

## Time Reversal Invariance and Irreversibility

### 1. The standard approach to time reversal invariance for dynamical systems

State space  $\Gamma$ . A process  $P$  is a parameterized curve  $s(t)$  in  $\Gamma$ . So a process that describes the evolution from  $t_i$  to  $t_f$  is given by  $P : t \mapsto s(t)$ ,  $t_i \leq t \leq t_f$ . The “reversal” operation is an involution  $R : s \mapsto Rs$  ( $R^2s = s$ ). Then the time reversed process is  $P^* : t \mapsto {}^T s(t) := (Rs)(-t)$ ,  $-t_f \leq t \leq -t_i$ . A theory is said to be time reversal invariant iff  $P^*$  satisfies the laws of the theory whenever  $P$  does.

Remarks: 1) Natural to call a process  $P$  irreversible when its time reverse  $P^*$  is forbidden by the laws. 2) Irreversibility is decided by the theory, not by what we can do by manipulation. 3) The above analysis does not apply to thermodynamics since it gives no dynamics. 4) The above analysis is empty until the reversal operation is specified. This has to be done on a case-by-case basis. The usual procedure is to start with classical mechanics and to try to handle the other cases by correspondence to the start case. 5) In the above approach the time orientation is left fixed. In the approach of Malement, *Stud. Hist. Phil. Mod Phys.* 35 (2004): 295-315, the time reversal operation involves reversing the time orientation. On this conception how does one test time reversal invariance in the lab?

*Example:* Newtonian mechanics.  $s(t) = (x(t), p(t))$  and  $(Rs)(t) = (x(t), -p(t))$ . With this definition, the time reverse process  $P^*$  is what you would see if you ran the movie of  $P$  backward. With the familiar velocity-independent force laws, Newtonian mechanics is time reversal invariant.

### 2. Thermodynamics

(a) Kelvin-Planck sense of reversibility: the idea is that a process  $P$  is reversible if it can be completely undone, i.e. there is another process that will restore the initial state of the system and of the auxiliary apparatus used to manipulate the system. Contrarywise, Kelvin-Planck irreversibility means that the ravages of time cannot be undone. Letting  $s$  stand for the thermodynamics state characterized by the values of macroscopic variables, e.g. pressure, volume, temperature) and  $z$  stand for the variables that characterize the auxiliary system, reversibility of  $P : (s_i, z_i) \rightarrow (s_f, z_f)$  means that there is a  $P' : (s_f, z_f) \rightarrow (s_i, z_i)$ . The implicit understanding is that  $P'$  need not retrace the intermediate states of  $P$ ; also  $P'$  should be “available in nature,” i.e. it is a process that, in principle, we could bring about.

(b) Sometimes a thermodynamic process is said to be reversible if it is quasi-static, i.e. it proceeds so slowly that the system can be considered, up to negligible error, to remain in equilibrium.

Remarks: 1) Senses (a) and (b) are not equivalent. 2) Suppose that the underlying microdynamics is time reversal invariant. Is this compatible with the existence of irreversible thermodynamical processes in the Kelvin-Planck sense?

### 3. Time reversal invariance in QM

The state is specified by a vector  $\psi$  (actually a ray) in a Hilbert space  $\mathcal{H}$ . The dynamics is given by the Schrödinger equation

$$\hat{H}\psi = i\hbar \frac{\partial\psi}{\partial t}$$

where  $\hat{H}$  is the Hamiltonian operator (the quantum counterpart of the energy of the classical system). The reversal operation  $R$  for simple systems is fixed by correspondence with classical mechanics. The quantum counterparts of the classical quantities  $x$  and  $p$  are operators  $\hat{x}$  and  $\hat{p}$  on  $\mathcal{H}$ . We want to implement  $R$  by finding an operator  $\hat{R}$  such that (i)  $\hat{R}\hat{x}\hat{R} = \hat{x}$  and  $\hat{R}\hat{p}\hat{R} = -\hat{p}$ , and (ii)  $\hat{R}$  preserves the commutation relations  $[\hat{x}, \hat{p}] = i\hbar$ . For single spinless particle, these requirements fix  $\hat{R}$  to have the form  $\hat{U}\hat{K}$  where  $\hat{U}$  is a unitary operator and  $\hat{K}$  is complex conjugation (this is representation dependent).

*Example:* Take the case of a single spinless particle moving in one dimension, and use the wave function representation, i.e.  $\psi \in L^2_{\mathbb{C}}(\mathbb{R})$  (square integrable complex valued function on  $\mathbb{R}$ ). Then  $\hat{R}$  is just complex conjugation, and if  $P : \psi(0) \rightarrow \psi(t)$  describes a wave packet moving, say, to the right, then the time reverse process  $P^* : \psi^*(-t) \rightarrow \psi^*(0)$  describes a wave packet moving to the left.

*Lemma:* The following three conditions are equivalent:

- (i) if  $\psi(t)$  solves the Schrödinger equation, then so does  ${}^T\psi(t) := \hat{R}\psi(-t)$
- (ii)  $\hat{R}\hat{H} = \hat{H}\hat{R}$
- (iii) the transition probability from state  $\psi_1$  to state  $\psi_2$  in time  $\Delta t$  is always equal to the transition probability from  $\hat{R}\psi_2$  to  $\hat{R}\psi_1$  in the same  $\Delta t$ .

Property (iii) is useful in experimental tests of time reversal invariance.

#### 4. *Time reversal invariance in classical electromagnetic theory*

Maxwell's equations:

$$\nabla \times B - \frac{\partial E}{\partial t} = j, \quad \nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot E = \rho, \quad \nabla \cdot B = 0$$

Plausibility argument: The charge density shouldn't change under time reversal. So if  $E$  is created by charge, it shouldn't change either. Think of a current  $j$  due to the motion of a particle with charge  $q$ ; then  $j = qv$ . And since  $v$  changes sign under time reversal, so does  $j$ . Think of the  $B$  field created by the current flowing through a wire. Under time reversal the current moves in the other direction, so  $B$  should change sign under time reversal. Collecting these results ( $\rho \rightarrow \rho$ ,  $j \rightarrow -j$ ,  $E \rightarrow E$ ,  $B \rightarrow -B$ ) we see that Maxwell's equations are time reversal invariant.

Criticism: This procedure implicitly assumes that Maxwell's equations are time reversal invariant; we can infer from that assumption how  $E$  and  $B$  must transform. But if we weren't sure from the beginning that Maxwell's equations are time reversal invariant, how could we infer the transformation rules for the field? An elegant and tight argument is given by Malament, *op. cit.*

#### 5. *David Albert's challenge to the standard view of time reversal invariance*

Albert thinks that the accounts given in Secs. 3 and 4 are wrong: classical electromagnetic theory and elementary QM are not time reversal invariant and, therefore, "there have been dynamical distinctions between past and future written into the fundamental laws of physics for a century and a half now" (2000, p. 21).

If this were right it would be earth shaking. What is his analysis of time reversal invariance, and how does it permit him to draw this conclusion?

#### 6. *Implications of time reversal invariance*

Sachs, *The Physics of Time Reversal Invariance* cites only one example of how time reversal invariance figures in the solution of a problem in classical dynamics:

Painlevé's theorem (aka Bad News for Cat's Theorem). Suppose that a cat is composed entirely of particles between which only conservative forces act. If the initial state  $s(0)$  of the cat has it upside down and if the initial velocities of all of the cat's particles vanish (and, thus,  $(Rs)(0) = s(0)$ ), then the cat in free fall cannot land on its feet.

In QM and QFT time reversal invariance has important applications. Example: Kramer's degeneracy. From the involutory character of the reversal operator,  $\hat{R}\psi$  and  $\psi$  must be equal (up to a phase factor). This, together with  $\hat{R} = \hat{U}\hat{K}$ , imply that  $\hat{R}\hat{R} = \pm\hat{I}$ . Consider an eigenstate  $\psi$  of energy:  $\hat{H}\psi = \lambda\psi$ . By time reversal invariance,  $\hat{H}(\hat{R}\psi) = \hat{R}(\hat{H}\psi) = \lambda(\hat{R}\psi)$ , i.e.  $\hat{R}\psi$  is also an energy eigenstate with the same energy  $\lambda$ . But when  $\hat{R}\hat{R} = -\hat{I}$ ,  $\hat{R}\psi$  and  $\psi$  are orthogonal. Therefore, the eigenvalue  $\lambda$  is degenerate.

Time reversal invariance of laws  $L$  does not imply that in any history  $t \mapsto s(t)$  satisfying the laws the past and future are mirror images, i.e.  $s(-t) = (Rs)(+t)$ . Given determinism and the assumption that the initial state  $s(0)$  is reversal invariant, i.e.  $(Rs)(0) = s(0)$ , the implication does hold. In classical mechanics,  $(Rs)(0) = s(0)$  implies that the velocities of the particles vanish at  $t = 0$ . In classical electromagnetism it implies that  $B(0) = 0 = j(0)$ . In QM it implies that  $\hat{U}\hat{K}\psi(0) = \exp(i\alpha)\psi(0)$ . In the case of a spinless particle, this comes to  $\psi^*(0) = \exp(i\alpha)\psi(0)$ , which implies that the expectation value of  $\hat{p}$  at  $t = 0$  is 0.

Time reversal invariance implies that future pointing and past pointing determinism stand or fall together. Thus, if we can use the laws to make (deterministic) inferences about the past, then we can use them to make (deterministic) inferences about the future, and vice versa.

In classical physics one can give examples where time reversal invariance fails and where determinism holds, say, in the future direction but not in the past direction.

It is hard to see how such an asymmetry can hold in QM (sans state vector reduction). To do quantum dynamics we need an essentially self-adjoint operator  $\hat{H}$ . Then  $\hat{V}(t) := \exp(-\hbar\hat{H}t)$ ,  $-\infty < t < +\infty$ , is a strongly continuous one-parameter group of operators.  $\hat{V}(t)$  and  $\hat{V}^{-1}(t)$  are inference engines to the future and the past. Can there be an example of a time-independent self-adjoint  $\hat{H}$  such that  $\hat{R}\hat{H} \neq \hat{H}\hat{R}$ ?

### 7. *The failure of time reversal invariance in weak interactions*

In QFT the imposition of standard assumptions (Lorentz invariance, locality, ...) entails that the interactions are invariant under the combination of  $CPT$  ( $C$  = charge conjugation,  $P$  = parity, and  $T$  = time reversal). Experiments show that the decay of neutral kaons violates  $CP$ . Thus, assuming  $CPT$ ,  $T$  must be violated. More recently, it has been claimed that direct violations of  $T$  have been observed: they claim to observe differences in the transition probabilities in (iii) of the above Lemma.

### 8. *Really dumb questions*

All the basic laws of physics are invariant under spatial rotations. Why then is the state of the world rotationally asymmetric? Dumb question because for rotationally invariant laws expressed as differential equations it is typically the case that the set of rotationally invariant solutions is of “measure zero.” So it would be surprising to find ourselves in a rotationally invariant world.

All the basic laws of physics (with one exception mentioned below) are time reversal invariant. Why then are the temporal processes in our world so asymmetric? Dumb question for exactly parallel reasons.

Nevertheless, some temporal asymmetries seem so pervasive or important they seem to call for explanation—the so-called “arrows of time” being examples. If an asymmetry cannot be traced to an asymmetry in the laws it must be due, at least in some important ways, to initial/boundary conditions. The devil is in the details.

### 9. *Implications of the failure of time reversal invariance*

The failure of time reversal invariance in weak interactions is often brushed aside on the grounds that it cannot have connection with the temporal asymmetries we care about. Even if this is true (and it may not be), it ignores two profound implications:

(i) at least locally, the spacetime that supports such laws must be time oriented

(ii) assuming that laws are “universal,” the spacetime that supports such laws must be time globally time orientable.

[Glitch in the argument: Need for an “up to a  $CPT$  transformation,” unless one takes seriously the violation of  $CPT$  invariance by black hole evaporation.]

*10. Black hole evaporation*

Claim: Black hole evaporation leads to a pure-to-mixed state transition, a violation of both  $T$  and  $CPT$  invariance.

*11. Are anti-particles just particles traveling backward in time?*