Ehrenfest's "Adiabatic Hypothesis"

Ehrenfest, 1917, Van der Waerden Paper 2.
Longer version, Ehrenfest "On adiabatic changes of a system in connection with the quantum theory," Amsterdam Academy 1917 pp 576–97

[Adiabatic Hypothesis : Under reversible adiabatic transformations admissible motions are converted to admissible motions]

very slow, in equilibrium

no heat exchanged

Utility

system we know how to quantize

system we don't know how to quantize

Harmonic oscillator can have energies E:

Adiabatic transformation

Anharmonic oscillator can have energies

$\frac{E}{\nu} = h \longrightarrow \boxed{}$

$\frac{E}{\nu} = 2h \longrightarrow \boxed{}$

$\vdots$

: what the transformation gives

Ehrenfest's Result

For periodic systems governed by a Lagrangian $L = \underbrace{A(\underset{\sim}{q})\dot{\underset{\sim}{q}}^2}_{T = \text{kinetic energy}} - \Phi(\underset{\sim}{q}, \underset{\sim}{a})$

external parameter manipulated to do work on system

$2\frac{\bar{T}}{\nu}$ is an adiabatic invariant

$\bar{T}$ = mean kinetic energy $\quad \nu$ = frequency

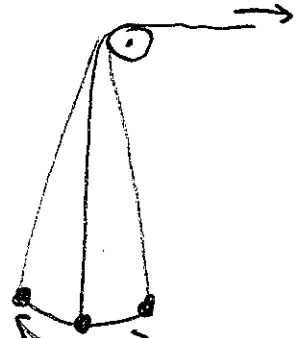
Applications

Simple harmonic oscillator

$\frac{\text{Energy}}{\text{frequency}}$ = invariant

Hence quantize as $0, h, 2h, \ldots$

classic, much mentioned implementation



Pull very slowly on string

= reversible adiabatic transformation

pendulum = simple harmonic oscillator for small amplitudes

Simple rotor

Angular momentum is an adiabatic invariant

Connection to Sommerfeld's quantization of action

$$\text{For}\quad L = \underbrace{\overset{\text{kinetic energy}}{T(q,\dot{q},a)}}_{\substack{T \text{ is quadratic in } \dot{q} \\ T = A(q,a)\dot{q}^2}} - \Phi(q,a)$$

$$\text{then } p\dot{q} = \frac{\partial L}{\partial \dot{q}}\dot{q} = 2A(q,a)\dot{q}^2 = 2T$$

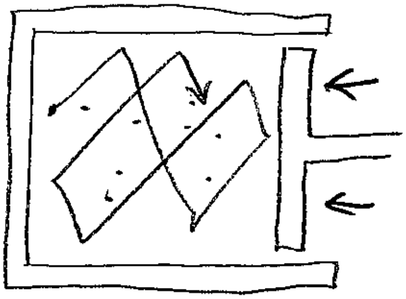
Hence

Ehrenfest's adiabatic invariant

$$\frac{2\overline{T}}{\nu} = \int_{\substack{\text{one period}\\ \tau}} 2T\,dt = \int p\dot{q}\,dt = \int p\,dq$$

$$\nu = \frac{1}{\tau = \text{period}}$$

connection to thermodynamics



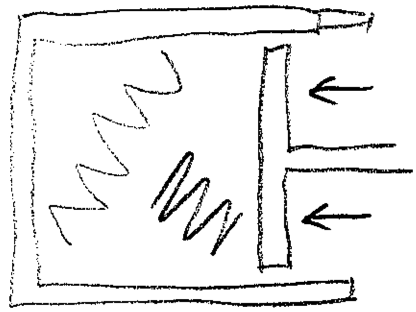
Reversible adiabatic compression of gas

↓

see simplest implementation as

1-D bouncing ball

$\frac{\text{Energy}}{\text{frequency}}$ is invariant



Reversible adiabatic compression of black body radiation in mirrored cavity

⇓ Derivation of "Wien displacement law"

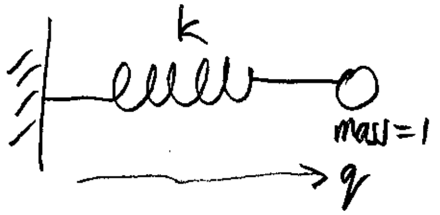
$\frac{\text{Energy in each frequency cut}}{\text{frequency}}$ = invariant

↑ No. of Einstein's quanta is fixed

Ehrenfest has a deeper connection to

$S = k \log W$ ... but it is eluding me

Simple harmonic Oscillator



$$\ddot{q} = \frac{d^2q}{dt^2} = -kq$$

solved by

$$q = \sin\sqrt{k}\,t$$

Harmonic frequency is $\sqrt{k}/2\pi = \nu$

Kinetic energy $T = \frac{1}{2}(\dot{q})^2 = \frac{1}{2}(\sqrt{k}\cos\sqrt{k}\,t)^2$

$$= \frac{k}{2}\cos^2\sqrt{k}\,t$$

Potential energy $U = \frac{1}{2}kq^2 = \frac{1}{2}k\sin^2\sqrt{k}\,t$

Total energy $E = U + T = k/2$

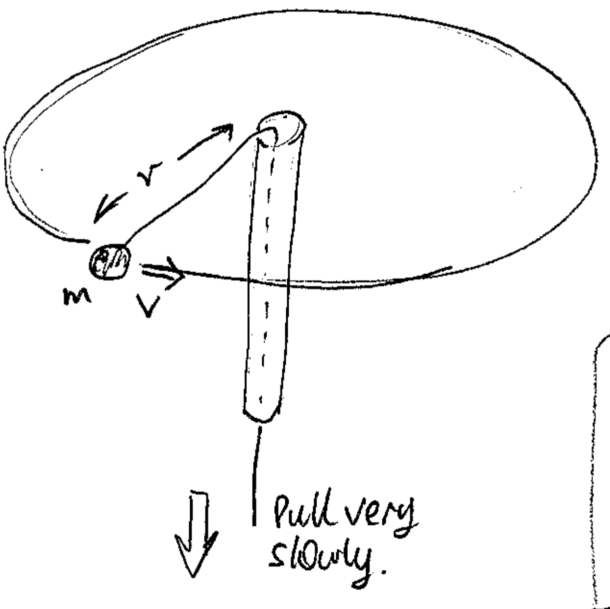
$$\oint T\,dt = \int_0^{2\pi/\sqrt{k}} \frac{k}{2}\cos^2\sqrt{k}\,t\,dt = \frac{k}{2}\left[\frac{2\sqrt{k}\,t + \sin(2\sqrt{k}\,t)}{4\sqrt{k}}\right]_0^{2\pi/\sqrt{k}}$$

$$= \frac{k}{2}\left[\frac{t}{2}\right]_0^{2\pi/\sqrt{k}} = \frac{k}{2}\cdot\frac{1}{2}\cdot\frac{2\pi}{\sqrt{k}} = \frac{\sqrt{k}\,\pi}{2} = \frac{1}{2}\,\frac{k/2}{\sqrt{k}/2\pi} = \frac{1}{2}\cdot\frac{E}{\nu}$$

Mean kinetic energy $\overline{T} = \frac{1}{2\pi/\sqrt{k}}\oint T\,dt = \frac{\sqrt{k}}{2}.\pi.\frac{\sqrt{k}}{2\pi} = \frac{k}{4} = \frac{1}{2}E$

$$\text{Action} = \oint p.dq = \oint \dot{q}\,dq = \oint \dot{q}^2\,dt = 2\oint T\,dt = \sqrt{k}\,\pi = E/\nu$$

Adiabatic Invariant for simple rotor



Rate change energy of mass = Work done against centrifugal force

$$\boxed{\frac{d}{dr}\left[\tfrac{1}{2}mv^2(r)\right] = -m\frac{v^2(r)}{r}}$$

since energy increases with decreasing r

solve

$$-m\frac{v^2}{r} = \frac{d}{dr}\left(\tfrac{1}{2}mv^2\right) = v\frac{d}{dr}[mv]$$

$$-mv = r\frac{d}{dr}[mv]$$

$$-mv\frac{dr}{dr}$$

$$r\frac{d}{dr}[mv] + mv\frac{dr}{dr} = 0$$

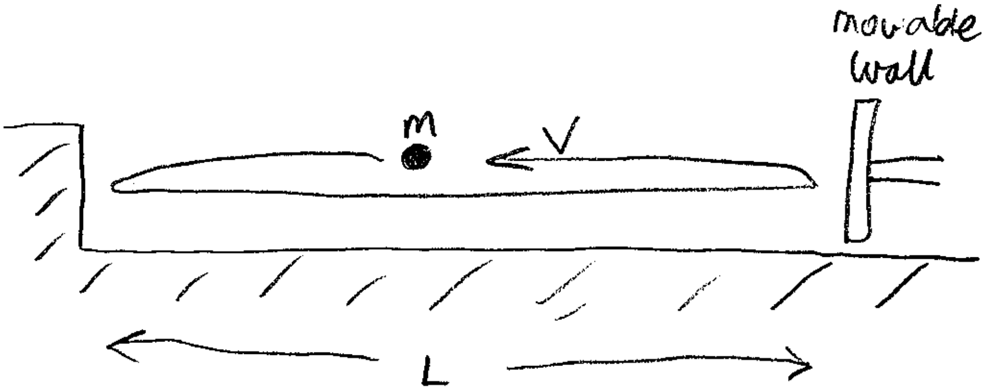
$$\frac{d\,mvr}{dr} = 0$$

Angular momentum
mvr
is an adiabatic invariant
... as expected.

Ehrenfest's

$$2\frac{\bar{I}}{\nu} = 2 \cdot \frac{\tfrac{1}{2}mv^2}{v/2\pi r} = \frac{mrv}{2\pi}$$

Adiabatic invariant.
1-D bouncing ball "Particle in a box"



Force on wall $=$ $\underbrace{\text{change momentum each collision}}_{2mV}$ $\times$ $\underbrace{\text{rate of collisions}}_{V/2L}$ $= mV^2/L$

$\underbrace{\text{Rate change of energy as wall moves slows}}$ $=$ Force on wall

$$\frac{d}{dL}\left(\tfrac{1}{2}mV^2\right) = -\frac{mV^2}{L}$$

$$\rightarrow \frac{d}{dL}\left(\tfrac{1}{2}mv^2\right) + \frac{mV^2}{L} = 0$$

$$V\frac{d}{dL}(mV) + V\frac{mV}{L} = 0$$

$$L\frac{d}{dL}(mV) + mV = 0$$

$$\frac{d}{dL}(mVL) = 0$$

Adiabatic invariant: $mVL = \dfrac{2\left(\frac{1}{2}mv^2\right)}{V/2L} = \dfrac{\text{Kinetic Energy} \times 2}{\text{Frequency}}$

Allowed motions for particle in a box

$$nh = \frac{2\ \text{kinetic energy}}{\text{frequency}} = \frac{2\left(\frac{1}{2}mv^2\right)}{V/2L} \qquad n = 0, 1, \cdots$$

$$\therefore \frac{nh}{2L} = \frac{mv^2}{V} = mV \qquad \frac{n^2h^2}{4L^2} = m^2v^2 = 2m \times \underbrace{\left(\tfrac{1}{2}mv^2\right)}_{\text{Energy } E_n}$$

$$\boxed{E_n = \frac{n^2h^2}{8mL^2}}$$

← same as eigenstates of corresponding Schrödinger equation for particle in a box

$$V_n = \frac{nh}{2m} \cdot \frac{1}{L}$$

adiabatic compression