# The Birthday/Lottery Ticket Problem

John D. Norton

There are *N* days in the year or *N* lottery ticket numbers available. We choose *n* days or *n* lottery ticket numbers, independently of each other, and with equal probability for each. What is the relationship between *n*, *N* and *p*, the probability that there are no duplications in the days or lottery tickets chosen?

## The Exact Calculation

The probability that there are no duplications is given by

(1)

## Approximation with Stirling’s Formula

We have from Stirling’s formula that large factorials are well approximated as:

Substituting into the expression for *p* we have

The first term above simplifies to

The second term simplifies to

The second and third terms together are

Combining we have

(2)

## Check formula

For *n* = 23 and *N* = 365, an exact calculation from (1) gives *p* = 0.492703. The formula (2) gives us *p* = 0.492710. The approximation is good to four significant figures.

## Simplification for small n/N

Collecting terms up to second order in *n*/*N*, we have

Approximating is more complicated. Write

Then we have

Recovering the expression from ln *Y*, we find

We now have

where the last approximation drops the third powers of *n*/*N*. Combining we find

(3)

## Approximation for *n*/*N* given *p* (for small *n/N*)

Inverting the approximation (3), we recover an expression for *n*/*N*. Raising (3) to the 1/*N* power, we have

Solving for *n*/*N*, we have

(4)

 Using (4) to compute *n*/*N* requires two steps, since the right-hand side of the equation also contains *n*/*N*. As long as the case is one of a small *n*/*N*, its value can be approximated by first computing *n*/*N* by assuming that *n*/*N* is zero in (4). That is, first compute

(5)

Then substitute the value recovered in (5) into (4) and use (4) to make the corresponding small adjustment to the value of *n*/*N*.

## Even Simpler Approximation for *n*/*N*

 If we approximate

we can recover a still simpler approximation for *n*/*N*. The approximation depends on taking a power series expansion in x for

where we set *x* = 1/*N*. We need the first derivative of *f*(*x*):

We form the power series expansion about *x* = 0, which is equivalent to *N* = ∞.

since *p*0 and *f*(0) = 0. Recalling that *x* = 1/*N*, we recover an approximation to first order in 1/*N*:

Substituting this approximation into (5), we recover[[1]](#footnote-1)

(6)

This last formula (6) gives a rough picture of how *n*/*N* grows with increasing *N*, when *p* has a fixed value:

Thus, after *N* is large, as N increases by a factor of 10 through 1000, 10,000, 100,000 etc., *n*/*N* *decreased* by a factor = 3.16 and *n* itself *increases* by a factor = 3.16. It follows that *n* can grow arbitrarily large with increasing *N*, but *n*/*N* will decreased arbitrarily close to 0.

1. Square root of a negative number? No. Since *p* < 1, log *p* < 0, so - log *p* > 0. [↑](#footnote-ref-1)