B \cos at + A \sin at$$

where A and B are arbitrary complex-valued constants. This solution corresponds to

$$C_1(t) = e^{-it}(A \cos at + B \sin at), \quad C_2(t) = e^{-it}(-B \cos at + A \sin at) \quad (18)$$

As a concrete illustration we take the case in which the first particle only is energized and communicates its energy to the remaining particles. This corresponds to the case of the initial condition $|\Psi(0)\rangle = |1\rangle = |1\rangle_1 \otimes |0\rangle_2$. The corresponding initial values for (18) are $C_1(0) = 1$ and $C_2(0) = 0$. This yields a time development $C_1(t) = e^{-it} \cos at$, $C_2(t) = e^{-it} \sin at$ so that the state evolves as

$$|\Psi(t)\rangle = e^{-it}(\cos at |1\rangle + \sin at |2\rangle) \quad (19)$$

In this time development, the excitation is communicated from particle 1 to particle 2. It then continues to oscillate back and forth between the two states $|1\rangle$, in which the first particle only is energized, and $|2\rangle$, in which the second particle only is energized. Measuring the degree of excitation as $|C_1(t)|$ and $|C_2(t)|$, this process is represented graphically in Fig. 5.

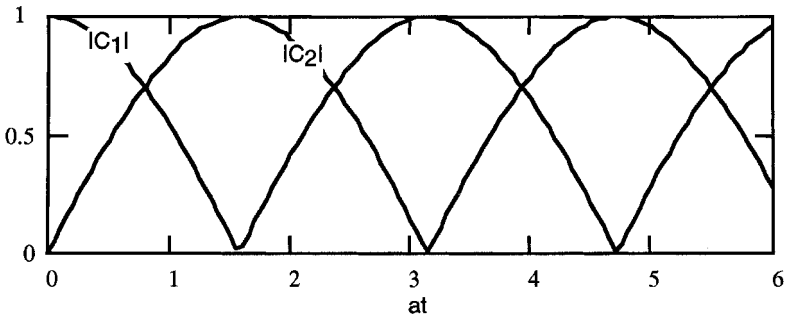


Fig. 5. Dynamics of a two-particle system.

4.2. $N=3$: The Case of Three Particles

For $N=3$, the Schrödinger equation (17), (17') reduces to

$$f_2 = -(1/a) f_1', \quad f_3 = f_1 - (1/a) f_2', \quad 0 = f_2 - (1/a) f_3'$$

The set can be reduced to a single equation in f_1 and its derivatives only,

$$f_1''' = +2 a^2 f_1' = 0$$

This equation can be solved for $f_1(t)$ and the reduced form (17) of the Schrödinger equation used to recover expressions for $f_2(t)$ and $f_3(t)$. The general solution is

$$\begin{aligned} f_1(t) &= A \cos \omega t + B \sin \omega t + D, & f_2(t) &= \sqrt{2} (A \sin \omega t - B \cos \omega t), \\ f_3(t) &= -A \cos \omega t - B \sin \omega t + D \end{aligned}$$

where $\omega = a\sqrt{2}$ and A , B , and D are arbitrary complex-valued constants. This solution corresponds to

$$\begin{aligned} C_1(t) &= e^{-it} (A \cos \omega t + B \sin \omega t + D) \\ C_2(t) &= e^{-it} \sqrt{2} (A \sin \omega t - B \cos \omega t) \\ C_3(t) &= e^{-it} (-A \cos \omega t - B \sin \omega t + D) \end{aligned} \quad (20)$$

As an initial state, consider the system in which the first particle only is energized so that we have $|\Psi(0)\rangle = |1\rangle = |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$. This state fixes $A = D = \frac{1}{2}$ and $B = 0$ so that the state evolves as

$$|\Psi(t)\rangle = \frac{1}{2} e^{-it} ((1 + \cos \omega t) |1\rangle + \sqrt{2} \sin \omega t |2\rangle + (1 - \cos \omega t) |3\rangle) \quad (21)$$

In this time development, the excitation is communicated from particle 1 to particle 2 and then to particle 3 and then continues to oscillate among the three excitations. Measuring the degree of excitation as $|C_1(t)|$, $|C_2(t)|$, and $|C_3(t)|$, this process is represented graphically in Fig. 6.

5. INDETERMINISM IN THE CASE OF INFINITELY MANY PARTICLES

The results of the last section indicate that the system of particles behaves standardly as long as we consider only finitely many particles. We now show that, in the case of infinitely many particles and for a particular

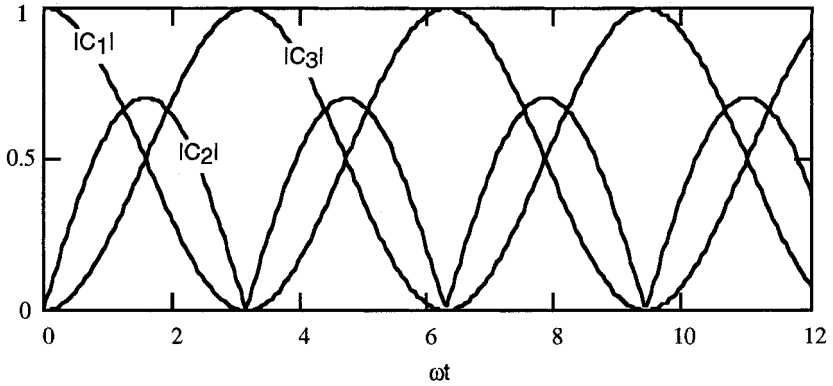


Fig. 6. Dynamics of a three-particle system.

supertask system, the time development specified by the Schrödinger equation is indeterministic as long as we require only that the system obey the Schrödinger equation in its differential form (14). That is, a fixed initial state $|\Psi(0)\rangle$ can evolve into more than one future state $|\Psi(t)\rangle$ at time t . The integral form of the Schrödinger equation specifies a time evolution given by $|\Psi(t)\rangle = U(t) |\Psi(0)\rangle$, where

$$U(t) = \exp(-iHt) \quad (14')$$

This time development will prove to be deterministic, even with unnormalizable initial states. The indeterminism of the time evolution of the Schrödinger equation in differential form (14) reverts to the deterministic evolution of (14') if we supplement the differential condition with the requirement that the state vector always be normalizable.

5.1. Propositions

The demonstration of these results will depend upon propositions about the supertask system derived in this section.

Proposition 1: No Unique Solution. If $|\Psi(t)\rangle$ satisfies the differential form of the Schrödinger equation (14) with Hamiltonian H (11) for infinitely many particles and initial condition $|\Psi(0)\rangle$, then there exists arbitrarily many more distinct¹⁰ solutions $|\Phi(t)\rangle$ with the same initial condition, $|\Phi(0)\rangle = |\Psi(0)\rangle$.

¹⁰ That is, the alternative solutions differ by more than a phase factor or arbitrary constant.

The proof of this proposition depends upon a special property of the solutions of the differential Schrödinger equation with Hamiltonian (11): each of the $f_n(t)$ of the equation set (17) are linear functions of $f_1(t)$ and its derivatives. To see this, one merely iterates through the equation set sequentially computing expressions for each $f_n(t)$ from the expressions computed for $f_{n-1}(t)$ and $f_{n-2}(t)$. One recovers

$$\begin{aligned}
 f_1 &= f_1 \\
 f_2 &= -(1/a) f_1' \\
 f_3 &= f_1 + (1/a^2) f_1'' \\
 f_4 &= -(2/a) f_1' - (1/a^3) f_1''' \\
 f_5 &= f_1 + (3/a^2) f_1'' + (1/a^4) f_1^{(4)} \\
 f_6 &= -(3/a) f_1' - (4/a^3) f_1''' - (1/a^5) f_1^{(5)} \\
 f_7 &= f_1 + (6/a^2) f_1'' + (5/a^4) f_1^{(4)} + (1/a^6) f_1^{(6)} \\
 &\vdots
 \end{aligned}
 \tag{22}$$

The exact expressions for the dependence of the f_n on f_1 and its derivatives turn out to be unimportant for the proposition.¹¹ All that matters is that the relationship is linear and that can be represented by writing

$$f_n(t) = \sum_{k=1, n-1} B_{n,k} f_1^{(k-1)}
 \tag{23}$$

for some set of real constants $B_{n,k}$, where $n = 1, 2, \dots$ and $k = 1, \dots, n - 1$.

Now assume that we have a solution $|\Psi(t)\rangle = \sum_{n=0, \infty} C_n(t) |n\rangle$ of the differential Schrödinger equation. The associated functions $f_n(t) = e^{it} C_n(t)$, for $n = 1, 2, \dots$, will solve the set of Eqs. (17). Now let $G(t)$ be a real-valued C^∞ function that is not everywhere zero but which satisfies the conditions

$$G(0) = 0, \quad G'(0) = 0, \quad G''(0) = 0, \quad G^{(3)}(0) = 0, \dots
 \tag{24}$$

¹¹ For $n = 1, 2, 3, \dots$, the general expression turns out to be

$$f_{2n} = \sum_{k=1, n} -\frac{1}{a^{2k-1}} \binom{n-1+k}{2k-1} f_1^{(2k-1)}, \quad f_{2n+1} = \sum_{k=0, n} \frac{1}{a^{2k}} \binom{n+k}{2k} f_1^{(2k)}$$

where $\binom{n}{k} = n!/(k!(n-k)!)$. These expressions can be proved by mathematical induction using the recursion relation $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$.

and so on for all orders of differentiation.¹² Define $g_1(t) = f_1(t) + G(t)$. This new function $g_1(t)$ can be inserted into the set of equations (17) in place of $f_1(t)$ and used to generate an infinite set of functions $g_2(t), g_3(t), \dots$, where $g_2 = (1/a) g'_1, g_3 = g_1 - (1/a) g'_2, \dots$. This set of functions, by construction, satisfies the set of equations (17) that is equivalent to the Schrödinger equation. Therefore if we define a new time-dependent state vector $|\Phi(t)\rangle = \sum_{n=0, \infty} D_n |n\rangle$, where $D_n(t) = e^{-it} g_n(t)$ for all n , then $|\Phi(t)\rangle$ will also satisfy the differential Schrödinger equation.

The initial condition $|\Phi(0)\rangle$ is specified by the values of $D_n(0)$ for all n and these are in turn specified by the values of $g_n(0)$. For each n we have from (23) that

$$\begin{aligned} g_n(0) &= \sum_{k=1, n-1} B_{n,k} g_1^{(n-1)}(0) \\ &= \sum_{k=1, n-1} B_{n,k} f_1^{(n-1)}(0) + \sum_{k=1, n-1} B_{n,k} G^{(n-1)}(0) \\ &= \sum_{k=1, n-1} B_{n,k} f_1^{(n-1)}(0) = f_n(0) \end{aligned}$$

where the term in G vanishes because of condition (24). For all n , if $f_n(0) = g_n(0)$, then $C_n(0) = D_n(0)$, so that $|\Psi(0)\rangle = |\Phi(0)\rangle$. The proof of the proposition is now complete. If $|\Psi(t)\rangle$ is a solution of the Schrödinger equation, then we have constructed a second solution that has the same state at $t = 0$. Since there are arbitrarily many nonzero functions $G(t)$ that satisfy conditions (24), there are arbitrarily many such alternative solutions.

Note that the proof of the proposition depends essentially of the infinity of the system. If the number of particles were finite, then, in addition to (17), the new solution would also need to satisfy the condition (17'). This would preclude insertion of an arbitrary $g_1(t)$ into scheme (17) and block the proof.

Proposition 2: Unique Analytic Solution. If $|\Psi(t)\rangle$ satisfies the differential form of the Schrödinger equation (14) with Hamiltonian H (11) for infinitely many particles and some chosen initial condition $|\Psi(0)\rangle$ and, if it is analytic, then $|\Psi(t)\rangle$ is unique.

¹² There are arbitrarily many such functions, although they cannot be analytic. Consider, for example, $G(t) = K(1/t^2) e^{-(1/t^2)}$, for any constant $K > 0$. This $G(t)$ satisfies (24) and also that $G(t) \geq 0$ for all t and $\text{Lim}_{t \rightarrow \infty} G(t) = 0$.

The proof of this proposition follows readily from the set of equations (23). The boundary condition fixes $|\Psi(0)\rangle$, so that all values of $f_1(0), f_2(0), f_3(0), \dots$ are fixed. We now iterate through the equation set (23). The second equation ($f_2 = -(1/a) f'_1$) fixes the value of $f'_1(0)$, the third equation the value of $f''_1(0)$, and so on. That is, the values of $f^{(n)}_1(0)$ for all n are fixed. Since we assume the solution is analytic, it follows that the function $f_1(t)$ is fixed. But once $f_1(t)$ is fixed, we can solve for all the remaining functions $f_2(t), f_3(t), \dots$ since, according to (23), each of these functions is expressible as a finite sum of terms in $f_1(t)$ and its derivatives.

Proposition 3. The unique analytic $|\Psi(t)\rangle$ of Proposition 2 (if it exists) satisfies the integral form of the Schrödinger equation (14'), $|\Psi(t)\rangle = \exp(-iHt) |\Psi(0)\rangle$.

Proof. Since the solution is analytic it can be written as a power series in t

$$|\Psi(t)\rangle = |\Psi(0)\rangle + (d/dt) |\Psi(0)\rangle t + \dots + (1/n!)(d^n/dt^n) |\Psi(0)\rangle t^n + \dots$$

We can use the Schrödinger equation in form (14) to reexpress each time derivative in terms of the Hamiltonian so that we have

$$|\Psi(t)\rangle = |\Psi(0)\rangle + (-iHt) |\Psi(0)\rangle + \dots + (1/n!)(-iHt)^n |\Psi(0)\rangle + \dots$$

But this series is just $|\Psi(t)\rangle = \exp(-iHt) |\Psi(0)\rangle$.

Proposition 4: Constancy of Finite Norm. If $|\Psi\rangle$ satisfies the differential Schrödinger equation (14) for the infinite particle system and its norm is finite over some time interval, then the norm is constant over this time interval.

To see this, consider the first n equations of the Schrödinger equation in form (17), multiply them by $f_1^*, f_2^*, \dots, f_n^*$, respectively, and sum. We recover

$$\begin{aligned} f'_1 f_1^* + f'_2 f_2^* + \dots + f'_n f_n^* \\ = -af_1^* f_2 + (af_1 f_2^* - af_2^* f_3) + \dots + (af_{n-1} f_n^* - af_n^* f_{n+1}) \end{aligned}$$

Its complex conjugate form is

$$\begin{aligned} f_1 f_1'^* + f_2 f_2'^* + \dots + f_n f_n'^* \\ = -af_1 f_2'^* + (af_1^* f_2 - af_2 f_3'^*) + \dots + (af_{n-1}^* f_n - af_n f_{n+1}^*) \end{aligned}$$

Most terms on the right-hand side cancel when we sum these two equations to recover

$$\frac{d}{dt} (f_1^* f_1 + \dots + f_n^* f_n) = -af_n f_{n+1}^* - af_n^* f_{n+1}$$

But for a quantum state $|\Psi(t)\rangle = \sum_{n=0, \infty} C_n(t) |n\rangle$, we have for the square of its norm that $\langle \Psi | \Psi \rangle = \sum_{n=0, \infty} C_n^* C_n = C_0^* C_0 + \sum_{n=1, \infty} f_n^* f_n$. Since $C_0^* C_0$ is constant with time from Eq. (16), we have

$$\frac{d}{dt} \langle \Psi(t) | \Psi(t) \rangle = - \lim_{n \rightarrow \infty} (af_n f_{n+1}^* + af_n^* f_{n+1}) \quad (25)$$

If the norm of the state vector is finite over some time interval, then the sum $\sum_{n=1, \infty} f_n^* f_n$ converges so that $\lim_{n \rightarrow \infty} f_n = 0$. Hence $\lim_{n \rightarrow \infty} (af_n f_{n+1}^* + af_n^* f_{n+1}) = 0$ and the norm of $|\Psi(t)\rangle$ is constant over the interval.

Proposition 5: Determinism for Normalizable State Vectors.

If $|\Psi(t)\rangle$ satisfies the differential Schrödinger equation and we also require that it be normalizable at all times t , then its time development is deterministic; that is, there is a unique $|\Psi(t)\rangle$ associated with each initial state $|\Psi(0)\rangle$.

Proof. First consider the case in which $|\Psi(0)\rangle = |0\rangle$. Its (norm)² is $\langle 0 | 0 \rangle = 1$. It is easy to see that the unique $|\Psi(t)\rangle$ that preserves this norm is $|\Psi(t)\rangle = |0\rangle$. Any other time development yields a state vector $|\Psi(t)\rangle = \sum_{n=0, \infty} C_n(t) |n\rangle$, for which $C_n(t) \neq 0$ for some $n > 0$ and for some $t > 0$. But we have from (16) that $C_0^*(t) C_0(t) = 1$ for all t . Thus these other nonzero coefficients $C_n(t)$ will lead to a norm greater than unity at some $t > 0$; that is, the norm of $|\Psi(t)\rangle$ will not be constant with time. But we have from Proposition 4 that such nonconstancy of norm arises only if the state vector has become unnormalizable. Therefore the only normalizable time development is $|\Psi(t)\rangle = |0\rangle$.

Now consider an arbitrary normalizable initial state $|\Psi(0)\rangle$. Assume, contrary to the proposition, for *reductio* purposes that there are two distinct, normalizable solutions of the differential Schrödinger equation $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ that agree on this initial condition, that is, for which $|\Psi(0)\rangle = |\Phi(0)\rangle$ but $|\Psi(t)\rangle \neq |\Phi(t)\rangle$ for some $t > 0$. Consider their difference, $|\Psi(t)\rangle - |\Phi(t)\rangle$. Since $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ are normalizable individually, it follows that their difference is normalizable too.¹³ Similarly

¹³ This follows from the triangle equality applied to Ψ and $-\Phi$, which tells us that $\|\Psi - \Phi\| \leq \|\Psi\| + \|-\Phi\|$. Since $\|\Psi\|$ and $\|-\Phi\|$ are each finite, so is $\|\Psi - \Phi\|$. (The norm $\|\Psi\|$ is defined by $\|\Psi\|^2 = \langle \Psi | \Psi \rangle$.)

the sum of normalizable solutions $|\Theta\rangle = |0\rangle + |\Psi(t)\rangle - |\Phi(t)\rangle$ is normalizable. Since the Schrödinger equation is a linear equation and $|\Theta\rangle$ is a linear combination of solutions, $|\Theta\rangle$ is also a solution of the Schrödinger equation. But now $|\Theta\rangle$ has impossible properties. Its initial state is $|\Theta(0)\rangle = |0\rangle + |\Psi(0)\rangle - |\Phi(0)\rangle = |0\rangle$. But $|\Theta(t)\rangle$ is not $|0\rangle$, the only normalizable solution with this initial condition since we have by supposition that $|\Psi(t)\rangle \neq |\Phi(t)\rangle$ for some $t > 0$. Therefore we must reject this supposition and conclude that there is only one $|\Psi(t)\rangle$ for each initial state $|\Psi(0)\rangle$.

Proposition 6: Determinism of the Integral Form of the Schrödinger Equation. If $|\Psi(t)\rangle$ satisfies the integral form of the Schrödinger equation (14'), then its time development is deterministic; that is, there is a unique $|\Psi(t)\rangle$ associated with each initial state $|\Psi(0)\rangle$, whether $|\Psi(0)\rangle$ is normalizable or not.

Proof. This integral form of the Schrödinger equation preserves the state vector norm since

$$\langle \Psi(t) | \Psi(t) \rangle = \langle \Psi(0) | \exp(iHt) \exp(-iHt) | \Psi(0) \rangle = \langle \Psi(0) | \Psi(0) \rangle \quad (26)$$

That is, if the state vector norm is finite at any time, it will retain this same value at all times. If it diverges at any time, it will diverge at all times. First, if the initial state vector is $|\Psi(0)\rangle = |0\rangle$, it has unit norm at $t = 0$. It is easy to see that the unique time development that preserves this norm is $|\Psi(t)\rangle = |0\rangle$. Any other time development yields a state vector $|\Psi(t)\rangle = \sum_{n=0, \infty} C_n(t) |n\rangle$ for which $C_n(t) \neq 0$ for some $n > 0$ and for some $t > 0$. Any solution of the integral Schrödinger (14') equation must also satisfy the differential Schrödinger equation (14). But we have from the differential Schrödinger equation via (16) that $C_n^*(t) C_0(t) = 1$ for all t . Therefore, if the time development is to preserve unit norm, all $C_n(t) = 0$ for $n > 0$. That is, $|\Psi(t)\rangle = |0\rangle$.

Now consider an arbitrary initial state $|\Psi(0)\rangle$, which may or may not be normalizable. Assume, contrary to the proposition, for *reductio* purposes that there are two distinct, solutions of the differential Schrödinger equation $|\Psi(t)\rangle$ and $|\Phi(t)\rangle$ that agree on this initial condition, that is, for which $|\Psi(0)\rangle = |\Phi(0)\rangle$ but $|\Psi(t)\rangle \neq |\Phi(t)\rangle$ for some $t > 0$. Consider $|\Theta\rangle = |0\rangle + |\Psi(t)\rangle - |\Phi(t)\rangle$. Since it is a linear combination of solutions of the integral Schrödinger equation and this equation is linear, $|\Theta\rangle$ is also a solution. But this solution has impossible properties. Its initial state is

$|\Theta(0)\rangle = |0\rangle + |\Psi(0)\rangle - |\Phi(0)\rangle = |0\rangle$. But $|\Theta(t)\rangle$ is not $|0\rangle$, the only solution with this initial condition that preserves unit norm, since we have by supposition that $|\Psi(t)\rangle \neq |\Phi(t)\rangle$ for some $t > 0$. Therefore we must reject this supposition and conclude that there is only one $|\Psi(t)\rangle$ for each initial state $|\Psi(0)\rangle$.

These propositions do not yet establish the final results sought. Proposition 1, for example, cannot demonstrate indeterminism non-vacuously until it is shown that there are any solutions at all. Proposition 2 cannot show that determinism is preserved nonvacuously with a restriction to analytic solutions until we show that there are analytic solutions. To show that the propositions are not just vacuously or trivially true, in the sections following we display several nontrivial¹⁴ solutions.

5.2. A Well-Behaved Solution

The indeterminism of the differential form of the Schrödinger equation allows the supertask system to exhibit unexpected behaviors. The analytic solution developed in this section exhibits the sort of behavior one intuitively expects. It is the quantum analogue of a classical supertask in which a single excited particle communicates its excitation to an infinite collection of initially unexcited particles. The excitation propagates through the infinitely many particles and dissipates.

The system's initial state is $|\Psi(0)\rangle = |1\rangle = |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 \otimes |0\rangle_4 \dots$, in which all but the first particle are in these ground states. This initial state admits a solution in terms of $J_n(x)$, the Bessel functions of the first kind, and is

$$|\Psi(t)\rangle = \sum_{n=1, \infty} e^{-it(J_{n-1}(2at) + J_{n+1}(2at))} |n\rangle \tag{27}$$

This solution is analytic since these Bessel functions are analytic. That this is a solution of the Schrödinger equation follows directly from the recursion relation satisfied by these Bessel functions of the first kind. That relation is

$$2 \frac{d}{dx} J_n(x) = J_{n-1}(x) - J_{n+1}(x)$$

¹⁴ There is of course, in addition, the simple case of $|\Psi(t)\rangle = |0\rangle$, which automatically solves the Schrödinger equation.

Setting $x = 2at$ and forming a sum of two Bessel functions, we recover

$$\frac{1}{a} \frac{d}{dt} (J_{n-1}(2at) + J_{n+1}(2at)) = (J_{n-2}(2at) + J_n(2at)) - (J_n(2at) + J_{n+2}(2at)) \quad (28)$$

Taking

$$f_n(t) = J_{n-1}(2at) + J_{n+1}(2at) \quad (29)$$

this expression (28) becomes

$$f_{n+1}(t) = f_{n-1}(t) - (1/a) f'_n(t) \quad (30)$$

for $n > 1$. For the case of $n = 1$, expression (28) reduces to

$$f_2(t) = -(1/a) f'_1(t) \quad (31)$$

What makes the case $n = 1$ exceptional is that the first term on the right-hand side of (28) reduces to $J_{-1}(2at) + J_1(2at)$ and this vanishes because of the symmetry property of Bessel functions of the first kind: for all n , $J_{-n}(x) = (-1)^n J_n(x)$. Expressions (30) and (31) are equivalent to form (17) of the Schrödinger equation and this shows that the identification for $f_n(t)$ in (29) satisfies the Schrödinger equation, so that the state vector (27) also satisfies the Schrödinger equation.

Since Bessel functions satisfy the identity $J_{\nu-1}(x) + J_{\nu+1}(x) = (2\nu/x) J_\nu(x)$, there is an alternate expression for $f_n(t)$,

$$f_n(t) = (n/at) J_n(2at) \quad (29')$$

To complete the demonstration that (27) is a solution of the Schrödinger equation for the initial condition $|\Psi(0)\rangle = |1\rangle$, we need only show that (27) is compatible with this initial condition. This compatibility follows directly from the properties of the Bessel function of the first kind. For the $C_n(0) = e^{-i0} f_n(0) = f_n(0)$ corresponding to the $f_n(t)$ of (29), we have

$$C_1(0) = J_0(0) + J_2(0) = 1, \quad C_n(0) = J_{n-1}(0) + J_{n+1}(0) = 0, \quad \text{for } n > 1$$

since $J_0(0) = 1$ and $J_n(0) = 0$ for $n > 0$.

The initial state of this solution is $|\Psi(0)\rangle = |1\rangle$, so its initial state has unit norm. We can quickly see that the system will retain this unit norm

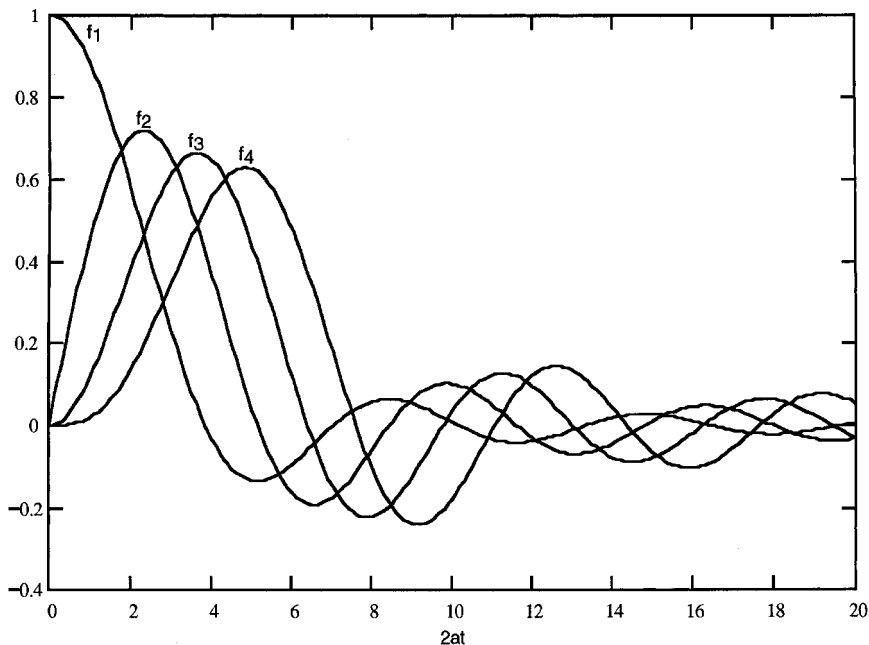


Fig. 7. A well-behaved solution.

throughout its time development. It is a standard property of the Bessel functions that, for fixed t , $\text{Lim}_{n \rightarrow \infty} J_n(at) = 0$, so that $\text{Lim}_{n \rightarrow \infty} f_n = 0$. It now follows from (25) of Proposition 4 that the time derivative of the norm vanishes for all t . Thus, if the state vector has unit norm at $t = 0$, it retains it for all time.

It is interesting to represent this solution (27) graphically. Measuring the degree of excitation of the particles as $f_n(t)$, we represent it in Fig. 7. The initial excitation of particle 1 drops asymptotically to zero. That excitation propagates through the remaining particles.

5.3. An Unnormalizable Solution

In the case of infinitely many particles, the energy eigenvector equation $H|E\rangle = E|E\rangle$ admits solutions for all values of E , positive, zero, and negative, so that the system has a continuous energy spectrum. The simplest energy eigenvector is for eigenvalue $E = 1$ and is

$$|E = 1\rangle = |1\rangle + |3\rangle + |5\rangle + \dots \quad (32)$$

up to constant phase factor. This vector cannot be normalized. But aside from this and the resultant complications for a theory of measurement,¹⁵ its behavior seems as respectable as the normalizable states of the previous section.

If we set the initial state of $|\Psi(t)\rangle = \sum_{n=1, \infty} C_n |n\rangle$ to $|\Psi(0)\rangle = |E=1\rangle$, we can find a set of coefficients $C_n(t) = e^{-it} f_n(t)$ that satisfy the differential Schrödinger equation by noting that form (17) of the Schrödinger equation is solved by

$$f_1(t) = f_3(t) = f_5(t) = \dots = 1, \quad f_2(t) = f_4(t) = f_6(t) = \dots = 0$$

This set of functions corresponds to a time evolution

$$|\Psi(t)\rangle = e^{-it}(|1\rangle + |3\rangle + |5\rangle + \dots) = e^{-it} |E=1\rangle \tag{33}$$

That is, the state remains an $E=1$ energy eigenstate. This is the unique analytic solution corresponding to this initial state. This time dependence is also compatible with the integral form (14') of the Schrödinger equation since it be written¹⁶

$$|\Psi(t)\rangle = \exp(-iHt) |E=1\rangle \tag{33'}$$

5.4. Spontaneous Excitation of the Ground State

We can illustrate the physical mechanism underlying the indeterminism allowed by Proposition 1 in the case of the differential form of the Schrödinger equation with a simple example. We take the first solution to be the ground state $|0\rangle$ so that $C_0 = 1$ and $C_n(t) = f_n(t) = 0$ for all $n > 0$. We form arbitrarily many further solutions by adding the function $G(t)$ to $f_1(t)$ as in the proof of Proposition 1 above, where

$$G(t) = 0, \quad t \leq 0$$

$$G(t) = K(1/t) e^{-1/t}, \quad t > 0$$

¹⁵ These complications do not seem any more severe than those encountered routinely in stochastic systems. Consider the selection of a single point on a dart board by the throwing of a dart with an ideally sharp point of zero size. The probability of selecting any one point is zero, but some point will be selected. We can at best specify nonzero probabilities for the point being in this or that set, and this is not usually deemed a fatal flaw of the analysis. Correspondingly the probability of measuring any particular state $|1\rangle, |3\rangle, \dots$ is zero. But there is unit probability that the measurement will be in the set $\{|1\rangle, |3\rangle, |5\rangle, \dots\}$ and zero probability that it will be in the set $\{|2\rangle, |4\rangle, |6\rangle, \dots\}$.

¹⁶ To see this, recall that $H|E=1\rangle = |E=1\rangle$ and $\exp(-iHt)|E=1\rangle = |E=1\rangle + (-iHt) \times |E=1\rangle + \dots + (1/n!)(-iHt)^n |E=1\rangle + \dots = |E=1\rangle + (-it)|E=1\rangle + \dots + (1/n!)(-it)^n \times |E=1\rangle + \dots = \exp(-it)|E=1\rangle$.

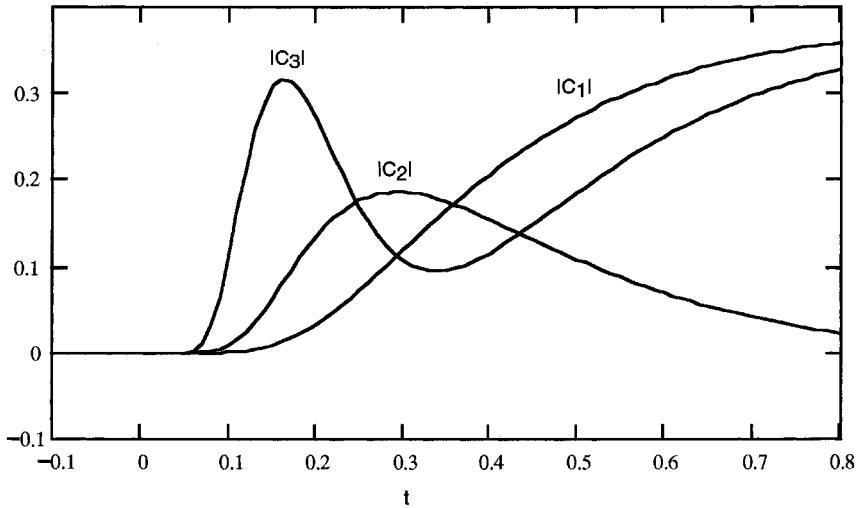


Fig. 8. Spontaneous energization of an infinite-particle system.

and K is an arbitrary constant. This yields new solutions of the differential Schrödinger equation with the same initial condition $|0\rangle$ in accord with the methods in Sec. 5.1. A plot of the excitations for $K=1$ shows that the mechanism of the indeterminism is the same as the one that yielded indeterminism in the case of the classical supertask of the masses and springs. See Fig. 8. After $t=0$, the first particle is excited by a faster excitation of the second particle, the second particle is excited by a faster excitation of the third, and so on.

These solutions are not analytic since the function $G(t)$ is not analytic. The solutions also fail to preserve the state vector norm. At $t=0$, the norm is just $(C_0^* C_0)^{1/2} = 1$. At times $t > 0$, the norm is greater than 1, since $C_0^* C_0$ remains constant at unity [see Eq. (16)], but nonzero coefficients C_1, C_2, \dots also appear.

We can infer that the norm diverges immediately with $t > 0$. Otherwise the norm would be finite for some interval of time beginning with $t=0$. But if it is finite, we have from Proposition 4 that it is constant in this interval, so that it would have to be unity throughout the interval. But we know it is greater than unity for $t > 0$.

These solutions fail to satisfy the integral form (14') of the Schrödinger equation. This integral form of the Schrödinger equation preserves the state vector norm if it is at any time finite, since, from (26), we have $\langle \Psi(t) | \Psi(t) \rangle = \langle \Psi(0) | \Psi(0) \rangle$. These solutions cannot satisfy this integral form of the equation since their initial (norm)² is $\langle \Psi(0) | \Psi(0) \rangle = 1$ but

they immediately develop to states at $t > 0$ with divergent (norms)², $\langle \Psi(t) | \Psi(t) \rangle = \infty$.

Finally, these solutions manifest indeterminism since the effect of different values of K is not just to multiply the same solution by a phase factor. All coefficients $C_n(t)$ for $n > 0$ contain K as a multiplicative phase factor. But $C_0(t)$ is independent of K . Thus different values of K alter the ratio between $C_0(t)$ and the remaining coefficients, thereby producing distinct solutions, all of which agree on the initial state $|0\rangle$.

5.5. Collection of Principal Results

We can now combine the propositions in Sec. 5.1 with the particular solutions in Secs. 5.2 to 5.4 to generate the following results for the case of infinitely many particles. The particular solutions in Secs. 5.2–5.4 instantiate the possibilities listed in Table I.

If the time development of the state vector is governed solely by the differential Schrödinger equation, then it is indeterministic. This restates Proposition 1. The indeterminism is illustrated most vividly by the spontaneous excitation in Sec. 5.4.

If the time development of the state vector is governed by (a) the differential Schrödinger equation and (b) the requirement that the state vector is always normalizable, then the time development is deterministic. This restates Proposition 5. Thus the well-behaved solution (27) of Sec. 5.2 is the unique, normalizable solution of the differential Schrödinger equation for the initial state $|\Psi(0)\rangle = |1\rangle$.

If the time development is governed by the differential Schrödinger equation and it is also analytic, then it is deterministic and satisfies the integral Schrödinger equation. This restates Propositions 2 and 3. It assures us that the well-behaved solution (27) is the unique analytic solution for the initial

Table I. Properties of the Particular Solutions

	Analytic	Normalizable	Indeterministic class of solutions	Solves differential Schrödinger equation	Solves integral Schrödinger equation
Well-behaved solution in Sec. 5.2	Yes	Yes	No	Yes	Yes
Unnormalizable solution in Sec. 5.3	Yes	No	No	Yes	Yes
Spontaneous excitation in Sec. 5.4	No	No	Yes	Yes	No

state $|\Psi(0)\rangle = |1\rangle$. It also assures us that the unnormalizable solution (33) is the unique analytic solution for the initial state $|\Psi(0)\rangle = |E=1\rangle$.

There are unnormalizable solutions that satisfy both the differential form (14) and the integral form (14') of the Schrödinger equation. The unnormalizable solution (33) in Sec. 5.3 illustrates this result. Relation (26) is usually taken to show that the integral form of the Schrödinger equation assures us of a constant state vector norm. It assures us of a little less. If the state vector norm is at any time finite, (26) assures us that it must be the same finite value at all times. If the norm diverges at any time, however, (26) assures us that it must diverge at all times—as is the case with (33).

If the time development of the state vector is governed by the integral Schrödinger equation (14'), then it is deterministic. This is a restatement of Proposition 6.

For any normalizable initial state vector $|\Psi(0)\rangle$, the unique time development satisfying the integral Schrödinger equation is the same as the unique, normalizable time development satisfying the differential Schrödinger equation. This follows immediately from the earlier results and the fact that solutions of the integral Schrödinger equation are both normalizable, if the initial state is normalizable, and solutions of the differential Schrödinger equation.¹⁷

The Hamiltonian H is unbounded. That is, there is no finite upper bound for $\langle H\Psi | H\Psi \rangle$, for the space of all state vectors $|\Psi\rangle$. In addition, we have that $\langle \Psi | H | \Psi \rangle$ diverges for some states $|\Psi\rangle$. While this result may be proved for a large class of states, it is sufficient to display one state for which both $\langle H\Psi | H\Psi \rangle$ and $\langle \Psi | H | \Psi \rangle$ diverge. That is the eigenstate $|E=1\rangle$ in Sec. 5.3. We have

$$\langle H\Psi | H\Psi \rangle = \langle \Psi | H | \Psi \rangle = \langle \Psi | \Psi \rangle = \infty \quad (34)$$

for $|\Psi\rangle = |E=1\rangle$ since $H|E=1\rangle = |E=1\rangle$ and $\langle E=1 | E=1 \rangle = \infty$.

6. AN ACCELERATED QUANTUM SUPERTASK

In the quantum supertask described so far, the worst pathology is a loss of determinism and a failure to preserve normalizability in the time development of the state vector. Both can be restored either by reverting to the integral form of the Schrödinger equation or by merely stipulating normalizability of the state vector. Might we find a variant form of the supertask for a which neither strategy succeeds? So far we have assumed that the

¹⁷ For some solution $|\Psi(t)\rangle = \exp(-iHt)|\Psi(0)\rangle$ of the integral Schrödinger equation, we have $(d/dt)|\Psi(t)\rangle = (d/dt)\exp(-iHt)|\Psi(0)\rangle = -iH\exp(-iHt)|\Psi(0)\rangle = -iH|\Psi(t)\rangle$ so that $i(d/dt)|\Psi(t)\rangle = H|\Psi(t)\rangle$.

strength of the interaction between successive particles remains constant, as indicated by the invariability of the coupling constant a . What would happen, we might wonder, if we allow the strength of this interaction to increase without bound for successive particle interactions? This type of acceleration is familiar from classical supertasks. It is realized in a way, for example, in the case of the classical supertask in Fig. 1 by allowing the bodies to be successively closer. In the quantum case, the effect of this acceleration would be that lower numbered excitations would arouse higher numbered excitations more rapidly—or so it would seem. Since the acceleration may be supposed to grow without bound with increasing excitation number, might it allow new pathological effects such as a normalizable initial state developing *only* into unnormalizable states? If this were to happen, neither of the above remedies of indeterminism could be applied and indeterminism becomes inescapable.

The purpose of this section is to review the structure of these accelerated supertasks. I conjecture that they produce no effects qualitatively different from those of the unaccelerated supertask, although I do not have a complete set of result analogous to those in Sec. 5 that would make the case. However, we shall see that the accelerated supertasks are more, not less, hospitable to normalizable solutions of the differential Schrödinger equation.

6.1. The Hamiltonian and the Schrödinger Equation

The vector spaces and Hamiltonian are as in the unaccelerated case, except that we replace the single coupling constant a with a family of real, positive coupling constants $a_n, n = 1, 2, 3, \dots$, where the a_n increase without bound with n , so that the interaction Hamiltonian becomes

$$H^{int} = \sum_{n=1}^{\infty} ia_n |n+1\rangle \langle n| - ia_n |n\rangle \langle n+1| \tag{13'}$$

The transition amplitudes of the resulting Hamiltonian (11) can be represented figuratively as in Fig. 9.

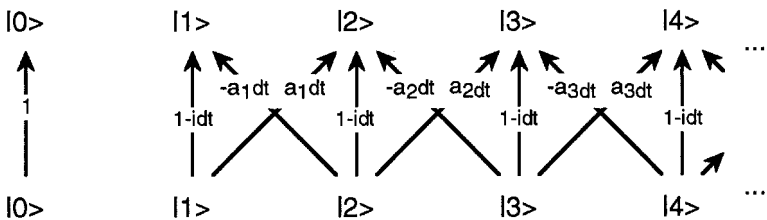


Fig. 9. Transition amplitudes for accelerated supertask Hamiltonian.

The differential Schrödinger equation (14) is now in component form

$$\begin{aligned}
 iC'_0 &= 0 \\
 iC'_1 &= C_1 - ia_1 C_2 \\
 iC'_2 &= C_2 + ia_1 C_1 - ia_2 C_3 \\
 &\vdots \\
 iC'_n &= C_n + ia_{n-1} C_{n-1} - ia_n C_{n+1} \\
 &\vdots
 \end{aligned}
 \tag{15''}$$

Defining $f_n = e^{it} C_n$ for $n > 0$, we recover in place of (17)

$$\begin{aligned}
 f_2 &= -\frac{1}{a_1} f_1' \\
 f_3 &= \frac{a_1}{a_2} f_1 - \frac{1}{a_2} f_2' \\
 &\vdots \\
 f_n &= \frac{a_{n-2}}{a_{n-1}} f_{n-2} - \frac{1}{a_{n-1}} f_{n-1}' \\
 &\vdots
 \end{aligned}
 \tag{17''}$$

6.2. Properties

The accelerated supertask is very similar in structure to the unaccelerated supertask and shares many of its properties. Indeed a review of the demonstrations of Propositions 1, 2, 3, and 6 in Sec. 5 shows that their demonstrations still succeed and they obtain for the accelerated case as well. Thus we have that the differential Schrödinger equation yields an indeterministic time development (Proposition 1). Any analytic solution for a given initial state is unique (Proposition 2) and satisfies the integral Schrödinger equation (Proposition 3). The time development of the integral form is deterministic (Proposition 6).

The demonstration of Proposition 4 does not carry over, so that of Proposition 5 fails as well. Thus I have no demonstration that adding the requirement of normalizability at all times to the state vector restores deterministic time development for the differential Schrödinger equation. I do conjecture, however, that this result still does obtain.

The failure of the demonstration of Proposition 4 derives from the modification of result (25) in the accelerated case. As a brief review of its derivation shows, in the accelerated case it is replaced by

$$\frac{d}{dt} \langle \Psi(t) | \Psi(t) \rangle = - \lim_{n \rightarrow \infty} (a_n f_n f_{n+1}^* + a_n f_n^* f_{n+1}) \tag{25'}$$

While normalizability of the state vector $|\Psi(t)\rangle$ still implies that $\lim_{n \rightarrow \infty} f_n = 0$, the latter limit is no longer sufficient to ensure the vanishing of the limit in Eq. (25'). The difficulty is that we have supposed $\lim_{n \rightarrow \infty} a_n = \infty$, so that a value for or even existence of the limit in (25') cannot be determined from normalizability of the state vector without further analysis.

6.3. Properties of Some Particular Solutions

The Initial State $|\Psi(0)\rangle = |1\rangle$. This initial state corresponds to the initial state of the well-behaved solution in Sec. 5.2. Because of the unbounded growth of the coupling constants a_n , one would expect that the higher numbered excitations would be aroused more rapidly than in the unaccelerated case. Compatible with this expectation is a computation of the time derivatives at $t = 0$ of f_n , where, as before, $|\Psi(t)\rangle = \sum_{n=0, \infty} C_n(t) |n\rangle = \sum_{n=0, \infty} e^{-it} f_n(t) |n\rangle$. A straightforward calculation using (17'') shows that the first nonzero time derivatives at $t = 0$ for f_n , where $n > 1$, are

$$f_2'(0) = a_1, \quad f_3''(0) = a_1 a_2, \quad f_4^{(3)}(0) = a_1 a_2 a_3, \dots, \\ f_n^{(n-1)}(0) = a_1 a_2 \dots a_{n-1}, \dots$$

With a_n growing without bound, these initial accelerations are arbitrarily greater with increasing n than the corresponding values for the unaccelerated case. If an analytic solution exists, the expression for $f_n(t)$ will be given by a Taylor series,

$$f_n(t) = \frac{1}{(n-1)!} a_1 a_2 \dots a_{n-1} t^{n-1} + \text{terms in higher powers of } t$$

If a_n grows faster than n , then, for some fixed t , this first term will diverge with increasing n . This suggests but does not show that normalizability might be hard to sustain in the case of accelerated supertasks. It does not show it since any divergence of this first term might be canceled by the behavior of the remaining terms of the sum. That precisely this is likely to

happen is strongly suggested by the remaining cases considered here. They will suggest that normalizability or even just solutions with $\text{Lim}_{n \rightarrow \infty} f_n = 0$ arise more easily than in the unaccelerated case.

The $E = 1$ Eigenstate. A short calculation shows that

$$|E = 1\rangle = |1\rangle + \frac{a_1}{a_2} |3\rangle + \frac{a_1 a_3}{a_2 a_4} |5\rangle + \dots + \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_4 \dots a_{2n}} |2n + 1\rangle + \dots \quad (35)$$

is an energy eigenstate with unit eigenvalue. Thus $|\Psi(t)\rangle = e^{-it} |E = 1\rangle$ is a solution of both differential and integral Schrödinger equations. Most important, if a_n grows rapidly enough with n (e.g., exponentially), the state vector $|E = 1\rangle$ will have a finite norm. Thus normalizability is achieved in the accelerated case where it was not achieved in the corresponding unaccelerated case of Sec. 5.3.

Spontaneous Excitation of the Ground State. The generation of this case proceeds analogously with that in Sec. 5.4 for the unaccelerated case. The resulting family of time developments is not analytic and solves the differential Schrödinger equation only. The principal qualitative difference between this family of solutions in the accelerated and unaccelerated cases concerns the behavior of the coefficients f_n in the limit of large n . For the unaccelerated case, since the solutions do not preserve the state vector norm, we have from (25) that $\text{Lim}_{n \rightarrow \infty} f_n \neq 0$. In the accelerated case, since (25) has been replaced by (25'), solutions with $\text{Lim}_{n \rightarrow \infty} f_n = 0$ no longer force constancy of the norm and thus are compatible with the unnormalizability of spontaneous excitations. Moreover, we see in the Appendix that this compatibility is realized in a large class of spontaneous excitations.

As we have seen, the mechanism of spontaneous excitation is that each excitation is aroused by a faster arousal of higher-numbered excitations. In the unaccelerated case, this arousal from infinity can be sustained only by nonzero excitations in the limit of large n . In the accelerated case, it can be sustained by excitations that vanish in the limit of large n .

The latter two examples suggest that the accelerated supertasks are more, not less, hospitable to normalizable solutions or at least solutions with lower magnitude excitations.¹⁸ For this reason I conjecture that no

¹⁸The natural intuition is that the ever-growing values of the coupling constant in the accelerated case will arouse greater not smaller magnitudes of excitations. While lower-numbered excitations will arouse the higher-numbered excitations more rapidly, the magnitude of the latter will not grow large since that magnitude is in turn drained away even more rapidly by the yet higher-numbered excitations.

worse pathologies will arise in the accelerated case than have already arisen in the unaccelerated case.

7. CONCLUSION

Classical supertasks have shown us that we cannot automatically assume that classical theory is the locus of benign behavior for physical systems. If we wish to ensure such behavior, we must be ready to restrict our classical systems in ways we may not have anticipated. We may have a classical interaction of finitely many bodies that retains determinism and the conservation of energy and momentum, no matter how many bodies we consider. But both can be lost in the passage from arbitrarily many bodies to infinitely many. If we are to preserve determinism and these conservation laws, we shall have to forgo that infinite limit.

Quantum mechanical supertasks carry a similar moral. A many-particle quantum interaction may be well behaved for any finite number of particles. But when we make the passage from arbitrarily many particles to infinitely many, we may lose the deterministic time development of the differential Schrödinger equation and find that normalized states evolve into unnormalizable states under a Hamiltonian that has become unbounded. In one sense, however, these quantum supertasks are better behaved than their classical counterparts. A natural added condition, that we require normalizability of the state vector, restores determinism and precludes those time evolutions that fail to preserve the state vector norm.¹⁹ Or we may secure determinism even for unnormalizable initial states if we require that the time development satisfy the integral form of the Schrödinger equation.

The pathologies of the quantum supertasks appear to be quite generic. They do not seem to depend upon some peculiarity of the system but upon the system's infinite degrees of freedom. We may construct supertasks based on more realistic interactions. For example, we can write down a

¹⁹ Might we also seek to tame classical supertasks by adding the requirement of conservation of energy and momentum? This would preclude the spontaneous excitation of the two classical systems of bodies considered in Section 1; both these systems begin with zero energy and momentum and the spontaneous excitations alter these quantities. But the requirement cannot preclude pathological behavior in these systems if they begin with infinite energies and momentum, such as can be the case with infinite systems of bodies all of which are in motion. It can also fail even if the initial energy and momentum are both finite. In the classical supertask in Fig. 1, in which infinitely many bodies at rest are struck by one moving with unit velocity, what final state can there be that preserves energy and momentum and respects the dynamics of the pairwise collisions?

Hamiltonian that governs the quantum analog of the classical masses and springs supertask. It seems reasonable to expect that the associated quantum system will be governed by an infinite system of equations qualitatively similar in structure to that governing the quantum supertask examined here and that it will exhibit indeterminism in analogous ways.

APPENDIX: SOLVING THE SCHRÖDINGER EQUATION

Solving the differential Schrödinger equation amounts to finding solutions of the infinite set of differential equations (17)/(17''), which can be quite challenging in nontrivial cases.²⁰ In the technique outlined here, I replace the functions $f_n(t)$ by their Laplace transforms

$$\mathcal{L}(f_n(t)) = F_n(s) = \int_0^\infty f_n(t) e^{-st} dt \tag{36}$$

The set of equations (17)/(17'') is reduced to an infinite set of algebraic equations in $F_n(s)$ which is far easier to solve. Laplace transforms are defined and invertible only for a narrow class of functions. But since this class includes bounded, C^∞ , real-valued functions, it contains just the sort of functions we seek as solutions. In the following we assume that all our functions $f_n(t)$ are bounded, C^∞ and real valued.

Recalling that $\mathcal{L}(f'_n(t)) = sF_n(s) - f_n(0)$, the Laplace transform of (17'') is

$$F_2(s) = \frac{1}{a_1} (-sF_1(s) + f_1(0)), \quad F_3(s) = \frac{a_1}{a_2} F_1(s) - \frac{1}{a_2} sF_2(s), \dots$$

$$F_n(s) = \frac{a_{n-2}}{a_{n-1}} F_{n-2}(s) - \frac{1}{a_{n-1}} sF_{n-1}(s), \dots \tag{37}$$

for the special case in which $f_1(0)$ is left undetermined and all remaining $f_n(0) = 0$ for $n > 1$. The set of Eqs. (37) admits a solution of the form

$$F_n(s) = \frac{A_n F_1(s) + B_n}{a_1 a_2 \dots a_{n-1}} \tag{38}$$

²⁰ Recall that equations (17)/(17'') do not include the trivial time dependence of C_0 , which is just $C_0(t) = C_0(0)$.

for $n > 0$, where the coefficients A_n and B_n for $n > 2$ are defined recursively by the relations

$$A_n = (a_{n-2})^2 A_{n-2} - sA_{n-1}, \quad B_n = (a_{n-2})^2 B_{n-2} - sB_{n-1} \quad (39)$$

The initial values of these coefficients are

$$A_1 = 1, \quad A_2 = s, \quad B_1 = 0, \quad B_2 = f_1(0) \quad (40)$$

We can apply this equation set to two special cases.

A1. The Well-Behaved Solution for the Unaccelerated Supertask

We can use the equation set to generate solution (27). As it happens I did not find the solution originally by this procedure but by fortuitously noting the similarity between the equation system (17) and the standard recursion relations for Bessel functions of the first kind. Since we now use the equation set (38), (39), (40) as a heuristic aid to find solution (27), we can simply assume that such a well-behaved solution exists along with the existence of whatever limits are needed to proceed to a final result. We can then confirm later that it does indeed have all the well-behaved properties we assumed. For this case, we have $a_n = a$, for all n . We assume that the solution sought is normalizable. Therefore $\text{Lim}_{n \rightarrow \infty} f_n = 0$. That is,

$$\text{Lim}_{n \rightarrow \infty} F_n(s) = \text{Lim}_{n \rightarrow \infty} \frac{A_n F_1(s) + B_n}{a \cdot a^2 \dots a^{n-1}} = 0$$

We can be assured that the latter result obtains if $F_1(s) = \text{Lim}_{n \rightarrow \infty} (-B_n/A_n)$. For the initial state $|1\rangle$, we have $f_1(0) = 1$, so that $B_2 = 1$. Inserting this value along with (40) into (39), we find that $B_n = A_{n-1}$ for $n > 1$. Therefore we have $F_1(s) = \text{Lim}_{n \rightarrow \infty} (-A_{n-1}/A_n) = \text{Lim}_{n \rightarrow \infty} [1/(s - a^2(A_{n-2}/A_{n-1}))]$, where the second equality derives from substituting for A_n using (39). If the $\text{Lim}_{n \rightarrow \infty} (-A_{n-1}/A_n)$ exists, it is $F_1(s)$ and from this second equality it satisfies $F_1(s) = 1/(s + a^2 F_1(s))$. The latter equation is a simple quadratic equation in $F_1(s)$. Solving, choosing the root that corresponds to a positive value of $F_1(s)$, we recover

$$F_1(s) = \frac{2}{s + \sqrt{s^2 + 4a^2}}$$

Consulting a standard table of Laplace transforms (Ref. 3, p.131), we invert the Laplace transform to recover $f_1(t) = (1/at) J_1(2at)$, which is just

the expression for $f_1(t)$ in (29') above. Once this expression for $f_1(t)$ is determined, the remaining expressions for $f_n(t)$ are recovered recursively from (17), and we can then confirm at our convenience that the solution has all the well-behaved properties assumed.

A2. Spontaneous Excitation for the Accelerated Supertask

We can now use the equation set (38), (39), (40) to affirm that there are many spontaneous excitations of the ground state for the accelerated supertask that satisfy the condition $\text{Lim}_{n \rightarrow \infty} f_n = 0$. For this case, we have $|0\rangle$ as the initial state at $t=0$ so that $B_2 = f_1(0) = 0$. Since now both $B_1 = B_2 = 0$, it follows from (39) that $B_n = 0$ for all n . Therefore we have

$$F_n(s) = \frac{A_n F_1(s)}{a_1 a_2 \cdots a_{n-1}} \quad (38')$$

The condition $\text{Lim}_{n \rightarrow \infty} f_n = 0$ is just

$$\text{Lim}_{n \rightarrow \infty} F_n(s) = \text{Lim}_{n \rightarrow \infty} \frac{A_n F_1(s)}{a_1 a_2 \cdots a_{n-1}} = 0$$

For simplicity, we now consider the case of a_n growing rapidly with n , that is, at least exponentially with n . In this case we find that

$$\text{Lim}_{n \rightarrow \infty} \frac{A_n}{a_1 a_2 \cdots a_{n-1}} = 0 \quad (41)$$

To see this, note that it follows from (39) and (40) that A_n alternates in sign such that $(-1)^{n+1} A_n$ is always positive. Thus a form of the first equation of (39) is

$$X_n = \frac{a_{n-2}}{a_{n-1}} X_{n-2} + \frac{s}{a_{n-1}} X_{n-1} \quad (42)$$

where $X_n = ((-1)^{n+1} A_n / a_1 a_2 \cdots a_{n-1})$ and we have for all n that $X_n > 0$. For each fixed value of s , there will be a value of n [call it $N(s)$] such that for all $n \geq N(s)$, a_n will have grown sufficiently large to ensure that $[(a_{n-2}/a_{n-1}) + (s/a_{n-1})] < 1$. Hence for all $n \geq N(s)$, X_n will be bounded from above by $M = \max(X_{N(s)-1}, X_{N(s)-2})$. From here it is straightforward but tedious to prove that $\text{Lim}_{n \rightarrow \infty} X_n = 0$. Informally, the result follows because the second term $(s/a_{n-1}) X_{n-1} < (s/a_{n-1}) M$ becomes arbitrarily small in the limit of large n , so that X_n must approach $(a_{n-2}/a_{n-1}) X_{n-1}$ in this limit. But since a_n grows at least exponentially with n , the ratio

(a_{n-2}/a_{n-1}) is always less than 1 and cannot approach 1 in the limit of large n . Thus X_n can approach $(a_{n-2}/a_{n-1}) X_{n-1}$ in the limit of large n only if $\text{Lim}_{n \rightarrow \infty} X_n = 0$, which is equivalent to (41).

In view of the limit (41), it would seem that any everywhere finite transform $F_1(s)$ can be inserted into (38') to recover a spontaneous excitation that satisfies $\text{Lim}_{n \rightarrow \infty} f_n = 0$. This appearance is almost correct. The freedom to choose $F_1(s)$ is large. The qualification is this. The Schrödinger equation (17'') with boundary conditions $f_n(0) = 0$ for $n > 1$ entails the system (38), (39), (40), but not conversely. It is this converse entailment that we need to assure us that solutions of (38), (39), (40) are also solutions of the original system (17'') with the appropriate boundary conditions. The converse entailment fails since (38), (39), (40) do not entail $f_n(0) = 0$ for $n > 1$, even though these conditions were used in the derivation of (38), (39), (40). Therefore we need to stipulate these conditions again, along with setting $f_1(0) = 0$ as a supplement to (38), (39), (40), so that the combined set becomes equivalent to the original equation (17'') with the corresponding boundary conditions. We have for Laplace transforms of functions with bounded derivatives that $f_n(0) = \text{Lim}_{s \rightarrow \infty} sF_n(s)$. Therefore we require that

$$\text{Lim}_{s \rightarrow \infty} sF_n(s) = \text{Lim}_{s \rightarrow \infty} \frac{A_n s F_1(s)}{a_1 a_2 \dots a_{n-1}} = 0 \quad \text{all } n > 0 \quad (43)$$

Any function $F_1(s)$ that satisfies condition (43) will generate a spontaneous excitation of the accelerated supertask for which $\text{Lim}_{n \rightarrow \infty} f_n = 0$ in this case of a_n growing at least exponentially with n .

It follows from the recursive definition (39) of A_n and the initial values (40) that $A_n s$ is a polynomial in s with the highest power s^n . Thus the limit (43) will obtain if $F_1(s)$ satisfies

$$\text{Lim}_{s \rightarrow \infty} s^n F_1(s) = 0 \quad \text{all } n > 0 \quad (44)$$

That is, $F_1(s)$ must drop to zero with increasing s faster than any $1/s^n$. This condition is actually a familiar one. We have for Laplace transforms that

$$\mathcal{L}(f_1^{(n)}(t)) = s^n F_1(s) - s^{n-1} f_1(0) - s^{n-2} f_1'(0) - \dots - f_1^{(n-1)}(0)$$

and for all bounded function $f_1^{(n)}(t)$ that $\text{Lim}_{s \rightarrow \infty} \mathcal{L}(f_1^{(n)}(t)) = 0$. So (44) is equivalent to

$$f_1(0) = 0, \quad f_1'(0) = 0, \quad f_1''(0) = 0, \dots, \quad f_1^{(n)}(0) = 0, \dots \quad (44')$$

We see immediately from (44') that a nontrivial $f_1(t)$ cannot be analytic.

It is easy to find functions $F_1(s)$ satisfying (44). It is assured if they decay exponentially with increasing s , for example. Here is one instance for which the inverse Laplace transformation is simple. Set $F_1(s) = \sqrt{(\pi/s)} e^{-2\sqrt{s}}$ for which (Ref. 3, p. 128), the inverse Laplace transform yields $f_1(t) = (e^{-1/t}/\sqrt{t})$. This choice of $f_1(t)$ generates the remaining $f_n(t)$ by means of (17''). For accelerated supertasks in which a_n grow at least exponentially with n , it constitutes a spontaneous excitation analogous to that in Sec. 5.4 but for which $\text{Lim}_{n \rightarrow \infty} f_n = 0$.

Presumably, a similar analysis will return many solutions with $\text{Lim}_{n \rightarrow \infty} f_n = 0$ for the initial condition $|\Psi(0)\rangle = |1\rangle$. Unlike the case of the unaccelerated supertask, the latter limit condition does not ensure the preservation through time of the finite norm of the initial state, since condition (25) no longer obtains. To be assured that the solution found preserves the norm of the initial state through time, we need to invoke the stronger limit condition of (25'), that is $\text{Lim}_{n \rightarrow \infty} (a_n f_n f_{n+1}^* + a_n f_n^* f_{n+1}) = 0$. But since this stronger limit condition is not linear in the functions f_n , it does not admit a simple Laplace transform.

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