Einstein's Arguments Against General Covariance

1. Argument from energy conservation

2. Hole argument

Einstein, "On the Foundations of the Generalized Theory of Relativity and the theory of Gravitation." Physikalische Z. 15 (1914), 176–80

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8. "Very true," thinks the reader, "but the fact that Messrs. Einstein and Grossmann are not able to give the equations for the gravitational field in generally covariant form is not a sufficient reason for me to agree to a specialization of the reference system." But there are two weighty arguments that justify this step, one of them of logical, the other one of empirical provenance:

The hole argument (easy to misunderstand version ...but see footnote)

a) *If the reference system is chosen totally arbitrarily, then the* $g_{\mu\nu}$ *can by no means be completely determined by the* $\mathfrak{T}_{\sigma\nu}$. [11] For imagine that the $\mathfrak{T}_{\sigma\nu}$ and $g_{\sigma\nu}$ are given everywhere, and that all the $\mathfrak{T}_{\sigma\nu}$ vanish in one part Φ of the four-dimensional space. I can now introduce a new reference system that agrees completely with the original one outside of Φ but differs from it (without a violation of continuity) within Φ. If one now refers everything to this new reference system, where the matter is represented by $\mathfrak{T}_{\sigma\nu}'$ and the gravitational field by $g_{\mu\nu}'$, then, even though we do have

$$\mathfrak{T}_{\sigma\nu}' = \mathfrak{T}_{\sigma\nu}$$

everywhere, the equations

$$g_{\mu\nu}' = g_{\mu\nu}$$

are certainly not all satisfied in the interior of Φ.[3] This proves the assertion.

Should one want to make it possible for the $g_{\mu\nu}$ (gravitational field) to be completely determined from the $\mathfrak{T}_{\sigma\nu}$(matter), then this could only be achieved by restricting the choice of the reference system.

The argument from energy conservation

b) In the original theory of relativity, the momentum and energy conservation law is expressed by an equation of the form [12]

$$\sum_{\nu} \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_{\nu}} = 0. \tag{3}$$

The corresponding equation obtained with the help of the absolute differential calculus is

$$\sum_{\nu} \frac{\partial \mathfrak{T}_{\sigma\nu}}{\partial x_{\nu}} = \frac{1}{2} \sum_{\mu\nu\tau} \frac{\partial g_{\mu\nu}}{\partial x_{\sigma}} \gamma_{\mu\tau} \mathfrak{T}_{\nu\tau}. \tag{4}$$ [13]

Equation (4) no longer has the form of a pure conservation law. This is understandable from a physical standpoint insofar as the matter, considered by itself, cannot satisfy the conservation laws in the presence of a gravitational field, because the gravitational field transfers momentum and energy to the matter. This is expressed on the right-hand side of equation (4). But if the conservation laws are to remain at all valid, we must demand that there be conservation laws of the form of (3) for the

[3] The equations are to be understood in such a way that the same numerical values are always assigned to the independent variables x'_{ν} on the left sides as to the variables x_{ν} on the right sides. ← !!

Big Gothic T → $\mathfrak{T}_{\sigma\nu} = \sum_{\tau} \sqrt{-g}\, \gamma_{\nu\tau} T_{\sigma\tau}$

Little Gothic t → $\mathfrak{t}_{\sigma\nu} = \sum_{\tau} \sqrt{-g}\, \gamma_{\nu\tau} t_{\sigma\tau}$

These are "tensor densities"

matter and the gravitational field taken together. Then there will have to be a system of equations of the form

[14]
$$\sum_{\nu} \frac{\partial(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu})}{\partial x_{\nu}} = 0 \tag{5}$$

where the $\mathfrak{t}_{\sigma\nu}$ depend only on the $g_{\mu\nu}$ and their derivatives. But there do not exist generally covariant systems of equations of the type of equations (5). Instead, closer examination shows that such systems are covariant only with respect to *linear* transformations. By demanding that the field equations of gravitation be formulated in such a manner that the validity of the conservation laws finds expression in this formulation, we therefore restrict the choice of the reference system in such a way that only *linear* transformations lead from one justified system to another one.

linear transformations only?

9. I have explained several times how the gravitational equations with respect to [15] reference systems specialized in this way are to be found. One asks: What kinds of differential expressions involving the $g_{\mu\nu}$ are the $\mathfrak{T}_{\sigma\nu}$ to be equated with in order for equations (4) to go over into equations (5) if I replace the $\mathfrak{T}_{\sigma\nu}$ on the right-hand sides of (4) with their expressions in terms of the $g_{\mu\nu}$? This question leads to the differential equations

[16]
$$\sum_{\alpha\beta\mu} \frac{\partial}{\partial x_{\alpha}}\left(\sqrt{-g}\,\gamma_{\alpha\beta}\,g_{\sigma\mu}\frac{\partial\gamma_{\mu\nu}}{\partial x_{\beta}}\right) = \kappa(\mathfrak{T}_{\sigma\nu} + \mathfrak{t}_{\sigma\nu}), \tag{6}$$

where we have set

$$-2\kappa\mathfrak{t}_{\sigma\nu} = \sqrt{-g}\left(\sum_{\beta\tau\rho}\gamma_{\beta\nu}\frac{\partial g_{\tau\rho}}{\partial x_{\sigma}}\frac{\partial\gamma_{\tau\rho}}{\partial x_{\beta}} - \frac{1}{2}\sum_{\alpha\beta\tau\rho}\delta_{\sigma\nu}\gamma_{\alpha\beta}\frac{\partial g_{\tau\rho}}{\partial x_{\alpha}}\frac{\partial\gamma_{\tau\rho}}{\partial x_{\beta}}\right).$$

Here $\delta_{\sigma\nu} = 1$ or 0, depending on whether $\sigma = \nu$ or $\sigma \neq \nu$. It is easy to show that these equations are covariant with respect to linear transformations.

It is beyond doubt that there exists a number, even if only a small number, of generally covariant equations that correspond to the above equations, but their derivation is of no special interest either from a physical or from a logical point of view, as the arguments presented in point 8 clearly show. However, the realization that generally covariant equations corresponding to equations (6) must exist is important to us in principle. Because only in that case was it justified to demand the covariance of the rest of the equations of the theory with respect to arbitrary substitutions. On the other hand, the question arises whether those other equations might not undergo specialization owing to the specialization of the reference system. In general, this does not seem to be the case.

10. One can see from the foregoing description of the foundations of the theory that no special assumptions need to be used to establish it. The reason why this is otherwise according to the account presented recently by Mie in this journal is

The argument against general covariance based on the conservation laws

Conservation of energy-momentum

$$T^{\mu\nu}{}_{;\nu} = 0$$

$T^{\mu\nu}$ represents non-gravitational energy

gravitational energy hidden in here

";" = covariant differentiation

Rewrite to make two types of energy appear symmetrically

$$\left(\sqrt{-g}\,T^{\mu\nu} + \sqrt{-g}\,t^{\mu\nu}\right)_{,\nu} = 0$$

"," = $\dfrac{\partial}{\partial x_\nu}$

ordinary energy-momentum — generally covariant tensor

gravitation energy momentum — IF this is a generally covariant tensor

THEN whole law is not generally covariant — not a generally covariant divergence

Expect it is since the two types of energy should be represented by invariant structures?

Einstein's error:

$t^{\mu\nu}$ need not be generally covariant

Then conservation can be generally covariant

Second Problem with argument from energy conservation:
Einstein erroneously asserts that it establishes the admissibility of LINEAR transformations only

Einstein does not give an explicit argument for the restriction to LINEAR transformations:
<u>my conjecture</u>:

Einstein rewrites conservation law

$$\sum_\tau \left[\sqrt{-g}\,\gamma_{\nu\tau}(T_{\sigma\tau}+t_{\sigma\tau})\right]_{,\nu}=0$$

as

$$0=\underbrace{\frac{1}{\sqrt{-g}}\sum_\tau\left[\sqrt{-g}\,\gamma_{\nu\tau}(T_{\sigma\tau}+t_{\sigma\tau})\right]}=\underbrace{\sum_\tau\left[\gamma\ (T_{\sigma\tau}+t_{\sigma\tau})\right]_{;\nu}}+\frac{1}{2}\sum_{\mu\nu}\underbrace{\frac{\partial g_{\mu\nu}}{\partial x_\sigma}}\left[\underbrace{\Theta_{\mu\nu}+\vartheta_{\mu\nu}}\right]$$

IF theory is generally covariant, then $t_{\sigma\tau}$ is a generally covariant tensor

THEN these are generally covariant

BUT this is covariant only under linear transformations

Therefore the entire law is covariant only under linear transformations

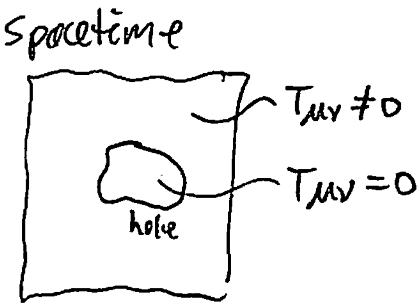
The Fallacy: Since Einstein has concluded his theory is not generally covariant, this antecedent fails.

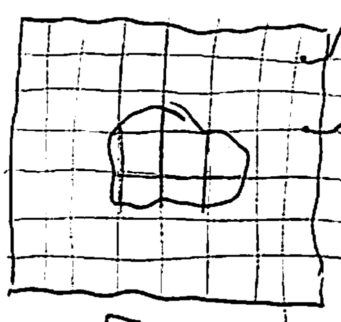
The hole argument:

claims to demonstrate:

$$\left(\begin{array}{l}\text{Generally} \\ \text{covariant} \\ \text{gravitational} \\ \text{field equations}\end{array}\right) \Longrightarrow \underbrace{\left(\begin{array}{l}\text{indeterminism} \\ T_{\mu\nu} \text{ everywhere, } g_{\mu\nu} \text{ outside} \\ \text{hole does not fix } g_{\mu\nu} \text{ within}\end{array}\right)}_{\text{physically unacceptable}}$$

The trivial reading of the hole argument (e.g. Pais)





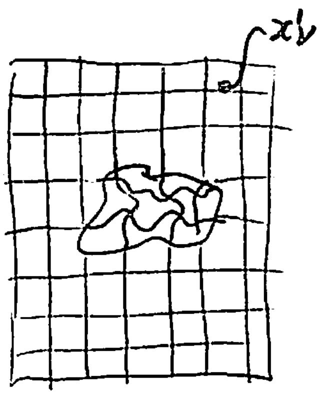
$T_{\mu\nu}$, $g_{\mu\nu}$ outside hole

? $\Downarrow$?

unique $g_{\mu\nu}$ inside hole

$\Downarrow$

transform to x'_ν
- $x'_\nu = x_\nu$ outside hole
- smoothly diverges inside



same $T^{\mu\nu}$, $g_{\mu\nu}$ outside hole

$\Downarrow$

different $g'_{\mu\nu}$ of x'_ν

($T'_{\mu\nu} = T_{\mu\nu} = 0$ within as before)

> Fallacy! $g'_{\mu\nu}$ is just different coordinate representation in x'_ν of $g_{\mu\nu}$ in x_ν

A cleaner statement by Einstein of hole argument

We consider a finite region of the continuum Σ, in which no material process takes place. Physical happenings in Σ are then fully determined, if the quantities $g_{\mu\nu}$ are given as functions of the x_ν in relation to the coordinate system K used for description. The totality of these functions will be symbolically denoted by $G(x)$.

Let a new coordinate system K' be introduced, which coincides with K outside Σ, but deviates from it inside Σ in such a way that the $g'_{\mu\nu}$ related to the K' are continuous everywhere like the $g_{\mu\nu}$ (together with their derivatives). We denote the totality of the $g'_{\mu\nu}$ symbolically with $G'(x')$. $G'(x')$ and $G(x)$ describe the same gravitational field. In the functions $g'_{\mu\nu}$ we replace the coordinates x'_ν with the coordinates x_ν, i.e., we form $G'(x)$. Then, likewise, $G'(x)$ describes a gravitational field with respect to K, which however does not correspond with the real (or originally given) gravitational field.

We now assume that the differential equations of the gravitational field are generally covariant. Then they are satisfied by $G'(x')$ (relative to K') if they are satisfied by $G(x)$ relative to K. Then they are also satisfied by $G'(x)$ relative to K. Then relative to K there exist the solutions $G(x)$ and $G'(x)$, which are different from one another, in spite of the fact that both solutions coincide in the boundary region, i.e., *happenings in the gravitational field cannot be uniquely determined by generally covariant differential equations for the gravitational field.*

"Die formal Grundlage der allgemeinen Relativitätstheorie" Preuss. Akad. der Wiss., Sitz., 1914, 1030–85 on pp. 1066-67

(Stachel) What the argument really says

Extra step: $x_\nu \longrightarrow x'_\nu$

$$g_{\mu\nu} \xrightarrow{\text{transforms}} g'_{\mu\nu}$$

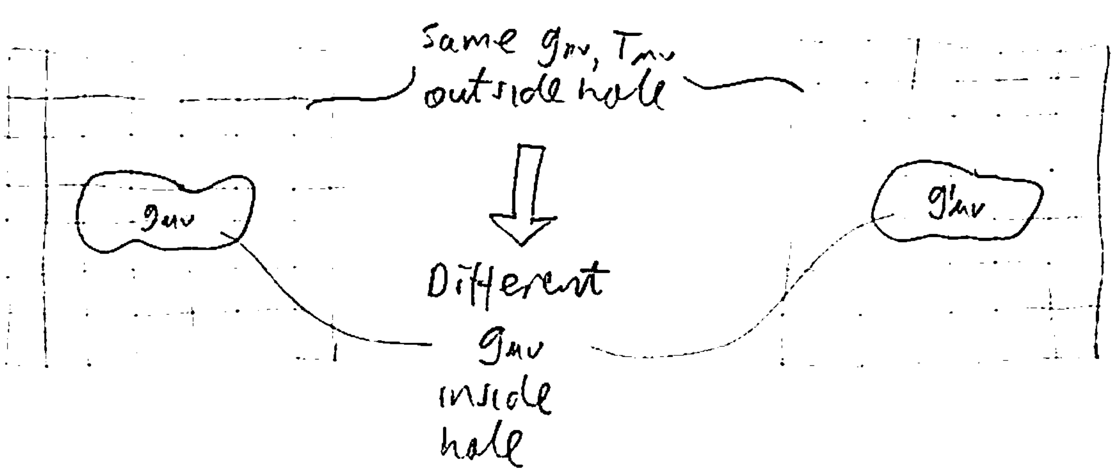
Both are solutions
of the generally
covariant field
equations (gcfe)

$g_{\mu\nu}(x_\nu)$ is ten functions x_ν that solves gcfe

$g'_{\mu\nu}(x'_\nu)$ " " " x'_ν " " "

Take these ten functions.
Use them to build <u>new</u> metric $g'_{\mu\nu}(x_\nu)$
in <u>original coordinate</u> system x_ν

Now have <u>two</u> solutions of gcfe in <u>one</u> coordinate system



Einstein's Later Diagnosis of the Error of the hole argument.

In §12 of my work of last year, everything is correct (in the first three paragraphs) up to the italics at the end of the third paragraph. One can deduce no contradiction at all with the uniqueness of occurrences from the fact that both systems G(x) and G'(x), related to the same reference system, satisfy the conditions of the grav. field. The apparent force of this consideration is lost immediately if one considers that

(1) the reference system signifies nothing real

(2) that the (simultaneous) realization of the two different g-systems (better said, two different gravitational fields) in the same region of the continuum is impossible according to the nature of the theory.

In the place of §12 steps the following consideration. The physical reality of world occurrences (in opposition to that dependent on the choice of reference system) consists *in spacetime coincidences*...

(Letter to Paul Ehrenfest, Dec. 26, 1915)

(1) Einstein had tacitly reified coordinate systems. He illicitly assumed it makes sense to image a coordinate system without metric as an independent reality.

Everything was correct in the hole argument up to the last conclusion. There is no physical content in two different solutions G(x) and G'(x) existing with respect to the *same* coordinate system K. To imagine two solutions simultaneously in the same manifold has no meaning and the system K has no physical reality. In place of the hole argument we have the following. *Reality* is nothing but the totality of space-time point coincidences... (PTO)

(Letter to Michele Besso, Jan 3, 1916)

(2) The point coincident argument establishes the physical equivalence of G(x) and G'(x They agree on all point coincidenc therefore they agree in physical conten

How Point coincidence argument escapes the hole argument.

Einstein to M. Besso, 3 Jan 1916:

Everything was correct in the hole consideration up to the last conclusion. There is no physical content in two different solutions $G(x)$ and $G'(x)$ existing with respect to the *same* coordinate system K. To imagine two solutions simultaneously in the same manifold has no meaning and the system K has no physical reality. In the place of the hole consideration we have the following. *Reality* is nothing but the totality of space-time point coincidences. If, for example, physical happenings could be built up out of the motion of material points alone, then the meetings of the points, i.e., the points of intersection of their world lines, would be the only reality, i.e., in principle observable. Naturally these points of intersection remain unchanged in all transformations (and no new ones are added) only if certain uniqueness conditions are preserved. Therefore it is most natural to require of laws that they determine no more than the totality of timespace coincidences. Following what has been said before, this is already achieved with generally covariant equations.

i.e. Point-coincidence argument assumes these two solutions are physically the same

modern language

coordinate Transformation $x_\nu \longrightarrow x'_\nu$ induces Point transformation (diffeomorphism)

h: p with coords x_ν $\longrightarrow$ p' with coords x'_ν

$\underbrace{\hspace{6em}}_{\text{in same co-ordinate system}}$

$$\left[\, g'_{\mu\nu} = h^* g_{\mu\nu} = \text{"carry along" of } g_{\mu\nu} \text{ under } h \,\right]$$

$g'_{\mu\nu}$ is just $g_{\mu\nu}$ "spread differently" on spacetime manifold.

modern version of hole argument (Earman, Norton)

substantivalism about manifold points $\Longrightarrow$ $\langle M, g\rangle$, $\langle M, g'\rangle$ represent different physical states

metrical properties spread differently and this matters

this is the substance view $\Big\{$ since points of M have properties independently of metric g

$\Big\Downarrow$ hole argument

indeterminism