Convergence of Proper Distance along a t = constant curve to Event Horizon

Einstein and Rosen were concerned that the metrical coefficient

$$g_{11} = \frac{1}{1 - 2m/r}$$

diverges as r →2m. We can see, however, that there is no divergence in the proper distances it assigns. On a curve of constant t, θ and ϕ, this metrical coefficient assigns a proper length to a coordinate difference dr

$$\frac{dr}{\sqrt{1 - 2m/r}} = \frac{r^2\, dr}{\sqrt{r - 2m}}$$

The proper distance along the curve from a point with $r = R > 2m$ to one with $r = 2m$ is

$$\int_{r=2m}^{R} \frac{r^2\, dr}{\sqrt{r - 2m}} = \int_{r=2m}^{R} \frac{r^2\, d(r - 2m)}{\sqrt{r - 2m}} =$$

We do not need to evaluate this integral exactly. Our real concern is whether the divergence of g_{11} leads the integral to diverge.

The fast way to see that this proper length does not diverge is just to look at the integral for R very close to $2m$. This is where the divergence would arise, if it is to happen. There we can approximate r^2 as $4m^2$ and, writing $x = r - 2m$ and $X = R - 2m$, the integral is approximately

$$\int_{r=2m}^{R} \frac{r^2\, d(r - 2m)}{\sqrt{r - 2m}} \approx 4m^2 \int_{x=0}^{X} \frac{dx}{\sqrt{x}} = 8m^2 \overline{\sqrt{x}}_{x=0}^{x=X} = 8m^2\sqrt{X} = 8m^2\sqrt{R - 2m}$$

which is finite.

A more precise calculation confines the integral to a finite interval. Since $2m < r < R$ for all but the end points of the interval of integration, we have

$$\int_{r=2m}^{R} \frac{4m^2\, d(r - 2m)}{\sqrt{r - 2m}} < \int_{r=2m}^{R} \frac{r^2\, d(r - 2m)}{\sqrt{r - 2m}} < \int_{r=2m}^{R} \frac{R^2\, d(r - 2m)}{\sqrt{r - 2m}}$$

The integrals at the extremes of this interval are easily computed through

$$\int_{r=2m}^{R} \frac{d(r - 2m)}{\sqrt{r - 2m}} = 2\overline{\sqrt{r - 2m}}_{r=2m}^{r=R} = 2\sqrt{R - 2m}$$

Hence the proper length from a point with $r = R > 2m$ to one with $r = 2m$ lies in the interval

$$8m^2\sqrt{R-2m} < \int_{r=2m}^{R} \frac{dr}{\sqrt{1-2m/r}} < 4R\sqrt{R-2m}$$

and is finite.

$$\int_{r=2m}^{r=R} dr/\sqrt{1-2m/r} = \int_{r=2m}^{r=R} d\sqrt{x} = \sqrt{r^2-2m}_{r=2m}^{r=R} = \sqrt{R^2-2m}$$