Tidal accelerations in Newtonian Gravitation

Basic equations

$$F = G\frac{m_1 m_2}{r^2} \qquad \Phi = -\frac{GM}{r}$$

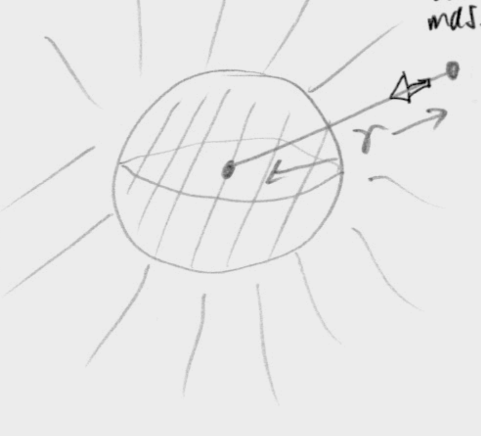
potential at r due to mass M

$$\underset{\sim}{F} = -m\underset{\sim}{\nabla}\Phi$$

$$F_x = -m\frac{\partial\Phi}{\partial x}$$

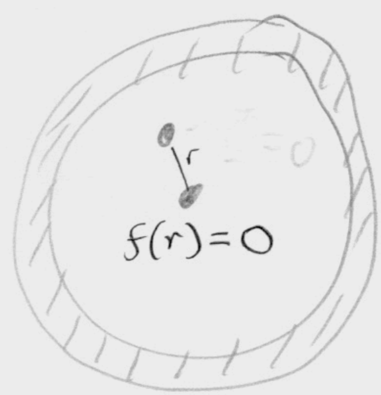
$$\vdots$$

Force exterior to sphere of uniform density and mass M



$$f(r) = -\frac{GM}{r^2}$$

Force inside spherical shell of uniform density



Force inside uniform mass distribution of density ρ

Force $f(r)$ at r

= force due to interior sphere

+

force due to outer shell $\Big\}$ zero!

$$= -\frac{G}{r^2}\left(\frac{4\pi}{3}\rho r^3\right)$$

$$= -\frac{4\pi}{3}G\rho\, r$$

mass of interior sphere is

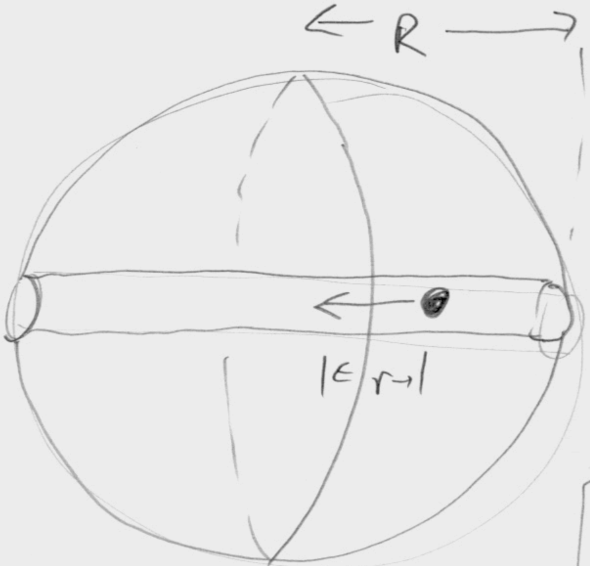
$\frac{4\pi}{3}\rho r^3$

$$f(r) = -\frac{\partial}{\partial r}\left(\quad\right) = -\frac{4\pi}{3}G\rho\, r$$

since then

$$\boxed{\text{Field } \phi(r) \text{ at } r: \quad \phi(r) = \frac{2\pi}{3}G\rho r^2}$$

Motion of free mass inside tube in earth



For a unit mass

$$f(r) = \frac{d^2 r}{dt^2} = -\frac{4\pi G\rho}{3}\, r$$

$$\Downarrow$$

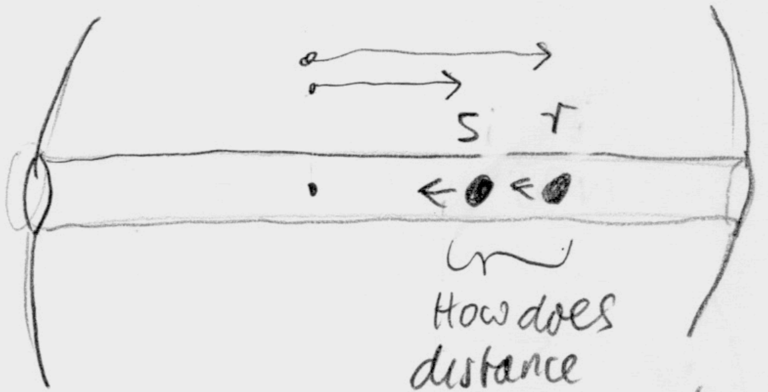
$$\boxed{\frac{d^2 r}{dt^2} + \underbrace{k^2}_{\frac{4\pi G\rho}{3}} r = 0}$$

simple harmonic oscillation

$$r(t) = R\cos(kt) \quad \text{if } r(0) = R$$

$$\text{where } k = \sqrt{\frac{4\pi G\rho}{3}}$$

Tidal acceleration



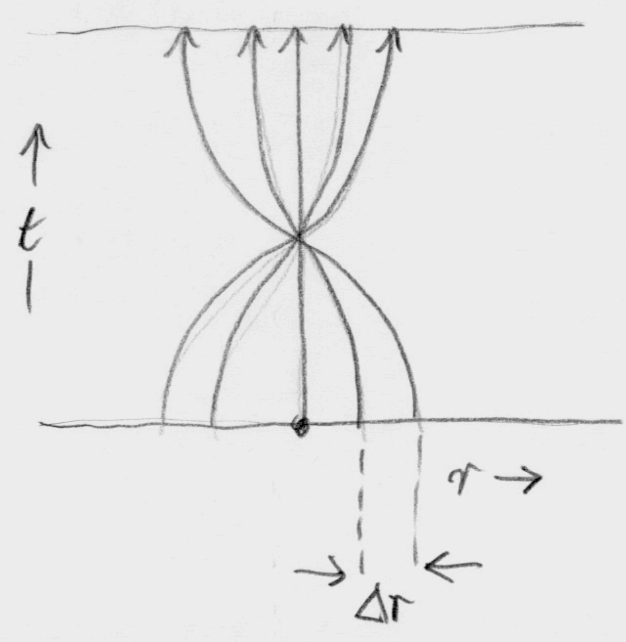
How does distance between nearby points vary?

$$\frac{d^2 s}{dt^2} = -\frac{4\pi G\rho}{3} s; \quad \frac{d^2 r}{dt^2} = -\frac{4\pi G\rho}{3} r$$

"Δr" $= (s - r)$ subtract

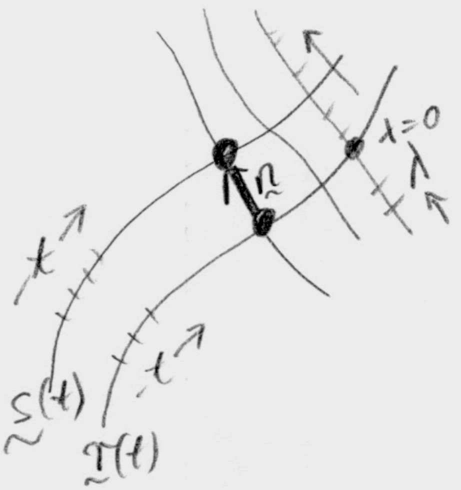
$$\frac{d^2}{dt^2}\Delta r = -\frac{4\pi G\rho}{3}\Delta r$$ Harmonic oscillator

$$\boxed{\Delta r = \Delta r(0)\cos kt \qquad k = \sqrt{\frac{4\pi G\rho}{3}}}$$



General Analysis of Tidal Acceleration

Two nearby bodies in free fall. Unit masses



Vector $\underset{\sim}{n}$, $n_i = (n_x, n_y, n_z)$

Points from $\underset{\sim}{r}$ to $\underset{\sim}{s}$

$$n_i \approx \frac{(s_i - r_i)}{\lambda} \qquad n_i = \lim_{s_i \to r_i} \left(\frac{s_i - r_i}{\lambda}\right)$$

$$\underbrace{\frac{d^2}{dt^2} s_i = -\frac{\partial \varphi(s_i)}{\partial x_i} \quad \frac{d^2}{dt^2} r_i = -\frac{\partial \varphi(r_i)}{\partial x_i}}_{\text{subtract}}$$

$$\lambda \frac{d^2 n_i}{dt^2} \approx \frac{d^2 s_i}{dt^2} - \frac{d^2 r_i}{dt^2} = -\left(\frac{\partial \varphi(s_i)}{\partial x_i} - \frac{\partial \varphi(r_i)}{\partial x_i}\right)$$

Power series expansion

$$\frac{\partial \varphi(s_i)}{\partial x_i} = \frac{\partial \varphi(r_i + n_i \lambda)}{\partial x_i} \approx \frac{\partial \varphi(r_i)}{\partial x_i} + \lambda \sum_{k=1,3} n_k \frac{\partial}{\partial x_k} \frac{\partial \varphi(r_i)}{\partial x_i} + \dots$$

$$\approx -\lambda \sum_k n_k \frac{\partial^2 \varphi(r_i)}{\partial x_k \partial x_i}$$

$$\boxed{\frac{d^2}{dt^2} n_i = -\sum_k n_k \frac{\partial^2 \varphi}{\partial x_k \partial x_i}}$$

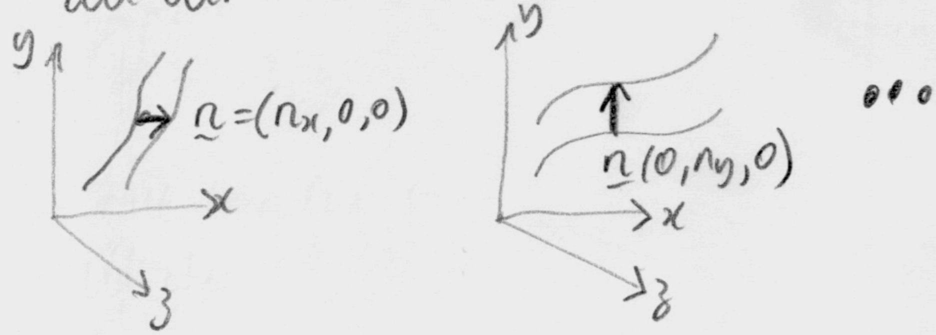
No tidal forces $\Longrightarrow$ Homogeneous field

Bodies in free fall see no tidal accelerations

$$0 = \frac{d^2 n_i}{dt^2} = -\sum_k n_k \frac{\partial^2 \varphi}{\partial x_k \partial x_i}$$

True for n_i oriented in all directions

Remove summation



For all combinations of i, k $\quad \frac{\partial^2 \varphi}{\partial x_k \partial x_i} = 0$ $\quad$ i.e. $\frac{\partial^2 \varphi}{\partial x_1 \partial x_1} = \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = \dots = \frac{\partial^2 \varphi}{\partial x_3 \partial x_3} = 0$

$$\varphi = A + B_1 x_1 + B_2 x_2 + B_3 x_3$$

A, B_i are constants

surfaces of constant φ are planes

Relation to Poisson's Equation

Field expression of inverse square law of gravity

$$\nabla^2 \phi = -4\pi G \rho$$

mass density

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = -4\pi G \rho$$

$$\sum_i \frac{\partial^2 \phi}{\partial x_i^2} = -4\pi G \rho \qquad x_i = (x, y, z) = (x_1, x_2, x_3)$$

$$\left(\begin{array}{ll} \delta_{ik} = 1 & i = k \\ \phantom{\delta_{ik}} = 0 & i \neq k \end{array} \right)$$

$$\sum_{i,k} \delta_{ik} \frac{\partial^2 \phi}{\partial x_k \partial x_i} \longleftarrow \text{"contract over } i, k\text{"} \longleftarrow \frac{\partial^2 \phi}{\partial x_k \partial x_i}$$

Tidal acceleration

$$\frac{d^2}{dt^2} n_i = -\sum_k n_k \frac{\partial^2 \phi}{\partial x_k \partial x_i}$$

extract field term