# Non-Existence of Limit Set

Supplement to Badly Behaved Curves

John D. Norton

Consider the sets of real numbers for all $n\in N$

*C*1 = (1/2, 1)
*C*2 = (1/4, 1/2) ∪ (3/4, 1)
*C*3 = (1/8, 1/4) ∪ (3/8, 1/2) ∪ (5/8, 3/4) ∪ (7/8, 1)

…

$C\_{n}= \bigcup\_{m=1}^{2^{n-1}}\left(\frac{2m-1}{2^{n}},\frac{2m}{2^{n}}\right)$ (1)

$$…$$

Here (*a*,*b*) = {*x*: *a* < *x* < *b*}, so that the interval is open. That is, it is the set of real numbers between *a* and *b*, *excluding* *a* and *b*. The result shown here is that there is no well-defined limit set formed by taking the limit at *n*🡪∞ of the sets *Cn*. The members of the limit set are defined by the condition

*x* ∈ *Clim* iff there is an *N* such that *x* ∈ *Cn* for all *n* > *N*. (2)

The non-existence of the set follows from two results:

(a) If *x* = *r*/2*N* for some *N* and natural number *r* ≤ *N*, then *x* ∉ *Cn* for all *n* > *N*. It follows from (1) that *x* ∉ *Clim*.

(b) If *x* ≠ *r*/2*N* for some *N* and natural number *r* ≤ *N*, then *x* never satisfies condition (2). Rather for every *x* and *n* such that *x* ∈ *Cn*, there is an *n’* > *n*, such that *x* ∉ *Cn’*; and for every *x* and *n* such that *x* ∉ *Cn*, there is an *n’* > *n*, such that *x* ∈ *Cn’*.

The result (a) is compatible with the limit set existing but being empty. Result (b) is more troublesome since (2) does not enable to say whether the values of *x* to which it applies are in the set or not. Hence the set is not empty, but not well defined.

## Proof of (a)

 To see (a), consider a real *x* such that *x* = *r*/2*N* for some natural number *N* and natural number *r* ≤ *N*. Then there is an *m* in the formula (1) such that *x* = *r*/2*N* = (2*m*+1)/2*N* or *x* = *r*/2*N* = 2*m*/2*N*. That is, *x* is one of the extremal reals in the specification of the open sets of (1). Since the intervals in (1) are open, it follows that *x* ∉ *CN*.

 It now also follows that *x* ∉ *CN*+1. For *x* will now be one of the extremal reals in the specification of the open sets of *CN*+1. For *x* = 2*r*/2*N*+1 = (4*m*+2)/2*N*+1 or *x* = 2*r*/2*N*+1 = 4*m*/2*N*+1. Iterating, it follows that *x* ∉ *CN*+2, *x* ∉ *CN*+3, … and so on for all *Cn* with *n* > *N.*

## Proof of (b)

 If *x* ≠ *r*/2*N* for some *N* and natural number *r* ≤ *N*, then, for any *n*, *x* is not one of the extremal reals used to specify the open sets in *Cn*. To proceed, pick any *n* > 1. (The choice will not affect the result.) There must exist some value of *m* in (1) such that

either (i) $x\in \left(\frac{2m-1}{2^{n}},\frac{2m}{2^{n}}\right)$ or (ii) $x\in \left(\frac{2m}{2^{n}},\frac{2m+1}{2^{n}}\right)$.

In case (i), we have that *x* ∈ *Cn*. The quick way to see this is to note that the open sets included in *Cn* have the form $\left(\frac{odd number}{2^{n}},\frac{even number}{2^{n}}\right)$. The sets excluded from *Cn* have the form $\left(\frac{even number}{2^{n}},\frac{odd number}{2^{n}}\right)$. Since *x* ≠ (4*m*-1)/2*n*+1, we must have that

either (i.a) $x\in \left(\frac{4m-2}{2^{n+1}},\frac{4m-1}{2^{n+1}}\right)$ or (i.b) $x\in \left(\frac{4m-1}{2^{n+1}},\frac{4m}{2^{n+1}}\right)$.

In case (i.a), we have that *x* ∉ *Cn*+1 since (i.a) has the form $\left(\frac{even number}{2^{n+1}},\frac{odd number}{2^{n+1}}\right)$. If, however, we have case (i.b), then *x* ∈ *Cn+1*, since *x* lies in an interval of the form $\left(\frac{odd number}{2^{n}},\frac{even number}{2^{n}}\right).$ In this case (i.b), we repeat the analysis and check whether *x* ∈ *Cn+*2; and so on for *Cn+*3 etc. Eventually we must find a *CN* with *N* > *n* such that *x* ∉ *CN*. For otherwise, *x* can be brought arbitrarily close to a real number of the form (even number / 2N) for some *N*. This can only be the case if *x* has the form (even number / 2N) for some *N*. However, by supposition of case (i), *x* does not have this form. Hence in either case (i.a) or (i.b), we eventually find a value of *N* > *n*, such that *x* ∉ *CN*. That is, if *x* ∈ *Cn*, there exists *N* > *n*, such that *x* ∉ *CN*.

 Case (ii) above is the case of *x* a member of an open set of the form $\left(\frac{even number}{2^{n}},\frac{odd number}{2^{n}}\right)$, so that *x* ∉ *Cn*. By reasoning analogous to that of case (i), we find that there exists *N* > *n*, such that *x* ∈ *CN*.

## The case of *x* = 1/3

 This is a simple case of a number for which there is no definite limiting fact over its membership in the limit set. This failure of the limit fact arises because *x* = 1/3 alternatives in its membership of the sets *Cn* indefinitely according to:

1/3 ∉ *C*1, 1/3 ∈ *C*2, 1/3 ∉ *C*3, 1/3 ∈ *C*4, …

That is, we have 1/3 ∉ *Cn*, when *n* is odd; and 1/3 ∈ *Cn*, when *n* is even.

## Approximations for 1/3

 To arrive at these results, we need some approximation formulae for 1/3. We have that

1/3 = 1/4 + 1/16 + 1/64 + … + 1/22*n* + …

We can split this series into two terms,

1/3 = *lower sum* + *error*

where

*lower sum* = 1/4 + 1/16 + 1/64 + … + 1/22*n* = $\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right)$

*error* = 1/22*n*+1 + 1/22*n*+2 + … = $\frac{1}{3}\left(\frac{1}{2^{2n}}\right)$

For the first approximation, we have that 0 < *error* = $\frac{1}{3}\left(\frac{1}{2^{2n}}\right)$ < $\left(\frac{1}{2^{2n}}\right)$. It follows that

$\frac{1}{3}\in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right),\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right)+\frac{1}{2^{2n}}\right)$ (3)

A tighter approximation arises from 0 < *error* = $\frac{1}{3}\left(\frac{1}{2^{2n}}\right)$ < $\frac{1}{2}\left(\frac{1}{2^{2n}}\right)$. It follows that

$\frac{1}{3}\in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right),\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right)+\frac{1}{2}\frac{1}{2^{2n}}\right)$ (4)

## 1/3 ∈ *Cn*, when *n* is even

 This result follows from approximation (3). To use it, we need to show that

(22*n* – 1)/3 is an odd number

To see this, sum the series

1 + 4 + 42 + … + 4*n*-1 = $\frac{2^{2n}-1}{2^{2}-1}=\frac{2^{2n}-1}{3}$

The sum on the left is a sum of *n*-2 even numbers and 1. Hence it is odd, as must be the term of interest on the right. Applying this to approximation (3), we have that

$$\frac{1}{3}\in \left(\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right),\frac{1}{3}\left(\frac{2^{2n}-1}{2^{2n}}\right)+\frac{1}{2^{2n}}\right)= \left(\frac{odd number}{2^{2n}},\frac{next even number}{2^{2n}}\right)$$

Hence it follows that 1/3 ∈ *C*2*n*, or that 1/3 ∈ *Cn*, when *n* is even.

## 1/3 ∉ *Cn*, when *n* is odd

 To see this, we use the approximation (4). If the fractions delimiting the open set are multiplied by 2/2, we recover:

$$\frac{1}{3}\in \left(\frac{2}{3}\left(\frac{2^{2n}-1}{2^{2n+1}}\right),\frac{2}{3}\left(\frac{2^{2n}-1}{2^{2n+1}}\right)+\frac{1}{2^{2n}}\right)$$

We know from earlier that (1/3)(22*n* – 1) is an odd number. Hence (2/3)(22*n* – 1) is an even number. Thus the approximation becomes

$$\frac{1}{3}\in \left(\frac{even number}{2^{2n+1}},\frac{next odd number}{2^{2n+1}}\right)$$

These open intervals are not subsets of *C*2n+1. It follows that 1/3 ∉ *C*2*n*+1, or that 1/3 ∈ *Cn*, when *n* is odd.