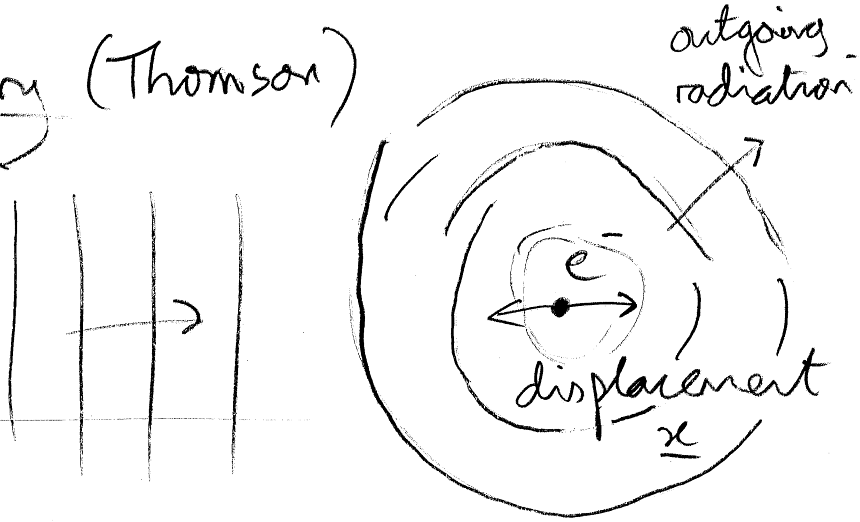
①

Arthur Compton 'A Quantum Theory of the Scattering of X-rays by Light Elements'

The Classical Theory (Thomson)

Incoming monochromatic plane wave



$$\underline{E} = \underline{\varepsilon}_i E_0 \cos(\underline{k}\cdot\underline{r} - \omega t)$$

polarisation ($\underline{\varepsilon}_i$) angular frequency ($\omega$)

Lorentz force $\quad \underline{F} = m_e\underline{\ddot{x}} = q\left[\underline{E} + \frac{1}{c}\,\underline{\dot{x}} \times \underline{B}\right]$

Assume $|\dot{x}|$ small, $|E|$, $|B|$ small

$$\Rightarrow \quad m_e\underline{\ddot{x}} = -q\underline{E}.$$

This describes a forced harmonic oscillator, angular frequency ω.

Only want time dependence (forget polarisation for now)

$$\underline{\ddot{x}}(t) = \frac{qE_0}{m_e}\sin(\omega t)$$

$$\Rightarrow \quad x(t) = \frac{qE_0}{m_e\omega^2}\sin(\omega t)$$

What does the emitted radiation look like?

(far away from source) Dipole approx. $\quad \lim_{r\to\infty} E_s(r) = \frac{-q^2E_0}{mc^2}\left[\underline{r}\times(\underline{r}\times\underline{\varepsilon}_i)\right]\frac{e^{i\frac{\omega}{c}r}}{r}$

Time averaged flux (energy per unit time per unit area) $\langle S \rangle = \frac{c}{8\pi} |E|^2$

for plane wave $= \lim_{r \to \infty} \frac{c}{8\pi} \left| E_s(r,\theta,\phi) \cdot \epsilon_f \right|^2 r^2 d\Omega$

Cross section is power per solid angle (divide by incident radiation) $\frac{d\sigma}{d\Omega} = \frac{\lim_{r\to\infty} \left| E_s(r,\theta,\phi) \cdot \epsilon_f \right|^2 r^2}{E_0^2}$ (solid angle)

use dipole approx. $\Rightarrow \frac{d\sigma}{d\Omega} = \underbrace{\frac{q^4}{m^2 c^4}}_{r_0^2} \left| \epsilon_i \cdot \epsilon_f \right|^2$

* Independent of wavelength

Classical electron radius $r_0 = \frac{e^2}{M_e C^2}$

Sum over polarisations, assume unpolarised

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{r_0}{2} \left[1 + \cos^2 \theta \right]$$

angle from incident

The Problem

'hard' X-rays

Scattering of X-rays with high energy off light elements (e.g. Carbon) shows
i) concentration in direction of incident rays
ii) loss of power overall.
(and now)
iii) reduction in wavelength off axis
experiments performed by Compton on graphite.

(2) What to do?

Compton (~1918): Large electrons would result in interference.

But Effect depends on wavelength (Compton (1922)) $\Rightarrow$ would entail a different r_0 for wavelength.

Compton (1923): Scattering of individual photons from electrons $\Rightarrow$ reduced wavelength.

Ingredients

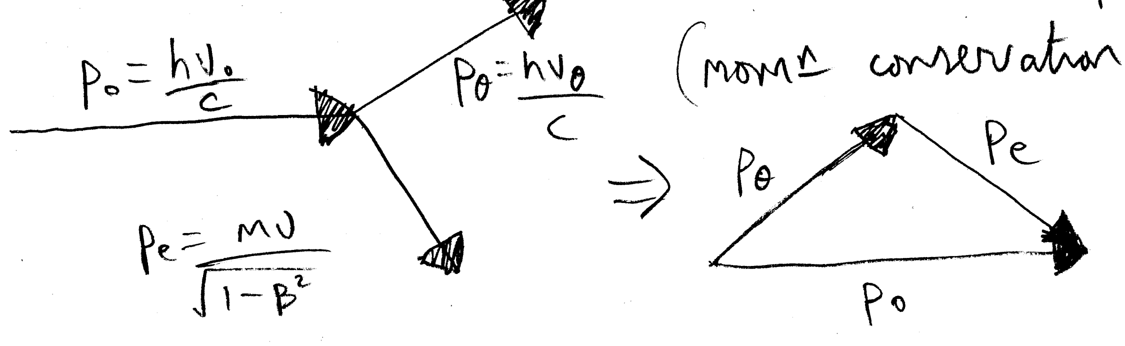
a) X-ray quanta (photons) have definite momentum vector $\frac{h\nu_0}{c}$ (and no mass).

b) Scattering involves a collision between one photon and one electron (or no interaction).

c) Energy lost in collision results in reduced wavelength off axis.

Collisions

$$\boxed{\begin{array}{l}\beta = \frac{v}{c} \\ v = \beta c\end{array}}$$



Mom$^{\underline{n}}$ conservation + cosine law

$$① \Rightarrow \left(\frac{M_e \beta c}{\sqrt{1-\beta^2}}\right) = \left(\frac{h\nu_0}{c}\right)^2 + \left(\frac{h\nu_\theta}{c}\right)^2 + 2\left(\frac{h\nu_0}{c}\right)\cdot\left(\frac{h\nu_\theta}{c}\right)\cos\theta$$

Energy conservation

$$② \Rightarrow \quad h\nu_\theta = h\nu_0 - mc^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)$$

$$① + ② \Rightarrow \quad \nu_\theta = \frac{\nu_0}{(1 + 2\alpha \sin^2 \frac{1}{2}\theta)}, \quad \alpha = \frac{h\nu_0}{mc^2} = \frac{h}{mc\lambda_0}$$

From ②

$$\beta = 2\alpha \sin\tfrac{1}{2}\theta \, \frac{\sqrt{1 + (2\alpha + \alpha^2)\sin^2\frac{1}{2}\theta}}{1 + 2(\alpha + \alpha^2)\sin^2\frac{1}{2}\theta}$$

So we know (i) frequency of deflected rays
(ii) velocity of deflected electron

__Now__ "it is of interest to notice" that the expression for ν_θ is "as if" the electron were moving in the direction of the incident beam with velocity $\bar{\beta} = \dfrac{\alpha}{1+\alpha}$

This is (of course) just the center of mass frame velocity!

$$\frac{V_{cm}}{c} = \frac{p_{TOT}\, c^2}{E_{TOT}\, c} = \frac{h\nu_0}{h\nu_0 + M_e c^2} = \frac{\left(\frac{h\nu_0}{M_e c^2}\right)}{\left(\frac{h\nu_0}{M_e c^2}\right) + 1}$$

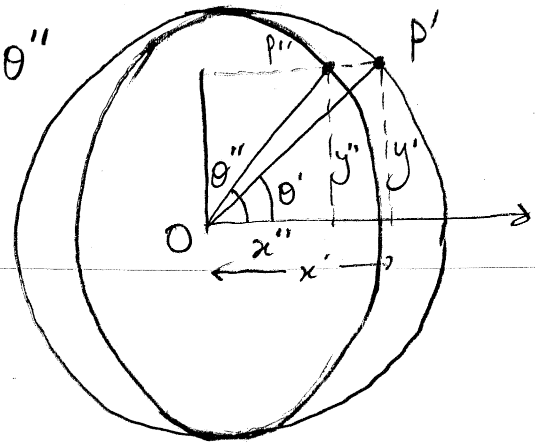
So (apparently without realising it) Compton will proceed to calculate in c. of m. frame.

(3) Relativistic Emission (Isotropic Source)

In the frame of the emitter, the radiation is isotropic. In the lab frame it is not due to length contraction. (NB we know from β velocity of electron may be $0.8c$)

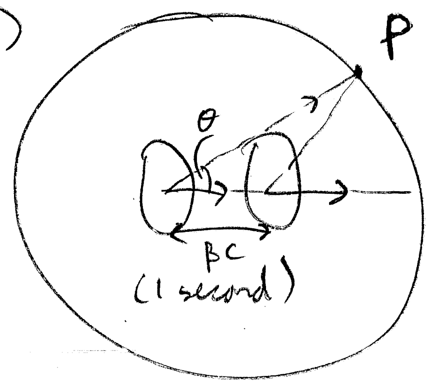
$\overrightarrow{v}$

$$y'' = y' = \sin\theta' = \sin\theta''$$
$$x' = \cos\theta'$$
$$x'' = \cos\theta''$$
$$\frac{x'}{x''} = \frac{1}{\sqrt{1-\beta^2}}$$



$$\tan\theta' = \frac{\sin\theta'}{\cos\theta'} = \frac{y'}{x'} = \left(\frac{y''}{x''}\right)\sqrt{1-\beta^2} = \tan\theta''\sqrt{1-\beta^2}$$

In the lab frame, the spheroid moves with velocity v so that contributions to a point P on an enveloping sphere were in fact emitted earlier. The correct transformation takes this into account.



$$\Rightarrow \quad \sin\theta' = \sin\theta \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}$$

$$\Rightarrow \quad d\theta' = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\, d\theta$$

(Here comes the Correspondence Principle (?))

Probability of emission in direction θ P_θ is isotropic in moving frame.

$$P_\theta\, d\theta = P_{\theta'}\, d\theta' = \frac{1}{2}\sin\theta'\, d\theta' = \frac{1-\beta^2}{(1-\beta\cos\theta)^2}\cdot\frac{1}{2}\sin\theta\, d\theta$$

$\left(\begin{array}{l}\text{After some}\\ \text{work}\end{array}\right)$ $$\frac{I_\theta}{I'} = \left(\frac{\nu_0}{\nu'}\right)^4$$ Agrees with Stefan-Boltzman Law $I \propto T^4$.

Now apply to scattering process (not isotropic)

Classical result: $I \propto (1 + \cos^2\theta')$ (moving frame)

Correspondence Principle (?): probability proportional to I

$$P_{\theta'}\, d\theta' = k(1+\cos^2\theta')\sin\theta'\, d\theta'$$

$\left[\int_0^\pi P_{\theta'}\, d\theta' = 1, \; = k\int_0^\pi (1+\cos^2\theta')\sin\theta' d\theta' \Rightarrow k = 3/8\right]$

Now transform to lab frame $\Rightarrow \boxed{P_\theta\, d\theta}$.

Energy scattered per second $n h\nu_0 P_\theta\, d\theta$.

$$\Rightarrow I_\theta = \frac{n h\nu_0 P_\theta d\theta}{2\pi R^2 \sin\theta d\theta}$$ surface area element

In forward direction there is no scattering.

③ $$I_0 = \frac{3}{8\pi}\frac{n h\nu_0}{R^2}(1+2\alpha)$$

But since no scattering, classical result applies

④ $$I_0 = I\left(\frac{Ne^4}{R^2 m^2 c^4}\right)$$ $N r_0^2$

Set ③ = ④

$$\Rightarrow n = \frac{8\pi}{3}\frac{INe^4}{h\nu_0 m^2 c^4(1+2\alpha)}$$

Energy removed per second $= \underline{n h\nu_0}$ (number of scattered photons)

Absoption coeff. $$\sigma = \frac{n h\nu_0}{I} = \frac{8\pi}{3}\frac{Ne^4}{m^2c^4}\cdot\frac{1}{1+2\alpha} = \frac{\overset{\text{classical}}{\sigma_0}}{1\cdot 2\alpha}$$

(4)

Integrate intensity over surface

$$\mathcal{E}_s = \int_0^\pi I_\theta \cdot 2\pi R^2 \sin\theta \, d\theta$$

$$= \frac{8\pi}{3} \frac{INe^4}{m^2c^4} \frac{1+\alpha}{(1+2\alpha)^2}$$

True scattering coeff.

$$\sigma_s = \sigma_0 \frac{1+\alpha}{(1+2\alpha)^2}$$

Coeff. of true absoption due to scattering

$$\sigma_a = \sigma - \sigma_s = \sigma_0 \left(\frac{1+2\alpha}{(1+2\alpha)^2} - \frac{1+\alpha}{(1+2\alpha)^2} \right)$$

$$\boxed{\sigma_a = \sigma_0 \frac{\alpha}{(1+2\alpha)^2}}$$

Experimental Results

[A]

$$\nu_\theta = \frac{\nu_0}{1+2\alpha \sin^2\frac{1}{2}\theta} \qquad \alpha = \frac{h\nu_0}{mc^2} = \frac{h}{mc\lambda_0}$$

$$\Rightarrow \lambda_\theta = \lambda_0 + \left(\frac{2h}{mc}\right) \sin^2\frac{1}{2}\theta$$

For known values

$$\lambda_\theta = \lambda_0 + 0.0484 \sin^2\frac{1}{2}\theta$$

Let $\theta = 90$. $\lambda_\theta - \lambda_0 = 0.024\,\text{Å}$

Experimental Result

At $\theta = 90$, $\lambda_\theta - \lambda_0 = 0.022\,\text{Å}$

$\boxed{B}$ Intensity nothing like classical.

$$\frac{I_\theta}{I_0} = \frac{1}{2}\left(1 + \cos^2\theta\right)$$

Fitted impressively well by "quantum" result

$$\frac{I_\theta}{I_0} = \frac{1}{2}\,\frac{1 + \cos^2\theta + 2\alpha(1+\alpha)(1-\cos\theta)^2}{\{1 + \alpha(1-\cos\theta)\}^5}$$

$\boxed{\text{Conclusion}}$

- Scattering is a collision process.
- Light contains quanta with directed momentum.
- Quantum of light (photon) gives all energy to a single electron
- (Only applies to light elements — free electrons)

Q: What does this have to do with the 'old quantum theory'?

– Anything required other than Einstein (1905)?