Maxwell's 1860 Factorization Condition

Maxwell assumes

for frequency distribution over molecular velocities

$$f(v_x, v_y, v_z) = \underbrace{f(v_x)\, f(v_y)\, f(v_z)}_{\text{i.e. components are probabilistically independent}}$$

x-component of velocity $\underset{\sim}{v}$ written "x" by Maxwell

Condition is not trivial!

Illustration.
Assume a dynamics with a maximum speed c.

Then

Independence fails

v_x is close to c
correlates with
v_y, v_z are close to 0

Illustration respects isotropy.
$f(\underset{\sim}{v})$ can still be a function of
$$v^2 = v_x^2 + v_y^2 + v_z^2$$
alone

This is the case of a relativistic gas!

Molecule in a relativistic gas at temperature T

Function of v^2

rest mass

$$f(v_x, v_y, v_z) \propto \exp\left(-\frac{\text{Energy}}{kT}\right) = \exp\left(-\frac{mc^2}{\sqrt{1-v^2/c^2}} \Big/ kT\right)$$

Power series $\dfrac{mc^2}{\sqrt{1-v^2/c^2}} = \underbrace{mc^2 + \frac{1}{2}mv^2}_{} + \frac{3}{8}m\frac{v^4}{c^2} + \frac{5}{16}m\frac{v^6}{c^4} + \dots$

If we approximate with these terms only, then the expression factorizes

$$\exp\left(-\frac{\text{Energy}}{kT}\right) \approx \exp\left(-\frac{mc^2 + \frac{1}{2}mv^2}{kT}\right) \qquad v^2 = v_x^2 + v_y^2 + v_z^2$$

$$= \exp\left(-\frac{mc^2}{kT}\right)\exp\left(-\frac{1}{2}\frac{mv_x^2}{kT}\right)\exp\left(-\frac{1}{2}\frac{mv_y^2}{kT}\right)\exp\left(-\frac{1}{2}\frac{mv_z^2}{kT}\right)$$

But factorization fails if we add more terms in the series

e.g. $\frac{3}{8}\frac{mv^4}{c^2}$ $\qquad v^4 = \left(v_x^2 + v_y^2 + v_z^2\right)^2$

$$= v_x^4 + v_y^4 + v_z^4$$

$$\left[+ 2v_x^2v_y^2 + 2v_x^2v_z^2 + 2v_y^2v_z^2\right]$$

These terms preclude factorization