Paradoxes of Newtonian Cosmology

* Strongest grounds for geometrizing Newtonian gravitation

* (relatively rare) instance of real symmetry

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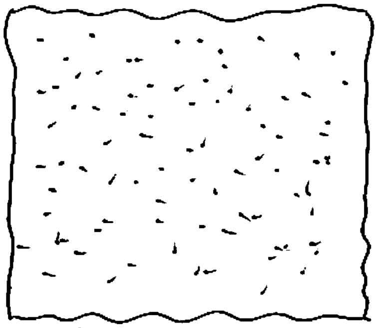
Set Up



Infinite Euclidean space

uniform matter distribution of density ρ

"Dust" = non-interacting particles in free fall

Newton's inverse square law of attraction

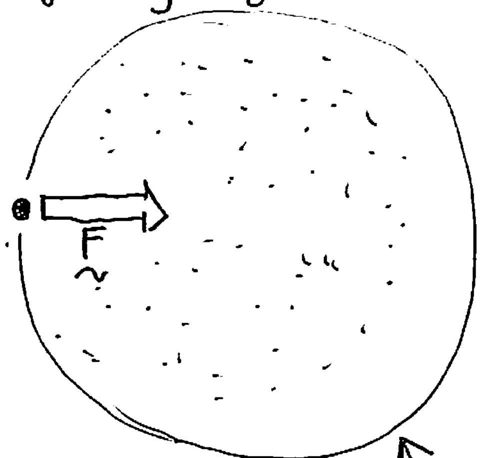
$$F = G\,\frac{m_1 m_2}{r^2}$$

Paradox

In strongest sense: Validly deduce A = force on test mass is $\underset{\sim}{F}$
and $\sim A$ = " " " " $\sim \underset{\sim}{F}$

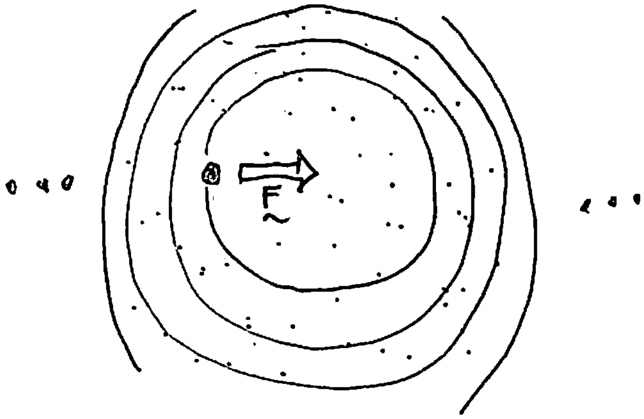
Version 1

① Choose arbitrary unit test mass and nominate force $\underset{\sim}{F}$ of any size and direction



② Add in masses in sphere on whose surface test mass sits of sufficient size to generate $\underset{\sim}{F}$

③ Add in remaining masses in concentric spherical shells ← Each exerts NO net gravitational force

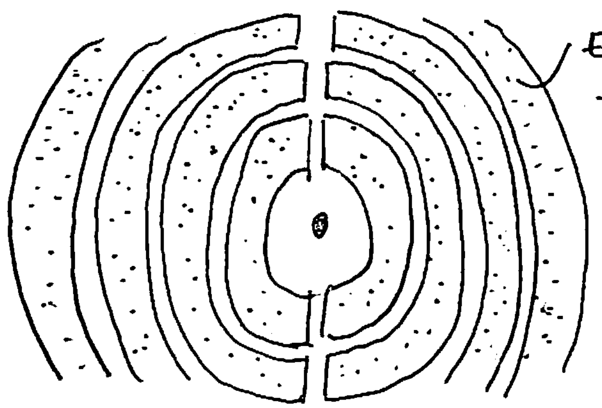


④ Conclude total force is $\underset{\sim}{F}$

⑤ Repeat for any other force $\underset{\sim}{F}'$

Version 2

① Divide all cosmic masses into hemispherical shells of thickness Δr centered on test mass



Each hemisphere exerts force $\underbrace{G\pi\rho\Delta r}_{\text{independent of size!}}$

$$\text{Net force} = -G\pi\rho\Delta r - G\pi\rho\Delta r - G\pi\rho\Delta r - \cdots + G\pi\rho\Delta r + G\pi\rho\Delta r + G\pi\rho\Delta r + \cdots$$

$=$ Non-convergent

$\boxed{\text{Newton-oops!}}$

more formally ...

Gravitational potential ϕ at origin due to masses in volume V $= -\underbrace{\iiint_V \frac{G\rho}{r}\, dV}_{\text{Diverges as } V \to \text{all space}}$

Gravitational force on unit mass at origin due to masses in volume V $= -\underset{\sim}{\nabla} \underbrace{\iiint_V \frac{G\rho}{r}\, dV}_{\text{Not uniformly convergent as } V \to \text{all space}}$

Tidal forces also diverge

Version 3 (malament)

Require <u>in addition</u> that gravitational forces everywhere conform to single ϕ by

$$\underset{\sim}{F} = -\underset{\sim}{\nabla}\phi$$

$$\nabla^2\phi = 4\pi G\rho$$

For uniform $\rho \neq 0$, no homogeneous, isotropic ϕ solves this equation
The closest are the "canonical"

$$\phi(\underset{\sim}{r}) = \frac{2}{3} G\pi\rho \left|\underset{\sim}{r} - \underset{\sim}{r}_0\right|^2$$

for which

any r_0

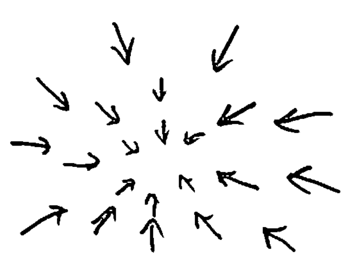
$$\underset{\sim}{F} = -\frac{4}{3} G\pi\rho \left(\underset{\sim}{r} - \underset{\sim}{r}_0\right)$$

choose different r_0

$\Downarrow$

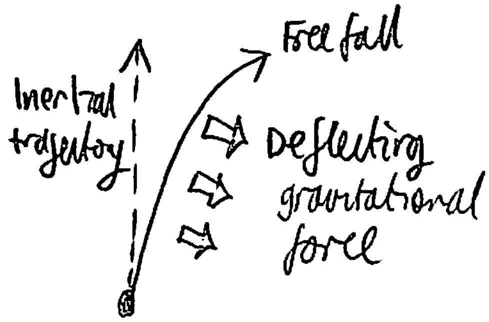
recover different forces

All force directed to ARBITRARY center r_0



Malament's solution

Newtonian view of free fall under gravity



* Split is arbitrary
* Gravitational force prevailing is fixed by convention by picking inertial motions conventionally.
* The physically real is just the free fall trajectories

  Agree in all canonical solutions on the observable relative accelerations

* Free fall trajectories ≡ Straights in spacetime of curved affine connection

More formally

Free Falls governed by

$$\boxed{\underset{\sim}{f} = \frac{d^2 \underset{\sim}{r}}{dt^2} = -\frac{4}{3} G\pi\rho(\underset{\sim}{x} - \underset{\sim}{x}_0)}$$

set $\underset{\sim}{x}_0 = 0$

$\underset{\sim}{x} = (x^1, x^2, x^3)$

$$\frac{d^2 \underset{\sim}{r}}{dt^2} + \frac{4}{3} G\pi\rho\, \underset{\sim}{x} = 0$$

$$\frac{d^2 x^i}{dt^2} + \frac{4}{3} G\pi\rho\, x^i = 0$$

compare

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{00}\left(\frac{dt}{dt}\right)^2 = 0$$

Non-trivial Γ^i_{km} are

$$\boxed{\Gamma^i_{00} = \frac{4}{3} G\pi\rho}$$

Tidal forces manifest as differential acceleration between test masses separated by small Δx^i

$$\frac{d^2 \Delta x^i}{dt^2} + \frac{4}{3}\pi G\rho \Delta x^i = 0$$

compare with equation of geodesic deviation

$$\frac{d^2 \Delta x^\alpha}{dt^2} + R_{\gamma\delta\beta}{}^{\alpha} \Delta x^\beta \left(\frac{dx^\gamma}{dt}\right)\left(\frac{dx^\delta}{dt}\right)$$

$\alpha, \beta, \ldots = 0,1,2,3$

* $R_{\gamma\delta\beta}{}^{\alpha}$ are constants in this coordinate system (suggests homogeneity)

* $R_{0ok}{}^{i} = \frac{4}{3}\pi G\rho\, \delta^i_k$

$$\boxed{R_{00} = 4G\pi\rho}$$

Analog of $\nabla^2\phi = 4G\pi\rho$

(From "The N-stein Family")

Properties of Canonical Solutions the Easy Way: Covariance Properties

Observable relative accelerations are independent of r_0

masses at $\underset{\sim}{r}_1, \underset{\sim}{r}_2$ in free fall

$$\frac{d^2 \underset{\sim}{r}_2}{dt^2} + 4\pi G\rho\,(\underset{\sim}{r}_2 - \underset{\sim}{r}_0) = 0$$

$$\frac{d^2}{dt^2}\underset{\sim}{r}_1 + 4\pi G\rho\,(\underset{\sim}{r}_1 - \underset{\sim}{r}_0) = 0$$

$$\frac{d^2}{dt^2}(\underset{\sim}{r}_2 - \underset{\sim}{r}_1) + 4\pi G\rho\,(\underset{\sim}{r}_2 - \underset{\sim}{r}_1) = 0$$

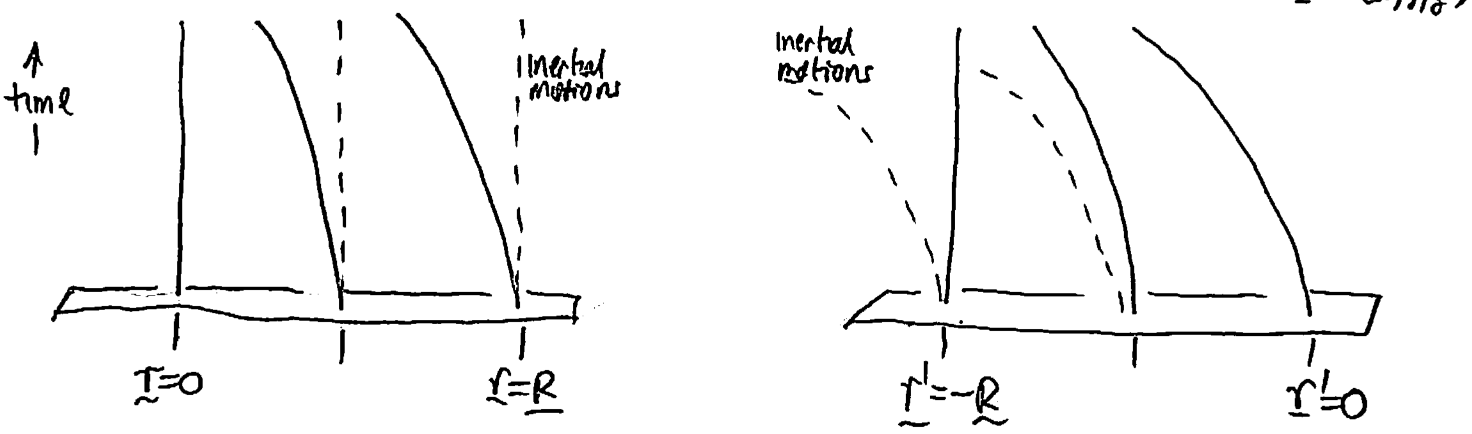
$\underset{\sim}{r}_0$ does not appear

Transformations between uniformly accelerated frames are SYMMETRIES of free fall motions

Frame $\underset{\sim}{r} = (x, y, z)$

$$t' = t$$
$$\underset{\sim}{r}' = \underset{\sim}{r} - \underset{\sim}{R}'(t)$$

Frame $\underset{\sim}{r}' = (x', y', z')$



$\underset{\sim}{R}(t)$ such that $\frac{d^2}{dt^2}R(t) + 4G\pi\rho\, R(t) = 0$

i.e. Free falls "look same" in both frames

since

$$\frac{d^2 \underset{\sim}{r}}{dt^2} + \frac{4}{3} G\pi\rho\, \underset{\sim}{r}(t) = 0 \iff \frac{d^2 \underset{\sim}{r}'}{dt^2} + \frac{4}{3} G\pi\rho\, \underset{\sim}{r}'(t) = 0$$

since

$$\frac{d^2 \underset{\sim}{r}'}{dt^2} = \frac{d^2}{dt^2}(\underset{\sim}{r} - \underset{\sim}{R}) = \frac{d^2 \underset{\sim}{r}}{dt^2} - \frac{d^2}{dt^2} \underset{\sim}{R}$$

$$\frac{4}{3} G\pi \underset{\sim}{r}' = \frac{4}{3}\pi G \underset{\sim}{r} - \frac{4}{3}\pi G \underset{\sim}{R}$$

equal

Derives from

$$\phi(\underset{\sim}{r}) = \frac{2}{3} G\pi \rho r^2$$

Derives from

$$\phi'(\underset{\sim}{r}') = \frac{2}{3} G\pi \rho' r'^2$$

where

$$\rho' = \rho$$

$$\phi' = \phi + \underbrace{\underset{\sim}{r} \cdot \frac{d^2 \underset{\sim}{R}}{dt^2}}_{\text{corresponds to adding homogeneous field}} + \underbrace{\phi(\underset{\sim}{R})}_{\text{Everywhere constant at one instant}}$$

corresponds to adding homogeneous field

$$-\nabla\phi' = -\nabla\phi - \frac{d^2}{dt^2} \underset{\sim}{R}$$

which is invisible to relative acceleration

Importance of SYMMETRY of transformation

Among class of uniformly accelerating frames, no invariant condition can pickout one as preferred

condition C holds of frame F

$\Downarrow$ symmetry

Equivalent condition C' holds of frame F'

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!!!!!

This symmetry property is very strong

Does not obtain in other cases

e.g. simultaneity in SR?
orthogonality $\rightarrow \varepsilon = \frac{1}{2}$

conventionality of geometry?
universal forces $= 0$

Field of sun?
Deflecting field isotropic about sun against flat inertial structure.

Analogy:
No invariant condition can pick out a rest frame in special relativity

] Teased out in my "The N-stein Family"