I. J. Good "The Paradox of Confirmation" BJPS 11(42) (1960) pp. 145-49

1

Bayesian Framework: Hypothesis H, Evidence E

Bayes theorem $\frac{P(H|E)}{P(\bar{H}|E)} = \underbrace{\frac{P(E|H)}{P(E|\bar{H})}}_{\text{"Bayes factor"}} \cdot \frac{P(H)}{P(\bar{H})}$

(not H)

Re-express in terms of odds ratios $O(P) = P/1-P$

$$\frac{P(E|H)}{P(E|\bar{H})} = \frac{O(H|E)}{O(H)}$$

since $O(H|E) = \frac{P(H|E)}{P(\bar{H}|E)} = \frac{P(E|H)}{P(E|\bar{H})} \cdot \underbrace{\frac{P(H)}{P(\bar{H})}}_{O(H)}$

Bayes factor measures weight of evidence E

= the factor that alters priors to posterior probabilities

$$\frac{P(H)}{P(\bar{H})} \xrightarrow[\times \text{ Bayes factor}]{} \frac{P(H|E)}{P(\bar{H}|E)}$$

Ravens (= crows) as a sampling problem

N objects overall →

c crows (N − c) non crows

i are black (b − i) black shoes } b black objects overall

!! ** Sampling $P = 1/N$ each object sampled ** !!

Possible worlds

H_0 = no black crows
H_1 = 1 black crow
⋮
H_c = c black crows = H "all crows are black"

} *** !! Assume all worlds equally probable *****!!

$\boxed{\text{Evidence } E\text{: A black crow is sampled}}$

$$P(E|H_i) = P(\text{black selected}|\text{crow})\,P(\text{crow}) = \frac{i}{c}\cdot\frac{c}{N} = \frac{i}{N}$$

$i = 1, \cdots, c$ since E rules out $i=0$

$$P(E|H_c) = c/N = P(E|H)$$

$$P(E|\bar H) = P(E|H_1 \vee \cdots \vee H_{c-1}) = P(H_1 \vee \cdots \vee H_{c-1})\frac{P(E)}{P(\bar H)}$$

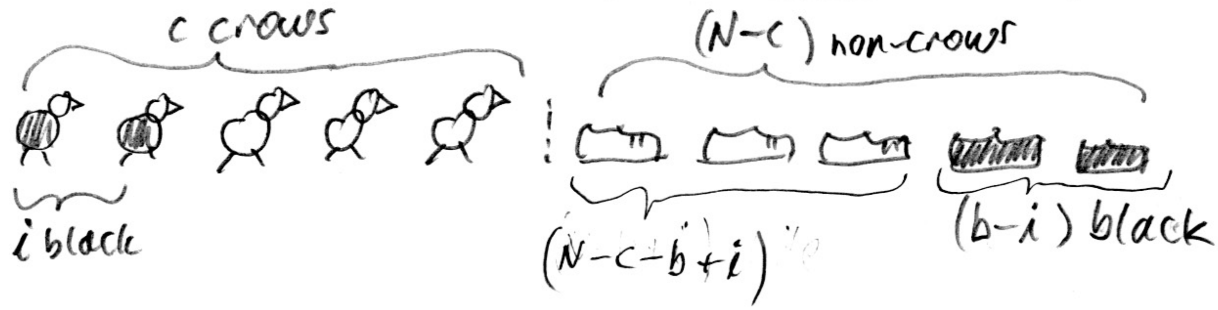
$$= \underbrace{P(H_1|E)\frac{P(E)}{P(H_1)}}_{P(E|H_1) = 1/N}\cdot\underbrace{\frac{P(H_1)}{P(\bar H)}}_{1/c} + \cdots + \underbrace{P(H_i|E)\frac{P(E)}{P(H_i)}}_{P(E|H_i) = i/N}\cdot\underbrace{\frac{P(H_i)}{P(\bar H)}}_{1/c} + \cdots + \underbrace{P(H_{c-1}|E)\frac{P(E)}{P(H_{c-1})}}_{P(E|H_{c-1}) = c-1/N}\cdot\underbrace{\frac{P(H_{c-1})}{P(\bar H)}}_{1/c}$$

I think this should be $1/c-1$, but then I don't get Black's result

$$= \underbrace{(1 + \cdots + c-1)}_{c(c-1)/2}\frac{1}{N}\cdot\frac{1}{c} = \frac{c-1}{2N}$$

$$\boxed{\text{Bayes factor } \frac{P(E|H)}{P(E|\bar H)} = \frac{c/N}{(c-1)/2N} = \frac{2c}{c-1} \longrightarrow 2 \quad \text{if } c \gg 1}$$

Evidence F : A white shoe is sampled



compute likelihoods individually

$$P(F \mid \underset{i=0}{H_0}) = P(\text{white} \mid \text{shoe})\, P(\text{shoe}) = \frac{N-c-b}{N-c} \cdot \frac{N-c}{N} = \frac{N-c-b}{N}$$

(If $0 \leq b < N-c$
Else $P(F \mid H_0) = 0$) since $b \geq N-c$ $\Downarrow$ Number white shoes $= N-c-b \leq 0$

$$P(F \mid \underset{i=0,\dots,c}{H_i}) = \frac{N-c-b+i}{N-c} \cdot \frac{N-c}{N} = \frac{N-c-b+i}{N}$$

(if $0 \leq b < N-c+i$
Else $P(F \mid H_i) = 0$ since No white shoes $= N-c-b+i \leq 0$ when $N-c+i \leq b$)

To avoid $P(F \mid H_i) = 0$ for all i, assume all inequalities satisfied: i.e.

$0 \leq b < N-c+i$ for $i = 0, \dots, c$

$\therefore \boxed{0 \leq b < N-c}$ ← i.e. Assume fewer black objects than non-crows

↑ Assume for simplification

$\boxed{\text{Compute likelihoods}}$

$$P(F|\underset{\substack{\uparrow \text{all crows}\\ \text{block}}}{H_c}) = P(F|H_c) = \left.\frac{N-c-b+i}{N}\right|_{i=c} = \frac{N-b}{N}$$

$$P(F|\bar{H}) = P(F \mid H_0 \vee H_1 \vee \cdots \vee H_{c-1})$$

$$= P(H_0 \vee H_1 \vee \cdots \vee H_{c-1} \mid F)\frac{P(F)}{P(\bar{H})}$$

$$= \sum_{i=0,c-1} \underbrace{\underbrace{P(H_i)\frac{P(F)}{P(H_i)}}_{P(F|H_i)}}_{\frac{N-c-b+i}{N}} \cdot \underbrace{\frac{P(H_i)}{P(\bar{H})}}_{1/c \text{ if all } H_0,\ldots,H_{c-1} \text{ equiprobable}}$$

$$= \sum_{i=1,c-1} \frac{1}{N}\cdot\frac{1}{c}(N-c-b+i)$$

Rescale index $i \to k$ using

$$\begin{aligned} i &= 0,\cdots,c-1 \\ -i &= 0,\cdots,1-c \\ \underbrace{c-i}_{k} &= c,\cdots,1 \\ k &\quad 1,\cdots,c \end{aligned}$$

$$= \sum_{k=1,\cdots,c} \frac{1}{N}\cdot\frac{1}{c}\underbrace{(N-b-k)}$$

sum is $[(N-b-1)+(N-b-c)]\,c/2$

$$= \left(N-b-\frac{c+1}{2}\right)c$$

$$= \frac{1}{N}\cdot\frac{1}{c}\left(N-b-\frac{c+1}{2}\right)c = \left(N-b-\frac{c+1}{2}\right)/N$$

Likelihood ratio   F = white shoe

$$\left[\frac{P(F|H)}{P(F|\bar{H})} = \frac{(N-b)/N}{\left(N-b-\frac{c+1}{2}\right)/N} = \frac{1}{1-\frac{1}{(N-b)}\cdot\frac{(c+1)}{2}}\right]$$

case I

$c \ll N-b$

very many more white objects than crows

$$\frac{P(F|H)}{P(F|\bar{H})} \approx 1 + \frac{1}{2}\underbrace{\frac{c+1}{N-b}}_{\substack{\text{small} \\ \text{“}\varepsilon\text{”}}}$$

$= 1 + 2\varepsilon$

↓

H = "all crows are black" gets a negligible boost in probability

case 2

$c \approx N-b$

about as many crows as non-black objects

$$\frac{P(F|H)}{P(F|\bar{H})} \approx \frac{1}{1-1\cdot\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

↓

H = "all crows are black" gets same boost in probability as provided by a black crow

[?HOW?! Informally:
Number of black objects FIXED at b
Hence finding a white non-crow means that more of b black property is available for crows]

Effect of relaxing simplifying assumption

$0 \leq b < N - C$ Assume fewer black objects than non-crows

Expectation: Since more black objects now available, expect "All crows are black" to be favored

This happens since

Likelihood Ratio: $\dfrac{P(F|H)}{P(F|\bar{H})} = \dfrac{(N-b)N}{\left(\text{sum of term, some of which are replaced by zero}\right)}$ → ∴ Likelihood ratio increases

Side light. White shoe favors
more crows are black

since

$$\frac{P(F \mid H_i)}{P(F \mid H_k)} = \frac{N-c-b+i}{N-c-b+k} > 1 \quad \text{when } i > k$$

Finding a white shoe means that
more of the black property b
can be held by ravens