Background to Blackbody Radiation Law

Wien displacement law

$$u_\nu(T) = \nu^3 f(\nu/T)$$

Derived from condition that reversible adiabatic compression preserves thermal equilibrium

Energy density in frequency ν at temperature T

Rayleigh-Jeans law (classical) 1900-1905

$$u_\nu(T) = \frac{8\pi\nu^2}{c^3} kT$$

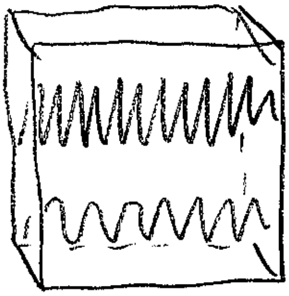
Wien distribution

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \exp\left(-\frac{h\nu}{kT}\right)$$

Planck distribution (fitted to best data in 1900)

$$u_\nu(T) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp(\frac{h\nu}{kT} - 1)}$$

Direct Pathway



Radiation consists of oscillators "normal modes" of all frequencies ν

Density of oscillators $d\nu$

$$n_\nu = \frac{8\pi\nu^2}{c^3}$$

Classical analysis

Each mode has 2 degrees freedom

+

Equipartition $\frac{1}{2}kT$ energy per degree freedom

$\Downarrow$

$$U_\nu(T) = n_\nu kT = \frac{8\pi\nu^2}{c^3} kT$$

Quantum analysis

Each mode is restricted to energies

$n\varepsilon = 0, h\nu, 2h\nu, \ldots$

with probability proportional to $\exp(-\frac{n\varepsilon}{kT})$

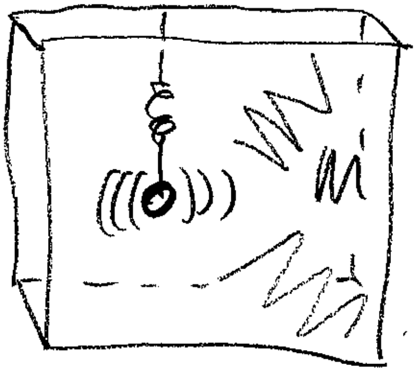
$\Downarrow$

mean energy per mode $\underline{E} = \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$

$\Downarrow$

$$U_\nu(T) = \frac{8\pi\nu^2}{c^3} \cdot \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

Indirect Pathway



Radiation is in thermal equilibrium with resonators of 2 degrees of freedom

$$\underbrace{u_\nu(T) = \frac{8\pi\nu^2}{c^3} U}_{\text{Planck. Derived classically before 1900}}$$

mean resonator energy at same frequency ν

classical analysis

Equipartition theorem

$$U = 2\left(\tfrac{1}{2}kT\right) = kT$$

$$u_\nu(T) = \frac{8\pi\nu^2}{c^3} kT$$

Quantum analysis

Resonators limited to energies

$0, h\nu, 2h\nu, \cdots$

$$U = \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

$$u_\nu(T) = \frac{8\pi\nu^2}{c^3} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

mean energy $\bar{E}$ of quantized system

Probability system has energy $n\varepsilon$ $\quad P(n\varepsilon) \propto \exp\left(-\frac{n\varepsilon}{kT}\right)$

$\uparrow$ unit, usually $h\nu$

$$kT\frac{d}{d\varepsilon}\underbrace{\left(\sum_{n=0}^{\infty}\exp\left(-\frac{n\varepsilon}{kT}\right)\right)}_{Z}$$

mean energy
$$\bar{E} = \frac{\sum_{n=0}^{\infty} n\varepsilon \exp\left(-\frac{n\varepsilon}{kT}\right)}{\underbrace{\sum_{n=0}^{\infty}\exp\left(-\frac{n\varepsilon}{kT}\right)}_{Z}}$$

$$= kT \cdot \frac{1}{Z}\frac{d}{d\varepsilon}Z$$

$$Z = \sum_{n=0}^{\infty}\exp\left(-\frac{n\varepsilon}{kT}\right) = \sum_{n=0}^{\infty}\left(\exp\left(-\frac{\varepsilon}{kT}\right)\right)^n = \frac{1}{1-\exp(-\varepsilon/kT)}$$

$$kT\frac{d}{d\varepsilon}Z = kT\frac{1}{\left[1-\exp(-\varepsilon/kT)\right]^2}\cdot\frac{1}{kT}\exp\left(-\varepsilon/kT\right) = Z^2\exp\left(-\varepsilon/kT\right)$$

$$\boxed{\bar{E} = \frac{Z^2\exp\left(-\frac{\varepsilon}{kT}\right)}{Z} = \frac{\exp\left(-\frac{\varepsilon}{kT}\right)}{1-\exp(-\varepsilon/kT)} = \frac{1}{\exp(\varepsilon/kT)-1}}$$