Particle Route to QED

- We will treat radiation as an assembly of particles obeying Bose-Einstein statistics.

Ingredients

(i) Number-Phase representation (new)

The number operator N_r says how many systems are in the r-th state.

(ii) Interaction Picture (new)

This combines the Schrödinger and Heisenberg pictures: all the time dependence goes into the perturbation due to interaction.

(iii) Perturbation Theory (not new)

Describes the change in a system due to a (weak) external influence

[From Dirac (192 'On the Theory...']

Schrödinger Eqn

$$(H - W + \overset{\text{perturbation}}{A})\psi = 0$$

Soln. $\psi = \sum_n a_n \psi_n$

$$\Rightarrow \sum_n (H - W + A) a_n \psi_n = 0$$

since $W a_n - a_n W = i\hbar \dot{a}_n$ so $a_n W = -i\hbar \dot{a}_n - W a_n$

we have $0 = \sum_n a_n (H - W + A)\psi_n - i\hbar \sum_n \dot{a}_n \psi_n$.

But ψ_n is a soln. $(H - W)\psi_n = 0$. Let $A\psi_n = \sum_m A_{mn} \psi_m$

then $0 = \sum_{mn} a_n A_{mn} \psi_m - i\hbar \sum_n \dot{a}_n \psi_n \Rightarrow \boxed{i\hbar \dot{a}_m = \sum_n a_n A_{mn}}$

§2

$$i\hbar \frac{\partial \psi}{\partial t} = (H_0 + V)\psi$$

Solution: $\psi = \sum_r a_r \psi_r$. $|a_r|^2$ is probability of system being in state r.

If $\sum_r |a_r|^2 = 1$ applies to a single system.

Normalize so that $\sum_r |a_r|^2 = N$ then we have N systems, and $|a_r|^2$ is the (likely) <u>number of systems</u> in state r.
(this idea appears in the previous paper)

We have $\quad i\hbar \dot{a}_r = \sum_s V_{rs} a_s \qquad (4)$

and $\quad -i\hbar \dot{a}_r^* = \sum_s a_s^* V_{sr} \qquad (4')$

At this stage a_r, a_r^* are complex coefficients in the linear expansion of ψ.

Now, says Dirac, we regard these as <u>canonical conjugates</u> (which we would regard as operators).

Define $F_1 = \sum_{rs} a_r^* V_{rs} a_s$ then

$$\frac{da_r}{dt} = \frac{1}{i\hbar}\frac{\partial F_1}{\partial a_r^*}, \quad i\hbar\frac{da_r^*}{dt} = -\frac{\partial F_1}{\partial a_r}.$$

These are <u>Hamilton's equations</u> for a_r, a_r^*.

Now, if a_r <u>was</u> a complex number it could be written as $\quad a_r = A e^{i\phi}$. $\qquad |a_r|^2 = N_r$

amplitude ↗ ↖ complex phase $\qquad \Rightarrow A = \sqrt{N_r}$

Writing $a_r = \sqrt{N_r}\, e^{-i\phi_r/\hbar}$; $a_r^* = \sqrt{N_r}\, e^{i\phi_r/\hbar}$,

we have
$$F_1 = \sum_{rs} a_r^* V_{rs} a_s$$
$$= \sum_{rs} \sqrt{N_r}\, e^{i\phi_r/\hbar} V_{rs} \sqrt{N_s}\, e^{-i\phi_s/\hbar}$$
$$= \sum_{rs} V_{rs} \underbrace{\sqrt{N_r}\sqrt{N_s}}_{\substack{\text{commuting}\\ \text{so c-numbers here}}} e^{i(\phi_r - \phi_s)/\hbar}$$

and
$$\dot{N}_r = -\frac{\partial F_1}{\partial \phi_r}, \quad \dot{\phi}_r = \frac{\partial F_1}{\partial N_1}$$

Hamilton's equations again i.e. a canonical transformation.
Now we include <u>time evolution</u> of the a_r's.

$$b_r = a_r e^{-iW_r t/\hbar}, \quad b_r^* = a_r^* e^{iW_r t/\hbar}.$$

$$i\hbar \frac{\partial}{\partial t}\left(a_r e^{-iW_r t/\hbar}\right) = a_r W_r e^{-iW_r t/\hbar} + \underbrace{e^{-iW_r t/\hbar} \frac{\partial a_r}{\partial t}}_{\text{(using (4))}}$$
$$i\hbar \dot{b}_r = W_r b_r + \sum_s V_{rs} b_s e^{i(W_s - W_r)t/\hbar}$$

Let $V_{rs} = v_{rs} e^{i(W_r - W_s)t/\hbar}$

then
$$i\hbar \dot{b}_r = W_r b_r + \sum_s v_{rs} b_s$$
$$= \sum_s H_{rs} b_s \qquad (5)$$

$$H_{rs} = W_r \delta_{rs} + v_{rs}$$

Now we have
$$F = \sum_{rs} b_r^* H_{rs} b_s \qquad (7)$$

We change variables to N_r, θ_r as before $(\phi_r \neq \theta_r)$

$$b_r = \sqrt{N_r}\, e^{-i\theta_r/\hbar} \,; \quad b_r^* = \sqrt{N_r}\, e^{i\theta_r/\hbar} \,,$$

writing $\quad F = \sum_{rs} H_{rs} \sqrt{N_r}\sqrt{N_s}\, e^{i(\theta_r - \theta_s)/\hbar}$

we have again

$$\dot{N}_r = -\frac{\partial F}{\partial \theta_r}, \quad \dot{\theta}_r = \frac{\partial F}{\partial N_r}.$$

$$F = \sum_r \underbrace{W_r N_r}_{b_r b_r^* = \sqrt{N_r}\sqrt{N_r}\, e^{i(\theta_r - \theta_r)/\hbar} = N_r} + \sum_{rs} v_{rs} \underbrace{\sqrt{N_r}\sqrt{N_s}\, e^{i(\theta_r - \theta_s)/\hbar}}_{b_r b_s^*} \qquad (9)$$

time dependence

Energy = energy of unperturbed system + energy due to perturbation

§3

So far we have considered N_r, θ_r and b, b^* to be c-numbers, but the form of the dynamical equations suggests that (like action-angle variables) they can be promoted to q-numbers (quantized).

(Since b, b^* already obey Schrödinger equations, this became known as <u>second</u> quantisation)

This allows us to obtain "the probability of any given distribution of the systems among the various states" since now <u>the number of systems in a given state</u> is the subject of the quantization scheme.

We have
$$b_r b_s^\dagger - b_s^\dagger b_r = \delta_{rs} \begin{cases} =1 & \text{when } r=s \\ =0 & \text{otherwise} \end{cases}$$

We will stipulate that N_r, θ_r are canonically conjugate

$$[\theta_r, N_s] = i\hbar\,\delta_{rs}$$

NB Turns out not to be strictly true!

In the number representation we replace θ_r by a differential operator (like $p \to -i\hbar \frac{\partial}{\partial x}$)

$$\theta_r \longrightarrow i\hbar \frac{\partial}{\partial N_r}$$

This is why it doesn't matter – we just work in number rep.

We use this to derive Dirac's (10)

$$b_r^\dagger = \sqrt{N_r}\, e^{i\theta_r/\hbar} \longrightarrow \sqrt{N_r}\, e^{\partial/\partial N_r}$$

Baker - Campbell $\quad e^A B e^{-A} = B + [A,B] + \frac{1}{2}[A,[A,B]] + \ldots$

$$e^{\partial/\partial N_r} f(N_r) e^{-\partial/\partial N_r} = f(N_r) + \underbrace{\left[\frac{\partial}{\partial N_r}, f(N_r)\right]}_{= \frac{df}{dN_r}} + \frac{1}{2}\underbrace{\left[\frac{\partial}{\partial N_r}, \left[\frac{\partial}{\partial N_r}, f(N_r)\right]\right]}_{= \frac{d^2}{dN_r^2}}$$

$$\left[\frac{\partial}{\partial N_r}, f(N_r)\right] g(N_r) = \left\{\frac{\partial}{\partial N_r} f(N_r) - f(N_r)\frac{\partial}{\partial N_r}\right\} g(N_r)$$

$$= \underbrace{\frac{\partial}{\partial N_r}(fg)}_{= \cancel{f\frac{\partial g}{\partial N_r}} + g\frac{\partial f}{\partial N_r}} - \cancel{f\frac{\partial g}{\partial N_r}} = g(N_r)\frac{\partial}{\partial N_r} f(N_r)$$

Taylor expansion of $f(x)$ about a

$$f(x) = f(a) + (x-a)\cdot f'(a) + \frac{1}{2}(x-a)^2 \cdot f''(a)$$

From above

$$e^{\partial/\partial N_r} f(N_r) e^{-\partial/\partial N_r} = f(N_r) + 1\cdot f'(N_r) + \frac{1}{2}\cdot 1^2 \cdot f''(N_r) + \ldots = f(N_r+1)$$

So we have

$$e^{\partial/\partial N_r}\sqrt{N_r}\,e^{-\partial/\partial N_r} = \sqrt{N_r+1}$$

$$\Rightarrow \quad e^{\partial/\partial N_r}\sqrt{N_r} = \sqrt{N_r+1}\,e^{\partial/\partial N_r}.$$

$$\left.\begin{aligned} b_r &= \sqrt{N_r+1}\,e^{-i\theta_r/\hbar} = e^{-i\theta_r/\hbar}\sqrt{N_r} \\ b_r^{\dagger} &= \sqrt{N_r}\,e^{i\theta_r/\hbar} = e^{i\theta_r/\hbar}\sqrt{N_r+1} \end{aligned}\right\} \quad (10)$$

The Hamiltonian becomes

$$F = \sum_{rs} H_{rs}\, b_r^{\dagger} b_s = \sum_{rs} H_{rs}\sqrt{N_r}\,e^{i\theta_r/\hbar}\sqrt{N_s+1}\,e^{-i\theta_s/\hbar}$$

$$\left(\begin{aligned} &\text{when } r=s \quad e^{i\theta_r/\hbar}\sqrt{N_s+1} = \sqrt{N_r}\,e^{i\theta_r/\hbar} \\ &\text{when } r\neq s \quad e^{i\theta_r/\hbar}\sqrt{N_s+1} = \sqrt{N_s+1}\,e^{i\theta_r/\hbar} \end{aligned}\right)$$

$$F = \sum_{rs} H_{rs}\sqrt{N_r}\sqrt{N_s+1-\delta_{rs}}\,e^{-i(\theta_r-\theta_s)/\hbar} \quad (11$$

$$= \sum_r W_r N_r + \sum_{rs} v_{rs}\sqrt{N_r}\sqrt{N_s+1-\delta_{rs}}\,e^{-i(\theta_r-\theta_s)/\hbar}$$

proper energy   interaction   time dependence

Now we label the no. of particles in the r th state with N_r and consider a many particle wave equation

$$i\hbar\frac{\partial}{\partial t}\psi(N_1', N_2', N_r', \ldots) = F\psi(N_1', N_2', N_r' \ldots)$$

one variable per state (not per particle)

The operators $e^{\pm i\theta_r/\hbar}$ work to <u>raise</u> or <u>lower</u> the number of particles in the r-th state.

When we apply the operator $e^{i(\theta_r - \theta_s)/\hbar}$

this lowers N_r by one and raises N_s by one.

$\Rightarrow$ acts to conserve occupation number
(a system leaves the state r and obtains state s)

Now define a Hamiltonian $H_A = \sum_n H(n)$ $\leftarrow H(n) = H_0 + V$

Many copies of the same system.

$$i\hbar \dot{b}(r_1, r_2 \ldots) = \sum_{s_1, s_2 \ldots} H_A(r_1, r_2 \ldots; s_1 s_2 \ldots) b(s_1, s_2 \ldots) \quad (14)$$

One particle dynamics, so stipulate:

$$H_A(r_1, r_2 \ldots; s_1 s_2 \ldots) = \begin{cases} 0 & \text{if } \exists i,j \text{ such that } r_i \neq s_i,\ r_j \neq s_j \\ H_{r_m s_m} & \text{if } r_j = s_j \text{ and } \forall k \neq j,\ r_k = s_k. \\ \sum_n H_{r_n r_n} & \text{if } \forall i,\ r_i = s_i. \end{cases}$$

This either leaves the state unchanged, or induces a transition in a single particle.

This simplifies (14), and now we require that $b(r_1, r_2, \ldots)$ is symmetric (for Bose-Einstein statistics).

The $b(r_1, r_2, \ldots)$ are normalized, and they must remain so in the occupation number representation

$$\sum_{r_1 r_2, \ldots} |b(r_1, r_2, \ldots)|^2 = \sum_{N_1, N_2 \ldots} |b(N_1, N_2, \ldots)|^2.$$

$$\Rightarrow \quad b(N_1, N_2, \ldots) = \sqrt{\frac{N!}{N_1! N_2! \ldots}}\ b(r_1, r_2, \ldots)$$

number of ways to have same occupation no.