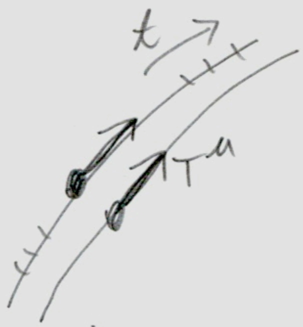
General Relativity

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

special relativity: g's must be geometrically flat

General relativity: Allow non-flat g's
Gravity resides in non-flatness.
Spatial slices also adopt non-Euclidean geometry

Tidal Acceleration



Neighboring particles in free fall in space<u>time</u>

spatial separation vector S^μ connects them

$$\frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$$

Riemann curvature tensor

$\frac{D}{dt}$ = "directional covariant derivative" ... same derivative used in geodesic equation

Acting on a vector V^μ

$$\frac{DV^\mu}{dt} = \frac{dV^\mu}{dt} + \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} V^\rho$$

looks like ordinary differentiation if we pick a coordinate system in which $\Gamma^\mu_{\sigma\rho} = 0$

After sean carroll, Spacetime & Geometry

$\boxed{\text{Flatness} \Longrightarrow \text{No tidal accelerations}}$

$$R^{\mu}{}_{\nu\rho\sigma} = 0 \quad \Longrightarrow \quad \frac{D^2}{dt^2} s^{\mu} = 0$$

special relativity

Neighboring particles in free fall have no relative acceleration $\longleftrightarrow$ Analog of homogeneous gravitational field

Gravitational field equations

Analogy to Newtonian case

$$\underbrace{R^{\mu}{}_{\nu\rho\sigma} \text{ contract over two indices}}$$

"Ricci tensor" $R_{\nu\rho} = R^{\mu}{}_{\nu\rho\mu} = R^{0}{}_{\nu\rho 0} + R^{1}{}_{\nu\rho 1} + R^{2}{}_{\nu\rho 2} + R^{3}{}_{\nu\rho 3}$

Analog to source free Poisson equation $\nabla^2\phi = 0$

$$\boxed{R_{\nu\rho} = 0}$$

Riemann curvature scalar $R = g^{\mu\nu} R_{\mu\nu}$

"Einstein tensor"

stress energy tensor of matter

Analog to full Poisson equation $\nabla^2\phi = -4\pi G\rho$

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = K T_{\mu\nu}}$$

Schwarzschild spacetime

$$ds^2 = \underbrace{-c^2\left(1-\frac{2GM}{c^2r}\right)dt^2}_{\text{Gives gravitation forces}} + \underbrace{\frac{1}{\left(1-\frac{2GM}{c^2r}\right)}dr^2}_{\text{Gives deviations from Euclidean geometry}} + r^2 \underbrace{d\Omega^2}_{\text{line element for a two sphere}}$$

line element for a two sphere $d\theta^2 + \sin^2\theta\, d\phi^2$

Gives gravitation forces

Gives deviations from Euclidean geometry

Recover gravitational forces in weak field:

$\varphi = -\frac{GM}{r}$ Consider small φ, neglect non-Euclidean geometry

$$ds^2 = -(c^2 + 2\varphi)dt^2 + \underbrace{dx^2 + dy^2 + dz^2}_{\text{Euclidean}}$$

Geodesic equation

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\cdot\frac{dx^\beta}{dt} = 0$$

$i = 1,2,3$

Only non-zero to lowest order are $\Gamma^i_{00} = \frac{\partial\varphi}{\partial x_i}$

$\Downarrow$

1 since $x^0 = t$

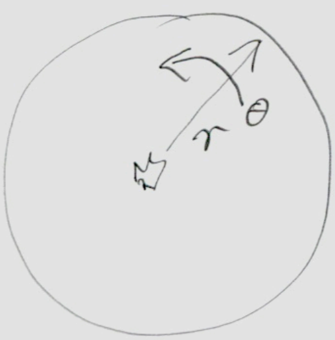
$$\frac{d^2x^i}{dt^2} + \Gamma^i_{00}\, \overset{1}{\cancel{\frac{dx^0}{dt}}} \cdot \overset{1}{\cancel{\frac{dx^0}{dt}}} = 0$$

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x_i} \quad \dots$$

compare Newtonian
Acceleration $= -\nabla\phi$

Non-Euclidean geometry

Take slice at $t = \text{constant}$, $\phi = 0$



$$ds^2 = \frac{1}{1 - \frac{2GM}{c^2 r}}\, dr^2 + r^2 d\theta^2$$

Integrate around circle $r = \text{constant}$
$\theta = 0$ to 2π

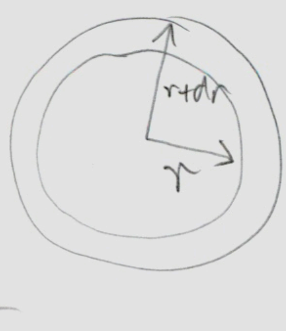
$$\text{circumference} = \int ds = \int_0^{2\pi} r\, d\theta = 2\pi r = \text{"}C(r)\text{"}$$

compare two circles at radii r, $r+dr$

$$C(r+dr) = 2\pi(r+dr)$$
$$C(r) = 2\pi r$$

Difference

$$dC = 2\pi \underbrace{dr}_{\text{coordinate difference. NOT true length}} = 2\pi \underbrace{\left(1 - \frac{2GM}{c^2 r}\right)}_{\text{slightly less than one.}} d_{\text{true distance}}$$



coordinate difference. NOT true length

slightly less than one. Euclidean geometry requires this to be exactly one

Computing Christoffel symbols

$$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda\sigma} \underbrace{[\mu\nu,\sigma]}_{\frac{1}{2}(g_{\mu\sigma,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})}$$

$$\left(\text{"}g_{\mu\nu,\sigma}\text{"} = \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$$

for $ds^2 = -(c^2+2\varphi)\underset{x^0}{dt^2} + \underset{x^1}{dx^2} + \underset{x^2}{dy^2} + \underset{x^3}{dz^2}$ $\quad \varphi = \varphi(x,y,z)$

Only non-zero $g_{00,i} = -2\varphi_{,i}$ $\quad i = 1,2,3$

Only non-zero $[\mu\nu,\sigma]$:

$[00,i] = -\frac{1}{2} g_{00,i} = \varphi_{,i}$ $\quad [0i,0] = [i0,0] = \frac{1}{2} g_{00,i} = -\varphi_{,i}$

<u>compute $g^{\mu\nu}$</u>

$g_{\mu\nu} \approx \begin{bmatrix} -\frac{1}{c^2} & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$ since $g^{\mu\nu} g_{\nu\sigma} = \delta^{\nu}_{\sigma}$

$$\begin{bmatrix} \frac{-1}{c^2(1+\frac{2\varphi}{c^2})} & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} -c^2(1+\frac{2\varphi}{c^2}) & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\frac{-1}{c^2(1+\frac{2\varphi}{c^2})} \approx \frac{-1}{c^2}\left(1-\frac{2\varphi}{c^2}\right) \approx -\frac{1}{c^2}$$

Only non-zero $\Gamma^{\lambda}_{\mu\nu}$

$$\Gamma^{i}_{00} = g^{ii}\{00,i\} = \varphi_{,i} \qquad i=1,2,3$$

$$\Gamma^{0}_{i0} = \Gamma^{0}_{0i} = g^{00} \cdot -\varphi_{,i} = \frac{1}{c^2}\varphi_{,i} \approx 0 \text{ to lowest order}$$

At lowest order, only non-zero $\Gamma^{\lambda}_{\mu\nu}$

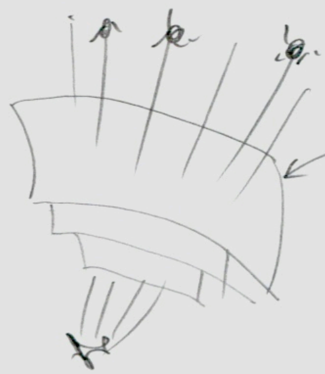
$$\boxed{\Gamma^{i}_{00} = \varphi_{,i} = \frac{\partial \varphi}{\partial x^{i}}}$$

Friedmann–Lemaître–Robertson–Walker spacetimes

$$ds^2 = -c^2 dt^2 + a^2(t) \underbrace{\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}$$

Dynamics lives in time dependence of scale factor

Geometry of:
$k=1$ 3-sphere
$k=0$ Euclid
$k=-1$ 3-hyperbolic



Solve Einstein equations for matter density ρ and pressure p

$$\dot{a}(t) = \frac{da(t)}{dt} \quad \ddot{a}(t) = \frac{d^2 a(t)}{dt^2}$$

with $c=1$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$